PROBLEM SET 7

Show **ALL WORK** to get full/partial credit. Begin each problem on a new page, and clearly label each part of the problem. (**Max Score:** 75)

Fetter & Walecka, Ch. 5

1) **(25 pts)** Using a coordinate system where the origin coincides with the center-of-mass (CM), calculate the moment of inertia about the $\hat{e}_3$ ($\perp$ to the plane of this paper) axis of the following objects:

a) Uniform solid rod of mass m and length ℓ

b) Uniform ring of radius a

c) Uniform disk of radius a

d) Using the *Parallel-Axis Theorem,* determine the moment of inertia of each of these objects about their end (or edge)

2) **(25 pts)** For a homogeneous cube of mass M and sides of length b

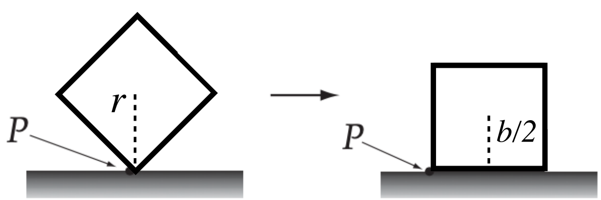
a) Calculate the inertia tensor I_{ij} about an edge of the cube (you may use your notes from class and write the result obtained, or re-calculate it)

b) Calculate the inertia tensor $I_{ij}^{(\text{cm})}$ about the center-of-mass using the *Parallel-Axis Theorem*

c) Now consider the cube initially in a position of unstable equilibrium with one edge in contact with a horizontal plane. The cube is then given a small displacement and allowed to fall. Show that the angular velocity of the cube when one face strikes the plane is given by:

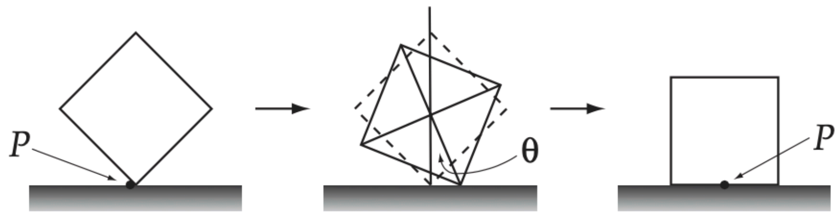
General Hint: The natural origin of the coordinate is at the center of the cube since in both cases, the rotation of the cube is about its CM; you may use the results from part (b) to determine the moment of inertia about the axis of rotation at the center of the cube

(i) $\omega^2 = \dfrac{3}{2}\dfrac{g}{b}(\sqrt{2} - 1)$ if the edge (point of contact) of the cube cannot slide on a plane (due to friction)



Hint (*i*): At any point of the extended object, the total velocity of an arbitrary particle α is given by: $v_\alpha = \mathbf{V}_{\text{CM}} + \omega \times \mathbf{r}_\alpha$. Now, if the cube topples over without sliding as in the figure above, the point of contact between the cube and the floor is stationary, yielding $v_\alpha = 0 \rightarrow \mathbf{V}_{\text{CM}} = -\,\omega \times \mathbf{r}_\alpha$ at point P. Therefore, in this particular case, $V_{\text{CM}} = \omega r$. Therefore the total kinetic energy has a translational + rotational component

(ii) $\omega^2 = \dfrac{12}{5}\dfrac{g}{b}(\sqrt{2} - 1)$ if the edge (point of contact) of the cube can slide without friction. Since there is no horizontal frictional force, the cube slides so that the CM falls directly downward along the vertical line



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original cube config.

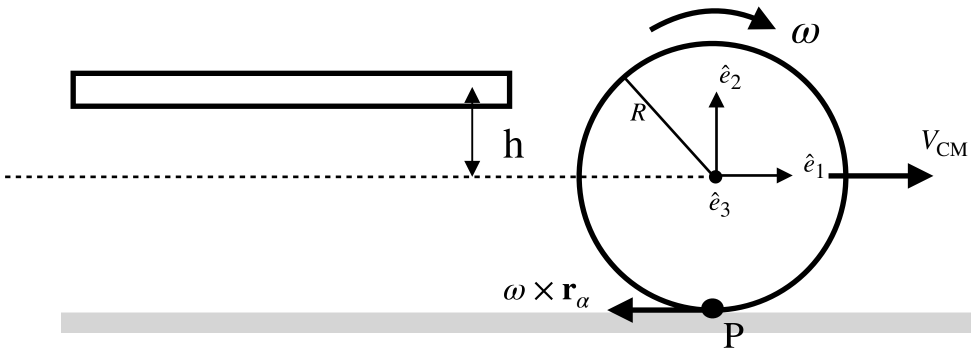
Hint (*ii*): In this scenario, there is no external force along the horizontal direction (no friction) therefore the cube slides such that the CM falls directly downward along a vertical line. In this scenario the total kinetic energy has a translational + rotational with the same rotational component as in part (i), but different translational component due to the fact that there is no friction)

The translational K.E. of the C.M. in this case can be calculated as follows:

- first need to determine the height y of the C.M. above the ground in terms of the angle θ [use simple trigonometry relation of the triangle shown in the middle picture]

- then, determine the velocity of the CM ($V_{\mathrm{CM}} = dy/dt$) [and use limits $\theta : 0 \to \pi/4$, which represent the angular change of the cube relative to the vertical.]
  Recall $\sin \pi/4 = \sqrt{2}/2$

3) **(25 pts)** The physics of billiards predicts a point on a ball which, when hit, develops no frictional force between the ball and the billiard table.



a) Find the moment of inertia of the billiard ball of mass M and radius R about the $\hat{e}_3$ axis.

b) Show that the height at which the billiard ball should be struck so that it will *roll without slipping* is $h = \frac{2}{5}R$

Hint: The rolling without slipping requires that the center-of-mass velocity be equal to the tangential velocity, since the point of contact with the ground is stationary. Mathematically, the total velocity of a particle α at the surface of the sphere is $v_\alpha = \mathbf{V}_{CM} + \omega \times \mathbf{r}_\alpha$. Then, for the particular point of contact P, $v_\alpha = 0$, and one can solve the problem using Newton's 2nd law for rotation ($\tau = I\alpha,\ \alpha \equiv d\omega/dt$)