Heep Least Squares Analysis For Coincidence Data

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System of Linear Equations for H(e,e'p)

$$
\begin{aligned}
dW^{calc} &= \frac{\partial W}{\partial E_b} E_b a_0 + \frac{\partial W}{\partial E_f} E_f a_1 + \frac{\partial W}{\partial \theta_e} a_2 = dW_{obs} + \epsilon_{dW} \\
dE_m^{calc} &= \frac{\partial E_m}{\partial E_b} E_b a_0 + \frac{\partial E_m}{\partial E_f} E_f a_1 + \frac{\partial E_m}{\partial \theta_e} a_2 = dE_{m_{obs}} + \epsilon_{dE_m} \\
dP_{mx}^{calc} &= \frac{\partial P_{mx}}{\partial E_b} E_b a_0 + \frac{\partial P_{mx}}{\partial E_f} E_f a_1 + \frac{\partial P_{mx}}{\partial \theta_e} a_2 = dP_{mx_{obs}} + \epsilon_{dP_{mx}} \\
dP_{mz}^{calc} &= \frac{\partial P_{mz}}{\partial E_b} E_b a_0 + \frac{\partial P_{mz}}{\partial E_f} E_f a_1 + \frac{\partial P_{mz}}{\partial \theta_e} a_2 = dP_{mz_{obs}} + \epsilon_{dP_{mz}}
\end{aligned}
$$

$$
a_0 = \frac{dE_b}{E_b} \qquad a_1 = \frac{dE_f}{E_f} \qquad a_2 = d\theta_e
$$

Calculated variations

Re-write derivative and parameters in matrix form

Measured variations

residual between predicted and observed

- **To further constrain the beam energy, final electron energy and angle, in addition to the variations in W, the variations in missing energy and momenta were also considered.**

- **ONLY variations in W from beam energy, e- momentum and angle were considered**

In Matrix Notation:
(Over-Determined System)

$$\begin{bmatrix} \sum\limits_{runs} \frac{\partial W}{\partial E_b} E_b & \sum\limits_{runs} \frac{\partial W}{\partial E_f} E_f & \sum\limits_{runs} \frac{\partial W}{\partial \theta_e} \\ \sum\limits_{runs} \frac{\partial E_m}{\partial E_b} E_b & \sum\limits_{runs} \frac{\partial E_m}{\partial E_f} E_f & \sum\limits_{runs} \frac{\partial E_m}{\partial \theta_e} \\ \sum\limits_{runs} \frac{\partial P_{mx}}{\partial E_b} E_b & \sum\limits_{runs} \frac{\partial P_{mx}}{\partial E_f} E_f & \sum\limits_{runs} \frac{\partial P_{mx}}{\partial \theta_e} \\ \sum\limits_{runs} \frac{\partial P_{mz}}{\partial E_b} E_b & \sum\limits_{runs} \frac{\partial P_{mz}}{\partial E_f} E_f & \sum\limits_{runs} \frac{\partial P_{mz}}{\partial \theta_e} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum\limits_{runs} dW_{obs} \\ \sum\limits_{runs} dE_{m_{obs}} \\ \sum\limits_{runs} dPm_{x_{obs}} \\ \sum\limits_{runs} dPm_{z_{obs}} \end{bmatrix} + \begin{bmatrix} \epsilon_{dW} \\ \epsilon_{Em} \\ \epsilon_{Pm_x} \\ \epsilon_{Pm_z} \end{bmatrix}$$

The coefficients are summed over all runs (only 3 in this case)

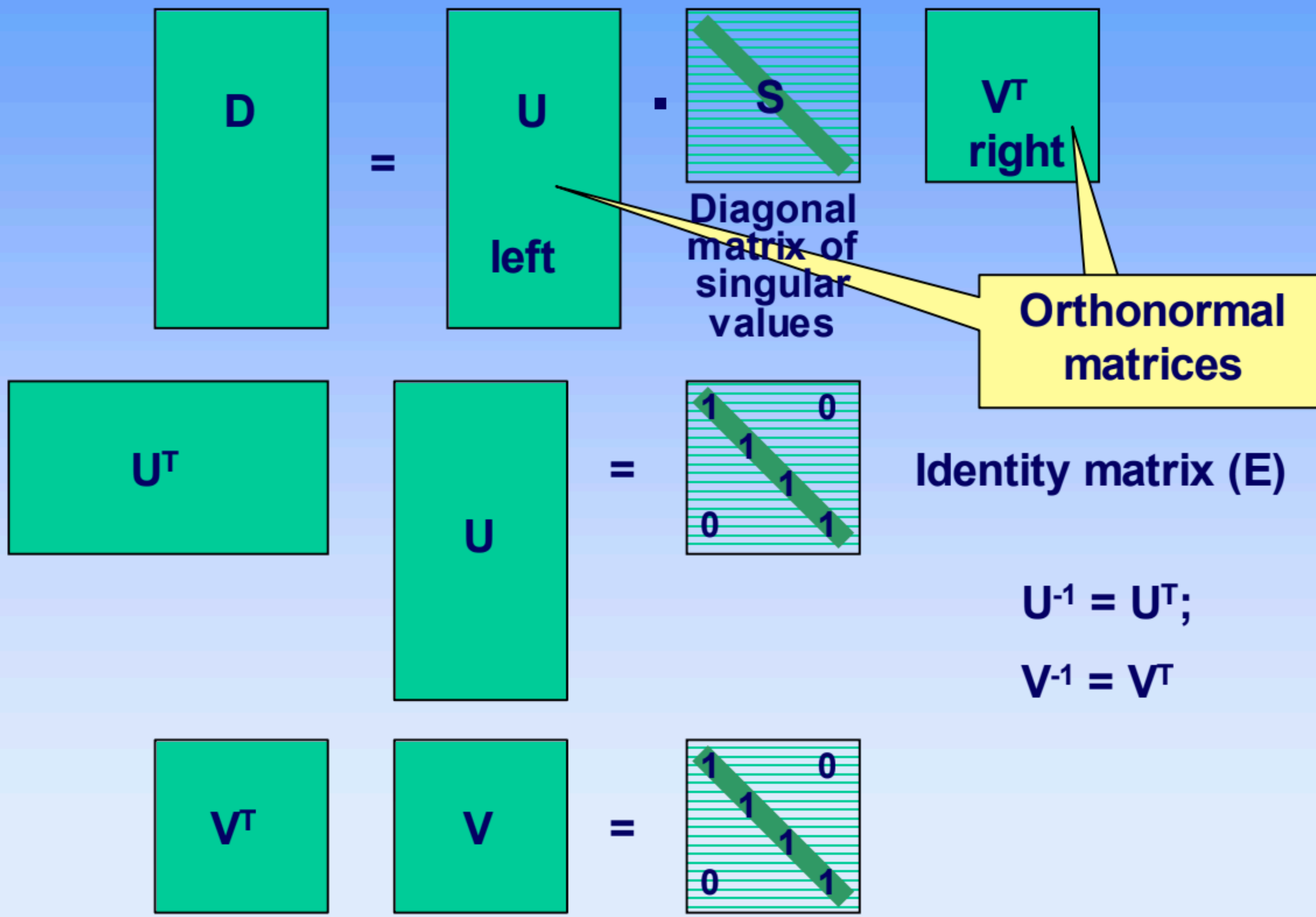
$$\mathbf{C}\overrightarrow{\mathbf{a}} = \overrightarrow{\mathbf{b}} + \hat{\epsilon}$$

Solve for: $\mathbf{C}\overrightarrow{\mathbf{a}} \cong \overrightarrow{\mathbf{b}}$ via Single-Value Decomposition

(SEE NEXT TWO SLIDES FOR OVERVIEW OF SVD)

Solution of systems of linear equations

Singular value decomposition (SVD)



Author: Vladimir Volkov

Solution of systems of linear equations

Solution by least squares using normal equations:

$D^T D\ A = D^T\ B,$

$A=(D^T D)^{-1}\ D^T\ B$

Solution by least squares using SVD:

$D = U\ S\ V^T;$

$U^T\ U\ S\ V^T\ A = U^T\ B;$ because $U^T U = E$, and E play role of a unity in multiplications:

$S\ V^T\ A = U^T\ B;$ $S^{-1}S = E$, then

$V^T\ A = S^{-1}\ U^T\ B;$ critical operation - inversion of S_{ii}

$A = V\ S^{-1}\ U^T\ B$ because $V\ V^T = E$

$$\mathbf{A} = \sum_{i=1}^{M}\left(\frac{\mathbf{U}_{(i)}\cdot\mathbf{B}}{S_i}\right)\mathbf{V}_{(i)} \pm \frac{1}{S_1}\mathbf{V}_{(1)} \pm \cdots \pm \frac{1}{S_m}\mathbf{V}_{(m)}$$

Errors

$$\sigma_{a_j}^2 = \sum_{i=1}^{M}\left(\frac{\mathbf{V}_{ji}}{S_i}\right)^2$$

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Getting the Residuals and Chi2

- **Once the optimum parameters, (vector a) have been determined by SVD, the residuals and chi2 can be calculated as follows:**

$$\hat{\epsilon} = \mathbf{C}\overrightarrow{\mathbf{a}} - \overrightarrow{\mathbf{b}} \longrightarrow$$ **Residuals**

One can think of this matrix as the inverse of the covariant matrix Of the measured quantities

$$\chi^2 \equiv \hat{\epsilon}^T \mathbf{N^{-1}} \hat{\epsilon} \longrightarrow \begin{bmatrix} \epsilon_{dW} \\ \epsilon_{Em} \\ \epsilon_{Pm_x} \\ \epsilon_{Pm_z} \end{bmatrix}$$

$$\mathbf{N^{-1}} = \begin{bmatrix} 1/\sigma^2_{dW_{obs}} & 0 & 0 & 0 \\ 0 & 1/\sigma^2_{dEm_{obs}} & 0 & 0 \\ 0 & 0 & 1/\sigma^2_{dPmx_{obs}} & 0 \\ 0 & 0 & 0 & 1/\sigma^2_{dPmz_{obs}} \end{bmatrix}$$

where the $\sigma_{i_{abs}}$ are the measured errors and assumed to be independent of each other

Expanding chi2 in matrix form, one obtains:

$$\chi 2 = \frac{\epsilon^2_{dW}}{\sigma^2_{dW_{obs}}} + \frac{\epsilon^2_{dE_m}}{\sigma^2_{dE_{m_{obs}}}} + \frac{\epsilon^2_{dP_{mx}}}{\sigma^2_{dP_{mx_{obs}}}} + \frac{\epsilon^2_{dP_{mz}}}{\sigma^2_{dP_{mz_{obs}}}}$$

Getting the Covariance Matrix of the Parameters

The covariance matrix can be determined from:

- **the initial matrix from the model** $\mathbf{C}\overrightarrow{\mathbf{a}} = \overrightarrow{\mathbf{b}} + \hat{\epsilon}$
- **the inverse of the data error matrix** $\mathbf{N^{-1}}$

$$\mathbf{COV_M = (C^T N^{-1} C)^{-1}}$$

The covariance matrix depends only on:

- **The measured data errors**
- **The experimental model**

Getting the Correlation Matrix

The correlation matrix can be determined by dividing the covariance by the diagonal elements as follows:

$$\mathrm{Cor}_{ij} = \frac{\boxed{\mathrm{Cov}_{ij}}}{\boxed{\mathrm{Cov}(i)\mathrm{Cov}(j)}}$$

(i,j) **element of the Covariance Matrix**

Covariance Diagonal elements ***(i), (j)***

For the special case of ***i=j*****, the diagonal elements are 1 which indicates they are 100% correlated with themselves**

Results from H(e,e'p) Coincidence Data

Measured data and error for each of the 4 elastic runs (only the 1st three runs considered)

```
//Define the measured DATA, SIMC variables
Double_t Wdata[4] = {9.42152e-01, 9.40229e-01, 9.43825e-01, 9.55773e-01};
Double_t Wdata_err[4] = {9.71583e-05,  3.77469e-04, 6.01623e-05, 4.13486e-05};

Double_t Wsimc[4] = {9.44690e-01, 9.44480e-01, 9.43591e-01, 9.42933e-01};
Double_t Wsimc_err[4] = {1.38566e-04, 6.68696e-05,  2.81437e-05, 2.23158e-05};

Double_t Emdata[4] = {6.58010e-03, 5.28983e-03, 6.26642e-03, 5.32021e-03};
Double_t Emdata_err[4] = {4.69566e-05,  1.47900e-04,  3.65940e-05, 4.14000e-05};

Double_t Emsimc[4] = {5.51066e-03, 6.06886e-03, 6.58531e-03, 6.47356e-03};
Double_t Emsimc_err[4] = {9.05496e-05, 3.84833e-05,  1.74941e-05, 1.37620e-05};

Double_t PmXdata[4] = {-1.29097e-03, -9.93560e-04, -5.53961e-04, 7.13300e-03};
Double_t PmXdata_err[4] = {3.30069e-05, 1.75464e-04, 2.87979e-05, 2.08787e-05};

Double_t PmXsimc[4] = {-5.88228e-04, -5.91870e-04, -6.80114e-04, -1.02935e-03};
Double_t PmXsimc_err[4] = {7.19549e-05, 2.48719e-05,  2.31853e-05, 1.11287e-05};

Double_t PmZdata[4] = {5.77521e-03, 5.42333e-03, 5.33585e-03, 3.43146e-03};
Double_t PmZdata_err[4] = {7.85239e-05, 1.13337e-04, 3.97398e-05, 3.45173e-05};

Double_t PmZsimc[4] = {5.82773e-03, 6.50794e-03, 6.35112e-03, 6.02246e-03};
Double_t PmZsimc_err[4] = {1.06200e-04, 2.89548e-05, 1.55699e-05, 1.62459e-05};
```

Data Errors Seem to be too small, which can make chi2 artificially large. Would it be better to consider the sigma from the gaussian fit as the error?

Results from H(e,e'p) Coincidence Data

```
****COVARIANCE MATRIX*****

3x3 matrix is as follows

     |      0    |      1    |      2    |
-------------------------------------------
   0 |  5.858e-08   7.221e-08  -1.409e-08
   1 |  7.221e-08   8.904e-08  -1.738e-08
   2 | -1.409e-08  -1.738e-08   3.419e-09

=== Optimized Parameters ===
total equations x total runs = 4x3 = 12 observations, # parameters = 3, dof = 9
chi2 = 126.145  chi2/dof = 14.0161

dEb / Eb = -0.000410038
dEf / Ef = -0.000448075
dth_e = 0.000164275


=== Uncertainty in Parameters ===
dEb / Eb = 0.000242029
dEf / Ef = 0.000298391
dth_e [rad] = 5.84754e-05


****CORRELATION MATRIX*****

3x3 matrix is as follows

     |      0    |      1    |      2    |
-------------------------------------------
   0 |          1      0.9999     -0.9958
   1 |     0.9999           1      -0.996
   2 |    -0.9958      -0.996           1
```

The error on the electron
Angle seems too good
(This is probably related to
The unrealistically small data errors)

Could we use the sigma from the
Fit as the error of the mean?