Advanced Classical Mechanics (PHYS 615) **Fall 2024**
Extra Credit Project 2 (due on date of Exam I)

The points obtained in this project will be added towards Exam I. To obtain full credit, your computer source code + calculations done on paper **MUST** be provided along with the plots.

At higher speeds, the magnitude of the drag force on an object is found to be proportional to the square of the speed of the object, $F_D \propto v^2$, and is best described by the drag equation (Prandtl expression) of the form

$$\mathbf{F}_D = -\frac{1}{2}c_w \rho A v^2 \hat{\mathbf{v}},$$

where c_w is a dimensionless drag coefficient, ρ is the air density, v is the speed and A is the cross-sectional area of the object (projectile) measured perpendicular to the velocity unit vector, $\hat{\mathbf{v}} = \mathbf{v}/v = (v_x\hat{\mathbf{v}}_\mathbf{x} + v_y\hat{\mathbf{v}}_\mathbf{y})/v$

Derive Equations of Motion **(5 pts)** Show that the equations of motion for a projectile of mass m moving in the x-y plane can be expressed as:

$$\frac{dv_x}{dt} = -bv_x\sqrt{v_x^2 + v_y^2}, \qquad \frac{dv_y}{dt} = -bv_y\sqrt{v_x^2 + v_y^2} - g$$

where $b \equiv c_w \rho A/(2m)$.

<u>Numerical Approach: Euler's Method</u> (Background Information)

These coupled ordinary differential equations are not straightforward to solve analytically. However, if the derivatives are known, given the initial conditions of acceleration, velocity and position, a numerical solution can be obtained via *Euler's Method*.

Euler's Method is a first-order numerical procedure for solving ordinary differential equations (ODEs). The main idea of Euler's method is that every function, if looked close enough, is really a line. We therefore use an initial condition and the known slope (i.e., differential equation) evaluated at the initial condition, and move in very small but finite steps away from the initial condition such as to still be close to the actual function. We then re-evaluate the slope at the new location, move again in a very small step, and repeat the procedure until the curve is fully reconstructed numerically.

Mathematically, let an ODE have the general form

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$

where $f(t, y)$ is a known function and the values in the initial condition are also known. In principle, the function can have more variables, but for simplicity of this argument, lets just use two. Then, the derivative at the initial condition is:

$$\left.\frac{dy}{dt}\right|_{t_0} = f(t_0, y_0)$$

which is sufficient information to write the equation of a tangent line to the solutions at $t = t_0$ by re-arranging the differential equation as follows:

$$y = y_0 + f(t_0, y_0)(t - t_0)$$

where $\Delta t = t - t_0$ is the step-size. If one takes a finite number of n step-sizes, this equation can then be generalized as

$$y_{n+1} = y_n + f(t_n, y_n) \cdot (t_{n+1} - t_n)$$

where for simplicity, we assume that the step-size is constant, $\Delta t = t_{n+1} - t_n$.

(25 pts) Use *Euler's Method* to solve the differential equations derived in the first part numerically.

- step 1: set the initial conditions to get the iteration process going

<u>initial conditions</u> ($t_0 = 0$):
$v_0 = 600\ \mathrm{m/s}$ # initial launch speed
$\theta = 60°$ # initial launch angle

$m = 30$ kg # projectile mass
$\rho = 1.293\ \mathrm{kg/m^3}$ # air density
$c_w = 0.35$ # drag coefficient
$A = 0.00785\ \mathrm{m^2}$ # cross-sectional area

$t = [0,\ 110],\ \Delta t = 0.1\ \mathrm{sec}$ # time range and step-size

-step 2: given the initial conditions (t=0), determine the initial acceleration in the x- and y- components and use that as an initial condition to calculate the position and velocity components in steps,

$$x_{n+1} = x_n + v_{x,n}\Delta t$$

$$v_{n+1} = v_n + a_n\Delta t$$

** note that even though an initial value of acceleration is used to calculate the velocities, since the acceleration itself depends on the v_x, v_y, once the v_{n+1} are calculated, they are used as input to update the acceleration itself.

At the end of the recurrence loop, the following numerical values should have been calculated and stored in a list: $(t, a_x, a_y, v_x, v_y, x, y)$ for the projectile motion with drag force

-step 3: include the general expression for the projectile motion with no drag force for comparison (these are analytical equations, and should be simple to add to your computer program): $a_x = 0,\ a_y = -g,\ x(t),\ y(t),\ y(x)$

-step 4: make overlay plots of the motion i) *with* and ii) *without* drag force for the following:

a) $y(x)$ vs. x (height vs. horizontal distance)
b) $y(t)$ vs. t (height vs. time)
c) $v_x(t)$ vs. t (horizontal velocity vs. time)
d) $v_y(t)$ vs. t (vertical velocity vs. time)

** note: for plots a), b) and c), set the lower limit of the y-axis to 0, to ignore unphysical negative values; for plot d), vertical velocity is expected to be positive for upward motion and negative for downward motion. Add proper labels and a legend to each of the plots.

Write a brief paragraph commenting on your results. Is it what you expected to happen ? Elaborate.