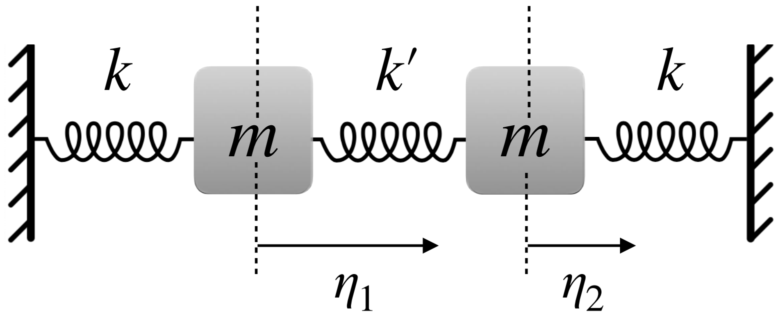
PROBLEM SET 6

Show **ALL WORK** to get full/partial credit. Begin each problem on a new page, and clearly label each part of the problem. (**Max Score:** 100)

Fetter & Walecka, Ch. 4

1) **(25 pts)** Reconsider the coupled oscillator example covered in class lecture.



a) Write the Lagrange equations of motions for this system

b) Try a real normal-mode solution of the type:

$$\eta_\lambda = C\rho_\lambda \cos(\omega t + \phi)$$

where (λ) represents the generalized coordinate index, ρ_λ represents the amplitude (eigenvector), ω is a normal-mode frequency, and (C, ϕ) represent overall constants determined from the initial conditions, $\eta(t=0)$, $\dot{\eta}(t=0)$, if they are given.

Substitute this solution to the coupled equations in part (a) and re-write the system of linear equations in matrix form:

$$(v_{\sigma\lambda} - \omega^2 m_{\sigma\lambda})\rho_\lambda = 0,$$

and explicitly identify the potential energy ($v_{\sigma\lambda}$) and mass ($m_{\sigma\lambda}$) matrices, as they will be used in the later parts.

c) To solve the homogeneous matrix equation in part (b), for a non-trivial solution $\rho_\lambda \neq 0$, requires $\det|v_{\sigma\lambda} - \omega^2 m_{\sigma\lambda}| = 0$, which yields a **characteristic** or **secular** equation of degree n (number of generalized coordinates, in this case is 2) in ω^2, meaning there are two eigenfrequencies or modes of oscillations. Determine each of these eigenfrequencies, ω_s^2.

d) Determine the relative amplitudes, $\rho_\lambda^{(s)}$, for each mode s, to show they can be expressed as:

$$\rho_1^{(1)} = -\rho_2^{(1)}, \quad \rho_1^{(2)} = \rho_2^{(2)}$$

Hint: Substitute the eigenfrequency, ω_s^2, back in the system of linear equations determined in part (b). Note that the system of n linear equations is **NOT** linearly independent, and therefore, either equation will give a ratio of the amplitudes, $\rho_1^{(s)}/\rho_2^{(s)}$ for each particular mode s.

NOTE: the ratio of amplitudes describes the relative motion of the blocks for a particular mode. For example, it might tell us whether the blocks are moving together in the same direction or in opposite direction.

e) Although not usually necessary to find the normalization factor in general, it is asked here for completion. Use the orthonormality condition, $\vec{\rho}^{(t)T}\mathbf{M}\vec{\rho}^{(s)} = \mathbf{I}$ or in index notation: $\sum_\lambda \sum_\sigma \rho_\sigma^{(t)} m_{\sigma\lambda} \rho_\lambda^{(s)} = \delta_{st}$ to calculate the normalization constant for each normal mode eigenvector, and show that the eigenvectors can be expressed as:

$$\vec{\rho}^{(1)} = \frac{1}{\sqrt{2m}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \vec{\rho}^{(2)} = \frac{1}{\sqrt{2m}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Hint: Insert the results of part (d) into the eigenvector: $\vec{\rho}^{(s)} = \begin{bmatrix} \rho_1^{(s)} \\ \rho_2^{(s)} \end{bmatrix}$, and along with the results of part (b), use the normalization condition, $\delta_{st} = 1,\ t = s$ to determine the normalization factor.

f) Construct the modal matrix (first read **Background**),

$$\mathbf{A} = \begin{bmatrix} \rho_1^{(1)} & \rho_1^{(2)} & \cdots & \rho_1^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_n^{(1)} & \rho_n^{(2)} & \cdots & \rho_n^{(n)} \end{bmatrix} = \left[\vec{\rho}^{(1)}, \vec{\rho}^{(2)}, \ldots \vec{\rho}^{(n)}\right]$$

from the generic column eigenvectors, (do not include the normalization factors from (e), but rather use the generic form) and use the relation, $\vec{\eta} = \mathbf{A}\vec{\zeta}$, along with the relative amplitudes relations [part (d)] to show that the generalized coordinates can be expressed in terms of the normal coordinates as a linear combination of the normal mode coordinates with the eigenvector coefficients

$$\eta_1 = \rho_1^{(1)}\zeta_1 + \rho_2^{(2)}\zeta_2, \quad \eta_2 = -\rho_1^{(1)}\zeta_1 + \rho_2^{(2)}\zeta_2$$

Then solve for (ζ_1, ζ_2) above to construct the normal-mode coordinates (ζ) in terms of the generalized coordinates to show that they have the generic form:

$$\zeta_1 = \frac{\eta_1 - \eta_2}{2\rho_1^{(1)}}, \quad \zeta_2 = \frac{\eta_1 + \eta_2}{2\rho_2^{(2)}}$$

Finally, use the normalization result from part (e) to show that the normal-mode coordinates take the form:

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \sqrt{\frac{m}{2}} \begin{bmatrix} \eta_1 - \eta_2 \\ \eta_1 + \eta_2 \end{bmatrix}$$

Background [see Fetter & Walecka Ch. 4 p.98-99]**:** The modal matrix is square matrix constructed from the column eigenvectors, $\mathbf{A} \equiv \left[\vec{\rho}^{(1)}, \vec{\rho}^{(2)}, \ldots \vec{\rho}^{(n)}\right]$, where the superscript represents the normal mode. It has the property of simultaneously *diagonalizing* the mass and potential energy matrices, as follows: $\mathbf{A}^T\mathbf{M}\mathbf{A} = \mathbf{I}$,

$$\mathbf{A}^T\mathbf{V}\mathbf{A} = \begin{bmatrix} \omega_1^2 & 0 \\ 0 & \omega_2^2 \end{bmatrix} \equiv \omega_D^2$$

The modal matrix also relates the original generalized coordinates, η_σ, to a new set of coordinates, ζ_λ, (normal-mode coordinates) thru the relation: $\eta_\sigma = \sum_\lambda A_{\sigma\lambda}\zeta_\lambda$, or in matrix notation $\vec{\eta} = \mathbf{A}\vec{\zeta}$

Furthermore, the normal coordinates relation to generalized coordinates can be obtained by making use of the diagonalizing property of the mass matrix, $(\mathbf{A}^T\mathbf{M})\vec{\eta} = \underbrace{\mathbf{A}^T\mathbf{M}\mathbf{A}}_{\mathbf{I}}\,\vec{\zeta}$ to obtain:

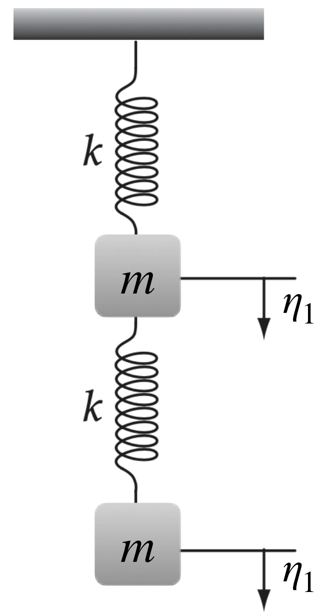
$\vec{\zeta} = (\mathbf{A}^T\mathbf{M})\vec{\eta}$, or in the more familiar notation: $\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \mathbf{A}^T\mathbf{M}\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}$,

where it is important to note that the index $\zeta_{1,2}$, represent the s-normal mode, whereas the index $\eta_{1,2}$, represents the generalized coordinate, e.g. in this example, describing the displacement of each block. In effect, each normal-mode coordinate is linear combination of the generalized coordinates.

g) Verify that the equations of motion in normal-mode coordinates are de-coupled, and thus have the form of a simple harmonic oscillator with frequency of the corresponding normal mode.

Hint: to verify this, take two time derivatives of each normal mode coordinate and show that: $\ddot{\zeta}_s = -\,\omega_s^2\zeta,\ s = 1{,}2$

2) **(25 pts)** A particle of mass m is attached to a rigid support by a spring with force constant k. At equilibrium, the spring hangs vertically downward. To this mass-spring combination is attached an identical oscillator, the spring of the latter being connected to the mass of the former.



a) Write the Lagrange equations of motion for this system.

b) Try a real normal-mode solution of the type:

$$\eta_\lambda = C\rho_\lambda \cos(\omega t + \phi)$$

into the Lagrange equations of motion and

(i) identify the potential energy ($v_{\sigma\lambda}$) and mass ($m_{\sigma\lambda}$) matrices

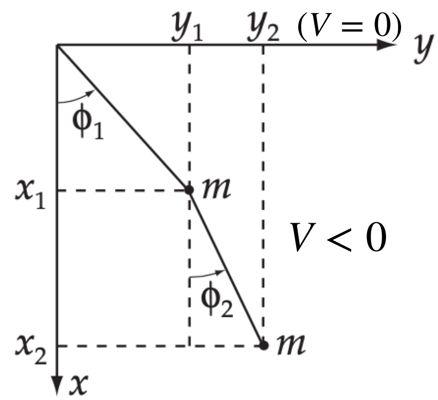
(ii) solve for the eigenfrequencies, ω_s^2, of each normal mode s to show they can be expressed

as: $\omega_1^2 = \dfrac{3+\sqrt{5}}{2}\dfrac{k}{m}, \quad \omega_2^2 = \dfrac{3-\sqrt{5}}{2}\dfrac{k}{m}$

c) Determine the relative amplitudes, $\rho_\lambda^{(s)}$, for each mode s to show they can be expressed as:

$$\frac{1-\sqrt{5}}{2}\rho_1^{(1)} = \rho_2^{(1)}, \quad \frac{1+\sqrt{5}}{2}\rho_1^{(2)} = \rho_2^{(2)}$$

3) **(25 pts)** A simple pendulum consists of a bob of mass m suspended by an inextensible (and massless) string of length ℓ. From the bob of this pendulum is suspended a second, identical pendulum. Consider the case of small oscillations ($\sin\phi \approx \phi, \cos\phi \approx 1 - \phi^2/2$)



a) Calculate the Lagrangian, $L = T - V$, for small angle oscillations, where
$T = \frac{1}{2}m\ell^2(2\dot{\phi}_1^2 + 2\dot{\phi}_1\dot{\phi}_2 + \dot{\phi}_2^2)$ and
$V = V_0 + \frac{1}{2}mg\ell^2(2\phi_1^2 + \phi_2^2)$, where
$V_0 = -3mg\ell$

b) Write the Lagrange equations of motion for (ϕ_1, ϕ_2) for this system using a small angle approximation. (From now on, let $\phi_\lambda \to \eta_\lambda$, for small displacements)

c) Try a real normal-mode solution of the type:

$$\eta_\lambda = C\rho_\lambda \cos(\omega t + \theta)$$

into the Lagrange equations of motion and

(i) identify the potential energy ($v_{\sigma\lambda}$) and mass ($m_{\sigma\lambda}$) matrices and show they take the form:

$$m_{\sigma\lambda} = m\ell^2 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad v_{\sigma\lambda} = mg\ell \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

(ii) solve for the eigenfrequencies, ω_s^2 of each normal mode s to show they can be expressed as

$$\omega_1^2 = \left(2 + \sqrt{2}\right)\frac{g}{\ell}, \quad \omega_2^2 = \left(2 - \sqrt{2}\right)\frac{g}{\ell}$$

d) Determine the relative amplitudes, $\rho_\lambda^{(s)}$, for each mode s to show they can be expressed as:

$$\rho_2^{(1)} = -\sqrt{2}\rho_1^{(1)}, \quad \rho_2^{(2)} = \sqrt{2}\rho_1^{(2)}$$

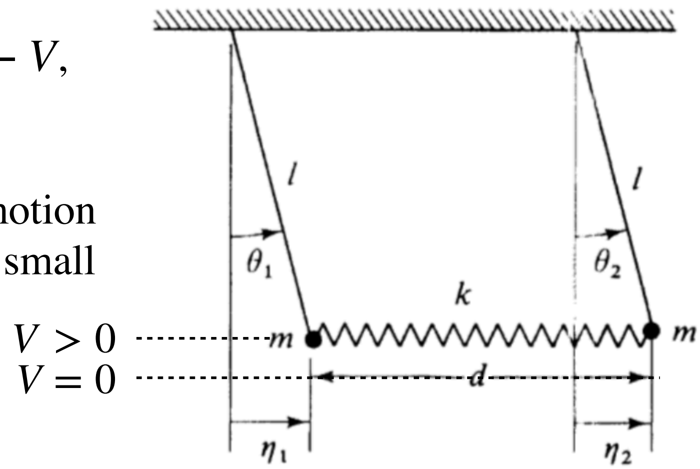
e) Construct the modal matrix, $\mathbf{A}$, and use the relation, $\vec{\eta} = \mathbf{A}\vec{\zeta}$, to show that the generalized coordinates can be expressed in terms of the normal coordinates as a linear combination of the normal mode coordinates with the eigenvector coefficients

$$\eta_1 = \rho_1^{(1)}\zeta_1 + \frac{1}{\sqrt{2}}\rho_2^{(2)}\zeta_2, \quad \eta_2 = -\sqrt{2}\rho_1^{(1)}\zeta_1 + \rho_2^{(2)}\zeta_2$$

then solve for the normal coordinates, ζ_1, ζ_2 to show:

$$\zeta_1 = \frac{\sqrt{2}\eta_1 - \eta_2}{2\sqrt{2}\rho_1^{(1)}}, \qquad \zeta_2 = \frac{\sqrt{2}\eta_1 + \eta_2}{2\sqrt{2}\rho_2^{(2)}}$$

4) **(25 pts)** Two pendula of equal length l and equal mass m are connected by a spring of force constant k. Consider the case of small oscillations ($\sin\theta \approx \theta, \cos\theta \approx 1 - \theta^2/2$), where $\eta = l\sin\theta \approx l\theta$ [see Fetter & Walecka, Ch. 4 example]



a) Calculate the Lagrangian, $L = T - V$, for small angle oscillations

b) Write the Lagrange equations of motion for (η_1, η_2) for this system using a small angle approximation.

c) Try a real normal-mode solution of the type:

$$\eta_\lambda = C\rho_\lambda \cos(\omega t + \phi)$$

into the Lagrange equations of motion and

(i) identify the potential energy ($v_{\sigma\lambda}$) and mass ($m_{\sigma\lambda}$) matrices and show they take the form:

$$m_{\sigma\lambda} = m\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad v_{\sigma\lambda} = \begin{bmatrix}\frac{mg}{l} + k & -k\\ -k & \frac{mg}{l} + k\end{bmatrix}$$

(ii) solve for the eigenfrequencies, ω_s^2 of each normal mode s to show they can be expressed

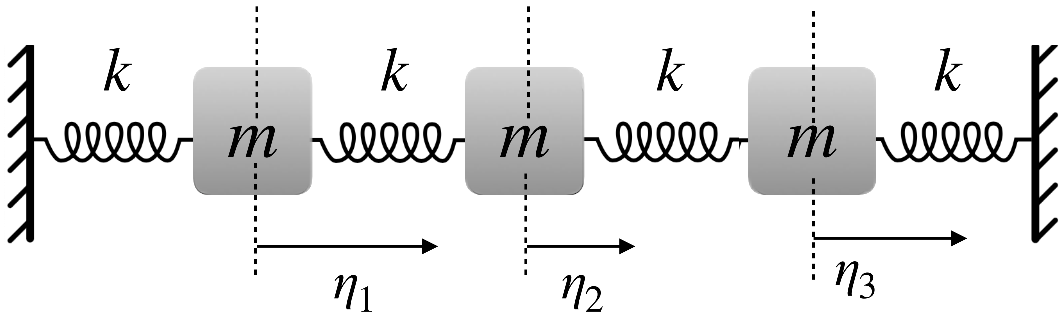
as: $\omega_1^2 = \dfrac{g}{l}, \quad \omega_2^2 = \dfrac{g}{l} + 2\dfrac{k}{m}$

d) Determine the relative amplitudes, $\rho_\lambda^{(s)}$, for each mode s to show they can be expressed as:
$\rho_1^{(1)} = \rho_2^{(1)}, \;\; \rho_1^{(2)} = -\rho_2^{(2)}$

e) Construct the modal matrix, $\mathbf{A}$, and use the relation, $\vec{\zeta} = (\mathbf{A}^T\mathbf{M})\vec{\eta}$, to show that the normal-mode coordinates (ζ) in terms of the generalized coordinates have the form:

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = \sqrt{\frac{m}{2}} \begin{bmatrix} \eta_1 + \eta_2 \\ \eta_1 - \eta_2 \end{bmatrix}$$

5) **(+10 pt extra credit Exam II)** Reconsider the coupled oscillator example covered in class lecture, but add an additional block + spring.



a) Calculate the Lagrangian, $L = T - V$

b) Write the Lagrange equations of motion for (η_1, η_2, η_3) for this system

c) Try a real normal-mode solution of the type:

$$\eta_\lambda = C\rho_\lambda \cos(\omega t + \phi)$$

into the Lagrange equations of motion and

(i) identify the potential energy ($v_{\sigma\lambda}$) and mass ($m_{\sigma\lambda}$) matrices and show they take the form:

$$m_{\sigma\lambda} = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad v_{\sigma\lambda} = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}$$

(ii) solve for the eigenfrequencies, ω_s^2 of each normal mode s to show they can be expressed

as: $\omega_1^2 = \dfrac{2k}{m}, \quad \omega_2^2 = (2+\sqrt{2})\dfrac{k}{m}, \quad \omega_3^2 = (2-\sqrt{2})\dfrac{k}{m}$

d) Determine the relative amplitudes, $\rho_\lambda^{(s)}$, for each mode s to show they can be expressed as:

$$\rho_1^{(1)} = -\rho_3^{(1)}, \quad \rho_2^{(1)} = 0$$

$$\rho_1^{(2)} = \rho_3^{(2)}, \qquad \rho_2^{(2)} = -\sqrt{2}\rho_3^{(2)}$$

$$\rho_1^{(3)} = \rho_3^{(3)}, \qquad \rho_2^{(3)} = \sqrt{2}\rho_3^{(3)}$$

e) Construct the modal matrix, $\mathbf{A}$. Then use the relation, $\vec{\eta} = \mathbf{A}\vec{\zeta}$, along with the relative amplitudes [part (d)] to show that the generalized coordinates can be expressed in terms of the normal coordinates as a linear combination of the normal mode coordinates with the eigenvector coefficients

$$\eta_1 = \rho_1^{(1)}\zeta_1 - \frac{1}{\sqrt{2}}\rho_2^{(2)}\zeta_2 + \rho_3^{(3)}\zeta_3$$

$$\eta_2 = \rho_2^{(2)}\zeta_2 + \sqrt{2}\rho_3^{(3)}\zeta_3$$

$$\eta_3 = -\rho_3^{(3)}\zeta_1 - \frac{1}{\sqrt{2}}\rho_2^{(2)}\zeta_2 + \rho_3^{(3)}\zeta_3$$

then solve for the normal coordinates, $(\zeta_1, \zeta_2, \zeta_3)$ to show:

$$\zeta_1 = \frac{\eta_1 - \eta_3}{2\rho_1^{(1)}}, \quad \zeta_2 = \frac{-\eta_1 + \sqrt{2}\eta_2 - \eta_3}{2\sqrt{2}\rho_2^{(2)}}, \quad \zeta_3 = \frac{\eta_1 + \sqrt{2}\eta_2 + \eta_3}{4\rho_3^{(3)}}$$