Written Assignment 5 – Official Solution

1. A retirement account is opened with an initial deposit of $8,500 and earns 8.12% interest compounded monthly. What will the account be worth in 20 years? What if the deposit was calculated using simple interest? Could you see the situation in a graph? From what point one is better than the other?

SOLUTION

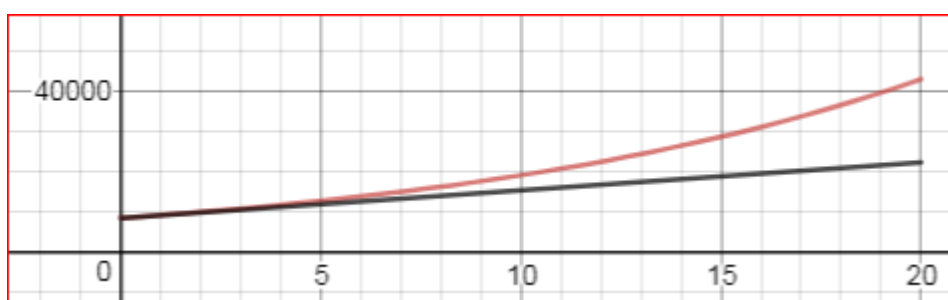We have

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where
- $A(t)$ is the account value =?,
- t is measured in years = 20 years,
- P is the starting amount of the account, often called the principal, or more generally present value = $8,500,
- r is the annual percentage rate (APR) expressed as a decimal = 0.0812, and
- n is the number of compounding periods in one year = 12.

Therefore

$$A(20) = 8,500\left(1 + \frac{0.0812}{12}\right)^{12\times20} = 8,500(1 + 0.00677)^{240} = 42,888.18.$$

If the deposit were compounded monthly with simple interest, we would have:

$$A(t) = P\left(1 + \frac{r}{n}nt\right) = P(1 + rt) = 8,500(1 + 0.0812\times20) = 8,500\times2.624 = 22,304.00.$$

Compound interest makes the principal grow much faster than simple interest as we can see by means of the graphs of both functions. To see the point from where this happens we have

$$8,500(1.00677)^{12x} = 8,500(1 + 0.0812x) \quad \rightarrow \quad (1.00677)^{12x} = 1 + 0.0812x$$

which is complicated, if not impossible, to solve. Using, for example, Excel we find

1	9.216,84	9.190,20
2	9.994,14	9.880,40
3	10.836,99	10.570,60
4	11.750,92	11.260,80
5	12.741,92	11.951,00
6	13.816,50	12.641,20
7	14.981,71	13.331,40
8	16.245,18	14.021,60
9	17.615,20	14.711,80
10	19.100,77	15.402,00
11	20.711,62	16.092,20
12	22.458,32	16.782,40
13	24.352,32	17.472,60
14	26.406,06	18.162,80
15	28.633,00	18.853,00
16	31.047,74	19.543,20
17	33.666,13	20.233,40
18	36.505,34	20.923,60
19	39.583,99	21.613,80
20	42.922,27	22.304,00

which shows that in some moment of the first year compounded interest starts to be more than simple interest.

2. Graph the function $f(x) = 5(0.5)^{-x}$ and its reflection about the line $y = x$ on the same axes, and give the x–intercept of the reflection. Prove that $a^x = e^{x\, lna}$. [Suggestion: type $y = 5(0.5)^{-x}$ $\{-7 < x < 2\}$ $\{0 < y < 7\}$ in desmos, and then type its inverse function.]
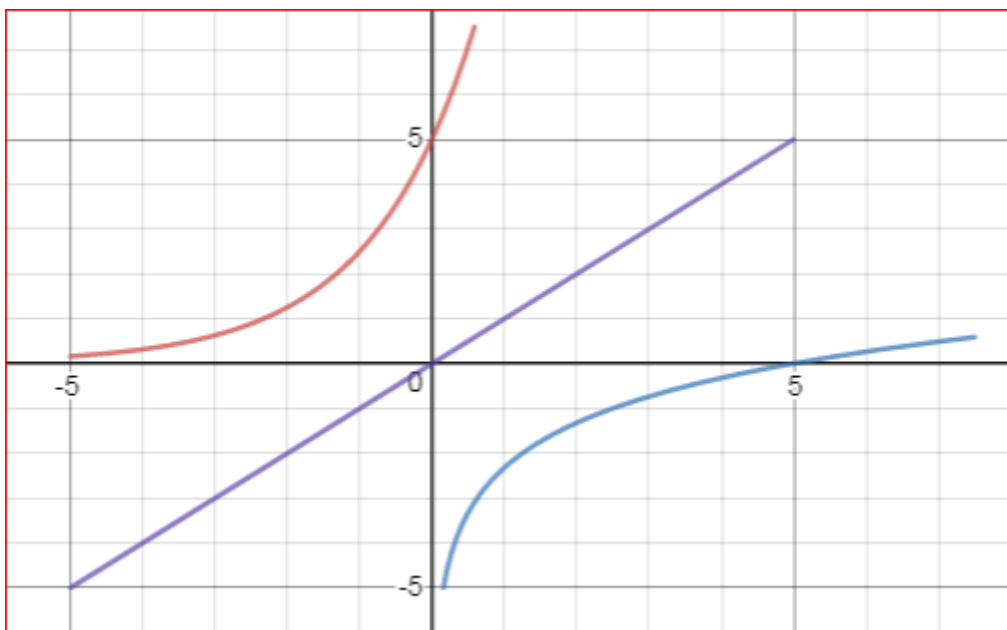
SOLUTION

We have,

$$a^x = e^{x\, lna} \quad \leftrightarrow \quad ln\, a^x = x\, lna$$

which is true by a property of logarithms. So, the first equality is also true. Therefore:

$$y = 5 \times 0.5^{-x} = 5e^{-x\, ln0.5} \quad \rightarrow \quad \frac{y}{5} = e^{-x\, ln0.5} \quad \rightarrow \quad ln\left(\frac{y}{5}\right) = -x\, ln0.5 \quad \rightarrow$$

$$\rightarrow \quad -\frac{ln\left(\frac{y}{5}\right)}{ln0.5} = x \quad \rightarrow \quad Y = -\frac{ln\left(\frac{X}{5}\right)}{ln0.5}$$

The x–intercept of $f^{-1}(x)$ is (5, 0).

3. Go to Example 13 on page 627. Do "Try It #13": How long will it take before twenty percent of our 1,000-gram sample of uranium-235 has decayed? Show all the steps carefully.

The decay equation is $A(t) = A_0\, e^{Kt}$, where t is the time for the decay, and K is the characteristic of the material. Suppose T is the time it takes for half of the unstable material in a sample of a radioactive substance to decay, called its half-life. Prove that $K = \frac{\ln 0.5}{T}$. What is K (approximately) for the uranium-235? Show the steps of your reasoning.

SOLUTION

T for the uranium-235 is 703,800,000 years.

$$800 = 1000e^{\frac{\ln 0.5}{703,800,000}t} \quad\leftrightarrow\quad 0.8 = e^{\frac{\ln 0.5}{703,800,000}t} \quad\leftrightarrow\quad \ln 0.8 = \frac{\ln 0.5}{703,800,000}t$$

$$\leftrightarrow\ 703,800,000\times\frac{\ln 0.8}{\ln 0.5} = t \ \leftrightarrow\ 703,800,000\times0.321928 = t \ \leftrightarrow\ 226,572,993 = t$$

About the constant K for the material:

$$A(t) = A_0\, e^{Kt} \quad\rightarrow\quad \frac{A_0}{2} = A_0\, e^{KT} \quad\leftrightarrow\quad \frac{1}{2} = e^{KT} \quad\leftrightarrow\quad \ln 0.5 = KT \quad\leftrightarrow\quad \frac{\ln 0.5}{T} = K$$

K for the uranium-235:

$$K = \frac{\ln 0.5}{T} = \frac{\ln 0.5}{703,800,000} \cong \frac{1}{100,000,000}\times\frac{\ln 0.5}{7} =$$

$$= \frac{1}{100,000,000}\times(-0{,}099021026) \cong \frac{1}{100,000,000}\times(-0.1) = -10^9.$$