

## SEMINAR II

# Structure of the deuteron

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### Abstract

*Basic properties of the deuteron are given. A simple model is established step-by-step, considering principles of quantum mechanics and symmetry. It turns out that the deuteron is a mixture of two angular momentum states  $L$ , which is an indication that the nuclear force is non-central. Discussion on the two-nucleon force gives some insight into the shape of the deuteron wave-function for both spin sub-states  $M = 0$  and  $M = \pm 1$ . The structure of the deuteron can be probed in elastic electron-deuteron ( $e, d$ ) scattering experiments. To obtain complete information on the deuteron's three form factors, we introduce tensor polarization of the deuteron target. The tensor asymmetry  $T_{20}$  between cross-sections for two different polarizations is an additional observable, that makes it possible to determine the form factors.*

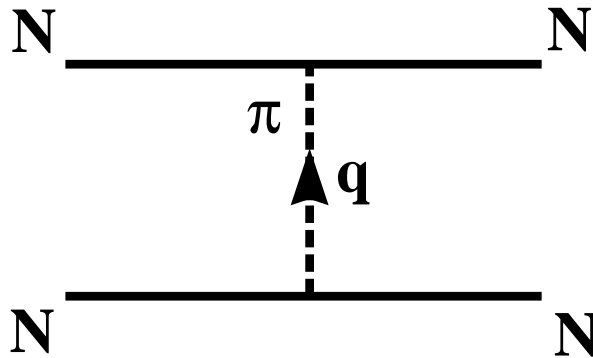
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# 1 Introduction

Apart from the proton, the nucleus of the hydrogen atom, the deuteron, consisting of one proton and one neutron, is the simplest nucleus bound together by the nuclear force. As such it is a testing ground for various models of nuclear structure.

The deuteron was discovered in 1932, in time when no satisfactory theory of nucleus was available. The nucleus was thought to be composed of protons and electrons since these were the only known charged particles and nuclei were seen to emit electrons ( $\beta$  decay) [1].



**Figure 1:** *Feynman diagram that generates the one-pion exchange potential, valid for large separations. [2]*

Today, nuclear force can be understood in terms of meson exchange. The first such attempt was Yukawa's suggestion that the force was mediated by a strongly interacting massive particle which later became known as the pion. We now know that mesons and nucleons themselves are composite particles, and meson-nucleon theories have been replaced by a more fundamental theory describing hadrons in terms of quarks and gluons. Thus nuclear force is an effect of the underlying strong interaction, based on exchange of gluons between quarks (quantum chromodynamics). However, the meson-nucleon exchange approach is still a strong element in nuclear physics as a basis for phenomenology and is making a potentially more rigorous comeback in the form of the effective field theories. The main reason for persistence of meson exchange theories is the complexity of quantum chromodynamics. [1]

## 2 Basic properties

We will obtain certain insights into the deuteron and the nucleon-nucleon (NN) interaction by using only quantum-mechanical principles. We shall not invoke field theory, instead a simple non-relativistic description will suffice.

### 2.1 Potential model

In a potential model we assume that the nuclear force can be represented by a two-body potential  $V$  [1]. We start with a Hamiltonian for general two-body problem in quantum mechanics.

$$H_{\text{total}} = \frac{\mathbf{p}_1^2}{2m_1} + \frac{\mathbf{p}_2^2}{2m_2} + V \quad (1)$$

Using a new set of coordinates, the relative displacement  $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ , and the center-of-mass point  $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2)$ , the relative movement becomes a one-body problem.

$$H = -\frac{\hbar^2}{2\mu}\nabla^2 + V \quad (2)$$

For the deuteron, the reduced mass is given by  $\mu = (m_p m_n)/(m_p + m_n)$ . The two-nucleon potential  $V$  cannot be assumed central, so it is dependent on the vector displacement  $\mathbf{r}$ . Spins  $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2$  and isospins  $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2$  of both nucleons should also be involved.

$$H |\psi_d\rangle = \varepsilon |\psi_d\rangle \quad (3)$$

### 2.2 Taking symmetries into account

The deuteron wave-function is composed of three parts – space, spin and isospin part. Experimental observation for total spin and parity gives  $J^\pi = 1^+$ , so we can conclude two facts from that. The total spin is a good quantum number i.e.  $[H, \mathbf{J}^2] = 0$ , while parity is positive only for eigenstates with even<sup>1</sup> relative angular momentum  $L = 0, 2, \dots$

Furthermore, there are only  $S = 0, 1$  possibilities for the total spin of two nucleons. Using the rule  $|L - S| \leq J \leq L + S$  leaves us with only two different states of the space

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<sup>1</sup> $Y_{LM}(\pi - \theta, \pi + \phi) = (-1)^L Y_{LM}(\theta, \phi)$

part appropriate.

$$\begin{array}{ccc} J & L & S \\ \hline 1 & = 0 + 1 \\ 1 & = 2 - 1 \end{array}$$

Now that we have reduced the Hilbert space for spin and spatial parts, we start with construction of isospin part. Proton and neutron are, if we ignore a slight mass difference, distinct only by the charge they carry and we treat them like two different aspects of the same particle, the nucleon. Since only two nucleon states exist ( $p, n$ ), we can assign isospin  $t = \frac{1}{2}$  to a nucleon, based on the analogy that a spin- $\frac{1}{2}$  system can have two different spin substates. The proton and neutron are members of an isospin doublet.

$$|p\rangle = |t = 1/2, t_3 = +1/2\rangle, \quad (4)$$

$$|n\rangle = |t = 1/2, t_3 = -1/2\rangle \quad (5)$$

Our two-fermion system's total wave-function has to be antisymmetric with respect to changing proton with neutron. Because spatial and spin parts are both symmetric, the isospin part has to be antisymmetric. The only antisymmetric combination against permutation of the nucleons is the one of the singlet coupling with total isospin  $T = 0$ .

$$|T = 0, T_3 = 0\rangle \sim \frac{1}{\sqrt{2}} \{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\} \sim \frac{1}{\sqrt{2}} \{|pn\rangle - |np\rangle\} \quad (6)$$

## 2.3 Wave-function

Summarizing our knowledge of the deuteron ground state, and omitting the isospin part for a moment, the wave-function can be written in the following way [3]:

$$\psi_M(\mathbf{r}) = R_0(r)\mathcal{Y}_{101}^M(\theta, \phi) + R_2(r)\mathcal{Y}_{121}^M(\theta, \phi) \quad (7)$$

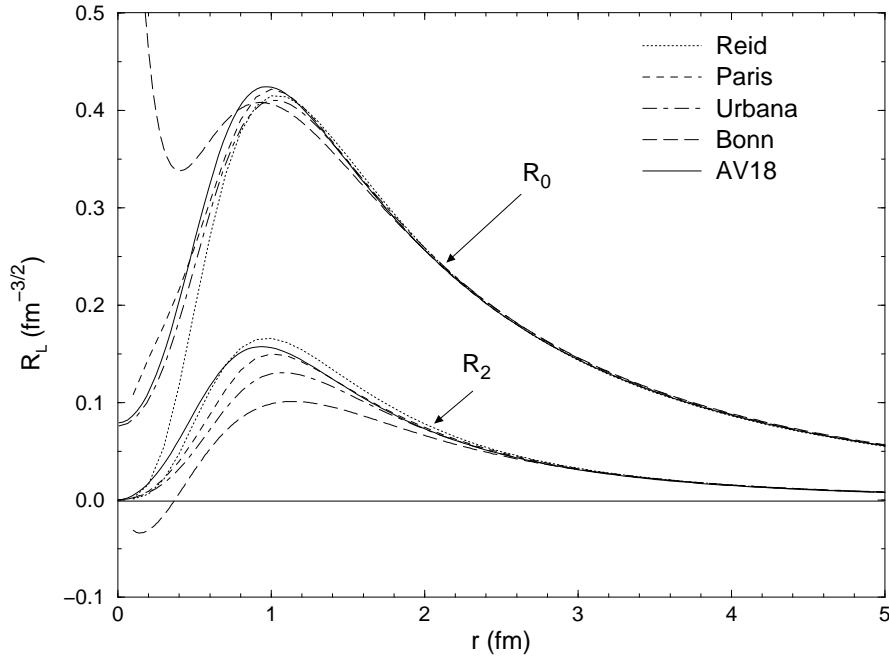
The spin-angle functions  $\mathcal{Y}_{JLS}^M$  are given by

$$\mathcal{Y}_{JLS}^M(\theta, \phi) = \sum_{M_L, M_S} \langle L, M_L, S, M_S | J, M \rangle Y_{L, M_L}(\theta, \phi) \chi_{S, M_S}. \quad (8)$$

The two terms in (7) are the so-called S- and D-wave components. Presence of a non-zero quadrupole moment (Table 1) suggests that a D-wave is present, because charge distribution of a pure S-wave component is spherically symmetric. This means that deuteron is a superposition of two states with different  $L$ . Probabilities for the two  $L$ -states are

$$P_S = \int_0^\infty R_0^2(r) r^2 dr \quad \text{and} \quad P_D = \int_0^\infty R_2^2(r) r^2 dr, \quad (9)$$

and normalization demands  $P_S + P_D = 1$ . Various models predict  $P_D$  to be  $\sim [5 \pm 2]\%$  [3]. This mixing of states is due to the non-central component of the NN force. As can be seen from Figure 2, various models predict equivalent behaviour at large separations ( $> 2$  fm), with large deviations below 1 fm. However, most models predict that wave-function diminishes at  $r = 0$ , and this is a strong hint that the NN potential is repulsive at short distances.

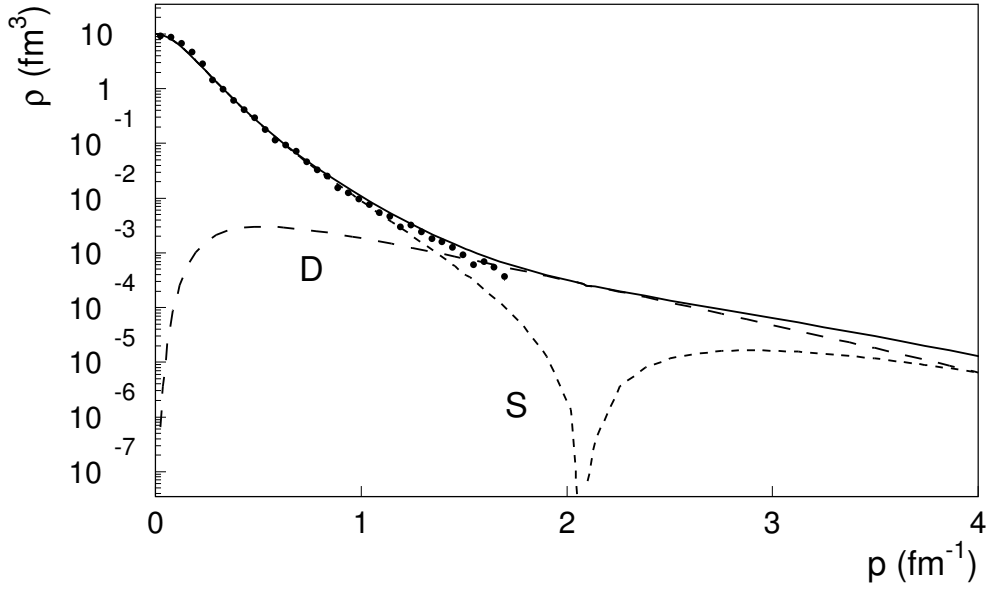


**Figure 2:** The S- and D-wave components of the deuteron wave-functions in  $r$ -space for various parametrizations (different dashing) of the NN potential. [4]

Now we apply a Fourier transform to the ground state wave-function (7)

$$\psi_M(\mathbf{p}) = \int \psi_M(\mathbf{r}) e^{-i\mathbf{p}\mathbf{r}} d\mathbf{r}, \quad (10)$$

and observe momentum distributions. In terms of S- and D-wave component contributions, results are seen on Figure 3. As can be seen non-spherical D-wave component is dominant in the high momentum region.



**Figure 3:** Deuteron *S*-wave(short-dashed), *D*-wave(long-dashed) and total (solid) momentum density distribution, derived from the Argonne *v*-18 model. [3]

Ground State Property	Value
Mass, $M_d$	1875.612762 (75) MeV
Binding energy, $\varepsilon$	2.22456612 (48) MeV
Spin and parity, $J^\pi$	$1^+$
Isospin, $T$	0
Magnetic dipole moment, $\mu_d$	0.8574382284 (94) $\mu_N$
Electric quadrupole moment, $Q_d$	0.2859 (3) $e \text{ fm}^2$
Matter radius, $r_m$	1.975 (3) fm

**Table 1:** Ground state properties of the deuteron, compiled from [1].

The measured properties of the deuteron are compiled in Table 1. Note that the binding energy per nucleon of the deuteron is 1.1 MeV. This is much less than for majority of nuclei, where a rule of thumb may be applied that each nucleon contributes 8 MeV to the binding energy [5]. Hence no excited states of the deuteron exist. Low binding energy makes the deuteron a very good substitute for free neutrons, since free neutrons cannot be used as targets because of their short lifetime in  $\beta$ -decay [5].

$$n \rightarrow p + e^- + \bar{\nu}_e, \quad \tau = (886.7 \pm 1.9) \text{ s}$$

### 3 The two-nucleon interaction in the $T, S = 0, 1$ state

Quantum chromodynamics (QCD) is the fundamental theory of quark and gluon interaction. However, we are still unable to derive the NN interaction from the first principles. This is why the description in terms of meson exchange (Figure 1) is still used.

Many realistic models exist, constructed by fitting the available two-nucleon scattering data. The way the deuteron looks like in the  $T, S = 0, 1$  state appears to be relatively model independent. The interaction in  $T, S = 0, 1$  state in most models can be expressed as

$$v = v^c(r) + v^t(r)S_{12} + v^{ls}(r)\mathbf{L} \cdot \mathbf{S} + v^{l2}(r)\mathbf{L}^2 + v^{ls2}(r)(\mathbf{L} \cdot \mathbf{S})^2. \quad (11)$$

The above form of the potential comes from considering other symmetries [5]. The functions  $v$  are fitted to experimental data. Basic features of the deuteron can be described already by dropping all  $L$  and  $LS$  terms and keeping only the static part of the interaction.

$$v^{\text{stat}} = v^c(r) + v^t(r)S_{12} \quad (12)$$

#### 3.1 Toroidal structures

It is instructive to study the expectation values of  $v^{\text{stat}}$  in eigenstates of the position operator  $\mathbf{r}$ . The static potential is dependent only on the mutual distance  $r$  and  $\theta$ , the polar angle of  $\mathbf{r}$  with respect to the spin-quantization axis  $z$ . For the two values of  $M$  we get

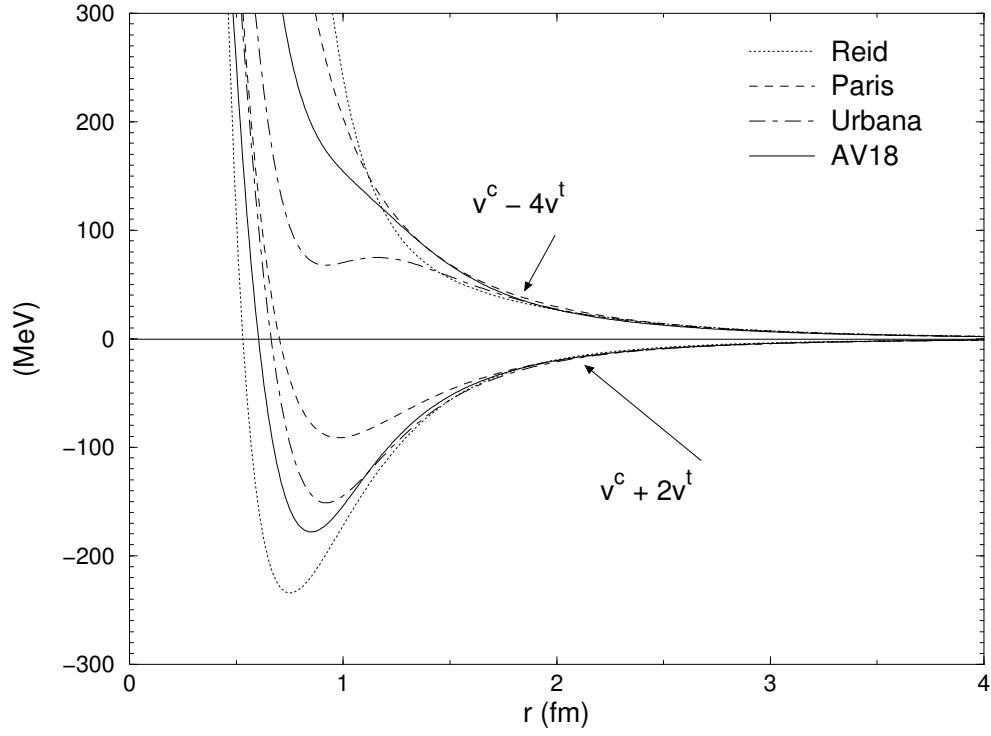
$$\langle M = 0 | v^{\text{stat}}(\mathbf{r}) | M = 0 \rangle = v^c(r) - 4v^t(r)P_2(\cos \theta), \quad (13)$$

$$\langle M = \pm 1 | v^{\text{stat}}(\mathbf{r}) | M = \pm 1 \rangle = v^c(r) + 2v^t(r)P_2(\cos \theta). \quad (14)$$

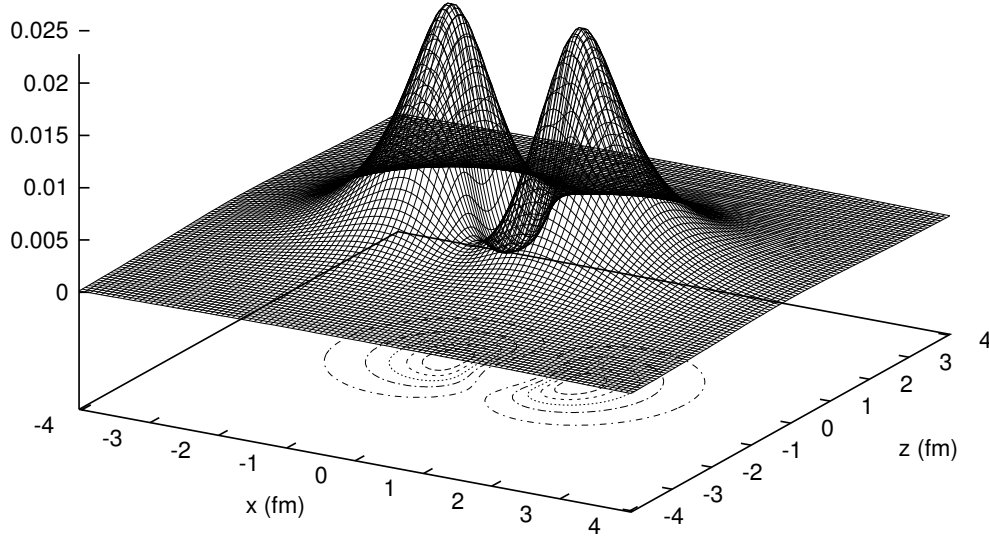
The potential for different polarizations behaves differently (Figure 4). For the  $M = 0$  state, the potential is repulsive in the vicinity of the quantization axis  $z$ , and attractive in the  $xy$ -plane, implying the preferred position of the nucleons on the  $xy$ -plane. The opposite is true for the  $M = \pm 1$  case, where the preferred orientation is along the quantization axis.

Taking a look at  $xz$ -cross-section of the probability densities of both possible polarizations (Figure 5 and 6) confirms the anticipated preferred positions of the nucleons. As the system possesses axial symmetry, the rotation of Figures 5 and 6 around the quantization axis generates structures shown in Figure 7. In terms of a maximum equi-density surfaces, the  $M = 0$  state generates a torus in the  $xy$ -plane, while  $M = \pm 1$  state generates

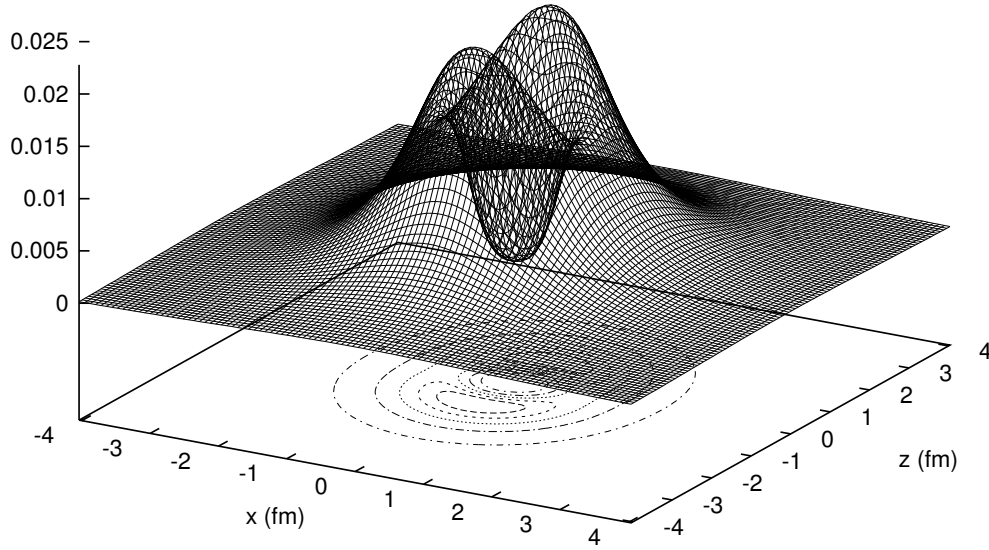




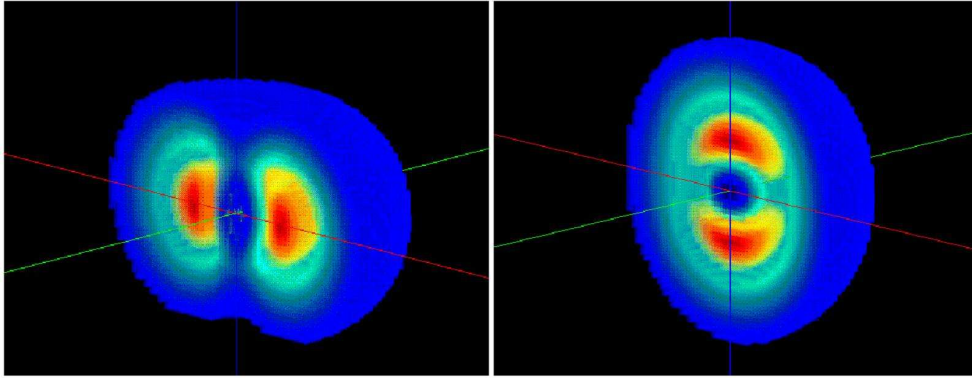
**Figure 4:** The upper four lines show expectation values of  $v^{stat}$  for  $M = 0, \theta = 0$ , and the lower four lines are for  $M = 0, \theta = \pi/2$  or equivalently  $M = \pm 1, \theta = 0$ . The expectation values for  $M = \pm 1, \theta = \pi/2$  (not shown) are half-way between. [4]



**Figure 5:** Deuteron density  $\rho_M [fm^{-3}]$  as a function of  $x, z$  for  $M = 0$ . [3]



**Figure 6:** Deuteron density  $\rho_M [\text{fm}^{-3}]$  as a function of  $x, z$  for  $M = \pm 1$ . [3]



**Figure 7:** Deuteron densities in  $M = 0$  (left) and  $M = \pm 1$  (right) magnetic substates. The red spots correspond to the maximal nucleonic densities, while the dark volumes correspond to lower densities (outer surface is for 10% of maximal density). [1]

a dumbbell<sup>2</sup> aligned along the  $z$ -axis. Averaging the density over the three polarizations ( $M = -1, 0, +1$ ), which is the case corresponding to an isotropic space i.e. in absence of quantization axis, gives a spherically symmetric distribution. The interesting fact is that the maximal nucleon density in the deuteron is about twice larger than the density in the centre of a typical nucleus.

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<sup>2</sup>A short bar with weights attached to both ends, used for weight lifting.

## 4 Elastic electron-deuteron scattering

### 4.1 Kinematics and observables

In elastic scattering of an ultra-relativistic electron of energy  $E_e$  from a deuteron with mass  $M_d$  at a laboratory angle  $\theta_e$ , the energy  $E_{e'}$  of the electron in the final state is given by

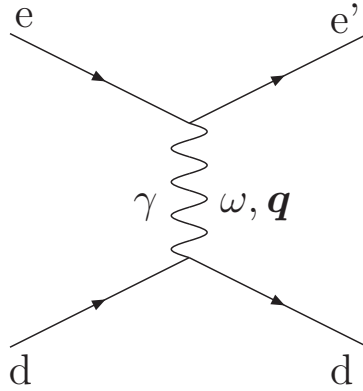
$$E_{e'} = E_e/f, \quad (15)$$

where  $f$  is the recoil factor. It has to be included, since deuteron's mass cannot be considered infinite.

$$f = 1 + \frac{2E_e}{M_d} \sin^2 \frac{\theta_e}{2} \quad (16)$$

Using the natural units  $\hbar = c = 1$ , the squared four-momentum transfer  $Q^2$  of the photon to the deuteron (Figure 8) is

$$Q^2 = \mathbf{q}^2 - \omega^2 = 4E_e E_{e'} \sin^2 \frac{\theta_e}{2}. \quad (17)$$



**Figure 8:** Feynman diagram that represents the scattering of an electron on the deuteron. Exchanged photon transfers energy  $\omega$  and momentum  $\mathbf{q}$ .

We start with the Rutherford scattering cross-section where we neglect both particle spins and structure.

$$\left( \frac{d\sigma}{d\Omega} \right)_{\text{Rutherford}} = \frac{\alpha^2}{4f E_e^2 \sin^4(\theta_e/2)} \quad (18)$$

When the spin of the electron is introduced, we obtain the Mott cross-section, describing a scattering of an electron from a point-like spinless particle. Neglecting electron's mass it

can be simplified a bit [6].

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} \left(1 - \beta^2 \sin^2 \frac{\theta_e}{2}\right) \approx \frac{\alpha^2 \cos^2(\theta_e/2)}{4fE_e^2 \sin^4(\theta_e/2)} \quad (19)$$

Final form has to include deuteron's electromagnetic structure effects.

As a reminder, consider a spatially extended particle with no particular direction in space. If a virtual photon with large momentum possesses a wavelength short enough to resolve the local charge distribution of the target particle, then the scattered electron will see only part of the charge. That is why scattering cross sections of spatially extended targets always decrease in comparison with the Mott cross-section (19). It turns out that the Mott cross section has to be corrected by factor  $|F(\mathbf{q}^2)|^2$ , i.e. the square of the charge form factor, to account for the structure of the probed particle. Form factor is defined as a Fourier transform of the normalized charge distribution  $\rho(\mathbf{r})$  [6].

$$F(\mathbf{q}^2) = \int e^{i\mathbf{q}\mathbf{r}} \rho(\mathbf{r}) d^3r \quad (20)$$

The  $\mathbf{q}^2$  dependency indicates that the charge form factor does not depend on the direction of momentum transfer. Obviously, in the  $\mathbf{q} \rightarrow 0$  limit form factor (FF) should just return the value of the charge. Particles with spin require several FFs, that is two for spin- $\frac{1}{2}$  and three for spin-1 particles, which is the case we shall encounter shortly [6].

In the case of the deuteron, the corrected Mott cross section is written like [3]

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} S, \quad (21)$$

where

$$S = A(Q^2) + B(Q^2) \tan^2 \frac{\theta_e}{2}. \quad (22)$$

All kinematics and information on the electron side of the reaction is packed in the Mott cross-section, while  $A(Q^2)$  and  $B(Q^2)$  contain information on the electromagnetic structure of the deuteron. The above expression for  $S$ , known as the Rosenbluth formula, allows separation of  $A(Q^2)$  and  $B(Q^2)$  by measuring the cross-section (21) at small scattering angles  $\theta_e$  to emphasize  $A(Q^2)$ , while backward scattering is almost entirely determined by  $B(Q^2)$ .

## 4.2 Requirement of polarized deuteron-target

Because the deuteron is a spin-one system, its structure functions  $A(Q^2)$  and  $B(Q^2)$  are given by three electromagnetic form factors  $G_C(Q^2), G_Q(Q^2), G_M(Q^2)$ , namely the charge

monopole, charge quadrupole, and magnetic form factor. These three are normalized in the static limit, so that they correspond to the static charge, charge quadrupole moment, and magnetic dipole moment [6].

$$G_C(0) = 1 \quad (23)$$

$$G_Q(0) = M_d^2 Q_d \quad (24)$$

$$G_M(0) = \frac{M_d}{m_p} \mu_d \quad (25)$$

$A$  and  $B$  can be given in terms of the three form factors, which all depend only on the  $Q^2$  [6].

$$A(Q^2) = G_C^2(Q^2) + \frac{8}{9}\eta^2 G_Q^2(Q^2) + \frac{2}{3}\eta G_M^2(Q^2), \quad (26)$$

$$B(Q^2) = \frac{4}{3}\eta(1+\eta)G_M^2(Q^2) \quad (27)$$

Here  $\eta = Q^2/4M_d^2$ . From the expressions for  $A$  and  $B$  we clearly see, that we cannot separate  $G_C$  and  $G_Q$ . To determine these two we need to introduce polarization degrees of freedom.

Polarized deuteron target has nonuniform distribution of the three spin substates which we describe in terms of polarizations. One is the vector polarization specifying the difference between relative populations of the deuterons polarized up and down.

$$P_z = n_{M=+1} - n_{M=-1}, \quad -1 \leq P_z \leq 1 \quad (28)$$

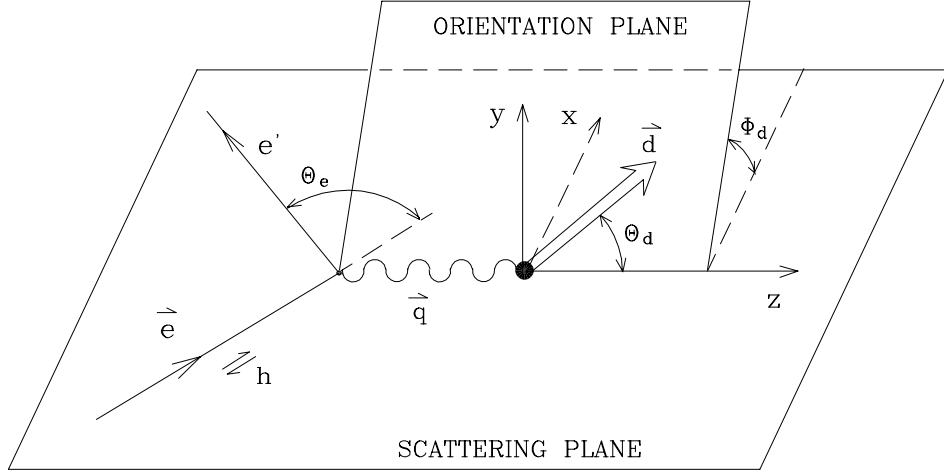
The presence of the third state  $M = 0$  requires another degree of freedom, which is called tensor polarization  $P_{zz}$ .

$$P_{zz} = 1 - 3n_{M=0}, \quad -2 \leq P_{zz} \leq 1 \quad (29)$$

### 4.3 Measuring the tensor asymmetry in elastic (e,d) scattering

We shall now consider a case where the vector polarization  $P_z$  is zero. The cross-section for a tensor-polarized deuteron contains an additional term with respect to (21). In the special case when the quantization axis is along the momentum transfer  $\mathbf{q}$ , it has the form:

$$\left(\frac{d\sigma}{d\Omega}\right)_{P_{zz}} = \frac{d\sigma}{d\Omega} \left(1 + \frac{P_{zz}}{\sqrt{2}} T_{20}(Q, \theta_e)\right) \quad (30)$$



**Figure 9:** Geometry of elastic  $e$ - $d$  scattering. [3]

$T_{20}(Q, \theta_e)$  is the elastic asymmetry and can be expressed in terms of the three form factors:

$$T_{20} = -\frac{1}{\sqrt{2}S} \left[ \frac{8}{3}\eta G_C G_Q + \frac{8}{9}\eta^2 G_Q^2 + \frac{1}{3}\eta \left( 1 + 2(2 + \eta) \tan^2 \frac{\theta_e}{2} \right) G_M^2 \right] \quad (31)$$

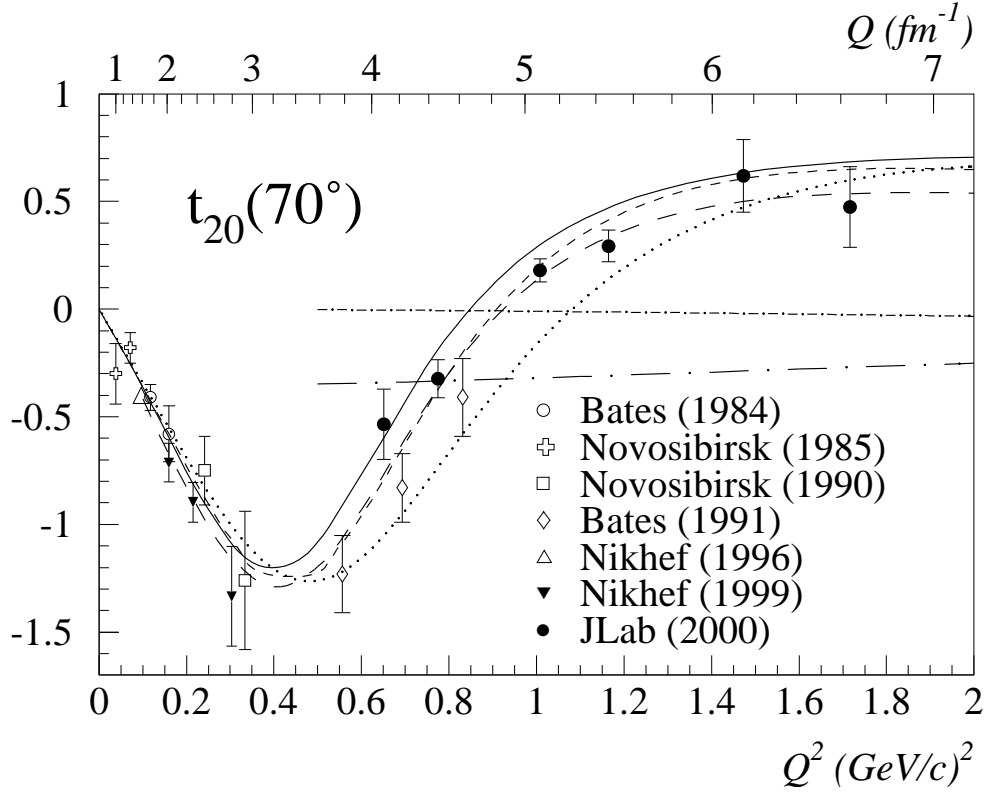
It contains a mixed term  $G_C G_Q$ . Hence, by measuring  $A$ ,  $B$ , and  $T_{20}$ , the three form factors  $G_C$ ,  $G_Q$  and  $G_M$  can be completely separated.

The actual experiment geometry is shown in Figure 9. We measure at fixed scattering angle  $\theta_e$  and momentum transfer  $Q$ . The polarized deuteron target is in a magnetic field that introduces the quantization axis  $z$  in the direction specified by the two angles  $\phi_d$  and  $\theta_d$ . In attempt to simplify extraction of the form factors, quantization axis is set in the direction of the momentum transfer  $\mathbf{q}$  and then cross-section (30) can be used.

When the deuteron is prepared with  $P_{zz}^- \equiv -2$  all the spin-vectors are in the  $xy$ -plane, according to convention in Figure 9. The other extreme case  $P_{zz}^+ \equiv 1$  has one half of spins up and the other half down the  $z$ -axis. The tensor asymmetry is actually an asymmetry between the event counts for the two extremal tensor polarizations  $P_{zz}^-$  and  $P_{zz}^+$ .

$$T_{20} = \sqrt{2} \frac{N^+ - N^-}{N^- \cdot P_{zz}^+ - N^+ \cdot P_{zz}^-} \quad (32)$$

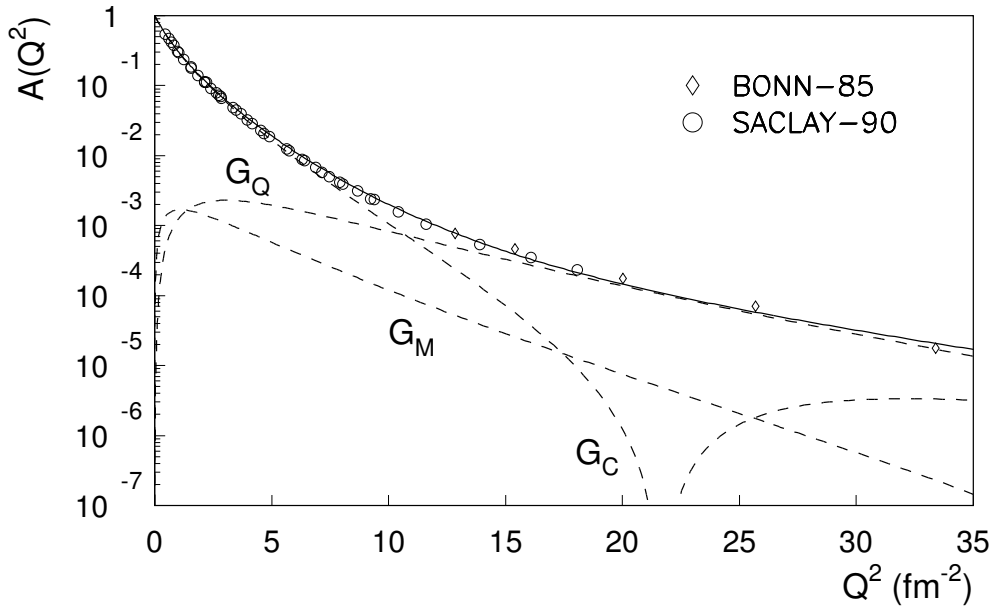
Figure 10 shows the momentum dependence of  $T_{20}$ .  $T_{20}$ 's first minimum is at  $Q \approx 700 \text{ MeV}/c$  with  $T_{20} \approx -\sqrt{2}$ , and from the expression (32) it can be seen that  $N^+$  is very close to zero at that point. That means deuteron polarized in the direction of momentum transfer is practically transparent for virtual photons with momentum  $Q \approx 700 \text{ MeV}/c$ .



**Figure 10:** The tensor asymmetry  $T_{20}$  dependence of the momentum transfer at fixed scattering angle  $\theta_e = 70^\circ$ . [7]

We can obtain dimensions of the toroidal structures directly from  $T_{20}(Q)$ . From  $T_{20}$ 's first minimum the distance between the dumbbell ends can be calculated, while the first maximum provides a measure of the torus thickness. The first minimum is in agreement with predicted distance, while the first maximum has not been measured yet.

Theories show large discrepancies in the high momentum transfer range (Figure 10), thus measurements at higher  $Q^2$  are welcome in the future.



**Figure 11:** The structure function  $A(Q^2)$  of the deuteron. The solid curve corresponds to the theoretical prediction. The dashed curves represent the individual contributions of the three deuteron form factors. [3]

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