PROBLEM SET 5

Show **ALL WORK** to get full/partial credit. Begin each problem on a new page, and clearly label each part of the problem. (**Max Score:** 100)

Relevant equations for this homework set, from Fetter & Walecka, Ch. 3

Lagrange Equations for Undetermined Multipliers

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\sigma} - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial q_\sigma} \qquad \sigma = 1, \ldots, n \tag{19.3}$$

$$f_j(q_1, \ldots, q_n, t) = c_j \qquad j = 1, \ldots, k \tag{19.4}$$

Hamiltonian

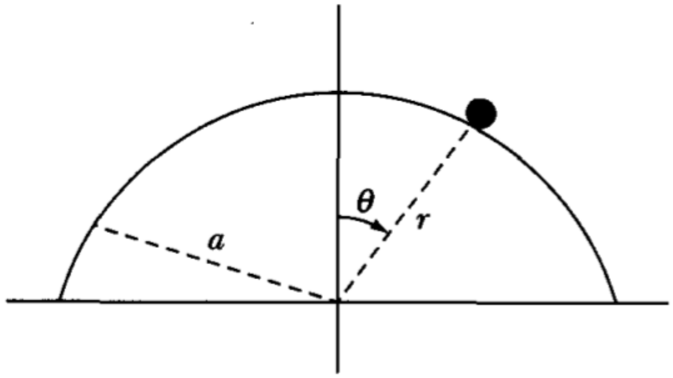
$$H \equiv \sum_\sigma p_\sigma \dot{q}_\sigma - L \tag{20.12}$$

Hamiltonian Equations of motion (not found in textbook)

$$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma}$$

$$-\dot{p}_\sigma = \frac{\partial H}{\partial q_\sigma}$$

1) **(30 pts)** A particle of mass m starts at rest on top of a smooth hemisphere of radius a.



a) **(5 pts)** Determine the lagrangian $L = T - V$, of this system.

b) **(5 pts)** Write down the equation of constraint, $f(r, \theta)$, and determine the Lagrange equations of motion for each generalized coordinate using the method of Lagrange multipliers. Show that the angular acceleration of the particle can be expressed as: $\ddot{\theta} = (g/a)\sin\theta$

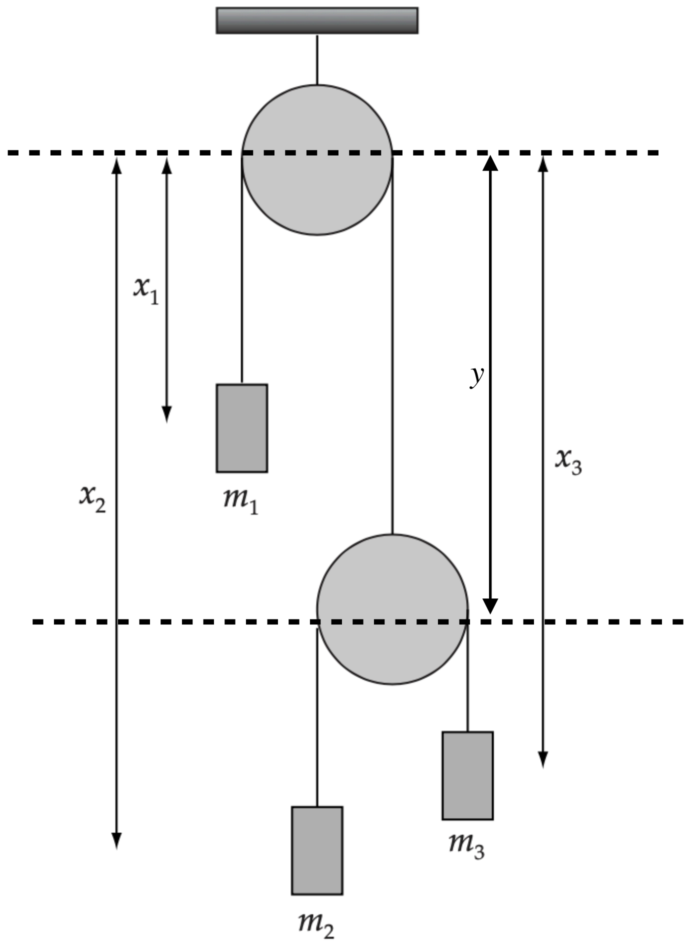
c) **(5 pts)** Use the fact that $\dot{\theta} = \dot{\theta}[\theta(t)]$, and the chain rule to show: $\ddot{\theta} = \dot{\theta}(d\dot{\theta}/d\theta)$. Use this result and the result from part b) to find a general expression for $\dot{\theta}^2$ as a function of θ. [**Hint**: recall, the particle starts from *rest* at the top of the hemisphere and falls by some arbitrary angle, θ]

d) **(5 pts)** Substitute the result derived in part c) into the equations of motion, solve for the Lagrange multiplier, λ and determine the forces of constraint, $Q_r^{(r)}$, $Q_\theta^{(r)}$.

e) **(5 pts)** Determine the angle θ_0 at which the particle leaves the hemisphere. [**Hint**: think what is the force of constraint when the particle leaves the hemisphere ?].

f) **(5 pts)** As a check of your general results obtained in part e), show that the force of constraint when the particle is at the top of the hemisphere is
$Q_\theta^{(r)} = mg$

2) **(25 pts)** Consider the double pulley system shown below. Use the coordinates indicated, and use the method of Lagrange's multipliers to determine the equations of motion and forces of constraint on the masses. The total length of the visible rope in the top and bottom pulley is l_1 and l_2, respectively.



a) **(5 pts)** Determine the lagrangian $L = T - V$, of this system

b) **(5 pts)** Write down the equations of constraint (total length of each rope in terms of the variables in the figure). Show that it can be reduced to a single equation of constraint. Then take two time derivatives to show:

$$2\frac{d^2x_1}{dt^2} + \frac{d^2x_2}{dt^2} + \frac{d^2x_3}{dt^2} = 0$$

c) **(5 pts)** Determine the Lagrange equations of motion for each of the three masses using the method of Lagrange multipliers.

d) **(5 pts)** Use the results from parts b) and c) to calculate the Lagrange multiplier λ.

e) **(5 pts)** Show that the constraint force (string tension T_1) that acts on mass m_1 is given by

$$T_1 = \frac{8g}{\frac{4}{m_1} + \frac{1}{m_2} + \frac{1}{m_3}}$$

3) **(15 pts)** A particle of mass m moves in one dimension under the influence of a force,

$$F(x,t) = \frac{k}{x^2} e^{-(t/\tau)}$$

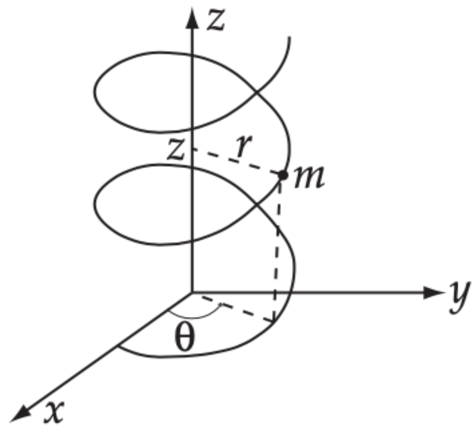
where k and τ are positive constants.

a) **(5 pts)** Compute the Lagrangian, $L = T - V$

b) **(5 pts)** Compute the Hamiltonian [**Hint**: start with the standard definition, $H \equiv \Sigma_\sigma\ p_\sigma \dot{q}_\sigma - L$] and show that it is equal to the total energy, $T + V$

c) **(5 pts)** Show mathematically whether the total energy H is a conserved or not.

4) **(15 pts)** A particle of mass m moves under the influence of gravity along the helix $z = k\theta$, r = constant, where k is a constant and z is vertical.



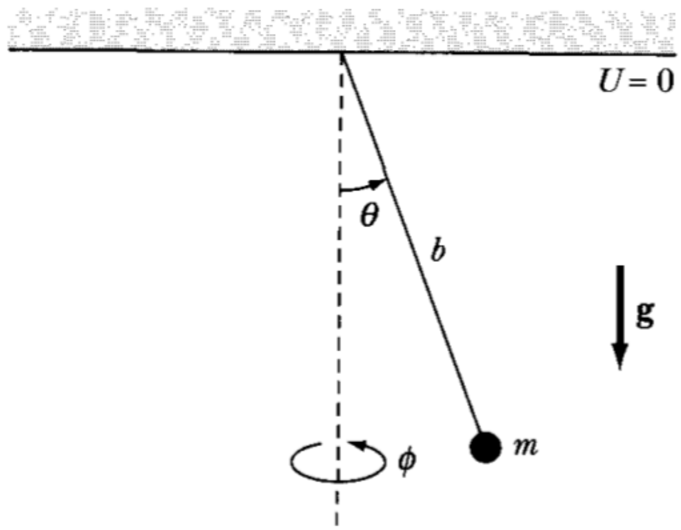
a) **(5 pts)** Compute the Hamiltonian [**Hint**: its easier to start with the Lagrangian, and use the definition $H \equiv \Sigma_\sigma \; p_\sigma \dot{q}_\sigma - L = T + V$]

b) **(5 pts)** Compute the Hamiltonian equations of motion

c) **(5 pts)** Show that the equation of motion of the particle can be expressed as

$$\ddot{z} = \frac{g}{\left[\frac{r^2}{k^2} + 1\right]}$$

5) **(15 pts)** Consider a spherical pendulum (e.g., Focault pendulum) in the figure below.



a) **(5 pts)** Compute the Hamiltonian $[H \equiv \Sigma_\sigma \; p_\sigma \dot{q}_\sigma - L = T + V]$

b) **(10 pts)** Compute the Hamiltonian equations of motion ($\dot{\theta}$, $\dot{\phi}$, $\dot{p}_\theta$, $\dot{p}_\phi$) for a spherical pendulum of mass m and length b.