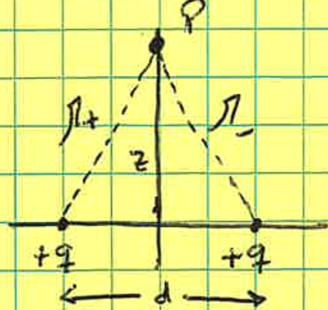
1) (a)

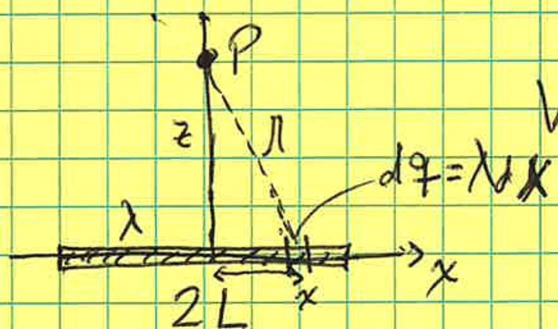


$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{2}\frac{q_i}{\mathscr{r}_i}$$

$$\mathscr{r}_\pm = \sqrt{z^2+\left(\frac{d}{2}\right)^2}$$

$$\boxed{V(z) = \frac{1}{4\pi\epsilon_0}\left[\frac{2q}{\sqrt{z^2+\left(\frac{d}{2}\right)^2}}\right]}$$

(b)



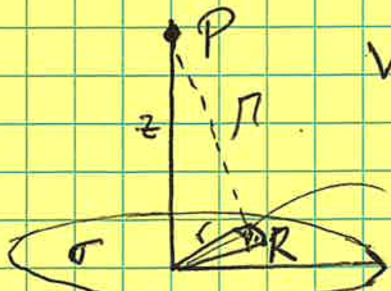
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda(\vec{r}')}{\mathscr{r}}dx$$

$$\mathscr{r} = \sqrt{x^2+z^2}$$

$$V(z) = \frac{\lambda}{4\pi\epsilon_0}\int_{-L}^{L}\frac{dx}{\sqrt{x^2+z^2}} = \frac{\lambda}{4\pi\epsilon_0}\underbrace{\left[\ln\left(x+\sqrt{x^2+z^2}\right)\Big|_{x=-L}^{L}\right]}_{\ln\left(L+\sqrt{L^2+z^2}\right) - \ln\left(-L+\sqrt{L^2+z^2}\right)}$$

$$\boxed{V(z) = \frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{L+\sqrt{z^2+L^2}}{-L+\sqrt{z^2+L^2}}\right]}$$

(c)



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\frac{\sigma(\vec{r}')\,da}{\mathscr{r}}$$

$$da = r\,d\phi\,dr,\quad \mathscr{r} = \sqrt{z^2+r^2}$$

$$V(z) = \frac{\sigma}{4\pi\epsilon_0}\int\frac{r\,dr\,d\phi}{\sqrt{z^2+r^2}} \Rightarrow \int_0^R\frac{r\,dr}{\sqrt{z^2+r^2}} = \sqrt{r^2+z^2}\Big|_0^R = \sqrt{R^2+z^2}-z$$

$$\Rightarrow \int_0^{2\pi}d\phi = 2\pi$$

$$V(z) = \frac{2\pi\sigma}{4\pi\epsilon_0}\left[\sqrt{R^2+z^2}-z\right] \Rightarrow \boxed{V(z) = \frac{\sigma}{2\epsilon_0}\left(\sqrt{R^2+z^2}-z\right)}$$

1) (d) $\vec{E} = -\nabla V = -\frac{\partial V}{\partial z}\hat{z}$

$$\Rightarrow \vec{E} = -\frac{\partial}{\partial z}\left(z^2+\left(\frac{d}{2}\right)^2\right)^{-1/2} \cdot \frac{2q}{4\pi\epsilon_0}$$

$$= -\frac{2q}{4\pi\epsilon_0}\left[-\frac{1}{\cancel{2}}\left(z^2+\left(\frac{d}{2}\right)^2\right)^{-3/2} \cdot \cancel{2}z\right]$$

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2qz\,\hat{z}}{\left(z^2+\left(\frac{d}{2}\right)^2\right)^{3/2}}}$$

(e) $$\vec{E} = -\frac{\lambda}{4\pi\epsilon_0}\frac{\partial}{\partial z}\underbrace{\left[\ln\left(L+\sqrt{L^2+z^2}\right)-\ln\left(-L+\sqrt{L^2+z^2}\right)\right]}_{\frac{2z\cdot\frac{1}{2}(L^2+z^2)^{-1/2}}{L+\sqrt{L^2+z^2}} - \frac{2z\cdot\frac{1}{2}(L^2+z^2)^{-1/2}}{-L+\sqrt{L^2+z^2}}}$$

$$= -\frac{\lambda}{4\pi\epsilon_0}\frac{z}{\sqrt{z^2+L^2}}\left\{\frac{-L+\sqrt{L^2+z^2}-L-\sqrt{L^2+z^2}}{(L^2+z^2)-L^2}\right\}\hat{z}$$

$$\Rightarrow \boxed{\vec{E} = \frac{2L\lambda}{4\pi\epsilon_0}\frac{1}{\sqrt{z^2+L^2}}\hat{z}}$$

(f) $$\vec{E} = -\frac{\sigma}{2\epsilon_0}\frac{\partial}{\partial z}\left(\sqrt{R^2+z^2}-z\right)\hat{z}$$

$$= -\frac{\sigma}{2\epsilon_0}\left(\frac{1}{2}(R^2+z^2)^{-1/2}\cdot 2z - 1\right)\hat{z}$$

$$\boxed{\vec{E} = \frac{\sigma}{2\epsilon_0}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right)\hat{z}}$$

2) PART A:

$$\rho_s = k/r^2 \quad (a \le r \le b)$$

(a) $r<a \Rightarrow \vec{E}\cdot d\vec{a} = \dfrac{Q_{encl}(r<a)}{\epsilon_0} = 0 \Rightarrow \boxed{\vec{E} = 0\hat{r}}$

(b) $a<r<b \Rightarrow \vec{E}\cdot d\vec{a} = \dfrac{Q_{encl}(a<r<b)}{\epsilon_0} \Rightarrow Q_{encl} = \int \rho_s \, dV$

$$\Rightarrow Q_{encl} = k\Big[\underbrace{\int_a^{r} dr}_{(r-a)} \underbrace{\int_0^{\pi} \sin\theta \, d\theta}_{2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}\Big] \qquad Q_{encl} = k\int \frac{r^2 \sin\theta \, dr \, d\theta \, d\phi}{r^2}$$

$$\Rightarrow Q_{encl} = 4\pi k(r-a)$$

$$\vec{E}\cdot d\vec{a} = E\cancel{4\pi} r^2 = \frac{\cancel{4\pi} k(r-a)}{\epsilon_0}$$

$$\Rightarrow \boxed{\vec{E} = \frac{k}{\epsilon_0}\left(\frac{r-a}{r^2}\right)\hat{r}}$$

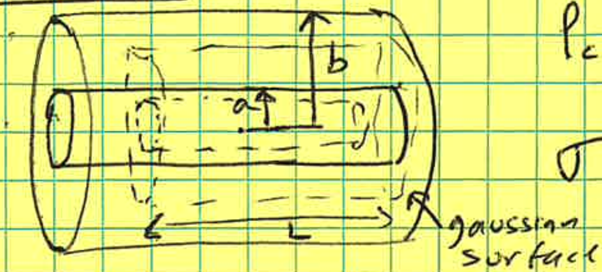
(c) $r>b \Rightarrow \vec{E}\cdot d\vec{a} = \dfrac{Q_{encl}(r>b)}{\epsilon_0} \Rightarrow Q_{encl} = \int \rho_s \, dV$

$$\Rightarrow Q_{encl} = k\Big[\underbrace{\int_a^{b} dr}_{(b-a)} \underbrace{\int_0^{\pi} \sin\theta \, d\theta}_{2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi}\Big] \Rightarrow Q_{encl} = 4\pi k(b-a)$$

$$\vec{E}\cdot d\vec{a} = E\cancel{4\pi} r^2 = \frac{\cancel{4\pi} k(b-a)}{\epsilon_0}$$

$$\Rightarrow \boxed{\vec{E} = \frac{k}{\epsilon_0}\,\frac{(b-a)}{r^2}\hat{r}}$$

2) PART B:



$\rho_c (s \le a) \rightarrow$ uniform volume charge density

$\sigma (s = b) \rightarrow$ uniform surface charge density

(d) $s < a \Rightarrow \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$, $\rho_c = \frac{Q}{\pi a^2 L} = \frac{Q_{enc.}}{\pi s^2 L}$

$\vec{E} \cdot d\vec{a} = E(s<a)\, 2\pi s L = \frac{\rho_c \pi s^2 L}{\epsilon_0}$ $\rightarrow Q_{encl} = Q\,(s/a)^2 = \rho_c \pi s^2 L$

$$\Rightarrow \boxed{\vec{E} = \frac{\rho_c s}{2\epsilon_0}\hat{s}}$$

(e) $a < s < b \Rightarrow \vec{E} \cdot d\vec{a} = \frac{Q_{en.}}{\epsilon_0}$ $\quad \rho_c = \frac{Q_{enc}}{\pi a^2 L}$

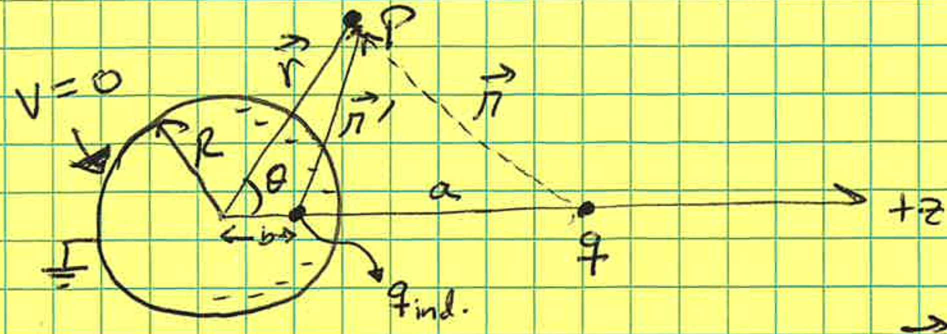
$E(a<s<b)\, 2\pi s L = \frac{\rho_c \pi a^2 L}{\epsilon_0}$

$$\Rightarrow \boxed{\vec{E} = \frac{\rho_c a^2}{2\epsilon_0 s}\hat{s}}$$

(f) $s > b \Rightarrow \vec{E} \cdot d\vec{a} = \frac{Q_{en}}{\epsilon_0}$ $\rightarrow Q_{en} = 0$ (total cable is electrically neutral)

$$\Rightarrow \boxed{\vec{E}(s>b) = 0\,\hat{s}}$$

3)



(a)

$$V(r>R) = \frac{1}{4\pi\epsilon_0}\left[\frac{q_{ind.}}{\mathcal{R}'} + \frac{q}{\mathcal{R}}\right] \qquad \vec{\mathcal{R}}' = r\hat{r} - b\hat{z}, \quad \vec{\mathcal{R}} = r\hat{r} - a\hat{z}$$

$$(\mathcal{R}')^2 = r^2 + b^2 - 2rb\cos\theta \qquad \mathcal{R}^2 = r^2 + a^2 - 2ra\cos\theta$$

<u>Boundary Condition</u>: $V(r=R,\ \theta=0) = 0$, $V(r=R,\ \theta=\pi) = 0$ } all "θ" at surface MUST give $V=0$ but select two strategic points

$$V(r=R,\ \theta=0) = \frac{1}{4\pi\epsilon_0}\left[\frac{q_{ind}}{\underbrace{\sqrt{R^2+b^2-2Rb}}_{\sqrt{(R-b)^2} \;=\; \pm(R-b)}} + \frac{q}{\underbrace{\sqrt{R^2+a^2-2Ra}}_{\sqrt{(R-a)^2} \;=\; \pm(R-a)}}\right] = 0$$

$\pm(R-a)$ → since $a>R$ take "−"

$$\Rightarrow \frac{q_{ind}}{(R-b)} + \frac{q}{(a-R)} = 0 \quad (1)$$

$$V(r=R,\ \theta=\pi) = \frac{1}{4\pi\epsilon_0}\left[\frac{q_{ind}}{\sqrt{R^2+b^2+2Rb}} + \frac{q}{\sqrt{R^2+a^2+2Ra}}\right] = 0$$

$$\Rightarrow \frac{q_{ind}}{(R+b)} + \frac{q_{ind.}}{(R+a)} = 0 \quad (2)$$

⇒ Since we have 2 eqs & 2 unknowns (q_{ind}, b) one can solve for the unknowns

solve (1) for q_{ind}: $q_{ind.} = -q\dfrac{(R-b)}{(a-R)} = q\dfrac{(R-b)}{(R-a)}$

solve (2) for $q_{ind.}$: $q_{ind.} = -q\dfrac{(R+b)}{(R+a)}$

set (1) = (2) $\Rightarrow \cancel{q}\dfrac{(R-b)}{(R-a)} = -\cancel{q}\dfrac{(R+b)}{(R+a)}$

$$\Rightarrow (R-b)(R+a) = -(R+b)(R-a) \Rightarrow R^2 + \cancel{aR} - \cancel{bR} - ab = \cancel{aR} + ab - R^2 - \cancel{bR}$$

3) (a) continued...

$$R^2 - ab = ab - R^2 \Rightarrow \cancel{2}R^2 = \cancel{2}ab$$

$$\Rightarrow \boxed{b = R^2/a}$$

Now substitute (b) into either (1) or (2) to solve q_{ind}.

$$(1) \quad q_{ind} = q\frac{(R-b)}{(R-a)} = q\frac{(R - R^2/a)}{(R-a)}$$

$$= \frac{q\,(aR - R^2)}{a\,(R-a)} = -\frac{qR}{a}\frac{\cancel{(R-a)}}{\cancel{(R-a)}}$$

$$\Rightarrow \boxed{q_{ind} = -qR/a}$$

$\Rightarrow$ Substitute (b, q_{ind}) into $V(r>R)$:

$$\boxed{V(r>R) = \frac{1}{4\pi\epsilon_0}\left[\frac{-q\,(R/a)}{\sqrt{r^2 + (R^2/a)^2 - 2r\,(R^2/a)\cos\theta}} + \frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}}\right]}$$

3) (b) $\sigma_{ch.} = -\epsilon_0 \left.\dfrac{\partial V}{\partial r}\right|_{r=R} = \epsilon_0 E_r (r=R)$

$$\Rightarrow E_r = -\frac{\partial V}{\partial r} = -\frac{1}{4\pi\epsilon_0}\left[\frac{-q(R/a)\cdot(-1/2)\cdot(2r-(2R^2/a)\cos\theta)}{\left[r^2+(R^2/a)^2-2r(R^2/a)\cos\theta\right]^{3/2}} - \frac{q(1/2)\cdot(2r-2a\cos\theta)}{(r^2+a^2-2ra\cos\theta)^{3/2}}\right]\Bigg|_{r=R}$$

$$E = \frac{q}{4\pi\epsilon_0}\left\{\frac{r-a\cos\theta}{(r^2+a^2-2ra\cos\theta)^{3/2}} - \frac{(R/a)\left[r-(R^2/a)\cos\theta\right]}{\left(r^2+(R^2/a)^2-2r(R^2/a)\cos\theta\right)^{3/2}}\right\}\Bigg|_{r=R}$$

$$\Rightarrow E(r=R) = \frac{q}{4\pi\epsilon_0}\left\{\frac{R-a\cos\theta}{(R^2+a^2-2Ra\cos\theta)^{3/2}} - \frac{(R/a)\left[R-(R^2/a)\cos\theta\right]}{\left[R^2+(R^2/a)-2R(R^2/a)\cos\theta\right]^{3/2}}\right\}$$

factor out $(R^2/a^2) \Rightarrow \left[\left(\frac{R}{a}\right)^2\cdot\left(a^2+R^2-2Ra\cos\theta\right)\right]^{3/2}$

$$\Rightarrow \left(\frac{R}{a}\right)^3\cdot\left(R^2+a^2-2Ra\cos\theta\right)^{3/2}$$

substitute back to denom.

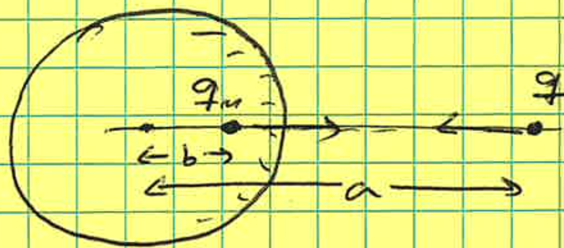
$$E_r(r=R) = \frac{q}{4\pi\epsilon_0}\left\{\frac{R-a\cos\theta}{(\ \text{''}\quad \text{''}\ )} - \frac{\cancel{(R/a)}\left[R-(R^2/a)\cos\theta\right]}{(R/a)^{\cancel{3}\,2}(\ \text{''}\quad\text{''}\ )}\right\}$$

$$E_r = \frac{q}{4\pi\epsilon_0}\left\{\frac{R-a\cos\theta-(a^2/R^2)\left[R-(R^2/a)\cos\theta\right]}{(R^2+a^2-2Ra\cos\theta)^{3/2}}\right\}$$

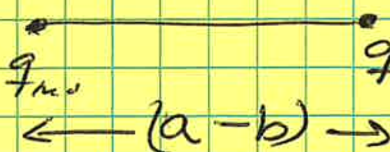
$$= \frac{q}{4\pi\epsilon_0}\left\{\frac{R-a\cos\theta-a^2/R+a\cos\theta}{\text{''}\quad\text{''}}\right\} = \frac{q}{4\pi\epsilon_0}\left\{\frac{(R^2-a^2)/R}{\text{''}\ \text{''}}\right\}$$

$$\Rightarrow \boxed{\sigma_{ch.} = E_r\epsilon_0 = \frac{q}{4\pi R}\frac{R^2-a^2}{(R^2+a^2-2Ra\cos\theta)^{3/2}}}$$

3) (C)



The electric force is the same as that between 2 point charges



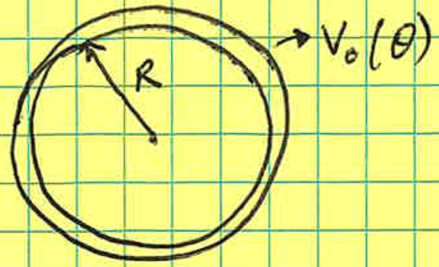
$$\vec{F}_e = \frac{1}{4\pi\epsilon_0}\frac{q \cdot q_{ind}}{(a-b)^2}\hat{z}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q(-qR/a)}{(a-R^2/a)^2}\hat{z}$$

$$= \frac{-q^2}{4\pi\epsilon_0}\frac{R/a \ \hat{z}}{(a^2-R^2)^2/a^2}$$

$$\boxed{\vec{F}_e = -\frac{q^2 R a \hat{z}}{4\pi\epsilon_0 (a^2-R^2)^2}}$$

4)



general solution in spherical coord.

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

(a) $V(r<R,\ \theta)$

Boundary conditions (1) & (2)

(1) $\lim\ r \to 0,\ V(0,\theta) \to \infty$ due to "$\frac{B_l}{r^{l+1}}$"

$\Rightarrow \boxed{B_l = 0 \text{ for all } l}$

"otherwise, the potential blows up"

$$\Rightarrow \boxed{V(r<R,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)}$$

(2) $V(r=R,\ \theta) = \sum A_l R^l P_l(\cos\theta) = V_0(\theta)$

To solve for A_l, use "Fourier's trick" on the Legendre polynomials since they are an orthogonal set

Multiply (2) by $P_{l'}(\cos\theta) * \sin\theta$ and integrate $\theta: 0 \to \pi$

$$\int_0^{\pi}\sum_l A_l R^l P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta = \int_0^{\pi} V_0(\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta$$

$$\int P_l P_{l'} \sin\theta\, d\theta = \begin{cases} 0 & l' \neq l \\ \frac{2}{2l+1} & l' = l \end{cases}$$

The infinite sum over l reduces to only a single value $l = l'$

$$A_l R^l \cdot \frac{2}{2l+1} = \int_0^{\pi} V_0(\theta)P_l(\cos\theta)\sin\theta\, d\theta$$

$$\Rightarrow \boxed{A_l = \frac{2l+1}{2R^l}\int_0^{\pi} V_0(\theta)P_l(\cos\theta)\sin\theta\, d\theta}$$

4) (b) $V(r>R,\theta)$

<u>Boundary Conditions:</u>

(1) $V(r=R,\theta)=V_0$

(2) $V(r\to\infty,\theta)\to 0$

As $r\to\infty$

$\frac{B_\ell}{r^{\ell+1}}\to 0$, $A_\ell r^\ell\to\infty$

$\Rightarrow \boxed{A_\ell = 0}$ → necessary condition for $V\to 0$ at $r\to\infty$ otherwise $V\to$ blows up

$$\Rightarrow \boxed{V(r>R,\theta)=\sum_\ell \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta)}$$

To determine "B_ℓ" use Fourier's trick on boundary condition (1)

$$\Rightarrow \sum_\ell \frac{B_\ell}{R^{\ell+1}} \underbrace{\int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta}_{\begin{cases} 0 & \ell'\neq\ell \\ \frac{2}{2\ell+1} & \ell'=\ell \end{cases}} = \int_0^\pi V_0(\theta) P_{\ell'}(\cos\theta)\sin\theta\, d\theta$$

$$\Rightarrow \boxed{B_\ell = \frac{2\ell+1}{2} R^{\ell+1} \int_0^\pi V_0(\theta) P_\ell(\cos\theta)\sin\theta\, d\theta}$$

4) (c) $V_0(\theta) = k\sin^2(\theta/2)$

(i) $V_0(\theta) = k\left(\sqrt{\frac{1-\cos\theta}{2}}\right)^2 = \frac{k}{2}(1 - \cos\theta)$

$1 \downarrow P_0(\cos\theta)$, $\cos\theta \downarrow P_1(\cos\theta)$

$$\Rightarrow \boxed{V_0(\theta) = \frac{k}{2}\left[P_0(\cos\theta) - P_1(\cos\theta)\right]}$$

(ii)

$$V(r=R,\theta) = \sum_\ell A_\ell R^\ell P_\ell(\cos\theta) = V_0(\theta)$$

$$\Rightarrow A_0 R^0 P_0 + A_1 R^1 P_1 = \frac{k}{2}P_0 - \frac{k}{2}P_1$$

$$\Rightarrow \boxed{A_0 = \frac{k}{2}} \qquad A_1 R = -\frac{k}{2} \Rightarrow \boxed{A_1 = -\frac{k}{2R}}$$

$$\Rightarrow V(r<R,\theta) = A_0 r^0 P_0 + A_1 r^1 P_1$$

$$= \frac{k}{2} - \frac{kr}{2R}\cos\theta$$

$$\Rightarrow \boxed{V(r<R,\theta) = \frac{k}{2}\left(1 - \frac{r}{R}\cos\theta\right)}$$

4) (c) $V_0(\theta) = k \sin^2 \theta/2$

(iii) $V(r=R,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) = V_0(\theta)$

$$\frac{B_0}{R} P_0 + \frac{B_1}{R^2} P_1 = \frac{k}{2} P_0 - \frac{k}{2} P_1$$

$$\Rightarrow \frac{B_0}{R} = \frac{k}{2} \Rightarrow \boxed{B_0 = \frac{kR}{2}}$$

$$\Rightarrow \frac{B_1}{R^2} = -\frac{k}{2} \Rightarrow \boxed{B_1 = -\frac{kR^2}{2}}$$

$$V(r>R,\theta) = \frac{B_0}{r} P_0 + \frac{B_1}{r^2} P_1$$

$$\boxed{V(r>R,\theta) = \frac{kR}{2r} - \frac{k}{2}\frac{R^2}{r^2}\cos\theta} \Leftrightarrow V_{out}$$

(iv) from (ii) & (iii) $V_{in} \equiv V(r<R,\theta) = \frac{k}{2}\left(1 - \frac{r}{R}\cos\theta\right)$

$V_{out} \equiv V(r>R,\theta) = \frac{k}{2}\left(\frac{R}{r} - \left(\frac{R}{r}\right)^2\cos\theta\right)$

$$\frac{\partial V_{out}}{\partial r} = \frac{k}{2}\left(-\frac{R}{r^2} - \frac{2R^2}{r^3}\cos\theta\right)$$

$$\frac{\partial V_{in}}{\partial r} = \frac{k}{2}\left(-\frac{\cos\theta}{R}\right)$$

$$\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r} = -\frac{kR}{2r^2} - \frac{kR^2}{r^3}\cos\theta + \frac{k\cos\theta}{2R} = -\frac{\sigma_0(\theta)}{\epsilon_0}$$

evaluating $\left(\partial V_{out}/\partial r - \frac{\partial V_{in}}{\partial r}\right)$ at $r=R$

$$\Rightarrow -\frac{k}{2R} - \frac{2k\cos\theta}{2R} + \frac{k\cos\theta}{2R} = -\frac{\sigma_0}{\epsilon_0} \Rightarrow \frac{-k - k\cos\theta}{2R} = -\frac{\sigma_0}{\epsilon_0}$$

$$\Rightarrow \boxed{\sigma_0(\theta) = \frac{\epsilon_0 k}{2R}(1+\cos\theta)}$$

5) $\rho(r', \theta) = \frac{kR}{(r')^2}(R-2r')\sin\theta$

(a) $V_{mon.} = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(r', \theta)\,d\tau'$

$$\downarrow$$

$$kR\int \frac{(R-2r')}{(r')^2}\sin^2\theta\,(r')^2\,dr'\,d\theta\,d\phi$$

$r: \rightarrow \int_0^R (R-2r')\,dr' = Rr'\Big|_0^R - \frac{2(r')^2}{2}\Big|_0^R = R^2 - R^2 = 0$

$$\rightarrow \boxed{V_{mon} = 0}$$

(b) $V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\alpha\,\rho(r')\,d\tau'$

$$\downarrow$$

$$kR\int (R-2r')\,r'\cos\theta\sin^2\theta\,dr'\,d\theta\,d\phi$$

$\theta: \rightarrow \int \sin^2\theta\cos\theta\,d\theta = \int_0^\pi \sin^2\theta\,d(\sin\theta)$

let $x \equiv \sin\theta \rightarrow$

$\theta = 0 \rightarrow \sin(0) = 0$, $x = 0$

$\theta = \pi \rightarrow \sin(\pi) = 0$, $x = 0$

$$\int_0^0 x^2\,dx = \frac{x^3}{3}\Big|_0^0 = 0$$

$$\boxed{V_{dip} = 0}$$

5) (c) $V_{quad.} = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int (r')^2 \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \rho(r') d\tau$

$\rightarrow kR \int \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)(R - 2r') \sin\theta (r')^2 \, dr' \, d\theta \, d\phi$

$r: \int_0^R (r')^2 (R - 2r') \, dr' = \frac{R(r')^3}{3}\Big|_0^R - \frac{2(r')^4}{4}\Big|_0^R$

$= \frac{R^4}{3} - \frac{R^4}{2} = \underline{\underline{-\frac{R^4}{6}}}$

$\theta: \frac{1}{2}\int (3\cos^2\theta - 1)\sin^2\theta \, d\theta$

$\downarrow$

$\int_0^\pi 3\cos^2\theta \sin^2\theta \, d\theta - \int_0^\pi \sin^2\theta \, d\theta$

$\downarrow$

$\int 3(1 - \sin^2\theta)\sin^2\theta \, d\theta$

$3\left(\int \sin^2\theta \, d\theta\right) - 3\int \sin^4\theta \, d\theta - \int \sin^2\theta \, d\theta$

$2\int \sin^2\theta \, d\theta - 3\int \sin^4\theta \, d\theta$

$\downarrow$

$2\left[\frac{1}{2}\theta - \frac{\sin 2\theta}{4}\Big|_0^\pi\right] - 3\left[\frac{3}{8}\theta - \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32}\Big|_0^\pi\right]$

$\downarrow$

$\pi \qquad - 3\left[\frac{3}{8}\pi\right] = \frac{8\pi}{8} - \frac{9}{8}\pi = \underline{\underline{-\frac{\pi}{8}}}$

$\phi: \int_0^{2\pi} d\phi = \underline{\underline{2\pi}}$

5) (c) continued...

Putting the integral results together...

$$V_{quad.} = \frac{1}{4\pi\epsilon_0} \frac{-kR}{r^3} \left(-\frac{R^4}{6}\right)\left(\frac{1}{\cancel{2}}\cdot\left(-\frac{\pi}{8}\right)\right) \cdot \cancel{2}\pi$$

$$V_{quad.} = \frac{1}{4\pi\epsilon_0} \frac{k\pi^2 R^5}{48 r^3}$$ for a point P on the z-axis $r \rightarrow z$

$$\boxed{V_{quad} \simeq \frac{1}{4\pi\epsilon_0} \frac{k\pi^2 R^5}{48 z^3}}$$

(d) $\vec{E} = -\nabla V$

$$\boxed{\begin{aligned} \vec{E}_{mon} &= -\nabla(0) = 0 \\ \vec{E}_{dp} &= -\nabla(0) = 0 \end{aligned}}$$

$$E_{quad} = -\nabla V_{quad.} = -\frac{\partial V}{\partial r}$$

$$= -\frac{1}{4\pi\epsilon_0} \frac{k\pi^2 R^5}{48} \underbrace{\frac{\partial}{\partial r}\left(r^{-3}\right)}_{-3r^{-4}}$$

$$\boxed{E_{quad} = \frac{1}{4\pi\epsilon_0} \frac{k\pi^2 R^5}{16 r^4}}$$