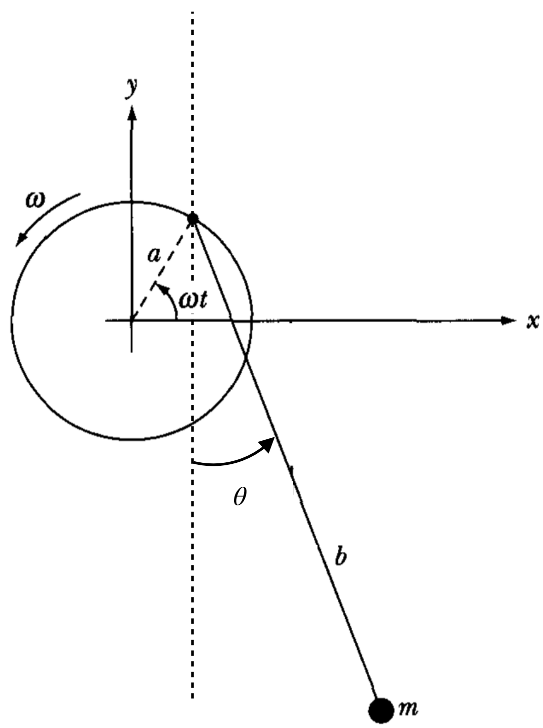
PROBLEM SET 4

Show **ALL WORK** to get full/partial credit. Begin each problem on a new page, and clearly label each part of the problem. (**Max Score:** 100)

1) **(20 pts)** The point of support of a simple pendulum of length b and mass m displaced at an arbitrary angle θ from its equilibrium position moves on a massless rim of radius a rotating with a constant angular velocity ω, counterclockwise.

a) Obtain a general *simplified* expression for the Cartesian components of the *i)* position, *ii)* velocity and *iii)* acceleration of the mass

b) Calculate the Lagrangian ($L = T - V$) of this system and determine a general expression for the angular acceleration of the angle θ, and show that in the limit $\omega \to 0$, the equation of motion reduces to that of a simple pendulum.



2) **(20 pts)** A simple pendulum of mass m and length ℓ, displaced at an arbitrary angle θ is placed inside a railroad car that has a constant acceleration a in the x-direction.

a) Choose a fixed (non-inertial) coordinate system with initial conditions for the moving train of $x_0 = 0,\ \dot{x} = v_0$ at $t = 0$. Obtain a general simplified expression (at some later time t) for the (x, y) Cartesian components of the *i)* position and *ii)* velocity of the pendulum.

b) Calculate the Lagrangian ($L = T - V$) of this system and determine a general simplified expression for the angular acceleration ($\ddot{\theta}$) of the pendulum.

c) Due to the railroad car's acceleration, the pendulum will be displaced to some *equilibrium* angle, θ_e from the vertical. Calculate the pendulum equilibrium angle, θ_e. Show that in the limit $a \to 0$, then $\theta_e \to 0$

d) For a small angle displacement η from the equilibrium position, show that the equation of motion becomes

$$\ddot{\eta} = -\omega^2 \eta$$

where $\omega^2 = \sqrt{a^2 + g^2}/\ell$, is the frequency of small oscillations of the simple pendulum about the equilibrium angle, θ_e.

Hint: Because oscillations are small and are about the equilibrium angle, let $\theta = \theta_e + \eta$, where η is a small angle. Substitute this angle in the general equation of motion determine in part b) . Also, when trying to show the general expression for ω, use the trigonometry of result of part c) to make a right triangle with angle θ_e, and that will help simplify the expression into the desired result.

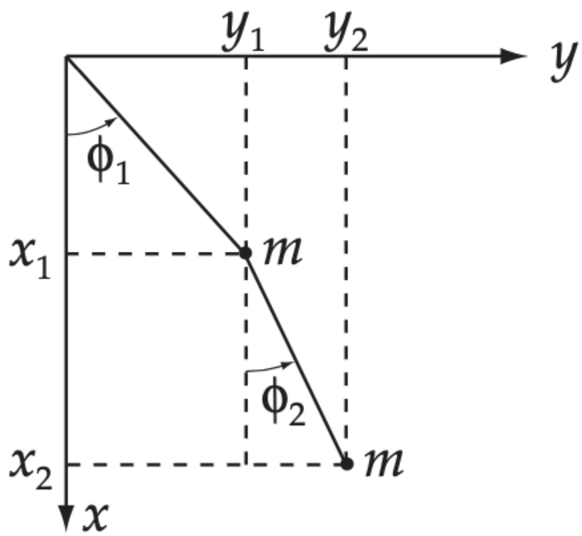
Useful trigonometric identities:

$\sin(a + b) = \sin(a)\cos(b) + \sin(b)\cos(a)$,

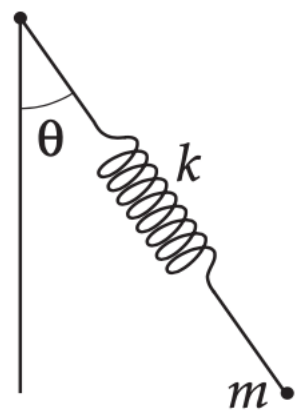
$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

and also use the small angle approximation, keeping only the first term in the Taylor series expansion, $\sin(x) \sim x,\ \cos(x) \sim 1$

3) **(20 pts)** A double pendulum consists of two simple pendula, displaced from equilibrium at arbitrary angles, ϕ_1 and ϕ_2, with one pendulum suspended from the bob of the other. If the two pendula have equal lengths ℓ and equal mass m, and if both pendula are confined to move in the same plane, find the Lagrange's equations of motion for the system. Do not assume small angles. Simplify your answer if possible, by using trigonometric identities.



4) **(20 pts)** A pendulum consists of a mass m suspended by a massless spring with unextended length b and spring constant k. The pendulum is then displaced by some angle θ and extended to an arbitrary length ℓ from its unextended spring length. Find the Lagrange's equations of motion.



5) **(20 pts)** A disk of mass M and radius R rolls without slipping down a plane incline from the horizontal by an angle α. The disk has a short weightless axle of negligible radius. From this axis is suspended a simple pendulum of length $l < R$ and whose bob has a mass m and displaced from the vertical by an arbitrary angle ϕ. Consider that the motion of the pendulum takes place in the plane of the disk, and find the Lagrange's equations of motion for the system.