Part IV

Definition of Nuclear Potential

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In the meson theory of nuclear forces at low energies, the concept of nuclear potential plays a very useful rôle. In this report we shall present an argument to define unambiguously the nuclear potential based on the meson theory. It is shown that unlike the real pion reactions various approximation methods lead to very similar potentials at the outer region, say $x \gtrsim 0.6 \sim 0.7$.

1. Introduction

The problem of nuclear forces in the meson theory is the oldest subject which has motivated Yukawa to conjecture the existence of a new field quantum, the meson. In his first attempt to this problem, he has formulated the problem by fully utilizing the concept of potential borrowed from electrodynamics. The concept of nuclear potential was born from the beginning in analogy to the Coulomb potential in electrodynamics. The latter was first invented in classical electrodynamics, remained rigorously valid also in quantum electrodynamics, and is of great use for practical purposes. Hence it is natural that the investigation of the former problem has been developed on postulating the validity of the potential concept. Furthermore, the nuclear potential is in practice the agency between the phenomenological theory of nuclear forces and the meson theory. It still remains, however, to be examined whether the potential approach is also justified or not in the meson theory of nuclear forces.

In the meson theory, the situation is much more complicated than in quantum electrodynamics because of the unavoidable occurrence of the strong correlations of the ordinary and isotopic spins which did not appear in the case of quantum electrodynamics. Unlike the Coulomb force, the nuclear forces between two nucleons cannot be expressed in the form of "potential" if a rigorous quantum mechanical treatment is adopted.

Since the nuclear potential is essentially an approximate concept in the meson theory, it is important to investigate how to define the most appropriate nuclear potential, otherwise we shall be unable to avoid confusions caused by lack of an unambiguous definition of the nuclear potential. In

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this report we shall define it so as to intermediate between the meson theory and phenomenological theory.

In what follows, we shall exhibit the definition and derivations of the nuclear potential and examine its validity. It is shown together with the phenomenological analysis that at low energies the potential approach is considerably useful and that the shape of the potential in the outer region is insensitive to the methods of derivation in contrast with the real pion reactions.

2. Necessary Conditions for Nuclear Potential

In this report we shall investigate the nature of such a nuclear potential that can be adopted to the Schrödinger equation in the phenomenological theory. For this purpose we shall first recaptulate the necessary conditions that the nuclear potential for phenomenological uses must satisfy.

A) The nuclear potential is an Hermitian operator defined below the threshold energy of the meson production.

B) The nuclear potential is an energy independent operator.

C) The solutions of the Schrödinger equation with the nuclear potential should give the correct phase shifts and binding energy of the deuteron.

By "correct" we mean that the results must agree with those obtained from the meson theory without passing through the nuclear potential.

D) The velocity dependence of the nuclear potential should be restricted so that the conventional outgoing wave boundary condition for the Schrödinger equation is enough to uniquely determine the solution.

This implies that the velocity dependence of the nuclear potential is at most linear in the relative momentum.

Since these conditions are very restrictive, it gives rise to a discussion whether such a nuclear potential that satisfies the above conditions can exist even approximately.

The purpose of the potential approach is *to decompose the whole problem of nucleon-nucleon interaction into two steps:* we first solve the problem of meson field to obtain the nuclear potential and then apply it to the problem of nucleon field. In general such a decomposition cannot be performed rigorously and we are obliged to make some approximations to carry out this program. Therefore the validity of each approximation should be examined case by case. At low energies this difficulty does not seem to be serious, but it turns out, as the energy increases, to be too serious to retain the potential approach. In such cases we must be careful to distinguish the difficulties due to the inadequacy of the potential approach from those characteristic of the meson theory, and other methods than the potential approach should be employed in order to avoid the inessential difficulties. In this report we shall study only the cases in which the potential approach is justified.

The necessary conditions stated above restrict very severely the method of approach. Therefore we shall discuss the limitations imposed by them.

A$'$) The condition A) requires that the wave function describing the nucleons phenomenologically should be properly normalized. Since the freedom of the meson field is frozen in the Schrödinger equation, the wave function must describe the state of dressed nucleons. The possibility of this interpretation is intimately connected to the decomposition problem discussed before.

B$'$) The condition B) requires that the Schrödinger equation should be of the form

$$H\psi=(\boldsymbol{p}^2/M+V)\psi=E\psi\,, \tag{2.1}$$

but not of the form

$$(\boldsymbol{p}^2/M+V(E))\psi=E\psi\,, \tag{2.2}$$

where $\boldsymbol{p}$ denotes the relative momentum and M the nucleon rest mass. This condition implies that the wave functions belonging to different energy eigenvalues should be orthogonal to each other so that we can regard the wave function ψ as representing the quantum mechanical probability amplitude for dressed nucleons. The equation (2.2) does not allow us to interpret the wave function ψ as a simple probability amplitude, but rather we have to regard it as a "distorted" probability amplitude.

D$'$) The condition D) imposes a severe restriction on the type of the meson-nucleon coupling so that the potential approach is granted. One of the most important differences between the Coulomb potential and nuclear potential consists in the difference between the rest masses of the exchanged Bosons. The former is due to the exchange of massless longitudinal and scalar photons and the latter is due to the exchange of mesons. The meson rest mass is considerably large to be neglected as compared with the nucleon rest mass, and consequently the recoil or retardation effects are generally expected to be important. The condition D) requires that the recoil effects must be small, otherwise the nuclear forces cannot even approximately be well expressed in the form of a potential in the above sense.

This condition requires that the meson field must be coupled to the nucleon field with the even Dirac matrices.

D$''$) The velocity dependent part of the nuclear potential that is linear in the relative momentum should involve a factor $\boldsymbol{L}\cdot\boldsymbol{S}$, where $\boldsymbol{L}$ isthe relative angular momentum and $\boldsymbol{S}$ the total spin of the two nucleon system defined by $\boldsymbol{S}=1/2\cdot(\boldsymbol{\sigma}^1+\boldsymbol{\sigma}^2)$.

This is a direct consequence of the invariance of the theory under th space reflection and time reversal as seen from Table 2-I.

Table 2-I. The parities of nucleon variables under space and time inversion.

	$\boldsymbol{p}$	$\boldsymbol{r}$	$\boldsymbol{\sigma}$
space	$-$	$-$	$+$
time	$-$	$+$	$-$

$\boldsymbol{p}$: relative momentum
$\boldsymbol{r}$: relative coordinate
$\boldsymbol{\sigma}$: nucleon spin

As discussed above, the potential concept remains valid only under very restricted conditions. Hence in some cases the potential approach might be completely useless, but we shall confine ourselves to the discussions of the potential approach and of its validity, that is, to separate the difficulties arising in the potential concept from those inherent in the conventional meson theory.

3. Singularities of Nuclear Potential and Cut-off Hypothesis

It was not later than three years after the birth of the Yukawa theory when we met the serious difficulty of the r^{-3} singularity in the meson potential. The behavior of the meson theoretical potential is too singular to allow the physically meaningful solutions of the Schrödinger equation. Since then, various methods were proposed to avoid the difficulty.

Møller and Rosenfeld (40) proposed the method of mixed meson fields, but this method cannot be free from objections since the cancellation of the singularity must lead to the cancellation of the tensor force that is needed experimentally.

Afterwards several authors (Nambu 48, Araki 49, Van Hove 49) argued that relativistic effects might diminish the order of the singularity.

However, many reasons suggest us that these methods cannot be the final answer. As stressed before the potential concept can be retained only under very severe conditions, and it can never be meaningful at the origin. The difficulties related to the potential concept must not be confused with those inherent in the meson field theory, but at very small distances the conventional meson theory itself might be invalid.

For these reasons, the discussions of the singularity does not seem to be meaningful at the present stage, and we are obliged to deal with the problem phehomenologically. In 1940, Bethe (40) proposed the method of cut-off. He adopted the second-order potential in the outer region where the second-order potential was supposed to be larger than the fourth-order one, and used the cut-off procedure in the inner region. At that time, only the first non-vanishing term in the perturbation theory was believed to be trustworthy in analogy to the quantum electrodynamics.

In 1950, Taketani, Nakamura, and Sasaki (50) proposed a new method by consciously recognizing various limitations inherent in the conventional meson theory. In the present report we shall base our discussions on their idea. The meson theoretical potential can be trusted even quantitatively at large distances, and as the distance becomes smaller it is only qualitatively valid, and finally within some critical distance it is meaningless and we cannot do better than phenomenological approach. It must, however, be noted that the quantitative behavior of the interaction between two nucleons at sufficiently low energies is determined by the shape of the nuclear potential at large distances and hence the meson theory can be examined by the experimental data.

4. Definition of Nuclear Potential, I

Roughly speaking, there are two ways to define the nuclear potential. The first one is to define the nuclear potential as the lowest eigenvalue of the Hamiltonian

$$H_a = H_M + H_{int}\,, \tag{4.1}$$

where H_M and H_{int} are the Hamiltonians of the free meson field, and the interaction between nucleon and meson fields, respectively. H_a is the adiabatic part of the total Hamiltonian

$$H_{tot} = H_N + H_a = H_N + H_M + H_{int}\,, \tag{4.2}$$

where H_N is the Hamiltonian for the free nucleon field.

The second one, sometimes referred to as the S matrix method, is to define the potential $\mathcal{V}$ by the equation

$$\begin{aligned} &< P[\exp(-i\textstyle\int H_{int}(t)\,dt)] >_{two\ \ nucleons} \\ &\qquad = P[\exp(-i\textstyle\int \mathcal{V}(t)\,dt)]\,. \end{aligned} \tag{4.3}$$

P is the chronological symbol defined by Dyson and both sides of the above equation refer to the interaction representation. In this method, the potential $\mathcal{V}$ is so defined as to give effectively the field theoretical S matrix for nucleon-nucleon scattering. This method is not so clear as the former, and we need some devices and modifications so that we can obtain the true potential satisfying the conditions given in **2.** We shall discuss on this method later in connection with the Tamm-Dancoff approximation.

In this section we investigate the former method. We are primarily concerned with the eigenvalue equation

$$H_a \Psi_0 = W_0 \Psi_0\,, \tag{4.4}$$

where W_0 is the lowest eigenvalue and Ψ_0 the corresponding eigenstate. Ψ_0 involves the information on the meson cloud around two fixed nucleons.

Since H_a commutes with the relative coordinate $\boldsymbol{r}$, the eigenvalue depends on $\boldsymbol{r}$ parametrically.

Once W_0 is obtained, the next step we must do is to solve the Schrödinger equation

$$(H_N + W_0(\boldsymbol{r}))\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}) \tag{4. 5}$$

This method is an analogue of the Born-Oppenheimer approximation in the theory of molecules. In the case of molecules, nuclei are much heavier than electrons and the period of electronic motion is much shorter than that of the nuclear motion, so that we can first solve the electronic motion by fixing the nuclear configuration and then solve the nuclear motion by adopting the eigenvalue of the former problem as the potential for the latter problem. This method is indeed successful as illustrated in the Heitler-London theory.

However, in the meson theory such an approximation turns out to be worse as the meson mass is much larger than the electron mass and the corrections to the adiabatic approximation becomes much more important. Nevertheless we are obliged to start from the adiabatic approximation because of the conditions given in **2.**

In what follows we shall exhibit a method to evaluate the non-adibatic correction. The magnitude of this correction differs from case to case, and we can employ the potential approach only if this correction is small.

The exact equation we must solve is

$$H_{tot}\Psi = E\Psi\,. \tag{4. 6}$$

Suppose that U is a unitary transformation which diagonalizes the Hamiltonian H_a, then we have

$$\Psi = U\Phi\,, \tag{4. 7}$$

$$U^{-1}H_a U = U^{-1}(H_M + H_{int})U = W, \text{ diagonal.} \tag{4. 8}$$

The equation for Φ is then given by

$$(U^{-1}H_N U + W)\Phi = E\Phi\,. \tag{4. 9}$$

The Schrödinger equation (4. 5) in the adiabatic approximation is obtained if we approximate as

$$U^{-1}H_N U \approx H_N\,. \tag{4. 10}$$

Then eq. (4. 9) is approximated as

$$(H_N + W)\Phi \approx E\Phi\,, \tag{4. 11}$$

and consequently this equation is reduced to eq. (4 .5) by the separation of variables

$$\Phi = \psi(\boldsymbol{r})\Phi_0\,, \tag{4. 12}$$

where $\psi(\boldsymbol{r})$ is the wave function for dressed nucleons and Φ_0 denotes the meson vacuum.

Thus it is clear that the non-adiabatic correction can be evaluated by improving the approximation (4.10) and that this correction arises due to the nucleon recoil. As has already been remarked, the meson field must be coupled to the nucleon field with the even Dirac matrices in order that the recoil correction be relatively unimportant and consequently the potential concept be useful at low energies. *For this reason, our discussions are limited to the pseudoscalar meson theory with pseudovector coupling throughout this report.*

5. Perturbation Theory

The oldest, simplest and most familiar method to solve the eigenvalue equation (4.4) is the perturbation theory.

For simplicity let us assume that the interaction Hamiltonian H_{int} is linear in the meson wave function and decompose it into creation and destruction parts, i. e.

$$H = h^{+} + h^{-}, \tag{5.1}$$

Then the second-order potential is given by

$$W^{(2)} = \langle h^{-} \frac{1}{-H_M} h^{+} \rangle_{meson\ vacuum}, \tag{5.2}$$

which yields, when applied to the neutral scalar meson theory and charge symmetric pseudoscalar meson theory, the following well known formulas:

Neutral scalar: $\quad -\left(\frac{g^2}{4\pi}\right)\mu\frac{e^{-x}}{x}, \qquad (x=\mu r),$

Symmetrical pseudoscalar:

$$\left(\frac{g^2}{4\pi}\right)\frac{\mu}{3}(\boldsymbol{\tau}^1.\boldsymbol{\tau}^2)\left[(\boldsymbol{\sigma}^1.\boldsymbol{\sigma}^2)+\left(1+\frac{3}{x}+\frac{3}{x^2}\right)S_{12}\right]\frac{e^{-x}}{x},$$

where S_{12} is defined by

$$S_{12} = \frac{3(\boldsymbol{\sigma}^1.\boldsymbol{r})(\boldsymbol{\sigma}^2.\boldsymbol{r})}{r^2} - (\boldsymbol{\sigma}^1.\boldsymbol{\sigma}^2).$$

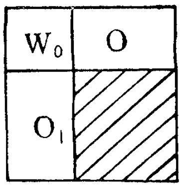
We shall not further discuss this method since it is very well known.

6. Method of Canonical Transformations

The method of canonical transformations is sometimes more convenient than the perturbation theory, especially in separating out the fourth-order potential from the repetition of the second-order. This method was originally applied to the problem of nuclear forces by Møller and Rosenfeld (40) and also fully utilized in quantum electrodynamics by many authors, especially by Bloch and Nordsieck (37) and by Tomonaga and Schwinger.

As mentioned in the previous section our task is the diagonalization of the Hamiltonian $H_a = H_M + H_{int}$. However, since we need only the lowest eigenvalue corresponding to the no (real) meson state, H_a need not completely be diagonalized but it is enough to decouple the no meson state from other states. Hence, in what follows, we use the term "diagonal" in this sense, and seek for such a transformation U that brings H_a into the following form:

$$U^{-1}H_aU = W, \tag{6.1}$$



where the lowest eigenvalue W_0 is the potential.

The transformation U is so determined as to eliminate the non-diagonal part which causes transitions between no meson and other states. Let us first eliminate the non-diagonal part H_{int} in the Hamiltonian H by the following transformation

$$U_1 = \exp(iS_1)\,. \tag{6.2}$$

where S_1 is determined by eq. (6.4) below. Then H_a is transformed into

$$\begin{aligned} U_1^{-1}H_aU_1 &= e^{-iS_1}H_ae^{iS_1} \\ &= H_M + i[H_M, S_1] + \frac{i^2}{2!}[[H_M, S_1]S_1] + \cdots \\ &\quad + H_{int} + i\,[H_{int}, S_1] + \cdots, \end{aligned} \tag{6.3}$$

The transformed Hamiltonian is diagonalized up to the first order in the coupling constant g if we put

$$H_{int} + i[H_M, S_1] = 0\,, \tag{6.4}$$

which determines the Hermitian operator S_1, and the transformed Hamiltonian takes the from

$$U_1^{-1}H_aU_1 = H_M + \sum_{n=1}^{\infty} i^n\left(\frac{1}{n!} - \frac{1}{(n+1)!}\right)\overbrace{[\ldots[H_{int}, S_1]\ldots S_1]}^{\leftarrow n\text{-brackets}\rightarrow} \equiv H_a'\,. \tag{6.5}$$

In order to diagonalize H_a' up to the second-order in the coupling constant, we shall apply the second transformation

$$U_2 = \exp(iS_2) \tag{6.6}$$

with the operator S_2 determined by

$$\frac{i}{2}[H_{int}, S_1]_{n.d.} + i[H_M, S_2] = 0\,, \tag{6.7}$$

where the subscript $n.\,d.$ refers to the non-diagonal part.

By the repetition of such transformations

$$U = \cdots U_2U_1\,, \tag{6.8}$$

the primary Hamiltonian H_a can be diagonalized to any desired order in the coupling constant. The potential obtained up to the fourth-order by this method is expressed as

$$W^{(2)}=\frac{i}{2}[H_{int}, S_1]_d\,, \tag{6.9}$$

$$W^{(4)}=\frac{i^3}{8}[[[H_{int}, S_1]S_1]S_1]+\frac{1}{2}[[H_M, S_2]S_2]_d\,.$$

There is a simple relation between the second and fourth-order static potentials due originally to Nambu (50) and the author (Nishijima 51). Let the second-order potential be expressed in the form

$$W^{(2)}=-\left(\frac{g^2}{4\pi}\right)\mu\, O_\alpha^{(1)}O_\alpha^{(2)}\frac{e^{-x}}{x}\,, \tag{6.10}$$

then the fourth-order static potential is generally given by

$$W^{(4)}=\frac{1}{2\pi}\left(\frac{g^2}{4\pi}\right)^2\mu[O_\alpha, O_\beta']^{(1)}[O_\alpha,O_\beta']^{(2)}\left(\frac{1}{x}+\frac{1}{x'}\right)K_0(x+x'),\ x'\to x\,. \tag{6.11}$$

O_α stands generally for the ordinary spin, isotopic spin, and differential operators. In the charge symmetrical pseudoscalar meson theory, O_α and O_α' are given respectively by

$$O_\alpha=\tau_\alpha(\boldsymbol{\sigma}\cdot\nabla),\ O_\alpha'=\tau_\alpha(\boldsymbol{\sigma}\cdot\nabla') \tag{6.12*}$$

In eq. (6.11) x and x' represent the distances travelled by the two exchanged mesons, and they are equated in the adiabatic limit.

Substituting the expressions (6.12) into eq. (6.11) one finds the so-called TMO potential (Taketani 52) which is also obtained by Nambu's method (Nambu 50) discussed later.

It is obvious from the expression (6.11) that the fourth-order potential remains only when the meson-nucleon interaction is spin and/or isotopic spin dependent and hence the fourth-order potential is caused by purely quantum mechanical effects. It is also clear, therefore, that the fourth-order potential vanishes in the neutral scalar meson theory. In fact, we get the exact potential in this case which is nothing but the lowest second-order potential. In this connection, Brueckner-Watson's theory (Brueckner 53) which contradicts the exact solution will be criticized later.

7. Some Remarks on Adiabatic Approximation

As repeatedly remarked the potential concept reserves its validity only when the adiabatic approximation holds. In the pseudoscalar meson theory with pseudoscalar coupling the validity of the potential concept must be precisely examined, since the meson field is coupled to the nucleon field

* More precisely written, we have $O_\alpha{}^i=\tau_\alpha{}^i(\sigma^i\cdot\nabla^i)$, $(i=1, 2)$. Notice $\nabla^2=-\nabla^1$.

with an odd Dirac matrix in this theory. For the present purpose the best way to deal with the theory might be to start after applying the Tani-Foldy transformation (Foldy 50, Tani 51) to the original Hamiltonian, for the adiabatic approximation can then be more easily applied. Such a program has been pushed through by Iwadare (55).

Even for even coupling theories, it is important to examine the validity of the adiabatic approximation by directly evaluating the non-adiabatic correction to the first order in the relative momentum. As discussed in 2, this correction give rise to the $\boldsymbol{L}\cdot\boldsymbol{S}$ coupling term.

In the non-relativistic approximation H_N assumes the form

$$H_N=(1/2M)\cdot(\boldsymbol{p}_1^2+\boldsymbol{p}_2^2)\,, \tag{7.1}$$

then we have instead of the approximate relation (4.10)

$$U^{-1}H_N U=H_N+\mathcal{W}\,, \tag{7.2}$$

where $\mathcal{W}$ represents the correction to eq. (4.10) and is given as

$$\mathcal{W}=(1/2M)(\{\boldsymbol{p}_1,\boldsymbol{D}_1\}+\{\boldsymbol{p}_2,\boldsymbol{D}_2\})+(1/2M)(\boldsymbol{D}_1{}^2+\boldsymbol{D}_2{}^2)\,. \tag{7.3}$$

The operator $\boldsymbol{D}$ is defined by

$$\frac{1}{i}U^{-1}\frac{\partial U}{\partial \boldsymbol{x}_s}=\boldsymbol{D}_s, \quad (s=1,2)\,. \tag{7.4}$$

(For the similar treatment, see Mott and Massey (49), p. 153.)

The unitary operator U depends on the relative coordinates $\boldsymbol{r}$ parametrically and determines the state of meson cloud around two fixed nucleons. Eq. (7.4) shows that $\boldsymbol{D}$ is the infinitesimal operator for the deformation of the meson cloud induced by the translation of the nucleon. In other words, $\boldsymbol{D}$ represents the momentum carried by the meson cloud.

The correct equation we have to discuss is then

$$(H_N+W+\mathcal{W})\Phi=E\Phi \tag{7.5}$$

instead of eq. (4.11). The operator $\mathcal{W}$ has non-diagonal matrix elements and consequently excites the proper field around nucleons, for instance, to isobaric states. Since, however, the rough order of magnitude of $\mathcal{W}$ is given by

$$\mathcal{W}\sim(\mu/M)W \tag{7.6}$$

as seen from eq. (7.3), eq. (7.5) is correctly approximated up to the first order in the ratio μ/M by

$$(H_N+W_o+\mathcal{W}_{oo})\psi(\boldsymbol{r})=E\psi(\boldsymbol{r})\,. \tag{7.7}$$

This is the Schrödinger equation with the lowest order recoil correction. In the above equation $\mathcal{W}_{oo}$ is the matrix element of $\mathcal{W}$ for the meson vacuum and the non-diagonal part of $\mathcal{W}$ contributes to the nuclear potential only in the second-order in μ/M.

The first term in $\mathcal{W}_{oo}$, i.e.,

$$(1/2M)\langle\{\boldsymbol{p}_1,\boldsymbol{D}_1\}+\{\boldsymbol{p}_2,\boldsymbol{D}_2\}\rangle_{meson\ vacuum} \tag{7.8}$$

gives rise to the $\boldsymbol{L}\cdot\boldsymbol{S}$ coupling correction, and the second term in $\mathcal{W}_{oo}$, i.e.,

$$(1/2M)<\boldsymbol{D}_1^2+\boldsymbol{D}_2^2>_{meson\ vacuum} \tag{7.9}$$

gives rise to the velocity independent correction which is first order in μ/M.

Galilei Invariance

If we calculate the non-adiabatic correction in the pseudoscalar meson theory starting from the non-relativistic Hamiltonian

$$H_{int}=\frac{g}{\mu}(\boldsymbol{\sigma}\cdot\nabla)(\boldsymbol{\tau}\cdot\boldsymbol{\phi}) \tag{7.10}$$

we find, besides the normal $\boldsymbol{L}\cdot\boldsymbol{S}$ term, an abnormal term proportional to

$$(\boldsymbol{r}_1-\boldsymbol{r}_2)\times(\boldsymbol{p}_1+\boldsymbol{p}_2)\cdot(\boldsymbol{\sigma}^{(1)}-\boldsymbol{\sigma}^{(2)}), \tag{7.11}$$

which is not Galilei invariant. In order to avoid this difficulty we must start from a Galilei invariant Hamiltonian. By taking the Pauli approximation of the Hamiltonian in the relativistic pseudoscalar theory with pseudovector coupling, S. Sato (Sato, S. 55; see also Sato, I. 55) improved (7.10) as

$$H_{int}=\frac{g}{\mu}(\boldsymbol{\sigma}\cdot\nabla)(\boldsymbol{\tau}\cdot\boldsymbol{\phi})+\frac{1}{2M}\left(\frac{g}{\mu}\right)\left\{(\boldsymbol{\sigma}\cdot\boldsymbol{p}),(\boldsymbol{\tau}\cdot\boldsymbol{\phi})\right\}+\frac{1}{2M}\left(\frac{g}{\mu}\right)^2\boldsymbol{\pi}^2, \tag{7.12}$$

where $\boldsymbol{\pi}$ is the canonical conjugate of ϕ. By this choice terms like (7.11) disappear and the resulting $\boldsymbol{L}\cdot\boldsymbol{S}$ coupling term is given by

$$\mathcal{W}_{LS}=-2\mu\left(\frac{\mu}{M}\right)\left(\frac{g^2}{4\pi}\right)^2\frac{(1-x)(1+x+x^2)}{x^6}e^{-2x}(3+2(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}))(\boldsymbol{L}\cdot\boldsymbol{S}). \tag{7.13}$$

It is characteristic of the $\boldsymbol{L}\cdot\boldsymbol{S}$ correction that ts range is half of the static one so that it is not appreciable at large distances compared with the adiabatic potential.

8. Definition of Nuclear Potential, II

In previous sections the nuclear potential is defined as the lowest eigenvalue of the Hamiltonian H_a. However, the S matrix method introduced in **4** can be also used to calculate the potential. The equation that defines the effective interaction $\mathcal{V}$ is given by

$$\begin{aligned}&<P[\exp(-i\int_{-\infty}^{t}dtH_{int}(t))]>_{meson\ vacuum}\\&=P\left[\exp(-i\int_{-\infty}^{t}dt\mathcal{V}(t))\right].\end{aligned} \tag{8.1}$$

The meaning of this equation has already been afforded in **4**. The effective interaction $\mathcal{V}$ is sometimes referred to as "the nuclear potential" in the Tamm-Dancoff method, but as we shall see later $\mathcal{V}$ does not satisfy the condition in **2**. Hence we cannot call $\mathcal{V}$ the nuclear potential although they are intimately connected with each other.

In the time independent representation, eq. (8.1) is rewritten

$$< \frac{1}{E+i\varepsilon - H_0 - H_{int}} H_{int} >_{meson\ vacuum}$$
$$= \frac{1}{E+i\varepsilon - H_N - \mathcal{V}} \mathcal{V}, \tag{8.2}$$

where $H_0 = H_N + H_M$. Eq. (8.2) is obtained from eq. (8.1) by the compari son of the scattered waves in both sides of eq. (8.1). Further it can readily be shown that $\mathcal{V}$ is the effective interaction appearing in the Tamm-Dancoff method (Tamm 45, Dancoff 50), i.e.,

$$(E_a - H_0) V_0 \Psi_a = \mathcal{V}(E_a) V_0 \Psi_a ,$$

where V_0 is the projection operator to two nucleon states. We shall reserve the derivation of the correct potential from $\mathcal{V}$ until the last section of this report.

9. Nambu's Method

There is another method due originally to Nambu (50) which is different from the previous two methods although the essential idea is very close to the S matrix method, and in what follows we shall briefly trace his idea.

So far we have calculated the nuclear potential in the adiabatic limit which is equivalently reproduced also by regarding the mesons to be instantaneously exchanged between two nucleons. Hence the inclined meson line in Fig. 9-1 is reduced to the horizontal line in Fig. 9-2 in the adiabatic limit. Hence, if we start with the adiabatic limit from the very beginning,

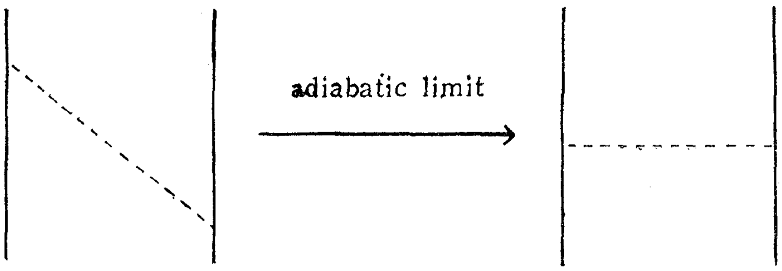


Fig. 9-1 Fig. 9-2

it is somewhat difficult to understand the mechanism of such a process as expressed by Fig. 9-3. The meaning of such a process turns out to be clear only when one takes account of the retardation effects. Such a viewpoint is realized only by tracing the temporal development of the system in the time dependent representation contrary to the adiabatic treatments in the time independent picture.

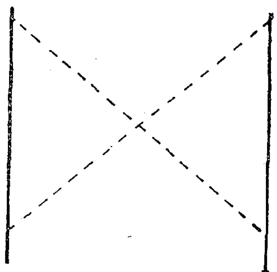


Fig. 9-3

In fact it was shown by Nambu that such a viewpoint lends itself to an intuitive derivation of the fourth-order nuclear potential. He investigated the difference between the fourth-order S matrix and the repetition of the second-order nuclear potential *with full retardation corrections* and considered that the difference is due to the *true* fourth-oreder nuclear potential, and then he adopted the adiabatic approximation.

It must be stressed that the repetition of the second-order potential on this method involves much more processes than those in other methods such as the Tamm-Dancoff approximation. For instance, the process represented by Fig. 9-4 can be considered to be the repetition of the second-order potential neither in the perturbation theory nor in the Tamm-Dancoff method, whereas it is involved in the repetition of the second-order potential with full retardation corrections.

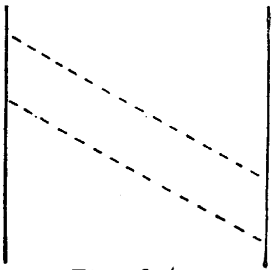


Fig. 9-4

His second-order potential differs from the second-order adiabatic potential but reduces to the latter in the adiabatic limit and the difference can, if necessary, be eliminated by a suitable canonical transformation.

Let us, then, trace his mathematical formulation. He started from the covariant Tomonaga-Schwinger equation

$$i\frac{\delta}{\delta\sigma(x)}\Psi[\sigma]=H_{int}(x)\Psi[\sigma] \tag{9.1}$$

with $$H_{int}(x)=gj_\lambda\varphi_\lambda\ ,$$

where g is the coupling constant, j_λ the source of the meson field being a bilinear form in ψ and $\bar{\psi}$, and λ an index denoting the ordinary spin and/or isotopic spin.

He applied the familiar canonical transformation,

$$\Psi[\sigma]=\exp\left[-\frac{i}{2}\int_{-\infty}^{\infty}\varepsilon(\sigma,\sigma')H_{int}(x')(dx')\right]\Psi_1[\sigma]\ , \tag{9.2}$$

and obtained the effective second-order Hamiltonian for systems without real mesons

$$H^{(2)}(x)=-\frac{1}{2}\left(\frac{g^2}{4\pi}\right)j_\lambda(\boldsymbol{r})\int O_{\lambda\mu}V(\boldsymbol{r}-\boldsymbol{r}')j_\mu(\boldsymbol{r}')d\boldsymbol{r}'\ , \tag{9.3}$$

where $O_{\lambda\mu}$ is defined by

$$[\varphi_\lambda(x),\varphi_\mu(x')]=i\Delta_{\lambda\mu}(x-x')=iO_{\lambda\mu}\Delta(x-x')\ , \tag{9.4}$$

and $V(\boldsymbol{r}-\boldsymbol{r}')$ by

$$V(\boldsymbol{r}-\boldsymbol{r}')=\begin{cases}\dfrac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\exp\left(-|\boldsymbol{r}-\boldsymbol{r}'|\sqrt{\mu^2+\dfrac{d^2}{dt^2}}\right), & \text{for } \mu\neq 0\,,\\ \dfrac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\dfrac{1}{2}\left\{\exp\left(-|\boldsymbol{r}-\boldsymbol{r}'|\dfrac{d}{dt}\right)+\exp\left(|\boldsymbol{r}-\boldsymbol{r}'|\dfrac{d}{dt}\right)\right\}, & \text{for } \mu=0\,.\end{cases} \tag{9.5}$$

The quantity V represents the potential operator with full retardation (or non-static) corrections as we shall see in the following. Since we are in the interaction representation, the operation d/dt on J_μ is expressed by the commutator with the free Dirac Hamiltonian i.e., if we rewrite eq. (9.3) in the configuration space

$$H^{(2)}(1,2)=-\frac{1}{2}\left(\frac{g^2}{4\pi}\right)\{j_\lambda(1)O_{\lambda\mu}V(12)j_\mu(2)+j_\mu(2)O_{\mu\lambda}V(21)j_\lambda(1)\}\,, \tag{9.6}$$

the differential operator d/dt_s can be replaced by

$$d/dt_s=i[H_s^{(0)},\ \ ]\,, \qquad (s=1,2)$$

where $H_s^{(0)}$ is the free Dirac Hamiltonian for the s-th nucleon, thus we can make all operators refer to the same time.

It is readily understood that the quantity V in (9.5) represents the potential operator with full retardation corrections only if one remembers the formula

$$\exp\left(\pm a\frac{d}{dt}\right)j(\boldsymbol{r},t)=j(\boldsymbol{r},t\pm a)\,,$$

and compares it with the latter case $\mu=0$ of eq. (9.5).

Next, suppose that the potential operator V is repeated for the calculation of a part of the fourth-order S matrix, then it will contain the contributions from such processes as expressed by Fig. 9-5 when t and $t+\Delta t$ are two adjacent times because of the retardation effects. Such a graph will contribute to the S matrix in the following form

$$(1-iH^{(2)}(t+\Delta t)dt)(1-iH^{(2)}(t)dt)\,. \tag{9.7}$$

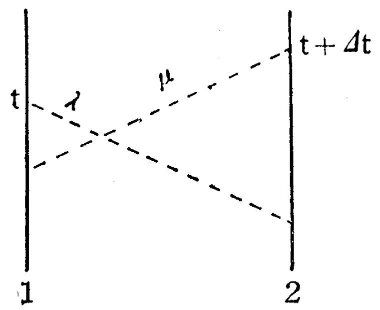


Fig. 9-5

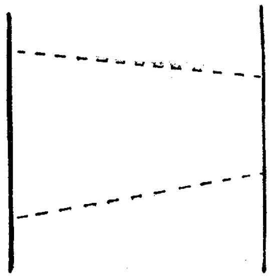


Fig. 9-6a

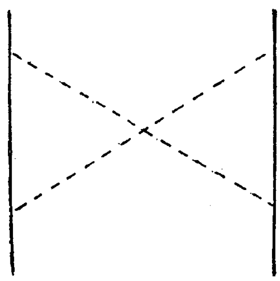


Fig. 9-6b

In the calculation of the fourth-order S matrix, we must take into account the wo diagrams Fig. 9-6a and Fig. 9-6b apart from the renormalization corrections which extend the source nucleons. The contribution from the diagram in Fig. 9-6a can completely be included in the repetition of he second-order potential in the case of Nambu's method contrary to the case of the Tamm-Dancoff method as touched upon in the beginning of this section. Furthermore, even the contribution from Fig. 9-6b is included although operators are in a wrong order, and the contribution from the "true" fourth-order potential is given as the difference of the matrix elements between that of Fig. 9-6b and (9.7).

In eq. (9.7) the nucleon current operators are arranged in the order

$$j_\lambda(1)j_\mu(1)\cdot j_\lambda(2)j_\mu(2) ,$$

where the figures in the parentheses are to discriminate the two nucleons and the time variables are not explicitly written. This statement is obvious from the order of H's in eq. (9.7), i.e., Fig. 9-5 stands for a process in which a meson λ is exchanged by $H^{(2)}(t)$ and then another meson μ is exchanged by $H^{(2)}(t+\Delta t)$.

In the above expression, however, the order of operators and that of time coordinates are opposite to one another as $j_\lambda(2)j_\mu(2)$, whereas they always appear in the same order, when one calculates the contribution of Fig. 9-6b, as $P(j_\lambda(2)j_\mu(2))$. Hence the contribution of the "true" fourth-order may be

$$\begin{aligned}\Delta S=&\frac{1}{2}g^4\int j_\mu(1)j_\nu(3)[j_\lambda(4),j_\rho(2)]\Delta_{F\mu\rho}(12)\Delta_{F\nu\lambda}(34)\\&\times\frac{1+\varepsilon(13)}{2}\cdot\frac{1+\varepsilon(42)}{2}(dx_1)(dx_2)(dx_3)(dx_4)\\&\equiv -i\int V^{(4)}dT ,\end{aligned} \tag{9.8}$$

where $V^{(4)}$ is the true fourth-order potential, Δ_F is defined by

$$\Delta_{F\mu\rho}(12)=\langle P(\varphi_\mu(1)\varphi_\rho(2))\rangle_{meson\ vacuum}$$

and T is the centre of gravity time.

The adiabatic approximation is achieved by neglecting the time-dependence of j's and gives

$$\begin{aligned}\Delta S_{adiab}=&\frac{g^4}{8}\int\frac{1+\varepsilon(13)\varepsilon(42)}{2}[j_\mu(1),j_\nu(3)][j_\lambda(4),j_\rho(2)]\\&\times\Delta_{F\mu\rho}(12)\Delta_{F\nu\lambda}(34)(dx_1)(dx_2)(dx_3)(dx_4) .\end{aligned} \tag{9.9}$$

Using this method, Taketani, Machida, and Onuma (52) calculated the fourth-order potential in the charge symmetric pseudoscalar meson theory. As seen from the appearance of commutator brackets in eq. (9.9), this result agrees with the $W^{(4)}$ given in eq. (6.11).

10. Expansion in Number of Exchanged Mesons

Our arguments so far doveloped are more or less based on the expansion in the meson-nucleon coupling constant. It is well known, however, that such a weak coupling approach is not generally justified in the meson theory, and we must prepare another more reliable argument. In this section we shall discuss the expansion in the mumber of exchanged mesons which is characteristic of the nuclear force problem.

First, the process of the one-meson-exchange is most generally expressed by Fig. 10-1, i.e. the nucleon lines are represented by the S_F' function, the meson line by the Δ_F' function, and the vertex part by Γ. The differences among various approximations to calculate this process consist in the treatments of these functions. When the two nucleons are far away from one another, only such a meson that has almost vanishing four momentum, will be interchanged provided that the nucleons are at very low energies.* And in such an idealized situation, the one-meson-exchange potential would, independently of the approximation methods employed, tend asymptotically towards

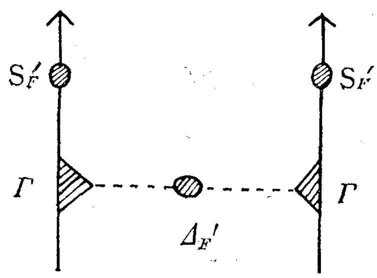


Fig. 10-1

$$\frac{1}{3}\left(\frac{g_e^2}{4\pi}\right)\mu(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})\left[(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})+\left(1+\frac{3}{x}+\frac{3}{x^2}\right)S_{12}\right]\frac{e^{-x}}{x}. \tag{10.1}$$

This form is completely identical with the second-order potential except that the coupling constant g is replaced by the effective or the renormalized coupling constant g_e. At very low energies, the main behavior of the nucleon-nucleon interaction would be determined by the asymptotic form of the true potential which is valid at very large distances of the order of the deuteron radius. Although we can adjust the inner potential phenomenologically, the strength of the outer potential cannot arbitrarily be changed since it should fit the low energy data. Hence, at low energies, the meson potential can be checked by the experimental data. In fact, it has been shown by Otsuki and Tamagaki (54) and Iwadare, Otsuki, Tamagaki, and Watari (56) that the meson potential is consistent with the low energy data and the effective coupling constant $g_e^2/4\pi$ is determined as 0.080 ± 0.010. Their phenomenological analyses can be regarded as a generalization as well as a more precise specification of the previous works by Taketani and his collaborators (Taketani 52, Fujii 54).

The importance of this new approach might be twofold; one is the recognition of the existence of a certain kind of low energy limit theorem that enables us to test the meson theory and determine the effective coupling constant, and the other is the agreement of the effective coupling constants between nuclear forces and real pion reactions such as the pion-nucleon scattering and photo-pion production. Indeed, Chew and Low (56)

* This means that two nucleons are very nearly free.

have reached the same value of the effective coupling constant, $\dot{g}_e^2/4\pi\sim 0.08$ on the analysis of real pion reactions by making use of low energy limit theorems.

This result might serve as an indication that the meson theory in the present stage involves much more truth than has been suspected so far.

As the two nucleons draw near to distances of the order of pion Compton wave length, the two-meson-exchange potential turns out to be appreciable. The diagrams Fig. 10-2, Fig. 10-3, and Fig. 10-4 contribute to the two-meson-exchange potential.

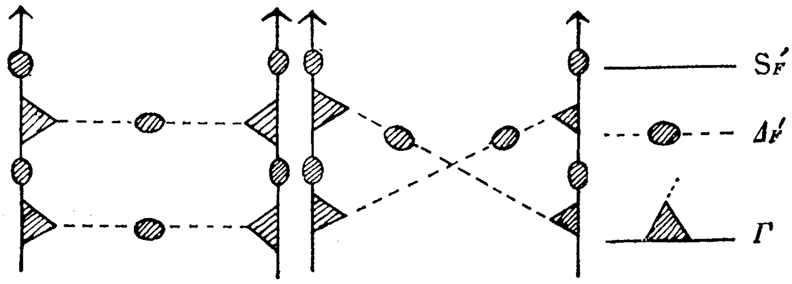


Fig. 10-2a Fig. 10-2b

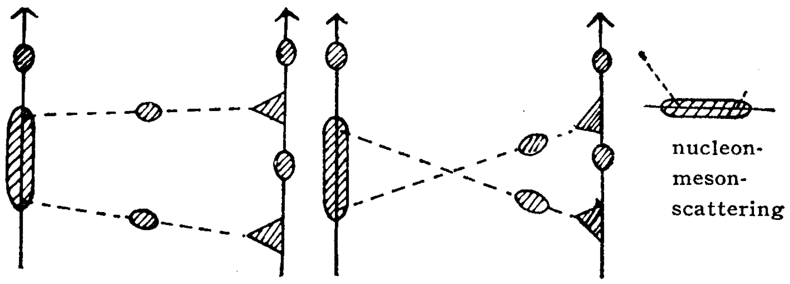


Fig. 10-3a Fig. 10-3b

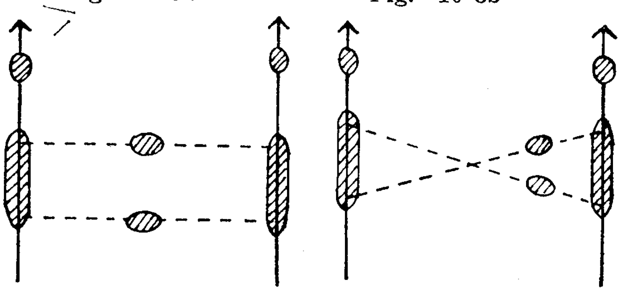


Fig. 10-4a Fig. 10-4b

The contributions of Fig. 10-2 are supposed to tend again at large distances towards the perturbation theoretical results with the coupling constant replaced by the renormalized one. Since, however, we are now interested in the behavior of these contributions at relatively small distances of the order of the pion Compton wave length, there might be small deviations from the perturbation theoretical result. These deviations are yet supposedly small at the force range provided that the distortion of the meson cloud around a nucleon by the presence of another nucleon is still small. A more precise argument on this point will be presented in the next section. If we adopt such a weak coupling viewpoint to the two-meson-exchange potential, the only correction that remains to be discussed of the three types of radiative corrections, (1) self energy part, (2) vertex part and (3) the meson-nucleon scattering, is the last one which is illustrated in Figs. 10-3 and 10-4. In fact, the evaluation of the last type of correction varies considerably from case to case according to the approximations employed, and the implication of this correction to the nuclear potential will also be discussed in the next section.

The precise examination of the two-meson-exchange potential will be given in the following two sections, and the remainder of this section will be devoted to the convergence problem for the expansion in the number of exchanged mesons. This is, of course, still an unsettled question, but higher order corrections in the $Ps(pv)$ theory calculated up to the three-meson-exchange (Machida 55) seem to support that it is really convergent at larger distances than a critical one, say $x_c \sim 0.6$. In order to clarify the situation we shall refer to another simple example discussed by Klein (53a). Applying the Dyson transformation, we can rewrite the Hamiltonian in the $Ps(ps)$ theory as

$$\begin{aligned} H_{int} &= ig\psi\gamma_5\tau_\alpha\psi\varphi_\alpha \\ &\rightarrow \left(\frac{g^2}{2M}\right)\bar{\psi}\psi\varphi_\alpha{}^2 + \left(\frac{g}{2M}\right)\bar{\psi}\boldsymbol{\sigma}\tau_\alpha\psi\cdot\nabla\varphi_\alpha + \cdots . \end{aligned}$$

Picking up only the first core term, Klein discussed the convergence problem and obtained an asymptotic convergence condition

$$\frac{g^2}{4\pi}\left(\frac{e^{-x}}{x}\right) < \frac{M}{\mu} .$$

This result suggests us that even for unrenormalizable interactions the present expansion might converge at least at distance larger than a certain critical distance.

In general, the convergence condition would take the form

$$\frac{g_e{}^2}{4\pi}\left(\frac{e^{-x}}{x^n}\right) < a , \qquad (10.2)^*$$

which results from the uncertainty principle argument as well as the dimensional consideration on the meson-nucleon interaction. In the above inequality, n is determined from the dimension of the coupling constant for the meson-nucleon interaction, and g is the dimensionless factor in the whole coupling constant.

Such a guess has already been implicitely exploited in the former part of this section, and if we choose $g_e{}^2/4\pi = 0.08$ in the $Ps(pv)$ theory the condition (10.2) would supposedly read

$$x > 0.6 \sim 0.7 .$$

* In Klein's example the charge renormalization is not taken into account, so that the bare coupling constant instead of the renormalized one appears.

Inside the critical distance, our simple weak coupling theoretical picture of nucleons would turn out to be false, so that we shall be obliged to refer to the phenomenological approach in the present stage of the meson theory. In addition to the failure of the present expansion method, there are many elements that prevent us to investigate the nature of the nucleon-nucleon interaction at small distances, hence it seems to the author that the calculation of the three-meson-exchange potential or higher ones could not be meaningful. For instance, the discussions in 7 shows us that the non-adiabatic correction of an appreciable order of magnitude arises already in the range of the two-meson-exchange potential.

11. Radiative Corrections

For the reason stated in the previous section our main subject in the present report is the investigation of the two-meson-exchange potential, and in order to accomplish this theme we must first investigate the effects of three types of radiative corrections; (1) the self energy part, (2) vertex part and (3) virtual meson-nucleon scattering. All these corrections refer to the one nucleon system, and we can utilize the results obtained in the study of the one nucleon system. Since we are discussing the nuclear potential in the static approximation, the same approximation has to be made also in one nucleon system. Furthermore, we must carefully take account of the renormalization procedures since these quantities are fundamental ones in the renormalization theory.

Fortunately these requirements are satisfied by Chew's static extended source theory (54) and we shall make use of his theory. It is readily noticed, however, that his theory cannot in its original form be applied to the two-nucleon system, i.e., the separation between observable and unobservable quantities cannot clearly be done as he achieved success in the case of the one-nucleon system.

The reason for this difficulty may be explained as follows. Chew has adopted the time independent description to the one-nucleon system, which is equivalent to another description where we trace the temporal development of the system by means of the Schrödinger equation. Now suppose that there is a two-nucleon system and that a meson is emitted virtually from a nucleon "1" as shown in Fig. 11-1. If the meson is absorbed by the same nucleon "1", it will contribute to the self energy effect the most part of which gives rise to unobservable mass renormalization. On the contrary, if it is absorbed by another nucleon "2", it will contribute to the nuclear force which gives rise to a completely observable effect.

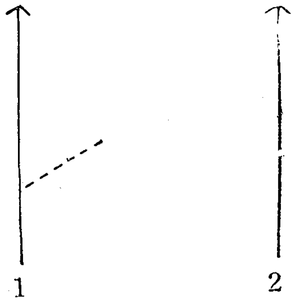


Fig. 11-1

In the Schrödinger picture, however, we cannot presume by which of the two nucleons the meson would be absorbed, and consequently the separation cannot clearly be performed. This is the reason for the difficulty of renormalization in his picture.

In spite of such a difficulty, however, it is still possible to carry out the renormalization program in the two-nucleon system by Chew's model. This is accomplished by referring to Feynman's over-all-space-time point of view (Feynman 49). In this picture, we sum up the amplitudes over all possible histories, and in each history it is possible to determine whether a virtual meson is a nuclear force meson or a self field meson.

Hence it is clear that we must refer to the over-all-space-time point of view in order to perform the renormalization program for the two-nucleon system, and in what follows we shall develop our theory on Feynman's viewpoint.

The meson-nucleon interaction in the static extended source theory is given by

$$H_{int}=\frac{g}{\mu}\tau_\lambda\int d\boldsymbol{r}\rho(\boldsymbol{r}-\boldsymbol{r}_N)\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\varphi_\lambda(\boldsymbol{r})\,, \tag{11.1}$$

where $\rho(\boldsymbol{r})$ is the form factor to cut off the high frequency components of the meson field. The total Hamiltonian of the system is then given by

$$H_{tot}=(M-\delta M)\int\psi^*\psi d\boldsymbol{r}+H_M+H_{int}\,. \tag{11.2}$$

The first term may be dropped from the beginning with the understanding that the observable nucleon mass is slided to zero. This is possible, of course, only in the no-pair approximation.

Let us consider the nucleon propagator

$$\begin{aligned}S_F(1-2)&=\langle vac|T[\psi(1)\psi^*(2)]|vac\rangle\\&=S(t)\delta(\boldsymbol{r})\,,\end{aligned}$$

where T denotes Wick's chronological product (Wick 50) and

$$t=t_1-t_2\,,\quad \boldsymbol{r}=\boldsymbol{r}_1-\boldsymbol{r}_2\,,$$

$$S(t)=\frac{i}{2\pi}\int\frac{\exp(-iEt)}{E-M+i\eta}dE=\begin{cases}\exp(-iMt), & \text{for } t\geq 0,\\ 0\,, & \text{for } t<0,\end{cases} \tag{11.3*}$$

η is a positive infinitesimal quantity to represent the boundary condition. The occurrence of $\delta(\boldsymbol{r})$ is characteristic of the static approximation and we can discuss physical problems by omitting this δ factor. The meson propagator is given by

* In this section we use η as the infinitesimal quantity to express the outgoing wave condition. ε is reserved for the relative energy.

$$\delta_{\lambda\mu}\Delta_F(1-2)=\langle vac|T[\varphi_\lambda(1)\varphi_\mu(2)]|vac\rangle$$
$$=\frac{i}{(2\pi)^4}\delta_{\lambda\mu}\int\frac{\exp[iq(x_1-x_2)]}{q^2+\mu^2-i\eta}(dq) \tag{11.4}$$

For the later purpose we shall also introduce

$$\langle vac|\varphi_\lambda(1)\varphi_\mu(2)|vac\rangle=\frac{1}{(2\pi)^3}\int\frac{d\boldsymbol{q}}{2\omega_q}\exp[i\boldsymbol{qr}-iq_0t]$$
$$\equiv\sum\frac{1}{2\omega_q}\exp[i\boldsymbol{qr}-iq_0t] \tag{11.5}$$
$$\equiv\Delta(1-2)\,.$$

Let us first consider the problem of the one-nucleon system. Then we easily have the Feynman rule:

vertex: $\quad -i(g/\mu)\tau_\lambda\sigma_\alpha\times$ (cut-off factor),

meson line: $\quad \nabla_\alpha^{(1)}\nabla_\beta^{(2)}\Delta_F(1-2)\cdot\delta_{\lambda\mu}$,

nucleon line: $\quad S(t_2-t_1)$.

In what follows we shall not explicitly write down the cut-off factor for the sake of simplicity.

For example, the self energy part is given by

$$\int\ldots dt_1dt_2\ldots(-1)(g/\mu)^2\tau_\lambda\sigma_\alpha S(t_2-t_1)\tau_\lambda\sigma_\beta$$
$$\nabla_\alpha^{(2)}\nabla_\beta^{(1)}\Delta_F(2-1)\cdots.$$

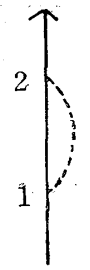


Fig. 11-2

In the one-nucleon system, the following relation is essentially important to prove the equivalence of Chew's method to Feynman's one.

$$S(t_2-t_1)\Delta_F(2-1)=S(t_2-t_1)\theta(t_2-t_1)\Delta_F(2-1)$$
$$=S(t_2-t_1)\Delta(2-1)\,, \tag{11.6}$$

where eqs. (11.3), (11.4) and (11.5) are utilized and θ denotes

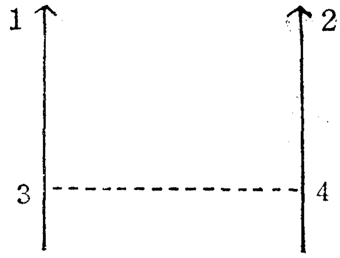
$$\theta(t)=\begin{cases}1, & \text{for } t\geq 0\,,\\ 0, & \text{for } t<0\,.\end{cases}$$

If we insert the expression (11.5) into the above integrand we have completely the same formula with Chew's one.

Next, let us consider the problem of the two-nucleon system. Then it is clear that the same argument as above can be applied to the propagator of a self field meson, but not to the propagator of a nuclear force meson. This means that the over-all description cannot be reduced to the time independent picture in the two-nucleon system, but one can apply the results obtained in Chew's theory to the one-nucleon parts of diagrams.

Our main subject in this section is the application of the above formalism to the derivation of the renormalized nuclear potential, and we shall solve this problem regarding the adiabatic nuclear potential as the self energy of the two-nucleon system. In Chew's theory the self energy of the nucleon is obtained as the pole of the nucleon propagator $S(E)$ with radiative corrections so that self energy of the two-nucleon system may be calculated with the same method. This is achieved by the calculation of the pole of the two-nucleon propagator. The two-nucleon propagator obeys the usual Bethe-Salpeter equation.

Since $\boldsymbol{r}$, $(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})$, $(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})$ and S_{12} are fixed in the static approximation there seems at first sight no freedom to describe the two nucleon system, but one must slove the relative energy dependence of the wave function characteristic of the Bethe-Salpeter method (Salpeter 51), which enables us to construct an eigenvalue equation. The integral equation corresponding to the simplest graph a in Fig. 11-3 is



Graph a
Fig. 11-3

$$\psi(t_1, t_2) = \left(\frac{g}{\mu}\right)^2 \int dt_3 dt_4 S(t_1 - t_3) S(t_2 - t_4)$$
$$\times (\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}) (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\nabla}) (\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{\nabla}) \tag{11.7}$$
$$\times \Delta_F(\boldsymbol{r}, t_3 - t_4) \psi(t_3, t_4) ,$$

where t_1, t_2 are the individual times. By applying the Fourier transformation, the above equation is rewritten as an integral equation with respect to the relative energy ε,

$$\psi(\varepsilon) = \int K_a(\varepsilon, \varepsilon', E) \psi(\varepsilon') d\varepsilon' , \tag{11.8}$$

where E is the total energy of the system to be determined as the eigenvalue and is involved as a parameter in the kernel K_a. The kernel K_a has the form

$$K_a(\varepsilon, \varepsilon', E) = \frac{2i}{4\pi}\left(\frac{g}{\mu}\right)^2 \frac{1}{(E/2)^2 - \varepsilon^2 + i\eta} O_a \int \frac{d\boldsymbol{q}}{(2\pi)^3} \cdot \frac{\exp(i\boldsymbol{q}\cdot\boldsymbol{r})}{\boldsymbol{q}^2 + \mu^2 - (\varepsilon - \varepsilon')^2 - i\eta} , \tag{11.9}$$

where O_a is defined by

$$O_a = (\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}) (\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\nabla}) (\boldsymbol{\sigma}^{(2)}\cdot\boldsymbol{\nabla}) , \quad \boldsymbol{\nabla} = \partial/\partial\boldsymbol{r} .$$

In what follows we shall omit the parameter E from the kernel.

The next kernel corresponding to the graph b in Fig. 11-4 is

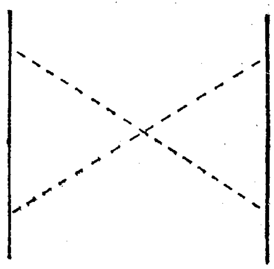
$$K_b(\varepsilon, \varepsilon') = -\frac{1}{(2\pi)^8}\left(\frac{g}{\mu}\right)^2 \int dq d\boldsymbol{q}_1 d\boldsymbol{q}_2 O_b \exp\{i(\boldsymbol{q}_1\boldsymbol{r} + \boldsymbol{q}_2\boldsymbol{r}')\}$$
$$\times \left[\left(\frac{E}{2} + i\eta\right)^2 - \varepsilon^2\right]^{-1} \left[\left(\frac{E}{2} + i\eta - q\right)^2 - \left(\frac{\varepsilon + \varepsilon'}{2}\right)^2\right]^{-1} \tag{11.10}$$
$$\times \left[(\omega_1 - i\eta)^2 - \left(q + \frac{\varepsilon - \varepsilon'}{2}\right)^2\right] \left[(\omega_2 - i\eta)^2 - \left(q - \frac{\varepsilon - \varepsilon'}{2}\right)^2\right]^{-1} ,$$

where ω's are given by $\omega_1 = \sqrt{\mu^2 + \boldsymbol{q}_1^2}$, $\omega_2 = \sqrt{\mu^2 + \boldsymbol{q}_2^2}$, and O_b by

$O_b=O_\alpha^{(1)}O_\beta'^{(1)}\cdot O_\beta'^{(2)}O_\alpha^{(2)}$. (cf. eq. (6.12).) Then we shall try to solve the eigenvalve equation (11.8) so that we can express E as a function of $\boldsymbol{r}$, $(\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)})$, $(\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})$ and S_{12}. This problem is reduced to a more simple equation

$$D(E)=0 \tag{11.11}$$

with the aid of Fredholm's theory. $D(E)$ is the Fredholm determinant associated with eq. (11.8), and is defined by



Graph b
Fig. 11-4

$$D(E)=1-\int K(\varepsilon,\varepsilon)d\varepsilon+\frac{1}{2!}\int\int\begin{vmatrix}K(\varepsilon,\varepsilon) & K(\varepsilon,\varepsilon')\\ K(\varepsilon',\varepsilon) & K(\varepsilon',\varepsilon')\end{vmatrix}d\varepsilon d\varepsilon'+\cdots\cdots, \tag{11.12}$$

We shall first calculate the Fredholm determinant $E(D)$ by taking account of the graph a alone. The second term of eq. (11.12) is

$$\int K_a(\varepsilon,\varepsilon)d\varepsilon=\frac{W^{(2)}}{E},$$

where $W^{(2)}$ is given by eq. (6.10)*.

The third term is given by

$$\frac{1}{2}\int\int d\varepsilon d\varepsilon'\begin{vmatrix}K_a(\varepsilon,\varepsilon) & K_a(\varepsilon,\varepsilon')\\ K_a(\varepsilon',\varepsilon) & K_a(\varepsilon',\varepsilon')\end{vmatrix}=-\frac{1}{4E}\left(\frac{g}{\mu}\right)^2 O_aO_a'G(r,r')_{r'\to r}+\cdots,$$

where G is defined by

$$G(r,r')=\frac{2}{(2\pi)^3}\left(\frac{1}{r}+\frac{1}{r'}\right)K_0(\mu r+\mu r'),$$

and we have used an expansion in E/ω in the calculation of the above integral and retained only the first term so that the above expression is correct only if $|E|<\mu$.

As a whole the contribution of the graph a to the determinant is

$$D_a(E)=1-\frac{1}{E}\left(-\frac{1}{4\pi}\left(\frac{g}{\mu}\right)^2 O_\alpha^{(1)}O_\alpha^{(2)}\frac{e^{-\mu r}}{r}+\frac{1}{4}\left(\frac{g}{\mu}\right)^4 O_\alpha^{(1)}O_\beta'^{(1)}O_\alpha^{(2)}O_\beta'^{(2)}\right.$$
$$\left.\times G(r,r')+\cdots\right)_{r'\to r}. \tag{11.13}$$

The contribution of the graph b to the fourth-order is given by

$$\int K_b(\varepsilon,\varepsilon)d\varepsilon=-\frac{1}{4E}\left(\frac{g}{\mu}\right)^2 O_bG(r,r')+\cdots,$$

where again the expansion in the ratio E/ω is applied.

Hence the contributions of the graph a together with those of b yield

$$D(E)=1-\frac{1}{E}(W^{(2)}+W^{(4)}+\cdots\text{power series in } E/\mu\ldots.), \tag{11.14}$$

* For simplicity we have dropped the cut-off factor in this calculation.

where $W^{(4)}$ is given by eq. (6.11). Due to this result we can solve eq. (11.11) and obtain the TMO potential

$$E = W^{(2)} + W^{(4)} + \cdots \tag{11.15}$$

The above approach is justified only when the expansion in the ratio E/μ rapidly converges. Such a correction occurs due to the effect that the meson cloud around a nucleon is distorted from the isolated one when another nucleon approaches. The weak coupling viewpoint adopted above remains correct only if the distortion of the meson cloud is rather small, i.e. if the cloud is nearly identical with the isolated case, otherwise we must calculate further terms in the expansion.

Next we shall study another approximation method. The physical requirement that $\psi^r(\varepsilon)$ in eq. (11.8) must be an even function of ε permits us to replace the kernel $K(\varepsilon, \varepsilon')$ by $K(\varepsilon, -\varepsilon')$ to obtain the same eigenvalue. If we adopt this approach, we have

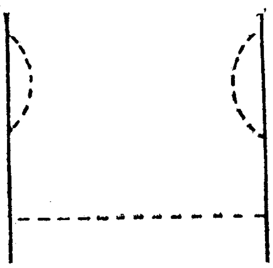
$$D(E) = 1 - \frac{1}{E}\left(W^{(2)} + \frac{O_a}{(2\pi)^3}\left(\frac{g}{\mu}\right)^2 K_0(\mu r) \cdot E + W^{(4)\prime} + \cdots\right),$$

where $W^{(4)\prime}$ is the same as the fourth-order potential by Brueckner and Watson (53). Unlike the previous case, even the second-order term becomes E dependent and we have retained the first term in the expansion. Then the solution of (11·11) takes the form

$$E = (\mathcal{V}_{BW} + \cdots) / \left(1 - \frac{O_a}{(2\pi)^3}\left(\frac{g}{\mu}\right)^2 K_0(\mu r) + \cdots\right), \tag{11.16}$$

where $\mathcal{V}_{BW}$ denotes Brueckner-Watson's potential and if one expands this expression in the coupling constant one finds again the TMO potential. Indeed the expression (11.16) is very like the FST potential (Fukuda 54). So far as the weak coupling point of view is taken for granted, these potentials must give approximately the same result at large distances. For this purpose we need that the term $(2\pi)^{-3} O_a (g/\mu)^2 K_0(\mu r)$ is very small compared with unity.

After these preliminaries we shall investigate the effects of radiative corrections which are the main subjects of this section. Let us first calculate the contribution of the graph a' to the one-meson-exchange potential. For this purpose, we need to find the modified nucleon propagator $S'(E)$. This has already been calculated by Chew and we can utilize his result. The renormalized expression of $S_r'(E)$ is given by Chew as



Graph a' Fig. 11-5

$$1/S_r'(E) = E\left[1 + \frac{g^2}{4\pi} \Delta(E)\right], \tag{11.17}$$

where

$$\Delta(E)=\frac{3}{\pi}\int_{\mu}^{\infty}d\omega_q\frac{q^3}{\mu^2\omega_q^2}\cdot\frac{E}{\omega_q-E}v^2(q)$$

$v(q)$ is the Fourier transform of the factor $\rho(\boldsymbol{r})$. From the above expression, it is clear that *such a method can be applicable only if* $E<\mu$ *because the expression* (11.17) *diverges at* $E=\mu$. *It is also the case for the Tamm-Dancoff method in the static limit as pointed out by Klein* (54).

The Fredholm determinant for the one-meson-exchange with the radiative corrections of the self-energy type becomes

$$D(E)=1-\frac{W^{(2)}}{E}\left(1-\frac{g^2}{4\pi}\Delta(E)+\cdots\right). \tag{11.18}$$

When E is small, one can expand $\Delta(E)$ as

$$\Delta(E)=c\left(\frac{E}{\mu}\right)+\cdots,$$

where

$$c=\frac{3}{\pi}\int_{\mu}^{\infty}d\omega_q\frac{q^3}{\mu\omega_q^3}v^2(q)\sim 1.$$

In this case the solution of the eigenvalue problem is

$$E=W^{(2)}\left(1+c\frac{g^2}{4\pi}\cdot\frac{W^{(2)}}{\mu}\right)^{-1}. \tag{11.19}$$

This result is reasonable in the sense that the self energy correction due to the distortion of the individual self fields of nucleons decreases exponentialy when the inter-nucleon distance increases.

The numerical estimation of this correction shows that it is very small, i.e. at most of the order of a few percent at the range, and we can safely discard this correction. According to Chew's theory, the vertex correction is even smaller than the self energy correction because of the correlations of ordinary and isotopic spins, and consequently the vertex correction can also be neglected in our discussions.

Finally the correction due to the scattering of virtual mesons by nucleons is of a rather complicated nature. This correction can be calculated also by means of Chew's solution for the meson-nucleon scattering such as represented by Fig. 11-6, and then inserting the resulting expression into the Fredholm determinant. However, we shall not calculate this correction explicitly because of the following reason.

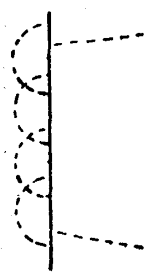


Fig. 11-6

It is well known that the meson-nucleon scattering is enhanced at high energies, especially in the $I=J=3/2$ states, but the virtual exchanged mesons have much lower energies than such real mesons so that they are rather weakly coupled to nucleons. In fact, Brueckner and Watson (53) have shown that such effects are sensible only at very small distances, $x<0.5$. As the two nucleons approach so that the scattering effects become important, the effects of multiple scattering of virtual mesons by two nucleons also turns out to be important. A typical diagram for the multiple scattering of a virtual meson is given by Fig. 11-7. The arguments in the previous section show, however, that this is an indication of the invalidity of the expansion in the number of exchanged mesons. Accordingly we may lose our basis that guarantees our approach. At such small distances, the concept of static nuclear potential becomes vaguous and the discussion of such effects is far beyond our ability.

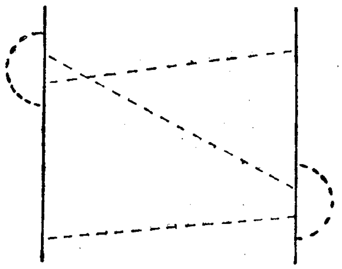


Fig. 11-7

12. Tamm-Dancoff Method and FST Potential

In connection to other branches of meson physics, the investigation of nuclear forces by means of the Tamm-Dancoff method is desirable. In the early stage, this method was invented to take account of the non-adiabatic effects such as the recoil correction so that it is often called the non-adiabatic method (Tamm 45, Dancoff 50). Later, this method was developed as an expansion in the number of meson, which is much more reliable than the conventional expansion in the coupling constant. Accordingly the term "non-adiabatic" is sometimes incorrectly used.

This method was first employed by Lévy (52) and Klein (53) to derive the adiabatic potential and further developed by many authors. The essence of this method consists in the following device.

In order to solve the Schrödinger equation

$$(H_0+H_{int})\Psi=E\Psi, \quad (H_0=H_N+H_M)\ , \tag{12.1}$$

one expands the state vector in the number of mesons (and/or nucleon pairs) as

$$\Psi=\Psi_0+\Psi_1+\Psi_2+\cdots. \tag{12.2}$$

Then the original equation (12.1) is transformed to an infinite set of simultaneous equations for $\Psi_0,\ \Psi_1,\ \Psi_2,\cdots$.

Assuming that Ψ_n's are negligible for $n>N$ if N is chosen suitably large, one eliminates all Ψ's but the lowest component, Ψ_0, leaving an equation of the form

$$(E-H_0)\Psi_0=\mathcal{V}(E)\Psi_0. \tag{12.3}$$

This $\mathcal{V}(E)$ is equivalent to the $\mathcal{V}$ introduced in **8** and represents the effective interaction between two nucleons. Many authors incorrectly referred to $\mathcal{V}(E)$ as the potential, but it can readily be shown that this $\mathcal{V}(E)$ never satisfies the conditions given in **2**, and this is the reason why the trivial conditions in **2** are recaptulated.

First, the left hand side of eq. (8.1) is not unitary since it is merely a submatrix of a unitary matrix. It is approximately unitary only if it is approximately diagonal in the sense of **6**, i.e., the dissociation probability of the meson around a nucleon is very small. Accordingly it is obvious that $\mathcal{V}(E)$ is not an Hermitian operator.

Secondly, $\mathcal{V}(E)$ is energy dependent as is clear from eq. (8.2). Thus $\mathcal{V}(E)$ cannot be called the nuclear potential.

Let us discuss the derivation of $\mathcal{V}(E)$, assuming that H_{int} is linear in the meson field. As an example, we shall cite here the method of Brueckner and Watson (53). Their method consists of successive elimination of redundant components from the left side of eq. (8.2). For this purpose they applied the algebraic relation

$$<\frac{1}{a-H_{int}}H_{int}>=<\frac{1}{a-L}H_{int}>+<\frac{1}{a-L}L>=<\frac{1}{a-L}L>,$$

where $<\ >$ denotes the meson-vacuum expectation value, and a and L are defined, respectively, by

$$a=E+i\varepsilon-H_0,$$

$$L=H_{int}\frac{1}{a}H_{int}.$$

In the above device, the components involving odd number of mesons are eliminated. By continuing similar procedures they eliminated further components successively. The solution of eq. (8.2) is then obtained in the form

$$\mathcal{V}(E)=<h^-\frac{1}{a}h^+>+<h^-\frac{1}{a}h^-\frac{1}{a-h^-\frac{1}{a}h^+}h^+\frac{1}{a}h^+>+\cdots\cdots. \tag{12.4}$$

They estimated first the effects of multiple scattering appearing in the denominator of the second term and showed that it is unimportant at large distances, $x>0.5$. Then they approximated the second term as

$$\frac{1}{a-h^-\frac{1}{a}h^+}\to\frac{1}{a}. \tag{12.5}$$

In order to obtain the adiabatic potential, they approximated the expression by replacing a by $-H_M$, i.e.,

$$a\to-H_M. \tag{12.6}$$

Thus they reached the potential called by their names

$$\mathcal{V}_{BW}=<h^-\frac{1}{-H_M}h^+>+<h^-\frac{1}{-H_M}h^-\frac{1}{-H_M}h^+\frac{1}{-H_M}h^+>. \tag{12.7}$$

As has been discussed, this cannot be a nuclear potential and in fact it differs from the TMO potential by

$$V_{TMO}=\mathcal{V}_{BW}-<h^-\frac{1}{-H_M}h^+><h^-\frac{1}{(-H_M)^2}h^+>. \tag{12.8}$$

The reason for this difference was investigated by many authors, especially by Sawada (53), Feldman (55), Klein (54) and lately by Fukuda, Sawada, and Taketani (Fukuda 54, Okubo 54) thoroughly. The BW potential is very similar to the TMO potential except for the triplet even states. The former gives an attractive central force in the state and seems to be more favorable than the latter. Therefore it must involve some truth in spite of the obscurity of the derivation and this point motivated many authors to examine the relation between these two potentials.

In the following, we shall briefly trace the method of Fukuda, Sawada, and Taketani.*

In order to solve eq. (12. 1) they decomposed the state vector Ψ as

$$\Psi = V_0\Psi + (1-V_0)\Psi\,, \tag{12.9}$$

where V_0 denotes the projection operator to the meson vacuum. Then Ψ can formally be expressed in terms of $V_0\Psi$

$$\Psi = \frac{1}{1-\dfrac{1}{E-H_0}(1-V_0)H_{int}} V_0\Psi = J(E)\,V_0\Psi\,. \tag{12.10}$$

This is a generalization of the relation between the free and true vacua obtained by Gell-Mann and Low (51).

Substituting eq. (12. 10) into (12. 1), one can readily write down the formal expression of $\mathcal{V}(E)$

$$\mathcal{V}(E) = V_0 H_{int} \frac{1}{1-\dfrac{1}{E-H_0}(1-V_0)\,H_{int}} V_0\,. \tag{12.11}$$

This $\mathcal{V}(E)$ is the effective interaction appearing in the equation

$$(E-H_0)\,V_0\Psi = \mathcal{V}(E)\,V_0\Psi\,. \tag{12.12}$$

Since $\mathcal{V}(E)$ is non-Hermitian and energy dependent, $V_0\Psi$'s are not properly orthonormalized. The correct orthonormalization condition is expressed by

$$\langle\Psi_i|\Psi_j\rangle = \langle V_0\Psi_i\,|V_0 J(E_i)^+ J(E_j)\,V_0\,|V_0\Psi_j\rangle = \delta_{ij}\,. \tag{12.13}$$

The energy dependence of the operator $J(E)$ can be eliminated with the help of the original equation of motion. We shall write the eliminated one as J, and introduce

$$\chi_i = V_0(J^+J)^{1/2}V_0\Psi_j\,. \tag{12.14}$$

Then χ's form a correct orthonormal set.

The equation for the χ function is then given by

$$E\chi = V_0(J^+J)^{-1/2}V_0{}^+J(H_0+H_{int})\,JV_0(J^+J)^{-1/2}V_0\chi\,. \tag{12.15}$$

* For further details, see Part III by Machida and Toyoda or the original paper (Fukuda 54, Okubo 54).

It is clear from the expression (12. 13) that the quantity

$$(J^+J)^{-1}=1-P_d(r) \tag{12.16}$$

represents the probability to find bare nucleons in a dressed state.

By equating the right hand side of (12. 15) to $(H_0+V)\chi$, one finds the correct expression for the potential. In the static limit $H_0 \to H_M$, one finds

$$V=\frac{J^+(H_M+H_{int})J}{J^+J}=\frac{<h^-\frac{1}{-H_M}h^+>+<h^-\frac{1}{-H_M}h^-\frac{1}{-H_M}h^+\frac{1}{-H_M}h^+>+\cdots}{1+<h^-\frac{1}{-H_M{}^2}h^+>+\cdots} \tag{12.17}$$

$$=\frac{\mathcal{V}_{BW}+\cdots}{1+<h^-\frac{1}{-H_M{}^2}h^+>+\cdots}=(1-P_d)\mathcal{V}_{BW}+\cdots. \tag{12.18}$$

This equation guarantees that the BW potential is correct if the dissociation probability is small in conformity with the general prediction mentioned in the beginning of this section. Furthermore, if one expands the denominater in the coupling constant, one arrives at the TMO potential. This is also justified provided that the dissociation probability is very small.

Thus we may conclude that so far as the weak coupling viewpoint is taken for granted, both the TMO and BW potentials are reliable. As stated before they differ markedly in the triplet even state, which is an indication that the above viewpoint is not completely justified and that the meson cloud around a nucleon is strongly distorted by the presence of another nucleon in the state. In such a case we must retain the higher order distortion effects correctly.

As a rule, however, many methods give completely the same one-meson-exchange potential and very similar two-meson-exchange potentials. Hence we may conclude that the meson theoretical potential is quantitatively trustworthy at large distances of the order of the deuteron radius and is qualitatively reliable at distances near the force range, and that the meson theory of nuclear forces is subject to experimental tests at least at low energies. Phenomenological analyses show that it is satisfactory at low energies.*

* For further details, see Part II by Iwadare, Otsuki, Tamagaki, and Watari.

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