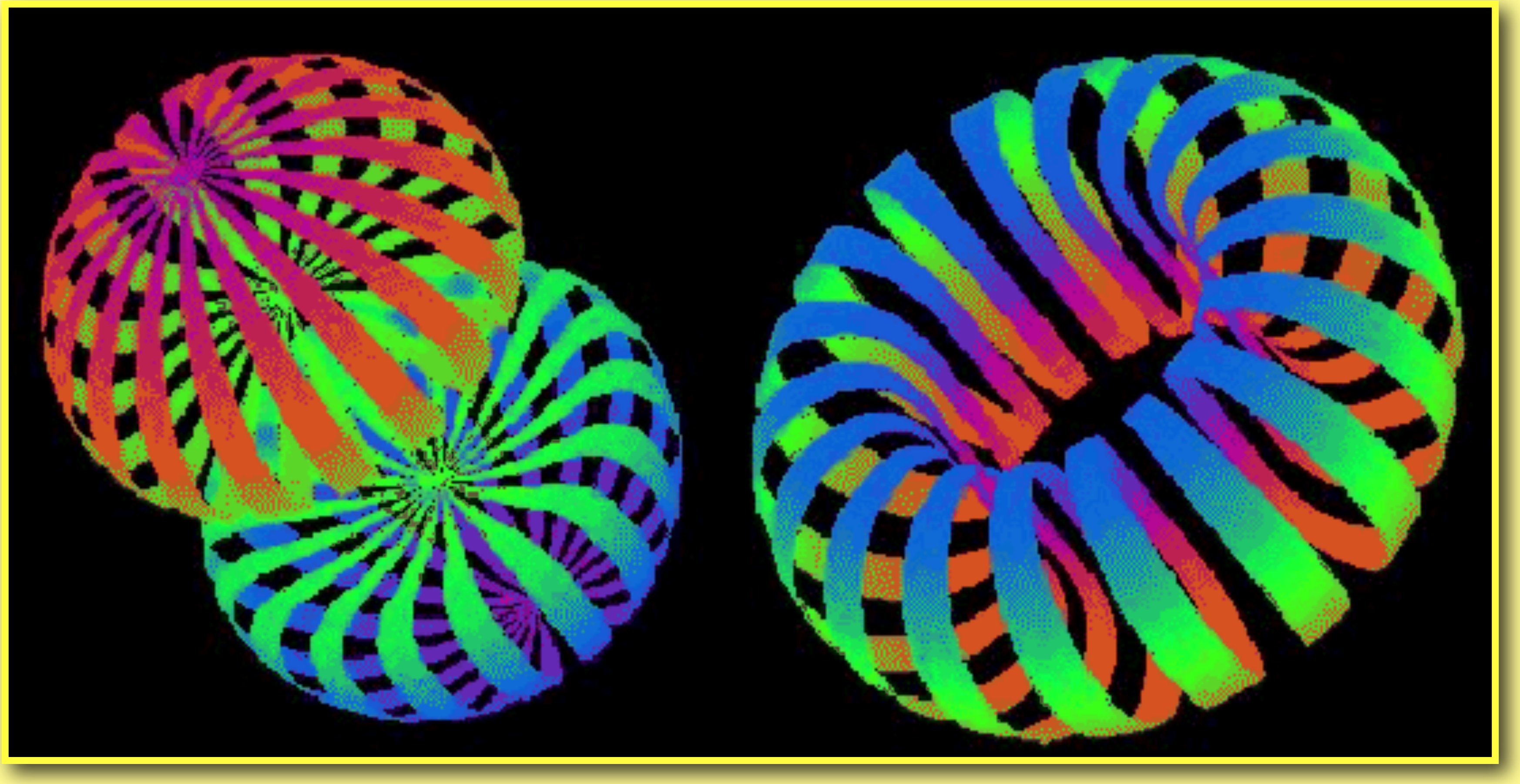
Some facts about the deuteron

1. A deuteron (^2H nucleus) consists of a neutron and a proton. (A neutral atom of ^2H is called deuterium.)
2. It is the simplest bound state of nucleons and therefore gives us an ideal system for studying the nucleon-nucleon interaction.
3. An interesting feature of the deuteron is that it does not have excited states because it is a weakly bound system.



Shapes of the deuteron in the laboratory reference frame. Stripes show surfaces of equal density for the $M_J = 1$ (left) and $M_J = 0$ (right) magnetic substates of the $J = 1$ ground state. From http://www.phy.anl.gov/theory/movie-run.html.

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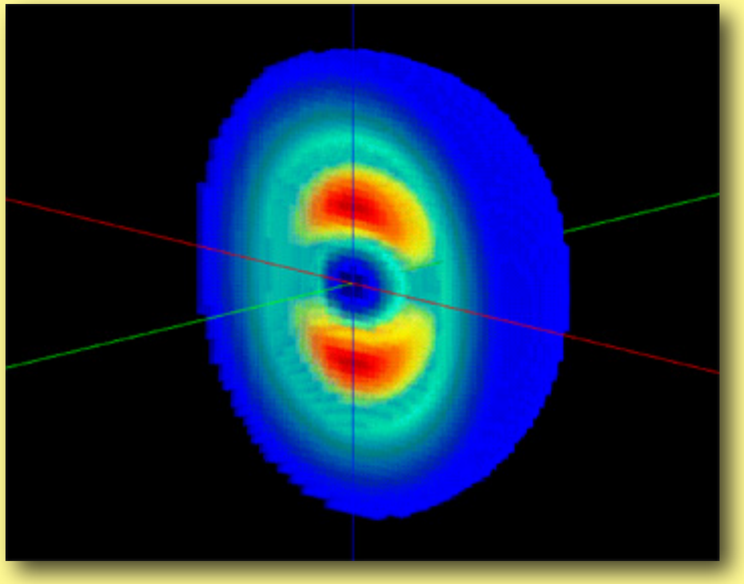

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3. An interesting feature of the deuteron is that it does not have excited states because it is a weakly bound system.

4. In analogy with the ground state of the hydrogen atom, it is reasonable to assume that the ground state of the deuteron also has zero orbital angular momentum **L = 0**

5. However the total angular momentum is measured to be **J = 1** (one unit of h/2π) thus it follows that the proton and neutron spins are parallel $s_n + s_p = 1/2 + 1/2 = 1$

6. The implication is that two nucleons are not bound together if their spins are anti-parallel, and this explains why there are no proton-proton or neutron-neutron bound states.

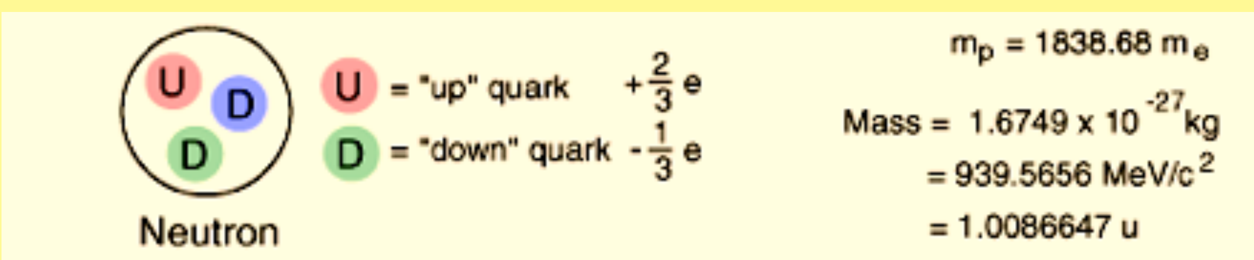
7. The parallel spin state is forbidden by the Pauli exclusion principle in the case of identical particles

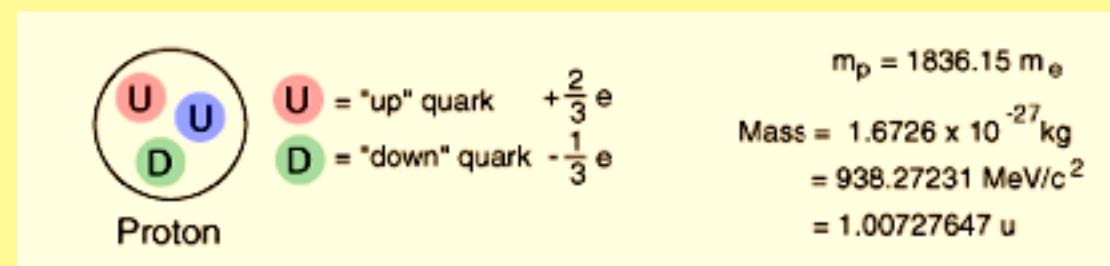
8. The nuclear force is thus seen to be **spin dependent**.

The deuteron is a stable particle.

Abundance of 1.5 x 10-4 compared to 0.99985 for ordinary hydrogen

Constituents	1 proton + 1 neutron
Mass	2.014732 u
Binding energy	2.224589 ± 0.000002 MeV
Angular momentum	1
Magnetic moment	0.85741 ± 0.00002 μ_N
Electric quadrupole moment	$+2.88 \times 10^{-3}$ bar





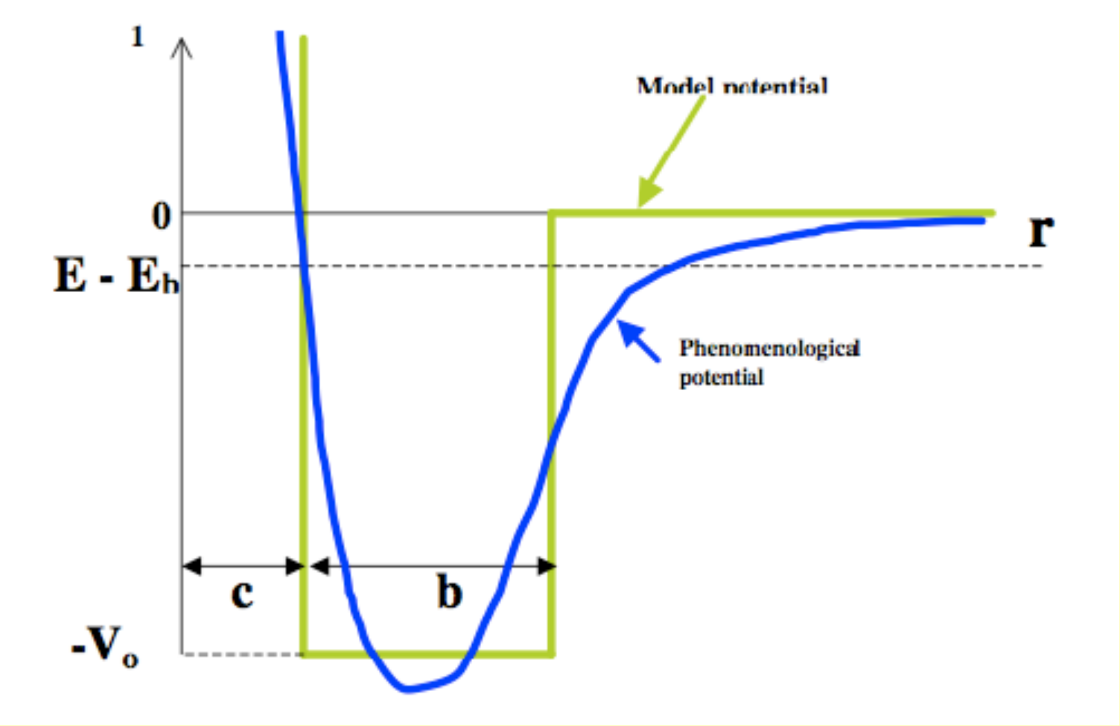
Binding energy of the deuteron is 2.2 MeV.

If the neutron in the deuteron were to decay to form a proton, electron and antineutrino, the combined mass energies of these particles would be 2(938.27 MeV) + 0.511 MeV = 1877.05 MeV

The mass of the deuteron is 1875.6 MeV

The average binding energy per nucleon is about 7 ~ 8 MeV for typical nuclei. The binding energy of the deuteron, B = 2.224 MeV, is small when compared with typical nuclei.

This means that the deuteron is very weakly bound.



To simplify the analysis of the deuteron, we could assume that the nucleon-nucleon potential is a three-dimensional square well

The measured spin of the deuteron is $\mathbf{J} = 1$.

By studying the reactions involving deuterons and the property of the photon emitted during the formation of deuterons, we know that its parity is **even**.

The total angular momentum $\mathbf{J}$ of the deuteron should be like

$$\mathbf{J} = \mathbf{s}_n + \mathbf{s}_p + \mathbf{l}$$

where $\mathbf{s}_n$ and $\mathbf{s}_p$ are individual spins of the neutron and proton.

The orbital angular momentum of the nucleons as they move about their common center of mass is $\mathbf{l}$

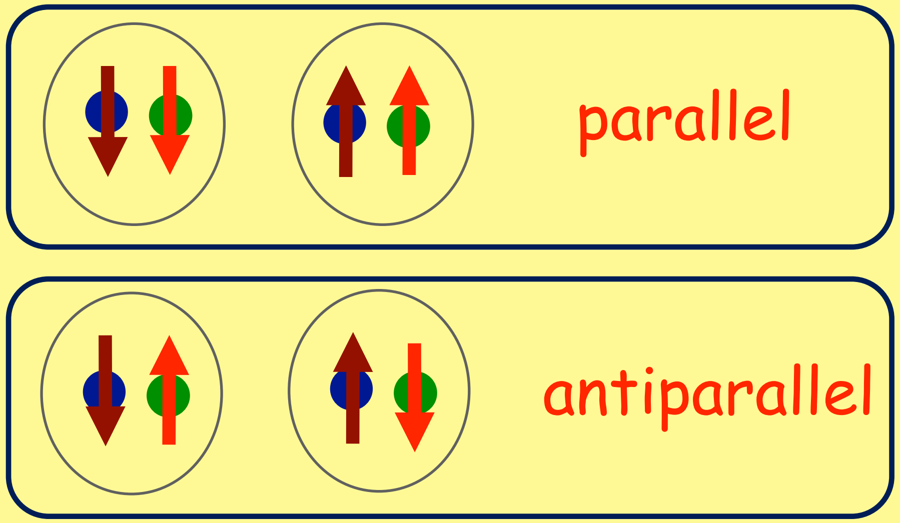
There are four ways to couple $\mathbf{s}_n$, $\mathbf{s}_p$, and $\mathbf{l}$ to get a total $\mathbf{J}$ of 1.

(a) $\mathbf{s}_n$ and $\mathbf{s}_p$ parallel with $\mathbf{l}$ = 0

(b) $\mathbf{s}_n$ and $\mathbf{s}_p$ antiparallel with $\mathbf{l}$ = 1

(c) $\mathbf{s}_n$ and $\mathbf{s}_p$ parallel with $\mathbf{l}$ = 1

(d) $\mathbf{s}_n$ and $\mathbf{s}_p$ parallel with $\mathbf{l}$ = 2



Since we know that the parity of the deuteron is **even** and the parity associated with orbital motion is determined by $(-1)^l$ we are able to rule out some options.

Orbital angular momentum l = 0 and l = 2 give the correct parity determined from experimental observations.

The observed even parity allows us to eliminate the combinations of spins that include l =1, leaving l = 0 and l = 2 as possibilities.

Deuteron Summary so far

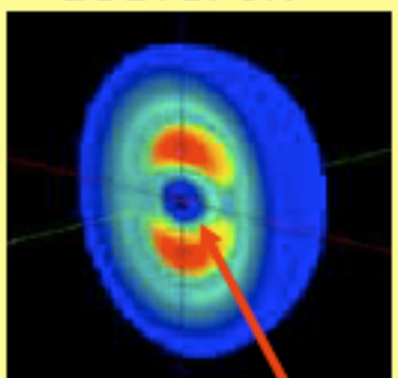
Quantum numbers: $(J^{\pi}, I) = (1^{+}, 0)$ favor a $^3S_1$ configuration (with $S = 1$, $L = 0$) as the lowest energy n-p bound state

Magnetic moment: $\mu = 0.857\ \mu_N$ is 2.6% smaller than for a pure $^3S_1$ state and is consistent with a linear combination of $L = 0$ and $L = 2$ components:

$$|\psi_d\rangle = a\,|{}^3S_1\rangle + b\,|{}^3D_1\rangle \quad with \quad a^2 + b^2 = 1 \quad and \quad b^2 = 0.04$$

If this is correct, we can draw two conclusions about the N-N force:

deuteron:



1. The $S = 1$ configuration has lowest energy (i.e., observed deuteron quantum numbers) , so the N-N potential must be more attractive for total spin $S = 1$ than for $S = 0$.

2. The lowest energy solution for a spherically symmetric potential is purely $L = 0$, and the $L = 2$ wave functions are orthogonal to $L = 0$ wave functions, so a physical deuteron state with mixed symmetry can only arise if the N-N potential is not exactly spherically symmetric!

Also: the "hole" in the middle of the deuteron means that at very short distances the N-N potential must be repulsive – the neutron and proton do not overlap!

Deuteron Quadrupole moment etc...

Formalism for electrostatic moments:

- Recall that an electric charge distribution of arbitrary shape can be described by an infinite series of multipole moments (spherical harmonic expansion), with terms of higher L reflecting more complicated departures from spherical symmetry

- The energy of such a distribution interacting with an external electric field $\vec{E} = -\vec{\nabla} V$ can be written as:

$$E_{\text{int}} = V(0)\, q + \left.\frac{\partial V}{\partial z}\right|_0 p_z - \frac{1}{4}\left.\frac{\partial^2 V}{\partial z^2}\right|_0 Q_{zz} + \ldots$$

where the charge distribution is located at the origin and has a symmetry axis along z;

q is the electric charge or `monopole moment'

p_z is the electric dipole moment

Q_{zz} is the electric quadrupole moment

and in general, higher order multipole moments couple to higher derivatives of the external potential at the origin.

Formalism, continued.... (axially symmetric case)

The multipole moments are defined in general by:

$$E_\ell = \int \rho(\vec{r})\, \hat{E}_\ell \, d^3 r$$

with corresponding multipole operators proportional to spherical harmonic functions of order ℓ:

$$\hat{E}_\ell = c_\ell \, r^\ell \, Y_{\ell 0}(\theta)$$

The first few multipole operators and associated moments are:

order, ℓ	Moment	Symbol	operator, $\hat{E}_\ell$
0	*monopole*	q	$1 = \sqrt{4\pi}\, r^0\, Y_{00}(\theta)$
1	*dipole*	p_z	$z = r\cos\theta = \frac{\sqrt{4\pi}}{3}\, r^1\, Y_{10}(\theta)$
2	*quadrupole*	Q_{zz}	$r^2(3\cos^2\theta - 1) = \frac{\sqrt{16\pi}}{5}\, r^2\, Y_{20}(\theta)$

Connection to nuclei and the deuteron problem:

For a nuclear system, the multipole moments are expectation values of the multipole operators, e.g. the electric charge:

$$q \;\equiv\; \int \rho(\vec{r})\, d^3r \;\equiv\; e \sum_{i=1}^{Z} \int \psi_i^*(\vec{r})\, \hat{E}_0\, \psi_i(\vec{r})\, d^3r \;=\; +Ze$$

The electric charge density is proportional to the probability density, i.e. the wave function squared, summed up for all the protons in the system !!!

We already know the electric charge of the deuteron, but what about higher moments?

Electric dipole moment, general case:

$$p_z \;=\; \int \Psi^* \hat{E}_1\, \Psi\, d^3r \;=\; \int |\Psi|^2\, r \boxed{\cos\theta}\, d^3r \;=\; 0$$

integral **must** vanish!

total wave function of the system

even under space reflection

odd, $\ell = 1$

continued...

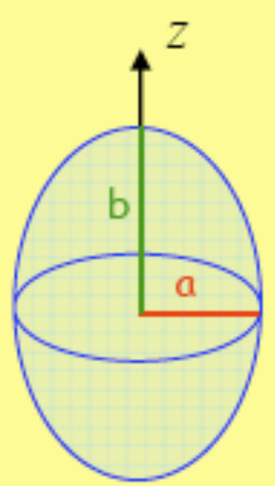
Basic symmetry property: no quantum system can have an electric dipole moment if its wave function has definite parity, i.e. if $|\Psi|^2$ is even.

(Incidentally, the possibility of a nonzero electric dipole moment of the neutron is of great current interest: so far the measured upper limit is $|p_n| < 10^{-25}$ e - cm ....)

In fact, all odd multipole moments must vanish for the same reason

First nontrivial case: Electric Quadrupole Moment Q_{zz}

$$Q_{zz} = \int \rho(\vec{r})\, r^2\, (3\cos^2\theta - 1)\, d^3r$$



example: uniform density object with ellipsoidal shape

$$Q_{zz} = \frac{2}{5}\, Ze\,(b^2 - a^2)$$

Q > 0 if b > a: "prolate"
Q < 0 if b < a: "oblate"

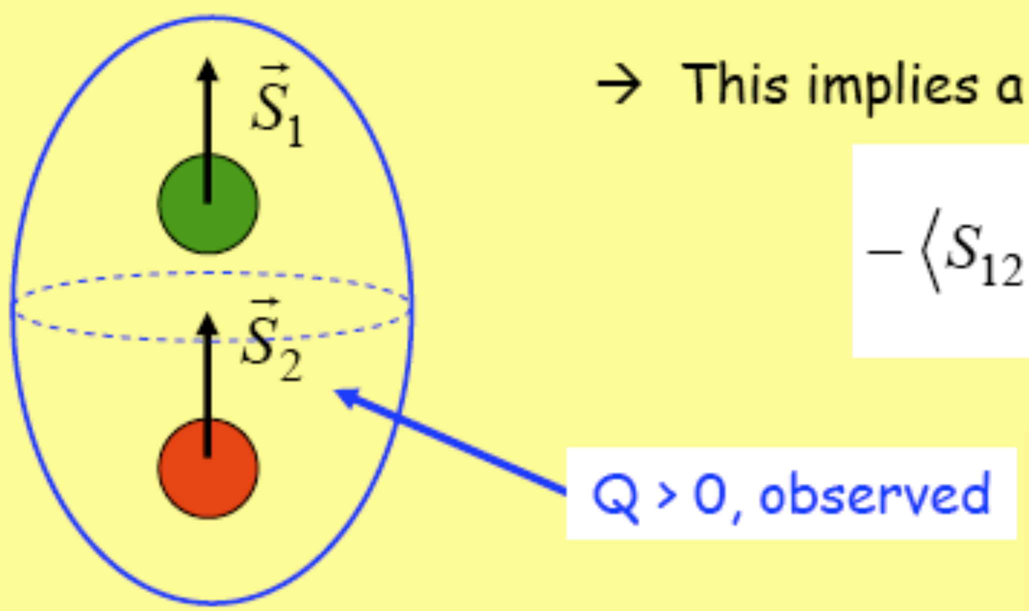
Basic features of the N-N potential, via the deuteron, etc.:

1. Independent of the value of L, the state with intrinsic spins coupled to S = 1 has lower energy

   → this implies a term proportional to:

$$-\left\langle \vec{S}_1 \bullet \vec{S}_2 \right\rangle = -\tfrac{1}{2}\left\langle S^2 - S_1^2 - S_2^2 \right\rangle = \begin{cases} -1/4, & S = 1 \\ +3/4, & S = 0 \end{cases}$$

2. The deuteron quadrupole moment implies a non-central component, i.e. the potential is not spherically symmetric. Since the symmetry axis for Q is along J, Q > 0 means that the matter distribution is stretched out along the J - axis:

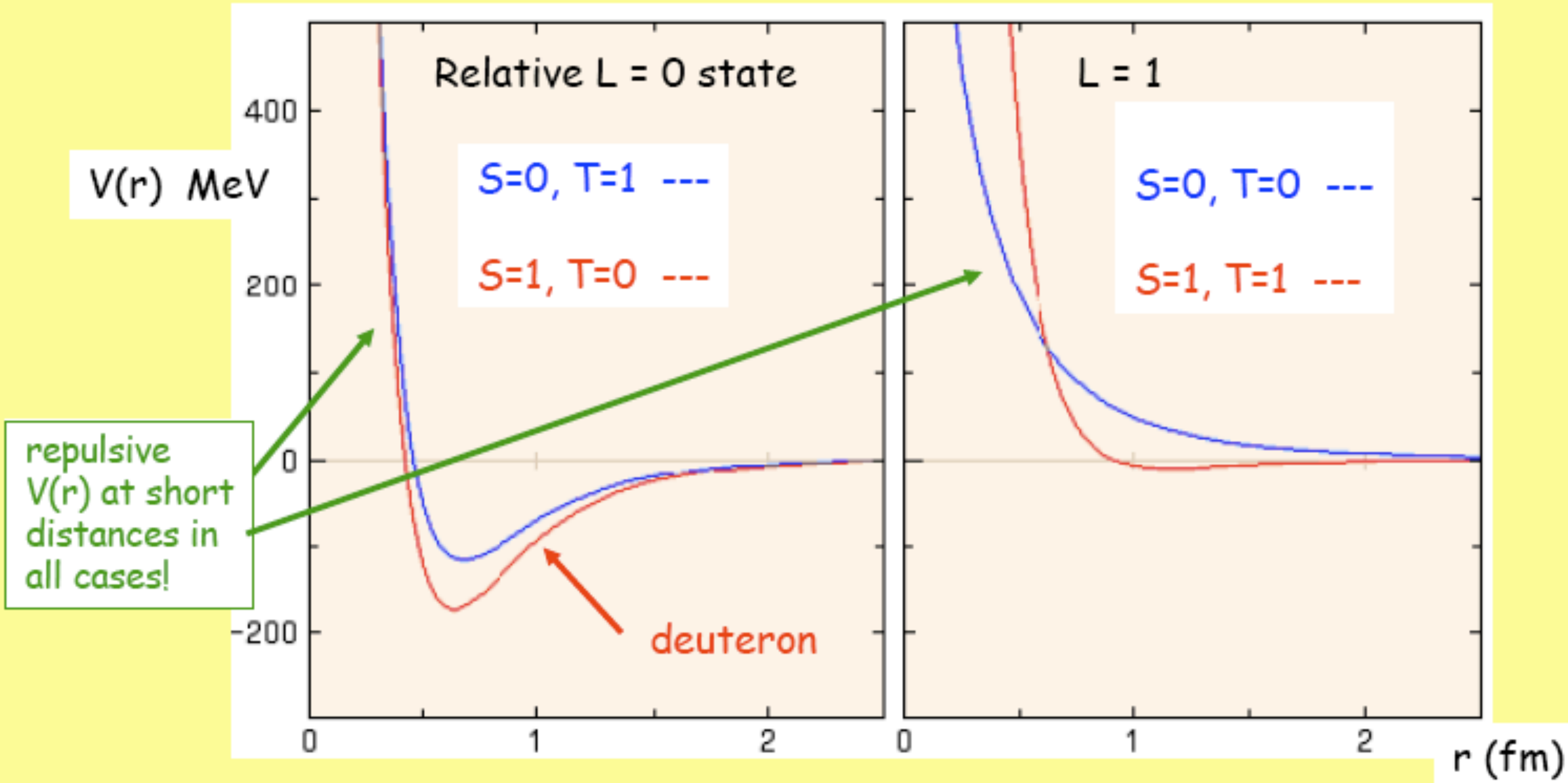


→ This implies a "tensor" force, proportional to:

$$-\left\langle S_{12} \right\rangle = -\left\langle 3\,\frac{\left(\vec{S}_1 \bullet \vec{r}\right)\left(\vec{S}_2 \bullet \vec{r}\right)}{r^2} - \vec{S}_1 \bullet \vec{S}_2 \right\rangle$$

What does the NN potential look like?

It looks different in different spectroscopic states of the 2N system!



Only the deuteron is bound! Its quantum numbers have the deepest potential well.

The dynamical behaviour of a nucleon must be described by the Schrödinger's equation:

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r}) + V(\vec{r})\Psi(\vec{r}) = E\Psi(\vec{r})$$

where m is the nucleon mass.

If the potential is rotationally invariant (a central potential) then the wave function solution can be separated into radial and angular parts:

$$\Psi(r) = R(r)Y_{lm}(\theta,\varphi)$$

Substitute R(r) = u(r)/r in to the Schrödinger's equation the function u(r) satisfies the following equation ;

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left\{V(r) + \frac{l(l+1)\hbar^2}{2mr^2}\right\}u(r) = Eu(r)$$

The solution u(r) is labeled by two quantum numbers n and l so that:

$$u(r) \rightarrow u_{nl}(r)$$

n: the principal quantum number which determines the energy of an eigenstate.

l: the orbital angular momentum quantum number.

m: the magnetic quantum number, $-l \leqq m \leqq l$.

The angular part of the solution $Y_{lm}(\theta,\varphi)$ is called the "spherical harmonic" of order l, m and satisfies the following equations:

$$\hat{L}_Z Y_{lm}(\theta,\varphi) = m\hbar Y_{lm}(\theta,\varphi) \qquad \hat{L}_Z \equiv -i\hbar \frac{\partial}{\partial\varphi}$$

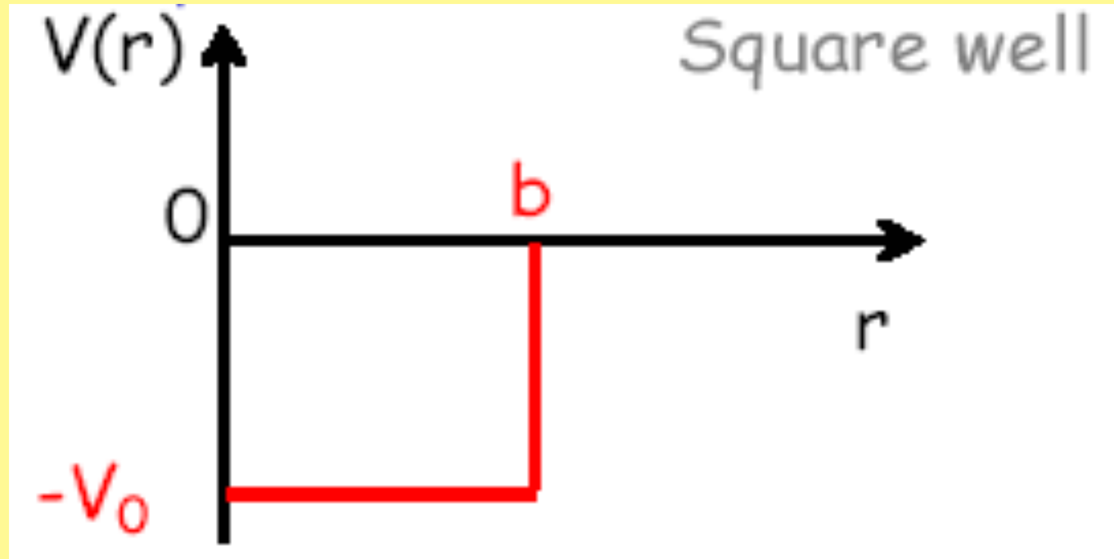
$$\hat{L}^2 Y_{lm}(\theta,\varphi) = l(l+1)\hbar^2 Y_{lm}(\theta,\varphi)$$

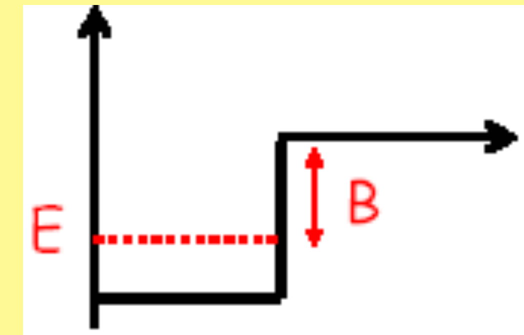
For the case of a three dimensional square well potential with zero angular momentum (l = 0), which we use as the model potential for studying the ground state of the deuteron, the Schrödinger's equation can be simplified into:

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} - V_0 u(r) = Eu(r) \qquad \text{for } r < R$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} = Eu(r) \qquad \text{for } r > R$$

b is the range of interaction





For a bound state E<0 (binding energy)

With L=0, the radial wavefunction satifies

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+V(r)u(r)=Eu(r)$$

We have two regions

$$\boxed{r<b \quad V=-V_0 \quad E=-B}$$

$$r>b \quad V=0 \quad E=-B$$

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2}+(B-V_0)u(r)=0$$

$$u(r)=A\sin kr+C\cos kr \qquad k^2=\frac{2m}{\hbar^2}(V_0-B)$$

We impose the condition that u(r)=0 if r=0 (otherwise $|R(r)|^2$ would explode)

$$u(r) = A\sin kr \quad r < b$$

Outside the range of the potential well

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + Bu(r) = 0 \longrightarrow \begin{array}{l} u(r) = De^{-k'r} + Fe^{k'r} \\ k'^2 = \dfrac{2mB}{\hbar^2} \end{array}$$

Physical solutions require F=0

$$u(r) = De^{-k'r} \quad r > b$$

Functions u(r) and their derivatives u′(r) have to be continuous

$$\begin{array}{ll} u(r) & A\sin kb = De^{-k'b} \\ du(r)/dr & kA\cos kb = -k'De^{-k'b} \end{array}$$

The ratio is (logarithmic derivative)

$$\cot kb = -\frac{k'}{k} = -\left(\frac{B}{V_0 - B}\right)^{1/2}$$

The unknown values are V_0 and b. If we assume V_0 >> Binding energy

$$\cot kb \approx 0, \quad kb = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots$$

minimum energy

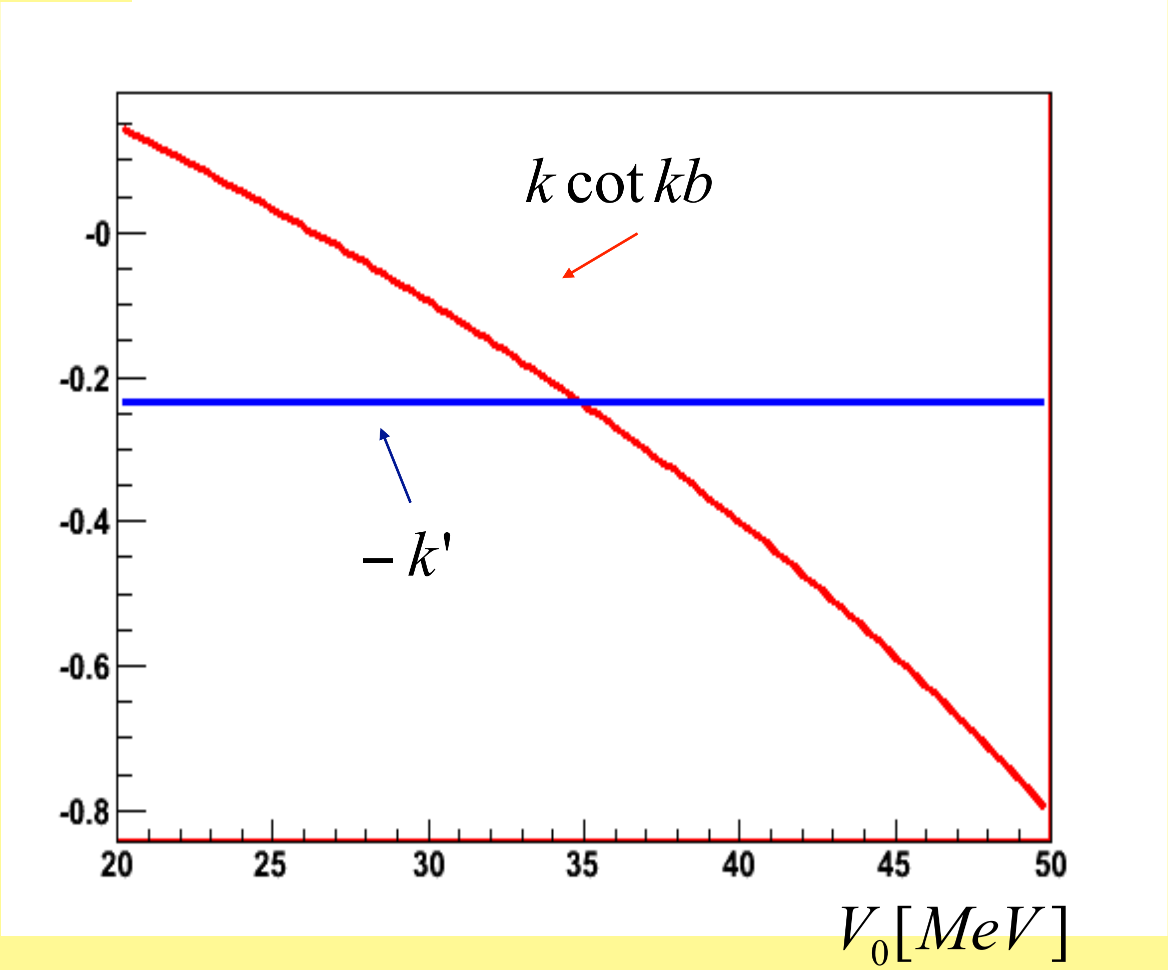
$$(kb)^2 \simeq \frac{2m}{\hbar^2} V_0 b^2 = \left(\frac{\pi}{2}\right)^2$$

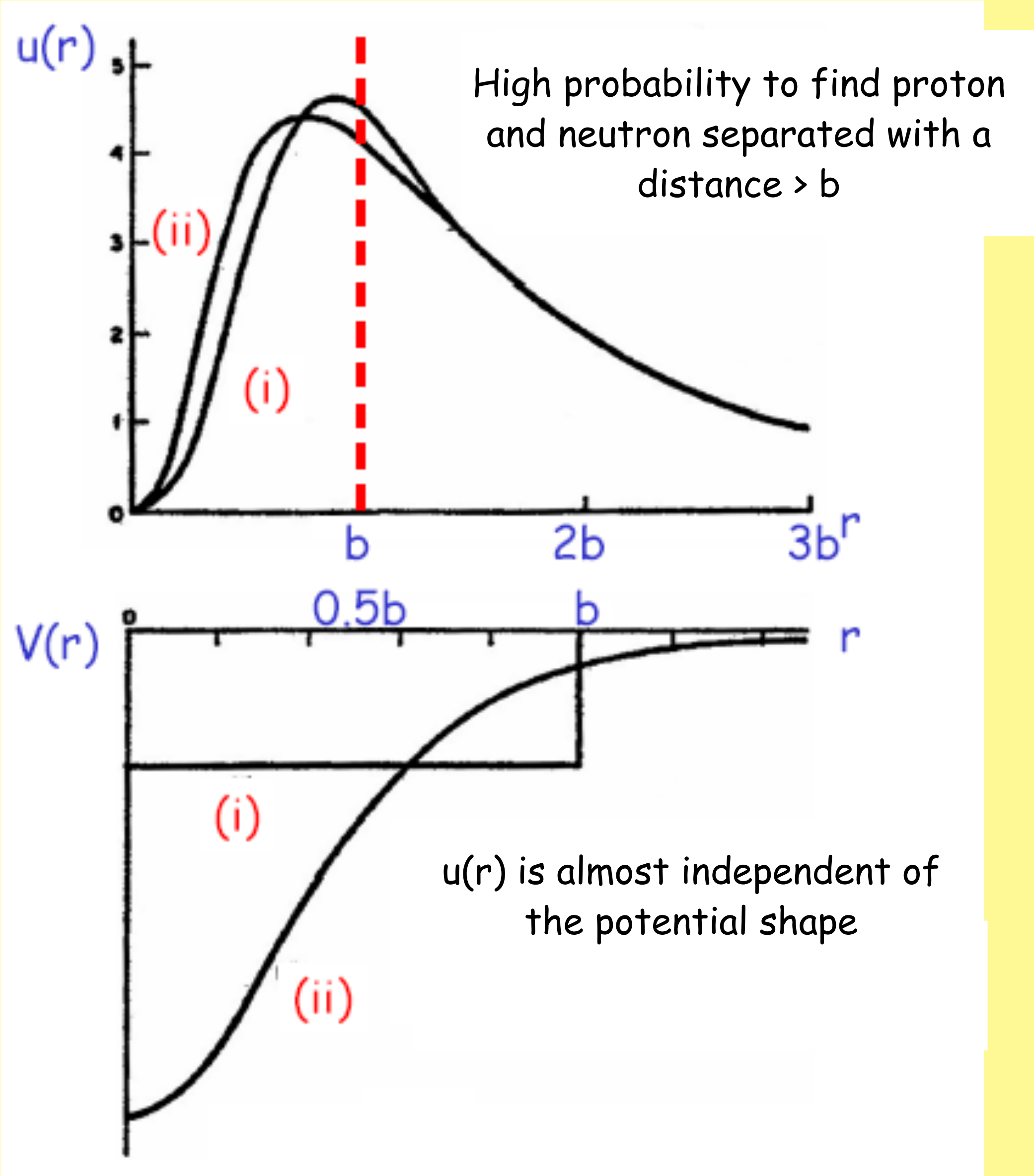
$$\Rightarrow V_0 \approx \frac{\pi^2 \hbar^2}{8mb^2} \approx 25\,\text{MeV}$$

If b = 2 fm

V_0 is the minimum depth for a bound state.

A more refined analysis will give 35 MeV





The size of the deuteron is determined by the binding energy and not by the range of the force

The characteristic length

$$R_D = \frac{\hbar}{\sqrt{2mB}} = 4.32 \text{ fm}$$

is the distance in which u(r) falls off by 1/e. It is called the deuteron radius (almost twice the potential range)

The nucleons have a large probability to stay outside the potential

A more refined approach

Assuming a central potential, the Schroedinger equation is

$$\left[-\hbar^2 \frac{1}{2\mu}\nabla^2 + V(r)\right]\psi(\vec{r}) = E\,\psi(\vec{r})$$

where

$$\psi(\vec{r}) = \frac{1}{kr}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{u_l(r)}{r}Y_{lm}(\Theta,\ \varphi)$$

The radial part would satisfy

$$\frac{d^2u_l(r)}{dr^2} + \left[k^2 - \frac{2\mu}{\hbar^2}V(r) - \frac{l(l+1)}{r^2}\right]u_l(r) = 0$$

The s-wave component can be written as

the spin component

$$\psi^{L=0}_{J=1,\,M} = \frac{1}{\sqrt{4\pi)}}\frac{u(r)}{r}\,\chi_{S=1,\,M_S}$$

In the triplet state, depending on M_S, it has the form

$$\chi_{S=1,\,M_S} = \begin{cases} \left|-\frac{1}{2}\ -\frac{1}{2}\right\rangle & M_S = -1 \\ \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\ -\frac{1}{2}\right\rangle + \left|-\frac{1}{2}\ \frac{1}{2}\right\rangle\right) & M_S = 0 \\ \left|\frac{1}{2}\ \frac{1}{2}\right\rangle & M_S = +1 \end{cases}$$

a state is represented by the angular momentum quantum numbers

$$|JM,\, LM_L,\, SM_S\rangle$$

Applying the tensor operator to a L=0 state leads to a d-wave

$$\hat{S}_{12}\,|1M,\,00,\,1M\rangle = \sqrt{8}\,|1M,\,20,\,1M\rangle$$

The total wave function can be written as

$$\begin{aligned}\psi &= \psi^{L=0}_{J=1,\,M} + \psi^{L=2}_{J=1,\,M} \\ &= \frac{1}{\sqrt{4\pi}}\left[\frac{u(r)}{r} + \frac{w(r)}{r}\frac{1}{\sqrt{8}}\hat{S}_{12}(\hat{r})\right]|\,\underbrace{1M,\,00,\,1M}_{\equiv\chi_{1\,M}} >\end{aligned}$$

The normalized condition is

$$\int_0^\infty dr\,\left[u^2(r) + w^2(r)\right] = 1$$

The radial equation is

$$u_l'' = \left[ME + MV_\pi + \frac{l(l+1)}{r^2}\right]u_l$$

$$k^2 = -ME$$

$$\hbar = 1$$

where

$$V_\pi = V_c + \hat{S}_{12}(\hat{r})V_T$$

The radial equation is decoupled

$$\begin{aligned}L = 0: &\quad u'' = [EM + MV_c]u + \sqrt{8}MV_T\,w \\ L = 2: &\quad w'' = [EM + \tfrac{6}{r^2} + MV_c - 2MV_T]\,w + \sqrt{8}MV_T\,u\end{aligned}$$

In a matrix notation, the total wave function can be written

$$\psi(r) = \frac{1}{\sqrt{4\pi}} \left[\frac{u}{r} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{w}{r} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv (L=0) \text{ and } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv (L=2)$$

The tensor operator transforms an s-state into a d-state

$$\hat{S}_{12} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sqrt{8} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Finally the radial equation becomes

$$\left[u'' + \frac{w''}{\sqrt{8}} \hat{S}_{12} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left[EM + MV_c + MV_T \hat{S}_{12} + \frac{6}{r^2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right] \left[u + \frac{w}{\sqrt{8}} \hat{S}_{12} \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

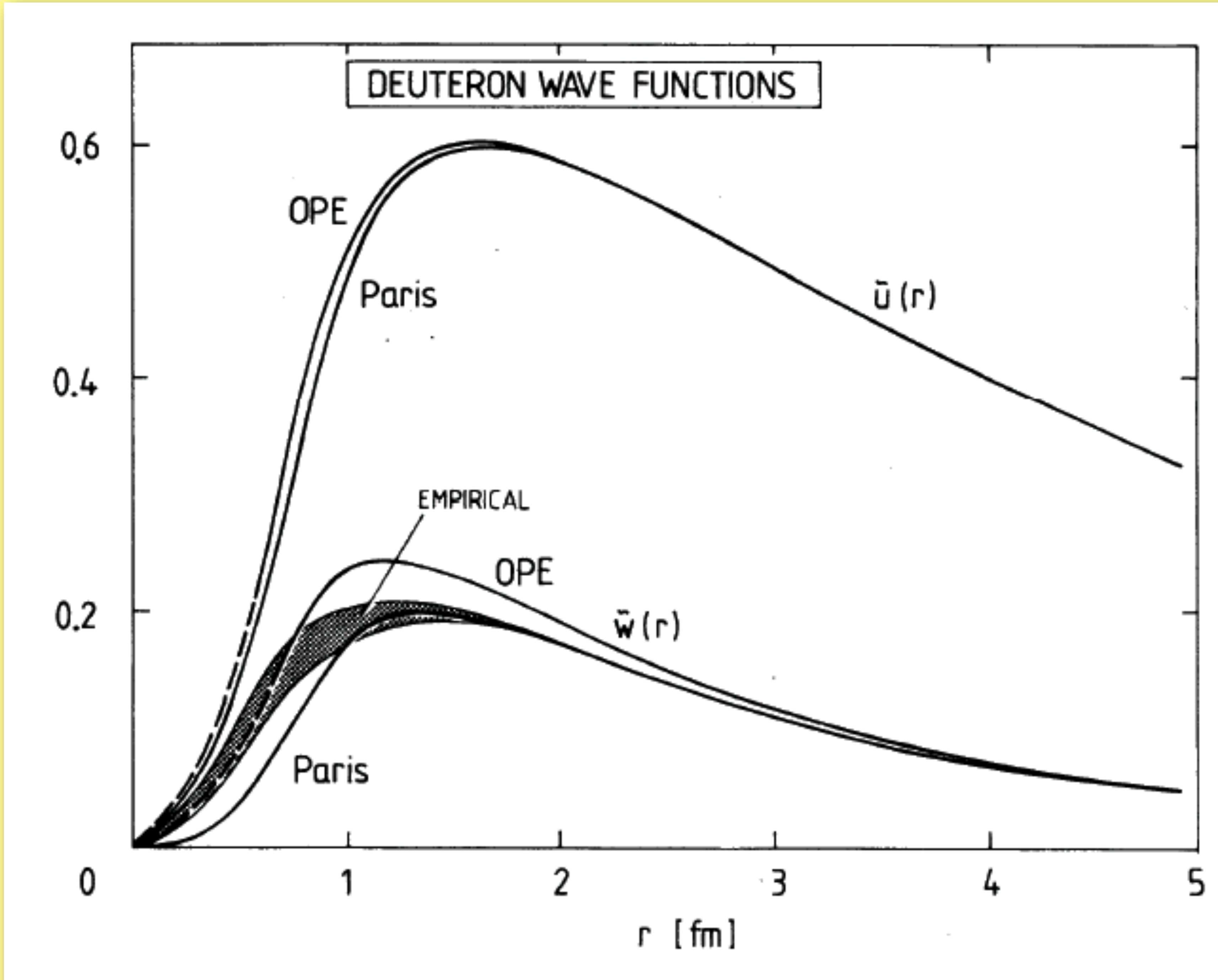
$$u'' \begin{pmatrix} 1 \\ 0 \end{pmatrix} + u'' \begin{pmatrix} 0 \\ 1 \end{pmatrix} = [EM + MV_c] u \begin{pmatrix} 1 \\ 0 \end{pmatrix} + [EM + MV_c]\, w \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$+ \frac{6}{r^2} w \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sqrt{8} MV_T u \begin{pmatrix} 0 \\ 1 \end{pmatrix} + MV_T \frac{w}{\sqrt{6}} \hat{S}_{12}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The set of coupled equations can be solved numerically

$$\hat{S}_{12}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 8 \begin{pmatrix} 1 \\ 0 \end{pmatrix} - 2\sqrt{8} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\hat{S}_{12}^2 = \begin{pmatrix} 0 & \sqrt{8} \\ \sqrt{8} & -2 \end{pmatrix} .$$


DEUTERON WAVE FUNCTIONS
0.6
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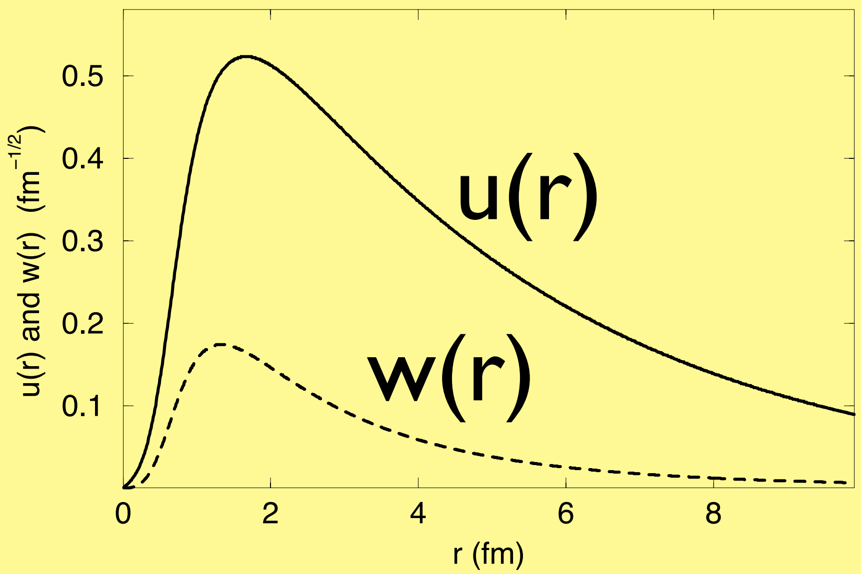
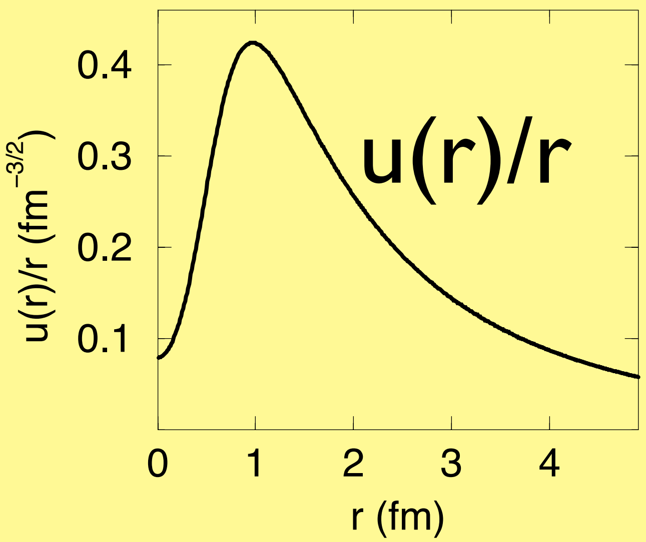
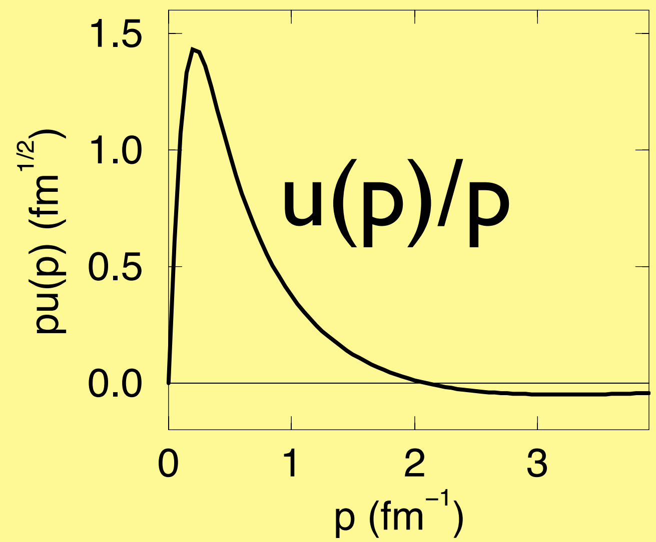
Additional slides



deuteron radial wavefunctions





Quantity	Most recent determination	Value
Mass M_d	[47, 48]	1875.612762 (75) MeV
Binding energy ε	[49]	2.22456612 (48) MeV
Magnetic dipole moment μ_d	[48]	0.8574382284 (94) μ_N
Electric quadrupole moment Q_d	[46, 50, 51]	0.2859 (3) fm^2
Asymptotic ratio $\eta_d = A_D/A_S$	[52]	0.0256 (4)
Charge radius r_{ch}	[53]	2.130 (10) fm
Matter radius r_m	[54, 55]	1.975 (3) fm
Electric polarizability α_E	[56, 57]	0.645 (54) fm^3

Experimental values

Deuteron intrinsic quadrupole moment:

recall our model for the deuteron wave function:

$$\left|\psi_d\right\rangle = a \left|{}^3S_1\right\rangle + b \left|{}^3D_1\right\rangle$$

result for the quadrupole moment:

note cancellation here

$$Q_{\text{int}} = \frac{\sqrt{2}}{10} \, |a^* b| \, \left\langle r^2 \right\rangle_{SD} - \frac{1}{20} \, b^2 \left\langle r^2 \right\rangle_{DD}$$

$$= +0.00286 \pm 0.00003 \ \text{bn}$$