Cross section, Flux, Luminosity, Scattering Rates

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Table of Contents

1 Introduction

Scattering experiments, where particles in a beam interact with target particles and a detector measures the scattered particles and/or new particles produced in the interaction, have produced a vast amount of information which has uncovered how the fundamental forces work, made possible the discovery of hundreds of particles and the understanding of their properties, and enabled precise measurements of systems ranging in size from molecules (10^{-9} m) to a fraction of a proton radius (10^{-18} m). In recognition of the fact that new phenomena appear whenever particle energies and measurement sensitivities cross new thresholds, physicists have continually pushed the energy and intensity frontiers of particle accelerators and developed extremely accurate and sensitive detectors and data acquisition systems that can record many petabytes of data per year. The process that started with the Cockroft-Walton accelerator in the early 1930s has culminated in today's colliding proton beams at the Large Hadron Collider at CERN, in Geneva, Switzerland.

In this note we will learn about the basic concepts used in scattering experiments, including cross section, flux and luminosity

2 Cross section, flux and scattering

Consider a typical fixed-target scattering experiment where a beam of particles strikes a fixed target. The following are the basic quantities involved in scattering experiments.
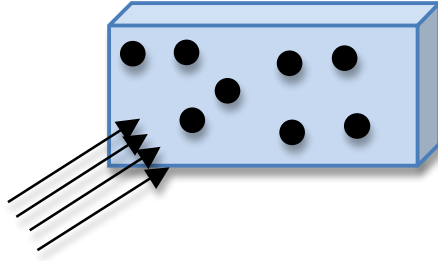
Cross section: Imagine a target slab of some material of thickness Δx [m], with an area A_t and a number density of small target scatterers (e.g., nuclei) n_t [m^{-3}]. Each scatterer has a projected cross-sectional area, or *cross section*, σ [m^2].

Flux: A beam of incoming pointlike particles has flux density j (#/m^2/sec) uniformly spread over an area A_b [m^2] with total flux $J = jA_b$ [#/sec]. The flux density can be written $j = n_b v_b$, where n_b [#/m^3] is the beam particle density and v_b [m/sec] is its velocity. In terms of these basic quantities the total beam flux is $J = A_b n_b v_b$. Within the target material, the number of scatterers within the beam area is $N_t = n_t A_b \Delta x$, where n_t is their number density and Δx is the target thickness.

Scattering: The beam particles are deflected only when they hit the target scatterers. The probability of scattering is the ratio of total target scatterer area within the beam to the overall area available for a beam particle. For a target thickness of dx the scattering probability dp is

$$dp = N_t \sigma / A_b = (n_t A_b dx)\sigma / A_b = \sigma n_t dx$$

Thus the number of interacting beam particles in the target is Ndp and the change in the number of beam particles is

$$dN = -Ndp = -N\sigma n_t dx$$

This equation can be integrated for a target of finite thickness x to find $N(x)$, the surviving number of beam particles vs x:

$$N(x) = N_0 e^{-\sigma nx} \equiv N_0 e^{-x/\lambda}$$

$\lambda \equiv 1/n_t\sigma$ is called the *scattering length* for a given cross section. When σ is the total proton cross section, the scattering length λ is known as the *nuclear collision length*. If σ is the inelastic cross section (where particles can be absorbed and new ones created), λ is called the *nuclear interaction length*. Both quantities are tabulated by the PDG.

3   Scattering length λ and λ_ρ

The number of surviving beam particles is $N(x) = N_0 e^{-x/\lambda}$, thus $\lambda = 1/n_t\sigma$ is the average distance that a beam particle travels in the target before scattering. Nuclear tables typically show the quantity $\lambda_\rho = \rho/n_t\sigma$ [g/cm^2] (which they call λ, so be careful), where ρ is the density of the material. Since $\rho = Am_0 n_t$, with A the atomic mass and m_0 the atomic mass unit (a carbon atom has a mass of exactly 12 mass units and $m_0 = 1.661 \times 10^{-24}$ g), we see that $\lambda_\rho = Am_0/\sigma$. Ex-

pressing the scattering length this way is useful because it varies very slowly with atomic mass A (the dependence on A is minimized because the total cross section increases $\propto A^{3/4}$). For high-energy protons or neutrons traveling through matter, typical values of λ_ρ are 50 – 110 g/cm^2.

Table 1 shows some values for the nuclear collision length for typical materials, taken from the Particle Data Group web page. Note the very slow dependence of λ_ρ on A.

Table 1: Nuclear collision length of common materials

Element	A	ρ (g/cm^3)	λ (cm)	λ_ρ (g/cm^2)
H	1	0.071 (liquid)	603	42.8
Be	9	1.848	29.9	55.3
C	12	2.21	26.8	59.2
Air (N$_2$, O$_2$, CO$_2$)	–	0.00121	50660	61.3
Al	27	2.70	25.8	69.7
Si	28	2.33	30.1	70.2
Cl	35.5	1.57 (liquid)	46.6	73.8
Fe	55.8	7.87	10.4	81.7
Cu	63.5	8.96	9.40	84.2
W (Tungsten)	183.8	19.3	5.72	110.4
Pb	207	11.35	10.1	114.1
U	238	18.95	6.26	118.6
Shielding concrete	–	2.30	28.3	65.1
Standard rock	–	2.65	25.2	66.8

When building radiation shielding, one prefers to use a substance with a small value of λ (tungsten) to reduce its actual thickness, but cost considerations normally force the adoption of cheaper materials like iron, lead or even concrete.

4   Cross section and scattering

For a beam of total flux J [#/sec], the rate of particle scattering (#scattered/sec) is merely $dJ_{\text{scat}} = Jdp = J\sigma n_t dx = j\sigma N_t$, where $N_t = n_t A_b dx$ is the number of scatterers in the area of the beam. Measuring the scattering rate thus provides an experimental way of determining the cross section σ:

$$\sigma = \frac{dJ_{\text{scat}}}{J n_t dx} = \frac{dJ_{\text{scat}}}{j N_t}$$

This formula permits us relate the experimentally measured cross section to theory. In the development of scattering theory in QM, $N_t = 1$ (one scatterer) and j represents the flux of an incoming plane wave of a single particle.

4.1   Cross section units: mb, μb, nb, etc.

The cross section is an area so it is measured in m^2; however, physicists commonly employ much smaller units such as the barn (b), mb, μb, nb, pb, fb, etc., 1 barn being 10^{-28} m^2, a typical size of heavy element nuclei. The proton-proton cross section is $\sim$ 40–100 mb, depending on energy.

This corresponds to a proton radius of proton of ~1.1–1.8 fm. At the LHC, cross sections for W and Z production are ~nb (10^{-37} m^2) and rare processes have cross sections ~pb (10^{-40} m^2) or even ~fb (10^{-43} m^2).

Note that the cross section merely defines an imaginary area in which the interaction takes place. The measured cross section can be much smaller than the actual geometric size of the scattering particles. For example, the neutrino-proton cross section for typical solar neutrinos of 1 MeV is around 10^{-45} m^2, vastly smaller than the proton "size".

The cross section of a point-like electron scattering in the coulomb potential of another point-like electron is infinite since no matter how large the impact parameter is, there is always sufficient force between the two particles that will result in some scattering (very small for large impact parameters, but non-zero nevertheless).

4.2 Dependence of σ on energy and spin

Cross sections change with the energy of an incoming particle (usually decrease because an interaction is less likely to change its direction at high energy) and its spin state (though we usually average over all spins). One should think of the cross section as a measure of the range and strength of the force between two particles.

Figure 1 shows the total cross section for various processes as a function of the center of mass energy $\sqrt{s}$. The pp, $\bar{p}p$, $K^{\mp}p$, $\pi^{\mp}p$ and $\Sigma^{-}p$ cross sections are mediated by strong interactions ($\sigma \approx 10-100$ mb) while the γp and $\gamma\gamma$ processes result from electromagnetic interactions (with $\sigma \approx 0.3-100 \ \mu$b). Figure 2 shows the cross section for $e^{+}e^{-} \rightarrow$ hadrons, mediated by electromagnetic and weak interactions (these become comparable near the Z mass ~ 91GeV). Figure 3 shows the cross sections for neutrino-nucleon and antineutrino-nucleon processes, which are mediated by weak interactions, as a function of neutrino energy. These weak interaction processes have cross sections in the range of $\sigma \approx 3-2000$ fb.
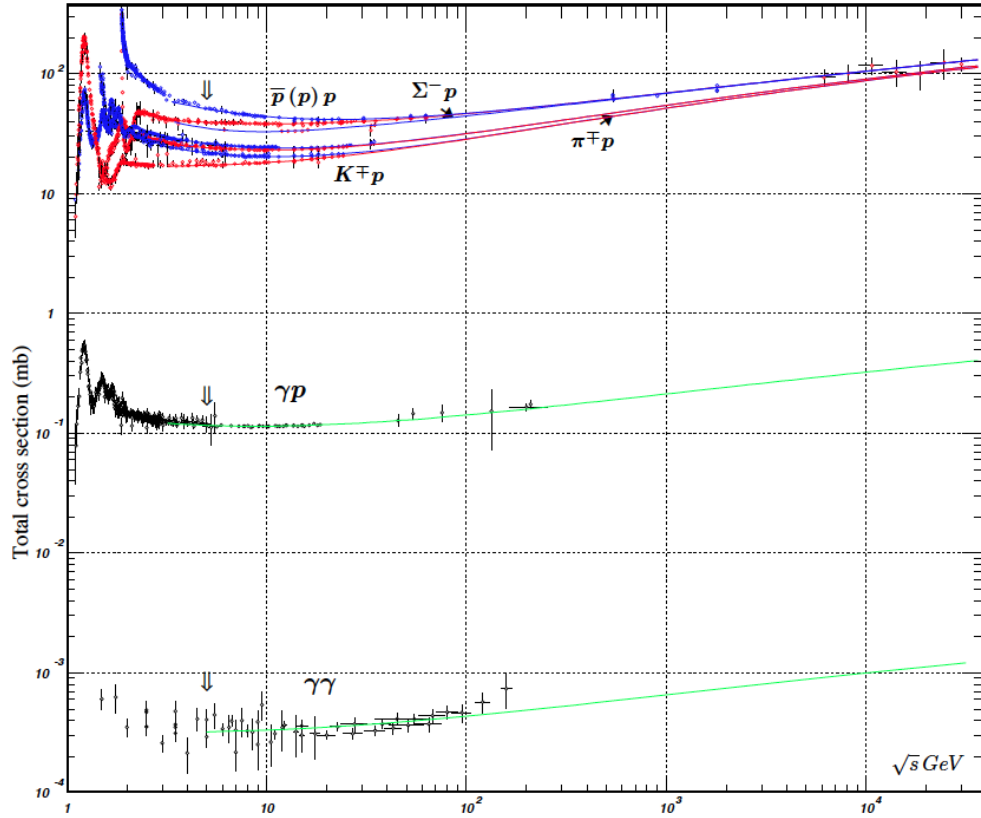
Figure 1: Graph showing various total cross sections in mb (from PDG)
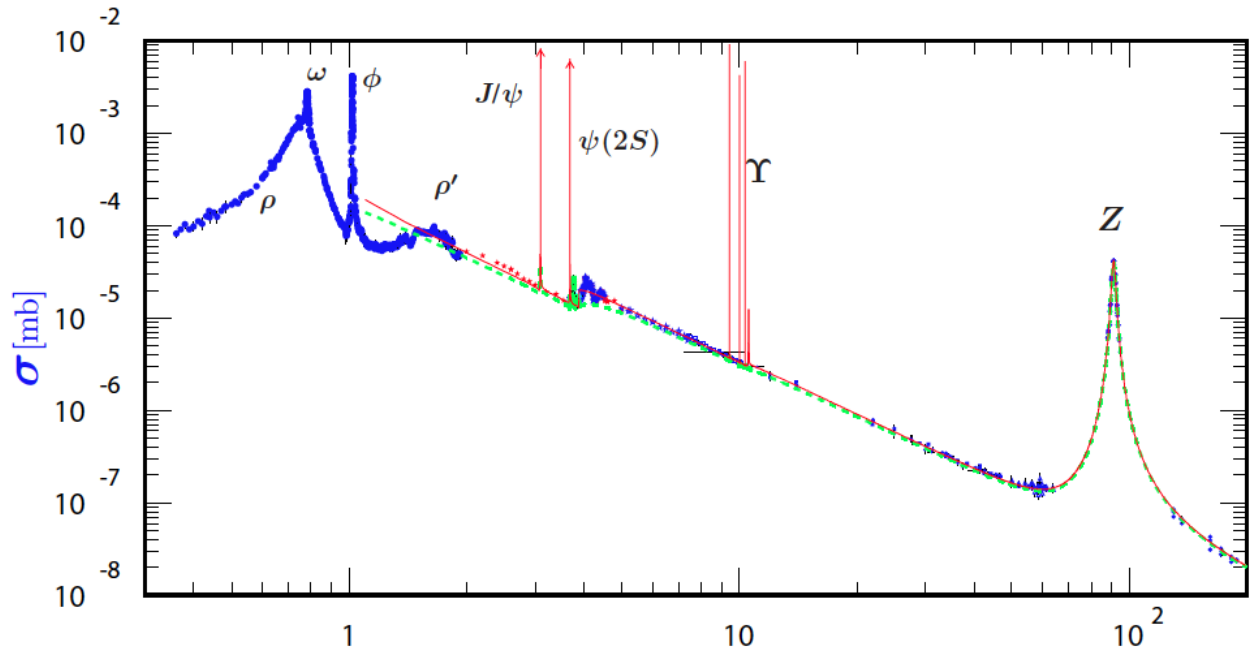

Figure 2: Plot of $e^+e^- \rightarrow$ hadrons **combining datasets from multiple experiments**
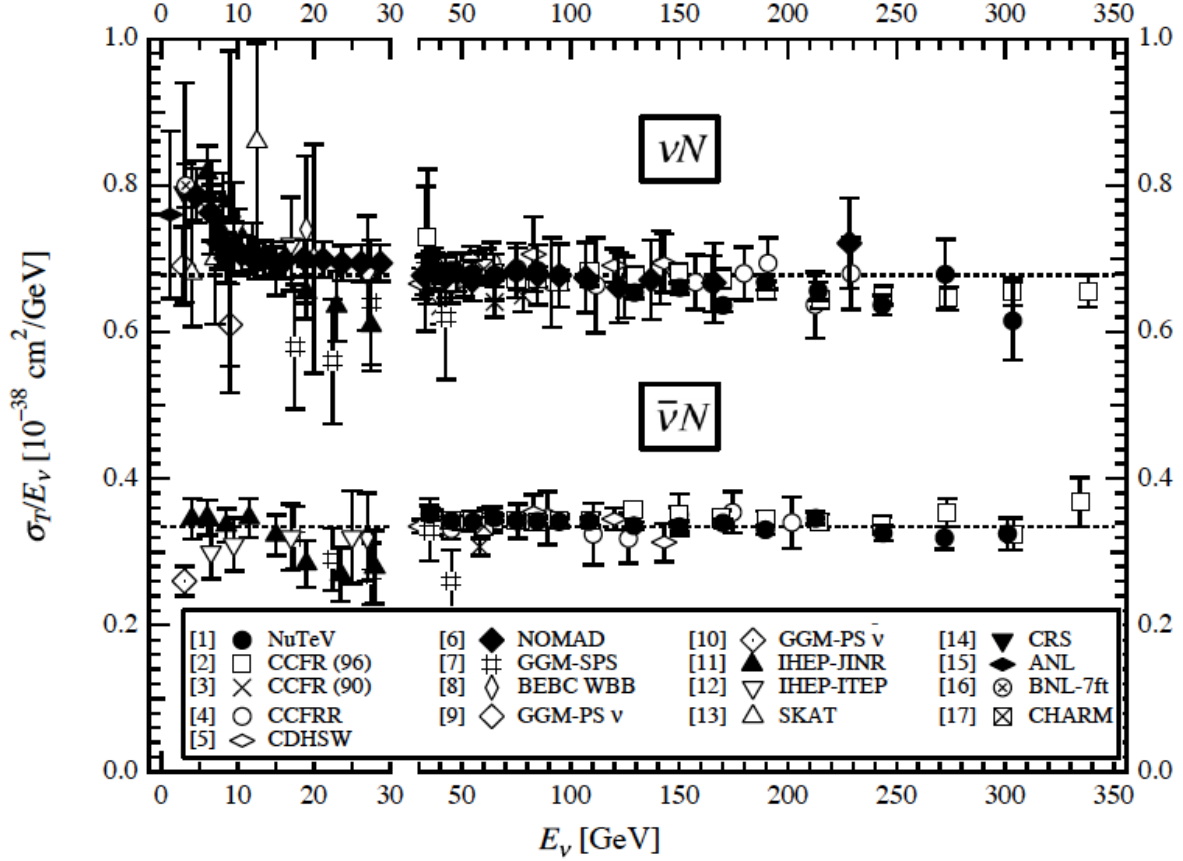
Figure 3: Plots of neutrino-nucleon and antineutrino-nucleon cross sections. What's actually plotted is σ_{vp} / E_v **(from PDG).**

5 Luminosity and integrated luminosity

For a given beam of flux J striking a target of number density n_t and thickness Δx, the rate of interactions for a process having a cross section σ is given by $J_{\text{scat}} = J\sigma n_t \Delta x \equiv \mathcal{L}\sigma$, where the factor $\mathcal{L} = Jn_t\Delta x = n_b v_b A_b n_t \Delta x$ multiplying the cross section is known as the *luminosity* [cm^{-2} sec^{-1}]. The luminosity is one of the fundamental quantities that governs how successful an experiment will be because it determines the scattering rate. Note that the scattering rate $J_{\text{scat}} = \mathcal{L}\sigma$ is the product of a quantity determined by physics (cross section) and a quantity determined by experimental setup (luminosity).

Integrated luminosity: The integrated luminosity $\mathcal{L}_{\text{int}} \equiv \int \mathcal{L}\,dt$ gives a measure of the total number of collisions over a period of time. For example, in Sep. 2012 the LHC had an average luminosity of $\sim 5 \times 10^{33}\,\text{cm}^{-2}\text{s}^{-1}$. The integrated luminosity over an hour was $\mathcal{L}_{\text{int}} = \left(5 \times 10^{33}\right) \times 3600 = 1.8 \times 10^{37}\,\text{cm}^{-2} = 18\text{pb}^{-1}$. $\mathcal{L}_{\text{int}}$ is most often expressed as pb^{-1}, fb^{-1}, etc. rather than cm^{-2}.

5.1 Example 1: Neutrino-nucleon event rate

The neutrino-nucleon total cross section at energy $E_v = 1$ GeV is $\sigma = 8 \times 10^{-39} \text{cm}^2 = 8$ fb. If a neutrino beam with total flux $J = 10^{10}$ / sec strikes an iron target, how thick does the target have to be in order to get 1000 interactions per day? Using $1000 = \mathcal{L}\sigma T$, where $T = 86400$ sec, we find that the luminosity is

$$\mathcal{L} = N_{\text{scat}} / \sigma T = 1000 / \left(8 \times 10^{-39} \times 86400\right) = 1.45 \times 10^{36} \text{cm}^{-2} \text{s}^{-1}$$

The integrated luminosity is $\mathcal{L}_{\text{int}} = \mathcal{L}T = 1.45 \times 10^{36} \times 86400 = 125 \times 10^{39} \text{ cm}^{-2} = 125 \text{fb}^{-1}$. The target nucleon density (protons and neutrons) for iron is

$$n_t = \rho / m_0 = \left(7.87 \text{ g/cm}^3\right) / \left(1.66 \times 10^{-24} \text{ g/nucleon}\right) = 4.7 \times 10^{24} / \text{cm}^3$$

This yields a target thickness of

$$\Delta x = \mathcal{L} / Jn_t = 1.45 \times 10^{36} / \left(10^{10} \times 4.7 \times 10^{24}\right) = 31 \text{ cm}$$

5.2 Example 2: Detecting a new particle in the CMS experiment

The average luminosity at CMS during Sep., 2012 was $5 \times 10^{33} \text{ cm}^{-2}\text{s}^{-1}$. Physicists are looking for a new particle with an estimated cross section of 0.25 pb and a detection efficiency (all decay products measured) of 12%. How long must the experiment run before CMS has collected 144 events where the particle is measured?

Since the detection efficiency is $\varepsilon = 0.12$, we must run long enough to accumulate 144 events where the particle is detected. Thus we have $144 = \mathcal{L}\sigma\varepsilon T$. Solving for T yields

$$T = 144 / \left(5 \times 10^{33} \times 0.25 \times 10^{-36} \times 0.12\right) = 0.9 \text{M sec} \approx 11 \text{ days}$$

The integrated luminosity is $\mathcal{L}_{\text{int}} = 5 \times 10^{33} \times 1.92 \times 10^6 = 9.6 \times 10^{39} \text{ cm}^{-2} = 9.6 \text{fb}^{-1}$.

6 Basic cross section definitions

Elastic: ingoing and outgoing particles are the same, e.g. $\sigma\left(e^+e^- \rightarrow e^+e^-\right)$ or $\sigma\left(pp \rightarrow pp\right)$.

Inelastic: opposite case, i.e., when either beam particle or target particle is changed or new particles are produced, e.g. $\sigma\left(e^+e^- \rightarrow \mu^+\mu^-\right)$ or $\sigma\left(pp \rightarrow p\Delta^+\right)$.

Total: sum of elastic and inelastic cross sections. The total pp cross section at 10 TeV CM energy is about 100 mb.

Differential: cross section for scattering within a restricted range of some observable, e.g. angle range $d\theta$ around some direction θ gives $d\sigma / d\theta$: Many kinds of differential cross sections are measured, e.g. $d\sigma / d\Omega$, $d\sigma / dE$, $d^2\sigma / dxdy$.

Production (e.g., $\sigma\left(pp \to W^+ + X\right)$) any process producing a W^+ in the final state. A production cross section is always inelastic.

Inclusive (e.g., $\sigma\left(pp \to \pi^+ + X\right)$): each π^+ adds to the cross section. This is also a production process.

Exclusive (e.g., $\sigma\left(pp \to pp\pi^0\right)$): an exactly defined final state (opposite of inclusive).

7 Cross section formulas for some processes

Using simple rules (applied to Feynman diagrams), cross sections of $2 \to 2$ processes can be calculated, at least in a first approximation, in theories with well-defined interactions. Below I list some standard cross sections in natural units as a function of s, the center of mass energy squared, and θ, the angle between one of the outgoing particles and the beam.

$G_F = 1.166 \times 10^{-5} \text{GeV}^{-2}$ is the weak decay constant, α is the familiar electromagnetic coupling constant, α_s is the QCD coupling constant and t and u are the other Mandelstam invariants. Recall that $t \equiv \left(p_1 - p_3\right)^2$ and $u \equiv \left(p_1 - p_4\right)^2$. At high energy where masses can be neglected $t = -\frac{1}{2}s\left(1 - \cos\theta\right)$ and $u = -\frac{1}{2}s\left(1 + \cos\theta\right)$. The total cross sections listed on the RHS are obtained by integrating over solid angle, $d\Omega = d\cos\theta d\phi = \sin\theta d\theta d\phi$.

$$\frac{d\sigma}{d\Omega}\left(e^+ e^- \to \mu^+ \mu^-\right) = \frac{\alpha^2}{4s}\left(1 + \cos^2\theta\right) \qquad \sigma\left(e^+ e^- \to \mu^+ \mu^-\right) = \frac{4\pi\alpha^2}{3s}$$

$$\frac{d\sigma}{d\Omega}\left(\nu_\mu d \to \mu^- u\right) = \frac{G_F^2 s}{4\pi^2} \qquad \sigma\left(\nu_\mu d \to \mu^- u\right) = \frac{G_F^2 s}{\pi}$$

$$\frac{d\sigma}{d\Omega}\left(\bar{\nu}_\mu u \to \mu^+ d\right) = \frac{G_F^2 s}{16\pi^2}\left(1 + \cos\theta\right)^2 \qquad \sigma\left(\bar{\nu}_\mu u \to \mu^+ d\right) = \frac{G_F^2 s}{3\pi}$$

$$\frac{d\sigma}{d\Omega}\left(e^+ e^- \to \gamma\gamma\right) = \frac{\alpha^2}{2s}\frac{u^2 + t^2}{tu}$$

$$\frac{d\sigma}{d\Omega}\left(q\bar{q} \to gg\right) = \frac{8\alpha_s^2}{27s}\left(t^2 + u^2\right)\left(\frac{1}{tu} - \frac{9}{s^2}\right)$$