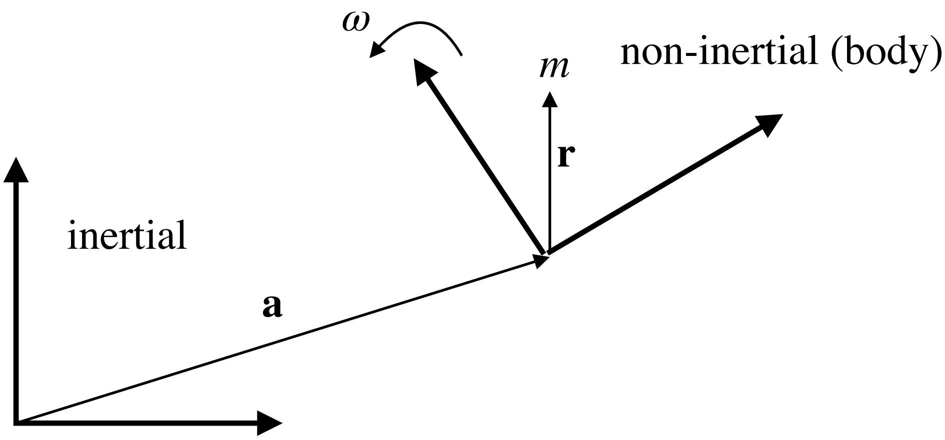
PROBLEM SET 3

Show **ALL WORK** to get full/partial credit. Begin each problem on a new page, and clearly label each part of the problem. (**Max Score:** 100)



$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{\text{body}} = \mathbf{F}^{(e)} - m\left(\frac{d^2\mathbf{a}}{dt^2}\right)_{\text{inertial}} - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{\text{body}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - m\frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}$$

general equation of motion from a non-inertial frame

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{e} = \mathbf{F}^{(e)} - \cancel{m\left(\frac{d^2\mathbf{a}}{dt^2}\right)_{\text{inertial}}} - 2m(\boldsymbol{\omega}_e + \cancel{\boldsymbol{\omega}_{se}}) \times \left(\frac{d\mathbf{r}}{dt}\right)_{e} - m(\boldsymbol{\omega}_e + \cancel{\boldsymbol{\omega}_{se}}) \times [(\boldsymbol{\omega}_e + \cancel{\boldsymbol{\omega}_{se}}) \times \mathbf{r}]$$

Equation of motion as seen by observer on Earth (some terms can be neglected)

gravity — Coriolis term — Centrifugal term

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right)_{e} = \mathbf{F}_g + \mathbf{F}' - 2m\boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt}\right)_{e} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$\boldsymbol{\omega} \approx \boldsymbol{\omega}_e \qquad \mathbf{F}_g = -GM_e m\frac{\mathbf{r}}{r^3}$$

Acceleration of objects as seen by an Earth observer

physical force (i.e. tension)

<u>Earth's constants</u>

$\omega_e \approx 7.29 \times 10^{-5}\ \text{s}^{-1}$

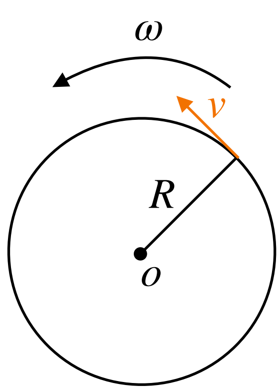
$R_e \approx 6.38 \times 10^{6}\ \text{m}$

$g_o = 9.8\ \text{m/s}^2$

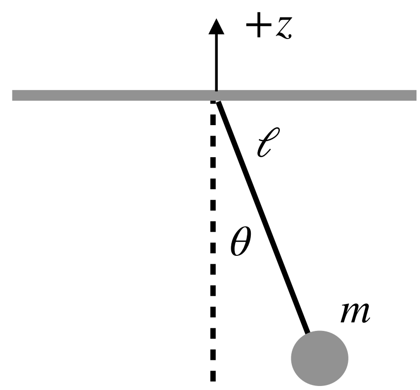
Most of the problems deal with gravity, Coriolis and/or centrifugal forces as seen by an Earth observer, where we place the origin of the non-internal frame at Earth's center

1) **(10 pts)** A pendulum bob of mass m hangs from a string of length ℓ attached to the ceiling of a carousel at a distance R from the center of the carousel. The carousel rotates at a constant angular speed ω which causes the pendulum to be displaced by an angle θ about the pivot, and stay in this equilibrium position while the carousel is rotating.

a) **(2 pts)** On the free-body diagram of the pendulum (right) label all the forces (physical + apparent) acting on the pendulum; label the horizontal and vertical components of the forces



(top view)

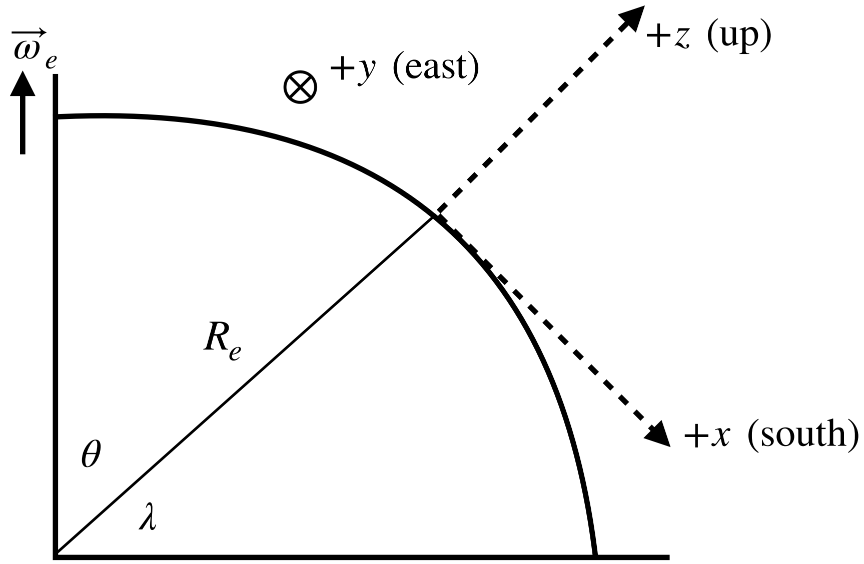


(side view; non-internal frame)

b) **(5 pts)** Apply Newton's 1st Law of motion on this system and show that the tension on the string can be expressed as $F_T = \dfrac{mv^2}{R^2}\ell$

c) **(3 pts)** Show that the equilibrium angle of the pendulum bob can be expressed as $\theta = \sin^{-1}(R/\ell)$

2) **(20 pts)** Earth's gravitational field is slightly offset from the radial direction due to the presence of a centrifugal force term which arises due to Earth's angular velocity $\vec{\omega}_e$.



a) **(10 pts)** Derive a general expression for the magnitude of the effective gravitational field in terms of the latitude angle λ

hint: first calculate $\vec{g}_{\text{eff}} = \vec{g}_o + \vec{g}_{\text{cf}}$ vector in terms of the local coordinates, where $\vec{g}_o$ is the radial gravitational field and $\vec{g}_{\text{cf}}$ is the centrifugal acceleration term; then calculate the magnitude $|\vec{g}_{\text{eff}}|$ which should be a function of λ.

b) **(10 pts)** Using the derived $g_{\text{eff}}(\lambda)$, show that

i) at the poles, $g_{\text{eff}}(\lambda = \pm 90^\circ) = g_o$

ii) at the Equator, $g_{\text{eff}}(\lambda = 0^\circ) = g_o - \omega^2 R$

iii) What is the numerical difference between g_{eff} at the poles and at the equator ?

3) **(20 pts)** Show that the small angular deviation (ϵ) of $\vec{g}_{\text{eff}}$ from $\vec{g}_{\text{o}}$ (the true vertical, i.e. towards the center of the Earth) at a point on Earth's surface at a latitude λ is

$$\epsilon = \frac{R_e \omega_e^2 \sin\lambda \cos\lambda}{g_o - R_e \omega_e^2 \cos^2\lambda}$$

where (R_e, ω_e) are the Earth's radius and angular speed, respectively.

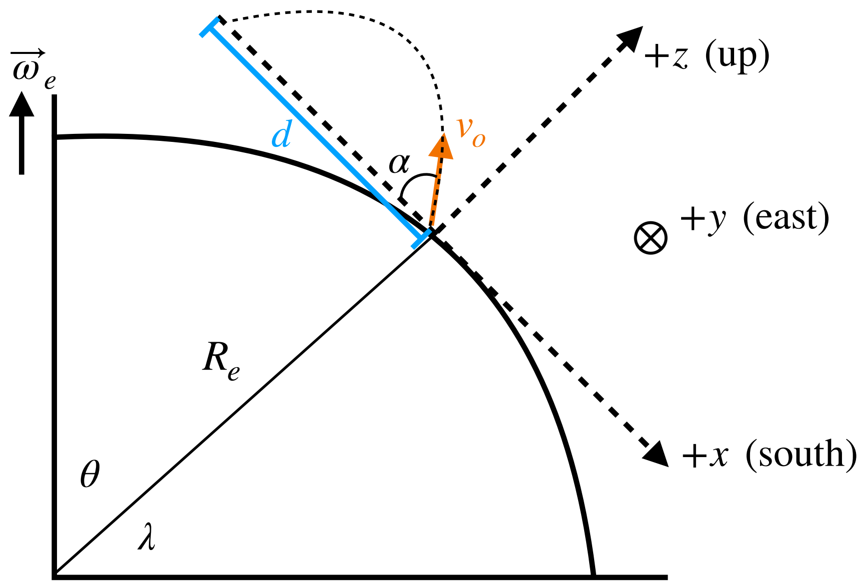
What is the value (in seconds of arc) of the maximum deviation, ϵ_{max}? Note that the entire denominator in the answer is the effective g, and g_o denotes the pure gravitational component.

hint: i) to derive the result, start with the vector components of $\vec{g}_{\text{eff}}$ derived in the previous problem; ii) to find ϵ_{max}, think of what angle λ would maximize ϵ

4) **(20 pts)** Shot towers were popular in the eighteen and nineteenth centuries for dropping melted lead down tall towers to form spheres for bullets. The lead solidified while falling and often landed in water to cool the lead bullets. Many such shot towers were built in New York State. Assume a shot tower was constructed at latitude $\lambda = 42°$ north, and the lead fell a distance of 27 m. In what direction (*north, south, east or west*) and how far did the lead bullets land from the direct vertical?

hint: You may use the local coordinate system set-up in problem 2) as reference to solve this problem. Ignore second order effects due to Earth's angular speed, since $\omega_e^2 \sim 0$ (i.e., Only consider the acceleration terms to be $\vec{a}_{\text{tot}} = \vec{g}_o + \vec{a}_{\text{coriolis}}$) and assume small deflections due to Coriolis term, $v_x = v_y \approx 0$, when computing the Coriolis acceleration.

5) **(30 pts)** A projectile is shot North ($-\hat{x}$ in local coordinates) with an initial speed of v_o and launch angle $\alpha = 45°$ at a latitude λ on Earth's surface.



a) **(25 pts)** Show that the deflection due to Coriolis effect can be expressed as

$$y(\lambda) = \omega_e d \sqrt{\frac{2d}{g_o}} \left(\sin \lambda - \frac{1}{3} \cos \lambda \right)$$

where $t_{\text{Fall}} \equiv \sqrt{2d/g_o}$ is the time flight time of the projectile and d is the horizontal range it falls.

b) **(5 pts)** Calculate the numerical value of the deflection for $\lambda = 30°, d = 15000$ m

hint: i) use the fact that $\alpha = 45°$, to express $\sin(\alpha) = \cos(\alpha) = \sqrt{2}/2$ during the derivation; ii) the flight time can be calculated by simply solving first for the vertical (z-component) as it is independent of the Coriolis term; iii) only consider the projectile subject to a gravitational and Coriolis acceleration, and assume $v_y \approx 0$ (small deflection) when computing the Coriolis acceleration.