Written Assignment 6

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MATH 1211: Calculus

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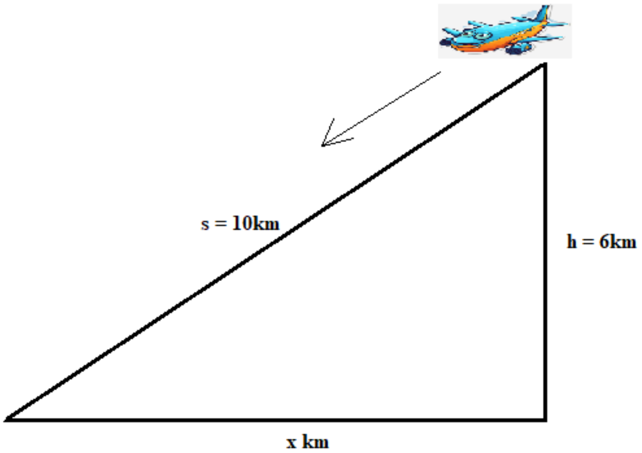
1. An airplane is flying towards a radar station at a constant height of 6 km above the ground. If the distance s between the airplane and the radar station is decreasing at a rate of 400 km per hour when $s = 10$ km, what is the horizontal speed of the plane? Make sure your answer includes units.



By Pythagoras theorem, $[s(t)]^2 = 6^2 + [x(t)]^2$ …………………..(i)

Performing derivative with respect to time on both sides

$\frac{d}{dt}[s(t)]^2 = \frac{d}{dt}6^2 + \frac{d}{dt}[x(t)]^2$

$2s\frac{ds}{dt} = 0 + 2x\frac{dx}{dt}$

$s\frac{ds}{dt} = x\frac{dx}{dt}$ ………………………...(ii)

$x = \sqrt{s^2 - h^2}$

$x = \sqrt{10^2 - 6^2}$

$x = \sqrt{64} = 8km$

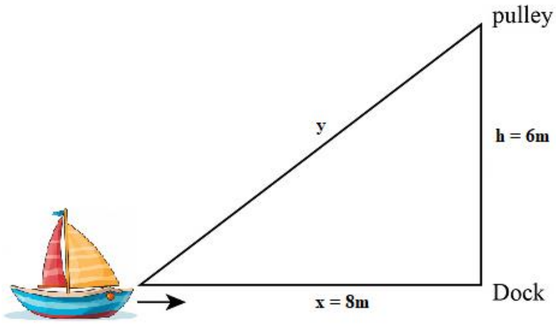
Substituting $s = 10km$, $\frac{ds}{dt} = -400\ km/h$, and $x = 8km$ into equation (ii)

$(10km)(-400km/h) = (8km)\frac{dx}{dt}$

$\frac{dx}{dt} = \frac{(\cancel{10km})(-400km/h)}{\cancel{8km}} = -500km/h$

Since speed is a scalar quantity, we take the absolute value of $-500km/h$. Therefore, the horizontal speed of the plane = $|-500km/h| = \mathbf{500km/h}$

2. A boat is being pulled into a dock by a rope attached to it and passing through a pulley on the deck, positioned 6 meters higher than the boat. If the rope is being pulled in at a rate of 3 meters/sec, how fast is the boat approaching the dock when it is 8 meters from the dock? Make sure your answer includes units.



By Pythagoras theorem, $[y(t)]^2 = 6^2 + [x(t)]^2$ …………………..(i)

Performing derivative with respect to time on both sides

$\frac{d}{dt}[y(t)]^2 = \frac{d}{dt}6^2 + \frac{d}{dt}[x(t)]^2$

$2y\frac{ds}{dt} = 0 + 2x\frac{dx}{dt}$

$y\frac{dy}{dt} = x\frac{dx}{dt}$ …………………………...(ii)

$y = \sqrt{x^2 + h^2}$

$y = \sqrt{8^2 + 6^2}$

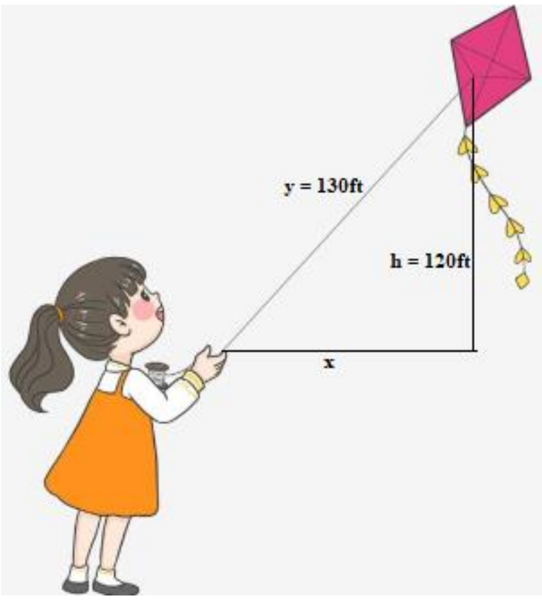
$y = \sqrt{100} = 10m$

Substituting $y = 10m, \frac{dy}{dt} = 3m/s$, and $x = 8m$ into equation (ii)

$(10m)(3m/s) = (8m)\frac{dx}{dt}$

$\frac{dx}{dt} = \frac{(\cancel{10m})(3m/s)}{\cancel{8m}} = \mathbf{3.75m/s}$

3. A girl is flying a kite on a string. The kite is 120 ft. above the ground and the wind is blowing the kite horizontally away from her at 6 ft/sec. At what rate must she let out the string when 130 ft of string has been released? Make sure your answer includes units.



By Pythagoras theorem, $[y(t)]^2 = 120^2 + [x(t)]^2$ …………………..(i)

Performing derivative with respect to time on both sides

$\frac{d}{dt}[y(t)]^2 = \frac{d}{dt}120^2 + \frac{d}{dt}[x(t)]^2$

$2y\frac{ds}{dt} = 0 + 2x\frac{dx}{dt}$

$y\frac{dy}{dt} = x\frac{dx}{dt}$ ………………………...(ii)

$x = \sqrt{y^2 - h^2}$

$x = \sqrt{130^2 - 120^2}$

$x = \sqrt{2500} = 50ft$

Substituting $x = 50ft$, $\frac{dx}{dt} = 6ft/s$, and $y = 130ft$ into equation (ii)

$(130ft)\frac{dy}{dt} = (50ft)(6ft/s)$

$\frac{dx}{dt} = \frac{(50\cancel{ft})(6ft/s)}{130\cancel{ft}} = \boldsymbol{2.31ft/s}$

4. Find the Linearization of $\boldsymbol{f(x) = sinx}$ at $\boldsymbol{a = \frac{\pi}{2}}$. Provide your answer as $\boldsymbol{L(x)} =$?

$L(x) = f(a) + f'(a)(x - a)$ …………………..(i)

$f(a) = \sin\left(\frac{\pi}{2}\right) = 1$

$f'(a) = cos\left(\frac{\pi}{2}\right) = 0$

Substituting a, $f(a)$ and $f'(a)$ into equation (i),

$$\boldsymbol{L(x) = 1 + 0\left(x - \frac{\pi}{2}\right) = 1}$$

5. Use Linear Approximation to estimate $\boldsymbol{e^{(-0.01)}}$. Provide your answer in 2 decimal places. Do not use a calculator. Show work for credit.

Given $a = 0$ and $f(x) = e^x$,

$L(x) = f(a) + f'(a)(x - a)$ …………………..(i)

$f(a) = e^0 = 1$

$f'(a) = e^0 = 1$

Substituting a, $f(a)$ and $f'(a)$ into equation (i),

$L(x) = 1 + 1(x - 0)$

$L(x) = 1 + x$

$$\boldsymbol{L(-0.01) = 1 + (-0.01) = 0.99}$$

6. Calculate the locations of maximums and minimums of the following functions: Show work in details.

- $f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3$

Taking $3x^2 - 3 = 0$, the critical point becomes $x = 1$ and $x = -1$.

Critical point (x)	$f(x)$	Conclusion
1	$f(1) = (1)^3 - 3(1) + 2 = 0$	Local minimum
-1	$f(-1) = (-1)^3 - 3(-1) + 2 = 4$	Local maximum

- $f(x) = x^4 - 8x^2 + 3$

$f'(x) = 4x^3 - 16x$

Taking $4x^3 - 16x \ = 0$, the critical point becomes $x = 0$, $x = 2$ and $x = -2$.

Critical point (x)	$f(x)$	Conclusion
-2	$f(-2) = (-2)^4 - 8(-2)^2 + 3 = -13$	Global minima
0	$f(0) = (0)^4 - 8(0)^2 + 3 = 3$	Local maximum
2	$f(2) = (2)^4 - 8(2)^2 + 3 = -13$	Global minima

7. Find the exact x-value where the function $\boldsymbol{f(x) = x + ln(x^2 - 1)}$ attains a maximum value. An estimated answer or a calculator answer will not earn any credit.

$$f'(x) = 1 + \frac{2x}{x^2 - 1}$$

$$= \frac{x^2+2x-1}{x^2-1} \text{ for } x \neq 1$$

$f(x)$ attains a maximum value at $f'(x) = 0$

$$\frac{x^2 + 2x - 1}{x^2 - 1} = 0$$

$$x^2 + 2x - 1 = 0$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{8}}{2}$$

$$= \frac{-\cancel{2} \pm \cancel{2}\sqrt{2}}{\cancel{2}}$$

$$= -1 \pm \sqrt{2}$$

Therefore, $f(x) = x + ln(x^2 - 1)$ will attain a maximum value at:

$$\boldsymbol{x = -1 + \sqrt{2}} \text{ and } \boldsymbol{x = -1 - \sqrt{2}}$$

8. Using the Mean Value Theorem and Rolle's Theorem, show that $\boldsymbol{x^3 + x - 1 = 0}$ has exactly one real root.

$$f'(x) = 3x^2 + 1$$

Setting $f'(x) = 0$,

$$3x^2 + 1 = 0$$

$$x^2 = -\frac{1}{3}$$

$$x = \sqrt{-\frac{1}{3}}$$

There is no solution in real numbers, hence the function neither has local maxim nor local minima. Now let us test with $x = 1$ and $x = -1$ to solve for $f(x)$ since we cannot work with $x = 0$ as the critical value.

$$f(1) = (1)^3 + (1) - 1 = 1$$

$$f(-1) = (-1)^3 + (-1) - 1 = -3$$

Based on the values of $f(1)$ and $f(-1)$ above, there must be a point c, $-1 < c < 1$ where $f(c) = 0$ since it has been proven that $f(-1) < 0$ and $f(1) > 0$. Thus, the function is continuous with only one real root.

9. If $\boldsymbol{f(1) = 10}$ and $\boldsymbol{f'(x) \geq 2}$ for $\boldsymbol{1 \leq x \leq 4}$, how small can $\boldsymbol{f(4)}$ possibly be?

Applying the mean value theorem,

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Taking $a = 1$ and $b = 4$, and $f'(c) \geq 2$,

$$\frac{f(4) - f(1)}{4 - 1} \geq 2$$

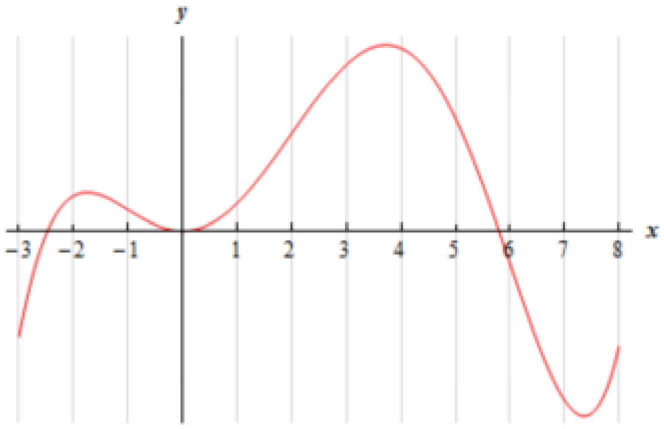
$$f(4) - f(1) \geq 6$$

But $f(1) = 10$,

$$f(4) - 10 \geq 6$$

$$f(4) \geq 16$$

10. The graph of a function is given below. Determine the intervals on which the function is concave up and concave down.



Concave down: (-3, -1) and (2, 6).

Concave up: (-1, 2) and (6, 8).