PROBLEM SET 2

Show **ALL WORK** to get full/partial credit. Begin each problem on a new page, and clearly label each part of the problem. (**Max Score:** 100)

Useful Definitions:

$\dot{r} \equiv dr/dt$, $\dot{\phi} \equiv d\phi/dt$

The general conic sections in polar form is

$$r(\phi) = \frac{a(1-\epsilon^2)}{1 \pm \epsilon \cos\phi},$$

where $\pm$ refer to the origin located at the right (+) and left (-) focal points, respectively; a is the semi-major axis and ϵ is the eccentricity.

Total energy in a central force field: $E = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} + V(r)$

Angular momentum in a central force field: $l = \mu r^2\dot{\phi}$

Useful relation: $\dot{r} = dr/d\phi \times \dot{\phi}$

Central force field: $F(r) = -\nabla V(r)$

Kepler's 3rd Law: $\tau^2 = \frac{4\pi^2 a^3}{G(M+m)}$

1) **(10 pts)** An Earth satellite has a perigee (nearest to Earth) of 300 km and an apogee (farthest from Earth) of 3,500 km from Earth's surface. With the Earth's center located at the right focal point of the ellipse, how far is the satellite *above* Earth (distance from Earth's surface to satellite) when

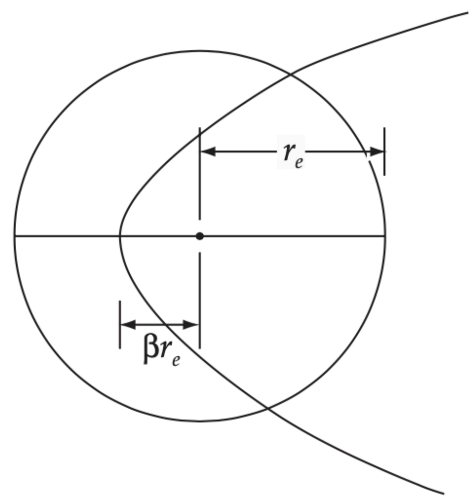
a) it has rotated $90°$ around Earth from perigee

b) it has moved *halfway* between perigee and apogee

Earth's radius is $r_E = 6.371 \cdot 10^6$ m.

2) **(15 pts)** Consider a comet moving in a parabolic orbit in the plane of the Earth's orbit. If the distance of closest approach of the comet to the Sun is βr_E, where r_E is the radius of the Earth's (assumed) circular orbit and where $\beta < 1$, show that the time the comet spends within the orbit of Earth is given by

$$\sqrt{2(1-\beta)} \cdot (1+2\beta)/3\pi \times \tau_E, \qquad \tau_E = 1\text{yr (Earth's period)}$$



hint 1: use the energy conservation formula to get a general expression for $dr/dt = f(r)$ and integrate, $dt = f(r)dr$ to obtain the time it takes the comet to move from βr_E to r_E (noting that this is half the time the comet spends in Earth's orbit, so naturally, one would multiply this time by 2.

hint 2: once a general expression is obtained, use Kepler's 3rd law to express the relation in terms of a multiple of Earth's orbital period, τ_E

3) **(15 pts)** A particle moves in a logarithmic spiral orbit given by $r = ke^{\alpha\phi}$, where k, α are positive constants.

a) find the force law $F(r)$ that allows the particle to move in a spiral
hint: need to first determine $V(r)$ and then can use $F(r) = -\nabla V(r)$

b) determine a general expression for $r(t)$ and $\phi(t)$

c) determine the numerical value of the total energy of this orbit

4) **(10 pts)** Find the force law for a central-force field that allows a particle to move in a spiral orbit given by $r = k\phi^2$, where k is a positive constant.

5) **(15 pts)** The Yukawa potential adds an exponential term to the long-range Coulomb potential, which greatly shortens the range of the Coulomb potential. It has great usefulness in atomic and nuclear calculations

$$V(r) = -\frac{k}{r}e^{-r/a}$$

Show that the particle's trajectory, $r(\phi)$, in a bound orbit of the Yukawa potential to first order in r/a (apply Taylor expansion on $e^{-r/a}$) has the form of a general conic section,

$$r(\phi) = \frac{\alpha}{1 \pm \epsilon \cos\phi} \qquad \text{or} \qquad r(\phi) = \frac{\alpha}{1 \pm \epsilon \sin\phi}$$

where $\alpha = \dfrac{l^2}{\mu k}$ and $\epsilon = \sqrt{1 + \dfrac{2l^2}{\mu k^2}\left(E - \dfrac{k}{a}\right)}$

The $\pm$ sign will depend on whether you chose the left of right focal point as the origin (i.e., see **Eq. 3.13** of Fetter & Walecka), and the "sin" or "cos" will depend on the initial condition, ϕ_0

Useful Integral: $\displaystyle\int \frac{du}{\sqrt{au^2 + bu + c}} = -\frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{2au + b}{\sqrt{b^2 - 4ac}}\right),$

given that $a < 0,\ b^2 > 4ac$

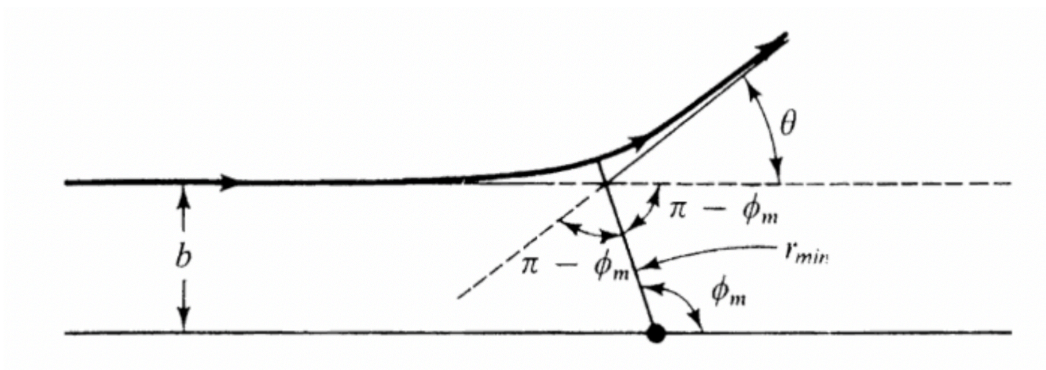
hint: first need to solve the integral to obtain $\phi(r)$, and then the expression can be inverted (similarly to the derivation of conic sections in polar form); use the substitution $u = 1/r$ to express the integral in the form above

6) **(20 pts)** The main problem in obtaining a general expression for Rutherford's differential cross section,

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

is to determine the angular dependence of the impact parameter, i.e. $db/d\theta$ or vice-versa

a) for an incident particle scattering off a repulsive potential, as shown in the figure below



show that the general relation between the scattering angle θ and the impact parameter b for a repulsive central potential V is given by

$$\theta(b) = \pi - 2b\int_0^{u_0} \frac{du}{[(1 - V/E) - b^2u^2]^{1/2}}$$

where $u = 1/r$ and u_0 is the classical turning point

hint 1: start from Eq. 3.13 of Fetter and Walecka and integrate from when the particle is at infinity ($r \to \infty, \phi \to \pi$) to its distance of closest approach ($r \to r_{\min}, \phi \to \phi_m$)

hint 2: energy and angular momentum (E, l) are constants of motion; therefore, obtain a an expression of the total energy and angular momentum when very far away $(r \to \infty)$ and combine them to obtain the impact parameter in terms of (E, l)

b) use the general relation from part (a) to re-derive the Rutherford scattering differential cross section (Eq. 5.28 of Fetter & Walecka) for a repulsive potential of the form $V(r) = k/r$, where $k = zZe^2 > 0$ and (ze) is the charge of the incident nucleus and (Ze) is the charge of the target nucleus

hint: note that the upper limit of integration (r_{min} or $u_0 = 1/r_{\text{min}}$) is the distance of closest approach, i.e. a turning point of motion where $dr/dt = 0$, therefore, by using the general expression for total energy of the system and setting $dr/dt|_{r=r_{\text{min}}} = 0$, one can solve for r_{min}

7) **(15 pts)** A uniform beam of particles with energy E is scattered by a repulsive central potential $V(r) = \gamma/r^2$. Derive the differential elastic cross section

$$\frac{d\sigma}{d\Omega} = \frac{\gamma\pi^2}{E\sin\theta}\frac{\pi - \theta}{\theta^2(2\pi - \theta)^2}$$