Written Assignment 4

Godfrey Ouma

University of the People

MATH 1211: Calculus

Rakesh Das

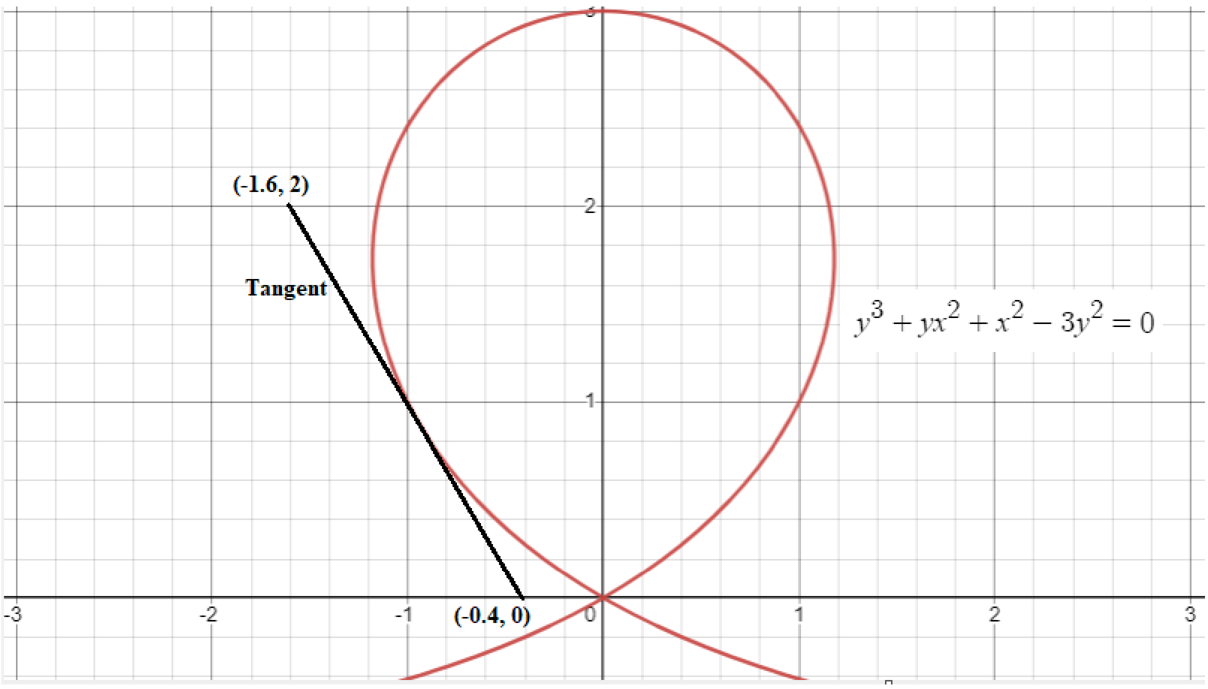
December 13, 2023

Written Assignment 4

1. Chains Inc. is in the business of making and selling chains. Let $C(t)$ be the number of miles of chain produced after t hours of production. Let $p(c)$ be the profit as a function of the number of miles of chain produced and let $q(t)$ be the profit as a function of the number of hours of production. Suppose the company can produce 3 miles of chain per hour and suppose their profit on the chains is $4000 per mile of chain. Find and interpret (use complete sentences) each of the following (include units), $c'(t)$, $p'(c)$, and $q'(t)$. How does $q'(t)$ relates to $p'(c)$ and $c'(t)$?

- $c'(t)$ shows the rate of change of the number of miles of chain produced with respect to time (t). In the context above, $c'(t) = 3miles/hour$.
- $p'(c)$ shows the rate of change of profit with respect to the number of miles of chain produced (c). In the context above, $p'(c) = \$4000/mile$.
- $q'(t)$ shows the rate of change of profit with respect to time (t). In the context above, $q'(t) = p'(c) \times c'(t) = \frac{\$4000}{\cancel{mile}} \times \frac{3\ \cancel{miles}}{hour} = \$12{,}000/\text{hour}$.

2. Use Desmos to graph the function $y^3 + yx^2 + x^2 - 3y^2$ and estimate the slope of the tangent line at (-1,1). Then find $\frac{dy}{dx}$ using implicit differentiation and plug in $x = -1$ and $y = 1$. Compare and discuss the estimated slope with the slope you found analytically.



<u>Estimated slope:</u>

From the graph above, slope of the tangent can be estimated by finding its gradient using the two points (-16,2) and (-0.4,0) it passes through.

$$Estimated\ slope = \frac{\Delta y}{\Delta x} = \frac{2-0}{-1.6-(-0.4)} = -\frac{2}{1.2} = -\frac{5}{3} \approx -1.67$$

Actual slope:

The actual slope of the tangent can be found by determining $\frac{dy}{dx}$ using implicit differentiation and plug in $x = -1$ and $y = 1$.

$$\frac{d}{dx}(y^3 + yx^2 + x^2 - 3y^2) = 0$$

$$\frac{d}{dx}y^3 + \frac{d}{dx}yx^2 + \frac{d}{dx}x^2 - \frac{d}{dx}3y^2 = 0$$

$$3y^2\frac{dy}{dx} + \left(x^2.\frac{dy}{dx} + 2xy\right) + 2x - 6y\frac{dy}{dx} = 0$$

$$3y^2\frac{dy}{dx} + x^2.\frac{dy}{dx} - 6y\frac{dy}{dx} = -2x - 2xy$$

$$(3y^2 + x^2 - 6y)\frac{dy}{dx} = -2x - 2xy$$

$$\frac{dy}{dx} = \frac{-2x - 2xy}{3y^2 + x^2 - 6y}$$

$$slope = f'(-1,1) = \frac{-2(-1) - 2(-1)(1)}{3(1)^2 + (-1)^2 - 6(1)}$$

$$= -\mathbf{\frac{4}{2}} = \mathbf{-2}$$

Comparing estimated and actual slope:

It is evident from the calculations above that the estimate slope is almost equal to the actual slope. The reason for the deviation is that obtaining estimated slope involve drawing a tangent to the curve at (-1,1), which may be subject to error in drawing leading to inaccurate slope.

3. Let $f(x) = (3x^2 + 1)^2$. Find $f'(x)$ in 3 different ways by following the instructions below in parts a, b and c:

a) **Develop the identity $f(x)$ then take the derivative.**

$$f(x) = (3x^2 + 1)^2 = 9x^4 + 6x^2 + 1$$

$$\boldsymbol{f'(x) = 36x^3 + 12x}$$

b) **View $f(x)$ as $(3x^2 + 1)(3x^2 + 1)$ and use the product rule to find $f'(x)$.**

Using product rule, let $h(x) = 3x^2 + 1$ and $g(x) = 3x^2 + 1$

$$f'(x) = h(x)g'(x) + g(x)h'(x)$$

$$= (3x^2+1)\frac{d}{dx}(3x^2+1) + (3x^2+1)\frac{d}{dx}(3x^2+1)$$

$$= 6x(3x^2+1) + 6x(3x^2+1)$$

$$= 12x(3x^2+1)$$

$$= \mathbf{36x^3 + 12x}$$

c) Apply the chain rule directly to the expression $\boldsymbol{f(x) = (3x^2+1)^2}$.

Chain Rule Using Leibniz's Notation

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

Let $u = 3x^2 + 1$, therefore:

$$f(x) = u^2$$

$$\frac{d}{du}u^2 = 2u$$

$$\frac{d}{dx}(3x^2+1) = 6x$$

$$\frac{dy}{dx} = (2u).(6x)$$

$$= 2(3x^2+1).(6x)$$

$$= \mathbf{36x^3 + 12x}$$

d) Are your answers in parts a, b, c the same? Why or why not?

The answers in the three parts are equal, probably because the three formulae used, power rule, product rule, and Chain rule using Leibniz's notation have been proven to be true when performing differentiation of any function.

4. Find $\boldsymbol{\frac{dy}{dx}}$ for the equation $\boldsymbol{3y^2 - cosy = x^3}$.

Applying implicit differentiation:

$$\frac{d}{dx}3y^2 - \frac{d}{dx}cosy = \frac{d}{dx}x^3$$

$$6y\frac{dy}{dx} + siny\frac{dy}{dx} = 3x^2$$

$$(6y + siny)\frac{dy}{dx} = 3x^2$$

$$\boldsymbol{\frac{dy}{dx} = \frac{3x^2}{6y + siny}}$$

5. Find the equation of the tangent line that passes through point (1,2) to the graph $\boldsymbol{8y^3 + x^2y - x = 65}$.

Applying implicit differentiation:

$$\frac{d}{dx}8y^3 + \frac{d}{dx}x^2y - \frac{d}{dx}x = \frac{d}{dx}65$$

$$24y^2\frac{dy}{dx} + \left(2xy + x^2\frac{dy}{dx}\right) - 1 = 0$$

$$24y^2\frac{dy}{dx} + x^2\frac{dy}{dx} = 1 - 2xy$$

$$(24y^2 + x^2)\frac{dy}{dx} = 1 - 2xy$$

$$\frac{dy}{dx} = \frac{-2xy + 1}{24y^2 + x^2}$$

$$Slope\ of\ the\ tangent = f'(1,2) = \frac{-2(1)(2) + 1}{24(2)^2 + (1)^2} = -\frac{3}{97}$$

Equation of the tangent line:

$$y - 2 = -\frac{3}{97}(x - 1)$$

$$\boldsymbol{y = -\frac{3}{97}x + \frac{197}{97}}$$

6. Find $\boldsymbol{\frac{dy}{dx}}$ for the equation $\boldsymbol{xy^2 - x^2y = 2}$.

Applying implicit differentiation:

$$\frac{d}{dx}xy^2 - \frac{d}{dx}x^2y = \frac{d}{dx}2$$

$$\left(y^2.1 + 2xy\frac{dy}{dx}\right) - \left(2xy + x^2\frac{dy}{dx}\right) = 0$$

$$2xy\frac{dy}{dx} - x^2\frac{dy}{dx} = 2xy - y^2$$

$$(2xy - x^2)\frac{dy}{dx} = 2xy - y^2$$

$$\boldsymbol{\frac{dy}{dx} = \frac{2xy - y^2}{x(2y - x)}}$$

7. Find $f'(x)$ for the function $f(x) = \sqrt[4]{-3x^4 - 2}$.

Chain Rule Using Leibniz's Notation

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

Let $u = -3x^4 - 2$, therefore:

$$f(x) = u^{\frac{1}{4}}$$

$$\frac{d}{du}u^{\frac{1}{4}} = \frac{1}{4}u^{-\frac{3}{4}}$$

$$\frac{d}{dx}(-3x^4 - 2) = -12x^3$$

$$\frac{dy}{dx} = \left(\frac{1}{4}u^{-\frac{3}{4}}\right).(-12x^3)$$

$$= -\frac{12x^3}{4u^{\frac{3}{4}}} = -\frac{12x^3}{4(-3x^4 - 2)^{\frac{3}{4}}}$$

$$\boldsymbol{= -\frac{3x^3}{(-3x^4 - 2)^{\frac{3}{4}}}}$$

8. Find $f'(x)$ for the function $f(x) = (5x^2 + 3)^4$.
Chain Rule Using Leibniz's Notation

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

Let $u = 5x^2 + 3$, therefore:

$$f(x) = u^4$$

$$\frac{d}{du}u^4 = 4u^3$$

$$\frac{d}{dx}(5x^2+3) = 10x$$

$$\frac{dy}{dx} = (4u^3).(10x)$$

$$= 4(5x^2+3)^3.(10x)$$

$$= \mathbf{40}\boldsymbol{x}(\mathbf{5}\boldsymbol{x}^{\mathbf{2}}+\mathbf{3})^{\mathbf{3}}$$

9. Find $\boldsymbol{f'(x)}$ for the function $\boldsymbol{f(x) = sin^2(\cos(4x))}$
Chain Rule Using Leibniz's Notation

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

Let $u = \cos(4x)$, therefore:

$$f(x) = (sinu)^2$$

$$\frac{d}{du}(sinu)^2 = 2\sin(u).\cos(u)$$

Applying double angle identity formula, $2\sin(u).\cos(u) = \sin(2u)$

$$\frac{d}{dx}(\cos(4x)) = -4(\sin(4x))$$

$$\frac{dy}{dx} = \sin(2u).-4(\sin(4x))$$

$$\boldsymbol{-4sin2(\cos(4x)).\sin(4x)}$$

10. Find $\boldsymbol{f'(x)}$ for the following functions:

a) $\boldsymbol{f(x) = \cos(\ln(4x^3))}$

Chain Rule Using Leibniz's Notation

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

Let $u = \ln(4x^3)$, therefore:

$$f(x) = \cos(u)$$

$$\frac{d}{du}\cos(u) = -\sin(u)$$

$$\frac{d}{dx}\ln(4x^3) = \frac{1}{4x^3}.12x^2$$

$$= \frac{\cancel{4x^2}}{\cancel{4x^2}}\left(\frac{3}{x}\right) = \left(\frac{3}{x}\right)$$

$$\frac{dy}{dx} = -\frac{3\sin(u)}{x}$$

$$= -\frac{\mathbf{3\,sin(ln\,(4}\boldsymbol{x}^{\mathbf{3}}\mathbf{))}}{\boldsymbol{x}}$$

b) $\boldsymbol{f(x) = e^{(4x^3+5)^2}}$

Chain Rule Using Leibniz's Notation

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

Let $u = (4x^3 + 5)^2$, therefore:

$$f(x) = e^u$$

$$\frac{d}{du}e^u = e^u$$

$$\frac{d}{dx}(4x^3 + 5)^2 = 24x^2(4x^3 + 5)$$

$$\frac{dy}{dx} = e^u.\left(24x^2(4x^3 + 5)\right)$$

$$= e^{(4x^3+5)^2}.24x^2(4x^3 + 5)$$