Caltech

Machine Learning & Data Mining

CS/CNS/EE 155

Lecture 8:

Hidden Markov Models

Sequence Prediction
(POS Tagging)

- x = “Fish Sleep”
- y = (N, V)

- x = “The Dog Ate My Homework”
- y = (D, N, V, D, N)

- x = “The Fox Jumped Over The Fence”
- y = (D, N, V, P, D, N)

Challenges

- Multivariable Output
  - Make multiple predictions simultaneously
- Variable Length Input/Output
  - Sentence lengths not fixed

Multivariate Outputs

- x = "Fish Sleep"
- y = (N, V)

POS Tags:
Det, Noun, Verb, Adj, Adv, Prep

- Multiclass prediction:

Replicate Weights:

$$w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_K \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_K \end{bmatrix}$$

Score All Classes:

$$f(x|w,b) = \begin{bmatrix} w_1^T x - b_1 \\ w_2^T x - b_2 \\ \vdots \\ w_K^T x - b_K \end{bmatrix}$$

Predict via Largest Score:

$$\underset{k}{\operatorname{argmax}} \begin{bmatrix} w_1^T x - b_1 \\ w_2^T x - b_2 \\ \vdots \\ w_K^T x - b_K \end{bmatrix}$$

- How many classes?

Multiclass Prediction

POS Tags:
Det, Noun, Verb, Adj, Adv, Prep

L=6

- x = "Fish Sleep"
- y = (N, V)
- Multiclass prediction:
  - All possible length-M sequences as different class
  - (D, D), (D, N), (D, V), (D, Adj), (D, Adv), (D, Pr)
    (N, D), (N, N), (N, V), (N, Adj), (N, Adv), …
- **L^M classes!**
  - **Length 2: $6^2 = 36$!**

Multiclass Prediction

POS Tags:
Det, Noun, Verb, Adj, Adv, Prep

L=6

- x = “Fish Sleep”
- y = (N, V)
- M
  – … ass
  – …)

Exponential Explosion in #Classes!
(Not Tractable for Sequence Prediction)

- **L^M classes!**
  – **Length 2: 6^2 = 36!**

Why is Naïve Multiclass Intractable?

x="I fish often"

POS Tags:
Det, Noun, Verb, Adj, Adv, Prep

- (D, D, D), (D, D, N), (D, D, V), (D, D, Adj), (D, D, Adv), (D, D, Pr)
- (D, N, D), (D, N, N), (D, N, V), (D, N, Adj), (D, N, Adv), (D, N, Pr)
- (D, V, D), (D, V, N), (D, V, V), (D, V, Adj), (D, V, Adv), (D, V, Pr)
- …
- (N, D, D), (N, D, N), (N, D, V), (N, D, Adj), (N, D, Adv), (N, D, Pr)
- (N, N, D), (N, N, N), (N, N, V), (N, N, Adj), (N, N, Adv), (N, N, Pr)
- …

Assume pronouns are nouns for simplicity.

Why is Naïve Multiclass Intractable?

x="I fish often"

POS Tags:
Det, Noun, Verb, Adj, Adv, Prep

- (D, D, D), (D, D, N), (D, D, V), (D, D, Adj), (D, D, Adv), (D, D, Pr)

Treats Every Combination As Different Class
(Learn model for each combination)

Exponentially Large Representation!
(Exponential Time to Consider Every Class)
(Exponential Storage)

Assume pronouns are nouns for simplicity.

Independent Classification

x="I fish often"

POS Tags:
Det, Noun, Verb, Adj, Adv, Prep

- Treat each word independently (assumption)
  - Independent multiclass prediction per word
  - Predict for x="I" independently
  - Predict for x="fish" independently
  - Predict for x="often" independently
  - Concatenate predictions.

Assume pronouns are nouns for simplicity.

Independent Classification

x="I fish often"

POS Tags:
Det, Noun, Verb, Adj, Adv, Prep

- Treat each word independently (assumption)
  - Independent multiclass prediction per word

#Classes = #POS Tags
(6 in our example)

Solvable using standard multiclass prediction.

Assume pronouns are nouns for simplicity.

Independent Classification

x="I fish often"

POS Tags:
Det, Noun, Verb, Adj, Adv, Prep

- Treat each word independently
  - Independent multiclass prediction per word

P(y|x)	x="I"	x="fish"	x="often"
y="Det"	0.0	0.0	0.0
y="Noun"	1.0	0.75	0.0
y="Verb"	0.0	0.25	0.0
y="Adj"	0.0	0.0	0.4
y="Adv"	0.0	0.0	0.6
y="Prep"	0.0	0.0	0.0

Prediction: (N, N, Adv)

Correct: (N, V, Adv)

Why the mistake?

Assume pronouns are nouns for simplicity.

Context Between Words

x="I fish often"

POS Tags:
Det, Noun, Verb, Adj, Adv, Prep

- Independent Predictions Ignore Word Pairs
  - In Isolation:
    - "Fish" is more likely to be a Noun
  - But Conditioned on Following a (pro)Noun…
    - "Fish" is more likely to be a Verb!
  - **"1st Order" Dependence (Model All Pairs)**
    - 2^{nd} Order Considers All Triplets
    - Arbitrary Order = Exponential Size (Naïve Multiclass)

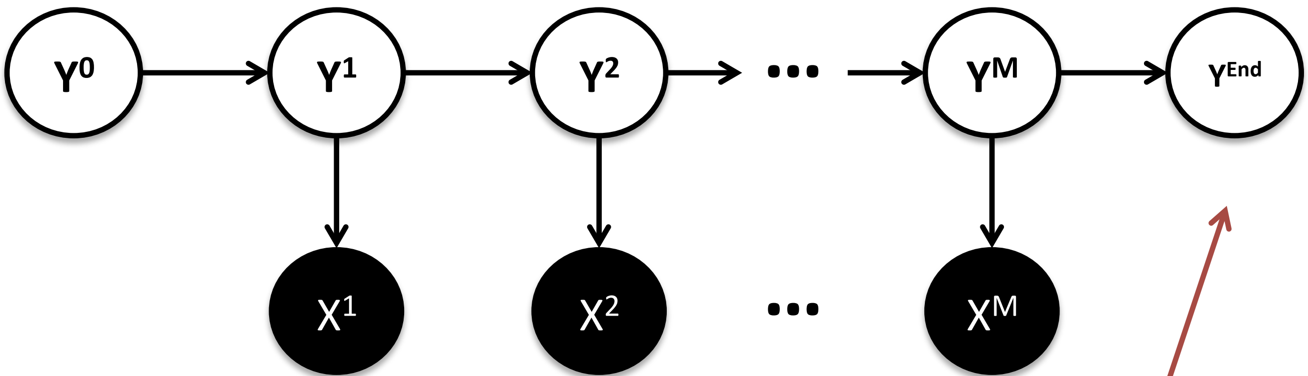
Assume pronouns are nouns for simplicity.

1^{st} Order Hidden Markov Model

- $x = (x^1, x^2, x^4, x^4, \ldots, x^M)$ (sequence of words)
- $y = (y^1, y^2, y^3, y^4, \ldots, y^M)$ (sequence of POS tags)

- $P(x^i | y^i)$ Probability of state y^i generating x^i
- $P(y^{i+1} | y^i)$ Probability of state y^i transitioning to y^{i+1}
- $P(y^1 | y^0)$ y^0 is defined to be the Start state
- $P(End | y^M)$ Prior probability of y^M being the final state
  - Not always used

Graphical Model Representation



Optional

$$P(x,y) = P(End \mid y^M)\prod_{i=1}^{M} P(y^i \mid y^{i-1})\prod_{i=1}^{M} P(x^i \mid y^i)$$

1^{st} Order Hidden Markov Model

$$P(x,y) = P(End \mid y^M)\prod_{i=1}^{M} P(y^i \mid y^{i-1})\prod_{i=1}^{M} P(x^i \mid y^i)$$

"Joint Distribution"

Optional

- $P(x^i | y^i)$ Probability of state y^i generating x^i
- $P(y^{i+1} | y^i)$ Probability of state y^i transitioning to y^{i+1}
- $P(y^1 | y^0)$ y0 is defined to be the Start state
- $P(End | y^M)$ Prior probability of y^M being the final state
  - Not always used

1^{st} Order Hidden Markov Model

$$P(x \mid y) = \prod_{i=1}^{M} P(x^i \mid y^i)$$

Given a POS Tag Sequence y:
Can compute each $P(x^i|y)$ independently!
(x^i conditionally independent given y^i)

"Conditional Distribution on x given y"

- $P(x^i|y^i)$ Probability of state y^i generating x^i
- $P(y^{i+1}|y^i)$ Probability of state y^i transitioning to y^{i+1}
- $P(y^1|y^0)$ y^0 is defined to be the Start state
- $P(End|y^M)$ Prior probability of y^M being the final state
  - Not always used

1^{st} Order Hidden Markov Model

Additional Complexity of (#POS Tags)^2

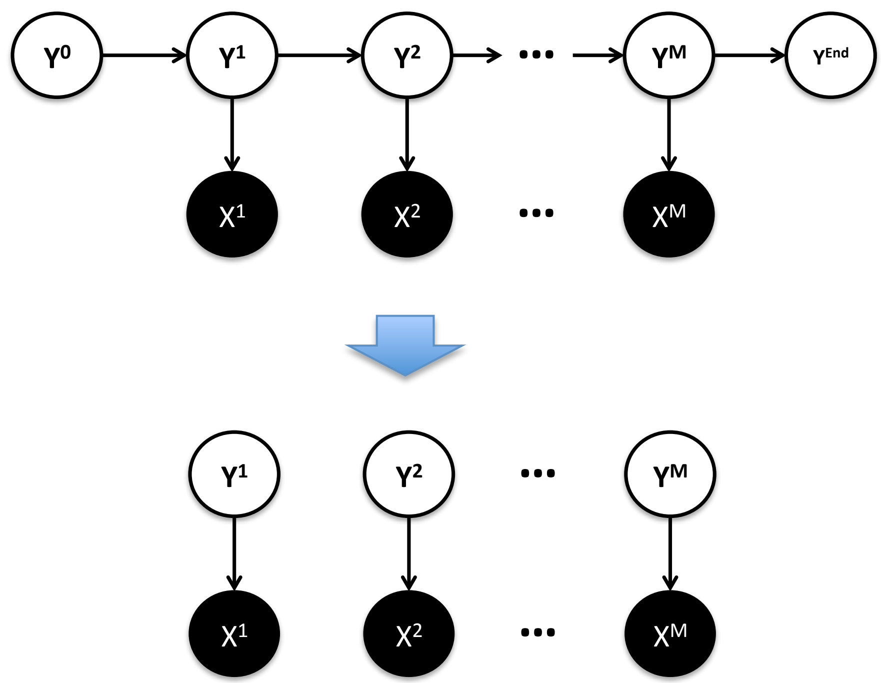
Models All State-State Pairs (all POS Tag-Tag pairs)

Models All State-Observation Pairs (all Tag-Word pairs)

Same Complexity as Independent Multiclass

- $P(x^i|y^i)$ Probability of state y^i generating x^i
- $P(y^{i+1}|y^i)$ Probability of state y^i transitioning to y^{i+1}
- $P(y^1|y^0)$ y^0 is defined to be the Start state
- $P(End|y^M)$ Prior probability of y^M being the final state
  - Not always used

Relationship to Naïve Bayes



Reduces to a sequence of disjoint Naïve Bayes models
(if we ignore transition probabilities)

P (word | state/tag)

- Two-word language: "fish" and "sleep"
- Two-tag language: "Noun" and "Verb"

P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5
x="sleep"	0.2	0.5

Given Tag Sequence y:

P("fish sleep" | (N,V)) = 0.8*0.5
P("fish fish" | (N,V)) = 0.8*0.5
P("sleep fish" | (V,V)) = 0.8*0.5
P("sleep sleep" | (N,N)) = 0.2*0.5

Slides borrowed from Ralph Grishman

Sampling

- HMMs are "generative" models
  - Models joint distribution P(x,y)
  - Can generate samples from this distribution
  - First consider conditional distribution P(x|y)

P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5
x="sleep"	0.2	0.5

Given Tag Sequence y = (N,V):

Sample each word independently:

Sample $P(x^1 | N)$ (0.8 Fish, 0.2 Sleep)

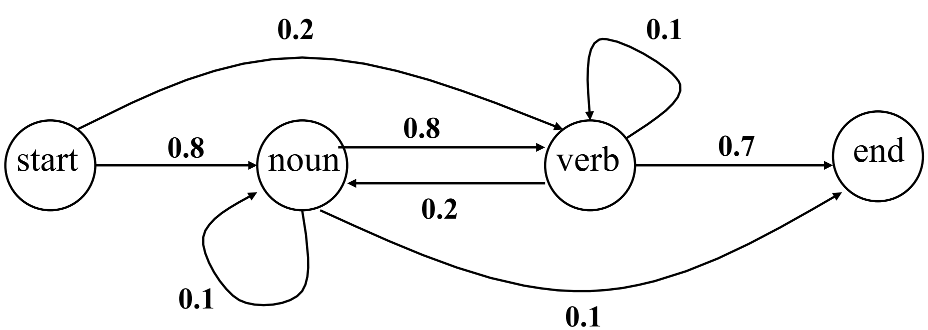
Sample $P(x^2 | V)$ (0.5 Fish, 0.5 Sleep)

  - **What about sampling from P(x,y)?**

Forward Sampling of P(y,x)

$$P(x,y) = P(End \mid y^M)\prod_{i=1}^{M} P(y^i \mid y^{i-1})\prod_{i=1}^{M} P(x^i \mid y^i)$$

P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5
x="sleep"	0.2	0.5



Initialize y^0 = Start
Initialize i = 0

1. i = i + 1
2. Sample y^i from $P(y^i|y^{i-1})$
3. If y^i == End: Quit
4. Sample x^i from $P(x^i|y^i)$
5. Goto Step 1

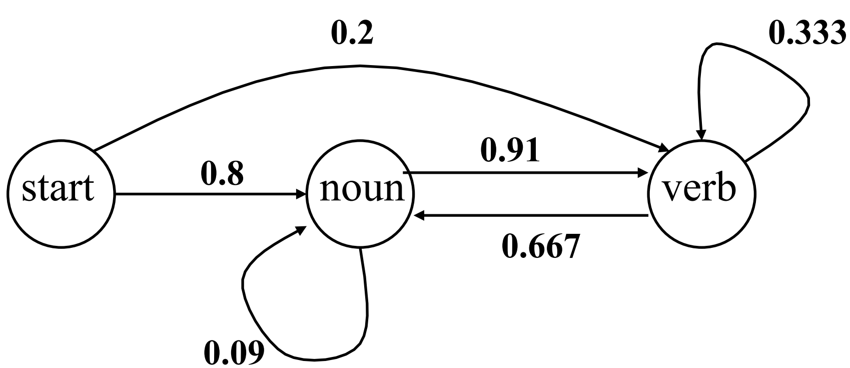
Exploits Conditional Ind.
Requires $P(End|y^i)$

Slides borrowed from Ralph Grishman

Forward Sampling of P(y,x|L)

$$P\left(x, y \mid M\right) = \cancel{P(End \mid y^M)} \prod_{i=1}^{M} P(y^i \mid y^{i-1}) \prod_{i=1}^{M} P(x^i \mid y^i)$$

P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5
x="sleep"	0.2	0.5



Initialize y^0 = Start
Initialize i = 0

1. i = i + 1
2. If(i == M): Quit
3. Sample y^i from $P(y^i|y^{i-1})$
4. Sample x^i from $P(x^i|y^i)$
5. Goto Step 1

Exploits Conditional Ind.
Assumes no $P(End|y^i)$

Slides borrowed from Ralph Grishman

1^{st} Order Hidden Markov Model

$$P\left(x^{k+1:M}, y^{k+1:M} \mid x^{1:k}, y^{1:k}\right) = P\left(x^{k+1:M}, y^{k+1:M} \mid y^{k}\right)$$

"Memory-less Model" – only needs y^k to model rest of sequence

- $P(x^i|y^i)$ Probability of state y^i generating x^i
- $P(y^{i+1}|y^i)$ Probability of state y^i transitioning to y^{i+1}
- $P(y^1|y^0)$ y0 is defined to be the Start state
- $P(End|y^M)$ Prior probability of y^M being the final state
  - Not always used

Viterbi Algorithm

Most Common Prediction Problem

- Given input sentence, predict POS Tag seq.

$$\operatorname*{argmax}_{y} P(y \mid x)$$

- **Naïve approach:**
  - Try all possible y's
  - Choose one with highest probability
  - **Exponential time: L^M possible y's**

Bayes's Rule

$$\operatorname*{argmax}_y P\left(y \mid x\right) = \operatorname*{argmax}_y \frac{P(y,x)}{P(x)}$$

$$= \operatorname*{argmax}_y P(y,x)$$

$$= \operatorname*{argmax}_y P(x \mid y)P(y)$$

$$P\left(x \mid y\right) = \prod_{i=1}^{L} P(x^i \mid y^i)$$

$$P\left(y\right) = P(End \mid y^L)\prod_{i=1}^{L} P(y^i \mid y^{i-1})$$

$$\operatorname*{argmax}_{y} P(y,x) = \operatorname*{argmax}_{y} \prod_{i=1}^{M} P(y^i \mid y^{i-1}) \prod_{i=1}^{M} P(x^i \mid y^i)$$

$$= \operatorname*{argmax}_{y^M} \operatorname*{argmax}_{y^{1:M-1}} \prod_{i=1}^{M} P(y^i \mid y^{i-1}) \prod_{i=1}^{M} P(x^i \mid y^i)$$

$$= \operatorname*{argmax}_{y^M} \operatorname*{argmax}_{y^{1:M-1}} P(y^M \mid y^{M-1}) P(x^M \mid y^M) P(y^{1:M-1} \mid x^{1:M-1})$$

$$P\left(y^{1:k} \mid x^{1:k}\right) = P(x^{1:k} \mid y^{1:k}) P(y^{1:k})$$

$$P\left(x^{1:k} \mid y^{1:k}\right) = \prod_{i=1}^{k} P(x^i \mid y^i)$$

$$P\left(y^{1:k}\right) = \prod_{i=1}^{k} P(y^{i+1} \mid y^i)$$

Exploit Memory-less Property:
The choice of y^L only depends on $y^{1:M-1}$ via $P(y^M | y^{M-1})$!

Dynamic Programming

- **Input:** x = $(x^1, x^2, x^3, \ldots, x^M)$
- **Computed:** best length-k prefix ending in each Tag:
  - Examples:

Best length-k prefix ending with tag V

$$\hat{Y}^k(V) = \left(\operatorname*{argmax}_{y^{1:k-1}} P(y^{1:k-1} \oplus V, x^{1:k})\right) \oplus V \qquad \hat{Y}^k(N) = \left(\operatorname*{argmax}_{y^{1:k-1}} P(y^{1:k-1} \oplus N, x^{1:k})\right) \oplus N$$

Sequence Concatenation

- **Claim:**

$$\hat{Y}^{k+1}(V) = \left(\operatorname*{argmax}_{y^{1:k} \in \left\{\hat{Y}^k(T)\right\}_T} P(y^{1:k} \oplus V, x^{1:k+1})\right) \oplus V$$

Pre-computed

Recursive Definition!

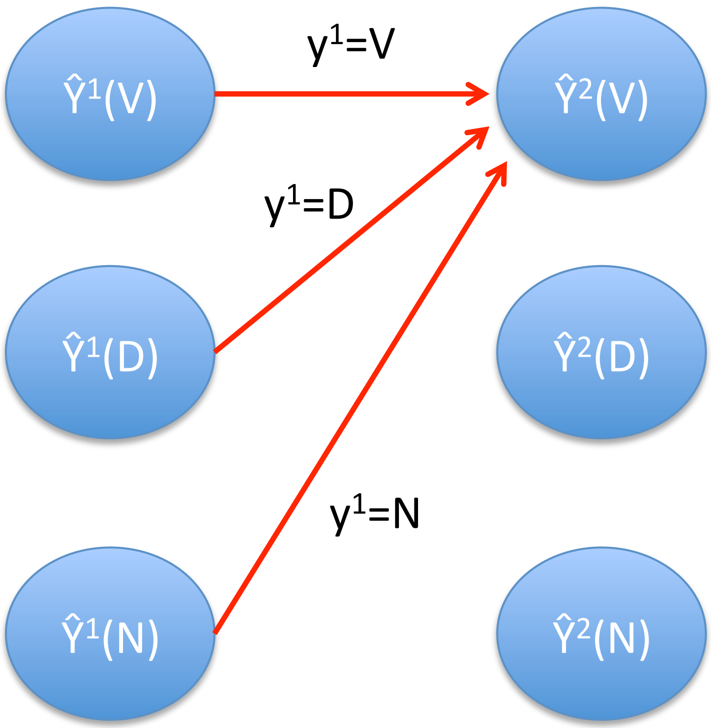
Solve:

Picking y1 from the set {V, D, N}

$$\hat{Y}^2(V) = \left(\underset{y^1 \in \{\hat{Y}^1(T)\}_T}{\operatorname{argmax}} \underbrace{P(y^1, x^1)} P(y^2 = V \mid y^1) P(x^2 \mid y^2 = V) \right) \oplus V$$

Store each
$\hat{Y}^1(Z)$ & $P(\hat{Y}^1(Z), x^1)$

Three types if tags: V, D, N

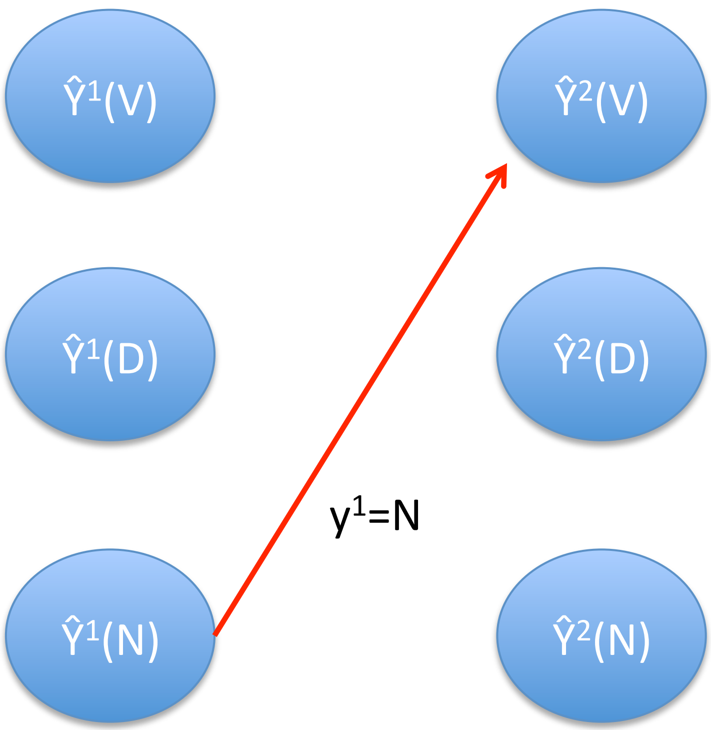


$\hat{Y}^1(Z)$ is just Z

Solve:

$$\hat{Y}^2(V) = \left(\operatorname*{argmax}_{y^1 \in \{\hat{Y}^1(T)\}_T} P(y^1, x^1) P(y^2 = V \mid y^1) P(x^2 \mid y^2 = V) \right) \oplus V$$

Store each
$\hat{Y}^1(Z)$ & $P(\hat{Y}^1(Z), x^1)$



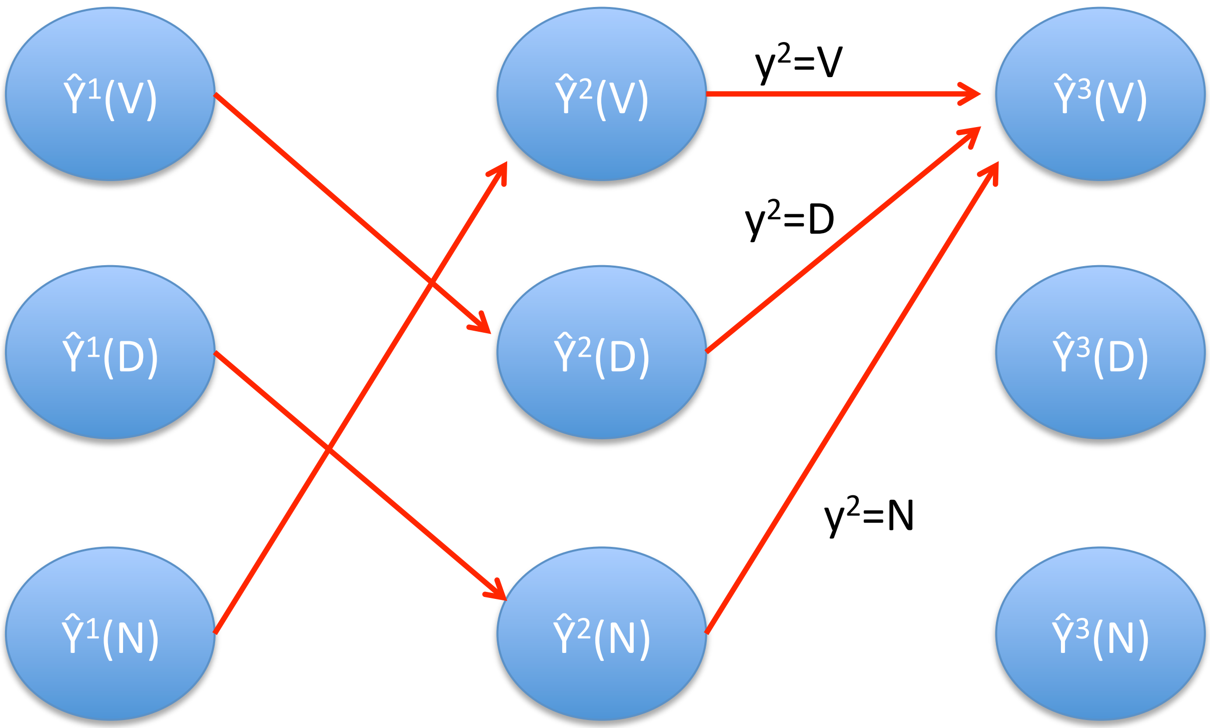
$\hat{Y}^1(Z)$ is just Z

Ex: $\hat{Y}^2(V) = (N, V)$

Solve: $$\hat{Y}^3(V) = \left(\underset{y^{1:2} \in \left\{\hat{Y}^2(T)\right\}_T}{\operatorname{argmax}} P(y^{1:2}, x^{1:2}) P(y^3 = V \mid y^2) P(x^3 \mid y^3 = V) \right) \oplus V$$

Store each
$\hat{Y}^1(Z)$ & $P(\hat{Y}^1(Z), x^1)$

Store each
$\hat{Y}^2(Z)$ & $P(\hat{Y}^2(Z), x^{1:2})$



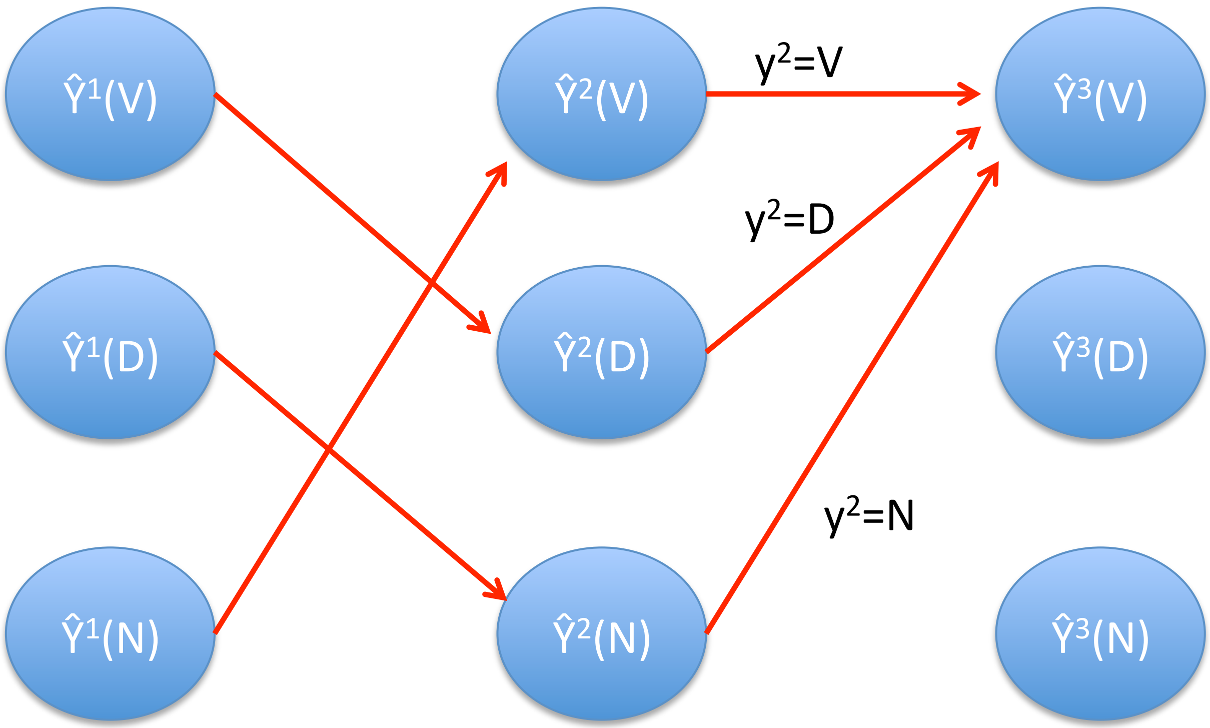
Ex: $\hat{Y}^2(V) = (N, V)$

Solve: $$\hat{Y}^3(V) = \left(\underset{y^{1:2} \in \{\hat{Y}^2(T)\}_T}{\text{argmax}} P(y^{1:2}, x^{1:2}) P(y^3 = V \mid y^2) P(x^3 \mid y^3 = V) \right) \oplus V$$

Store each
$\hat{Y}^1(Z)$ & $P(\hat{Y}^1(Z), x^1)$

Store each
$\hat{Y}^2(Z)$ & $P(\hat{Y}^2(Z), x^{1:2})$

Claim: Only need to check solutions of $\hat{Y}^2(Z)$, Z=V,D,N



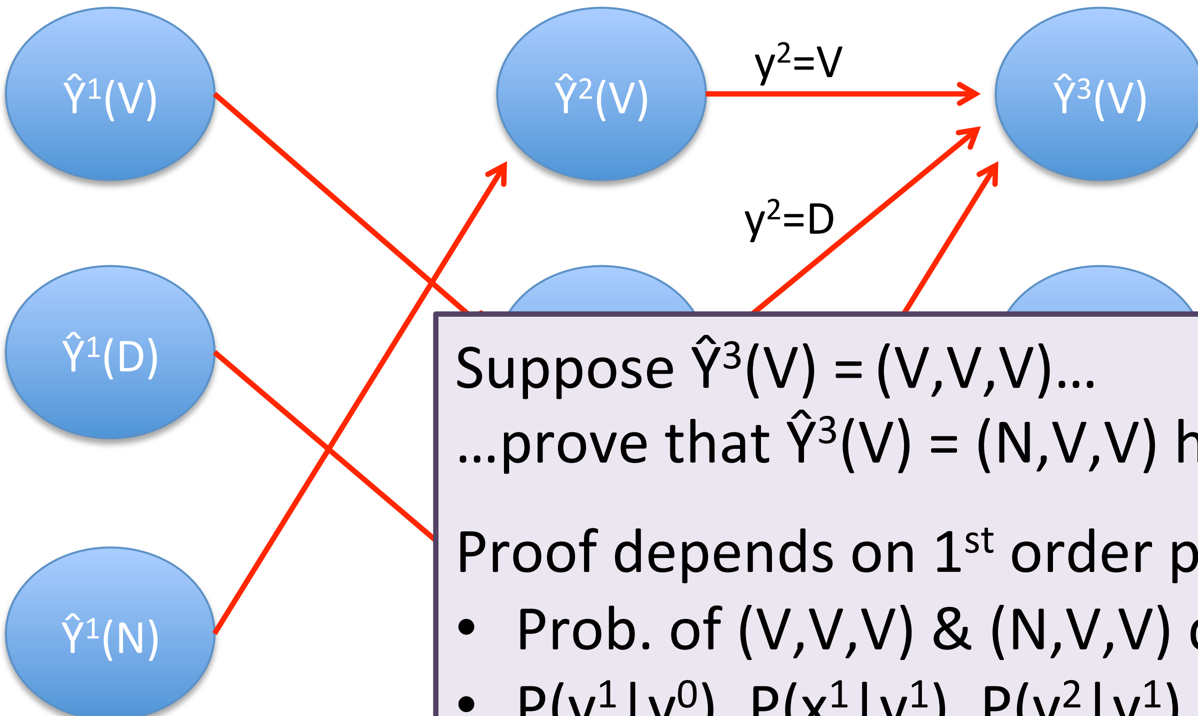
Ex: $\hat{Y}^2(V)$ = (N, V)

Solve: $$\hat{Y}^3(V) = \left(\underset{y^{1:2}\in\left\{\hat{Y}^2(T)\right\}_T}{\operatorname{argmax}} P(y^{1:2},x^{1:2})P(y^3=V\mid y^2)P(x^3\mid y^3=V) \right) \oplus V$$

Store each
$\hat{Y}^1(Z)$ & $P(\hat{Y}^1(Z),x^1)$

Store each
$\hat{Y}^2(Z)$ & $P(\hat{Y}^2(Z),x^{1:2})$

Claim: Only need to check solutions of $\hat{Y}^2(Z)$, Z=V,D,N



Suppose $\hat{Y}^3(V)$ = (V,V,V)…
…prove that $\hat{Y}^3(V)$ = (N,V,V) has higher prob.

Proof depends on 1st order property

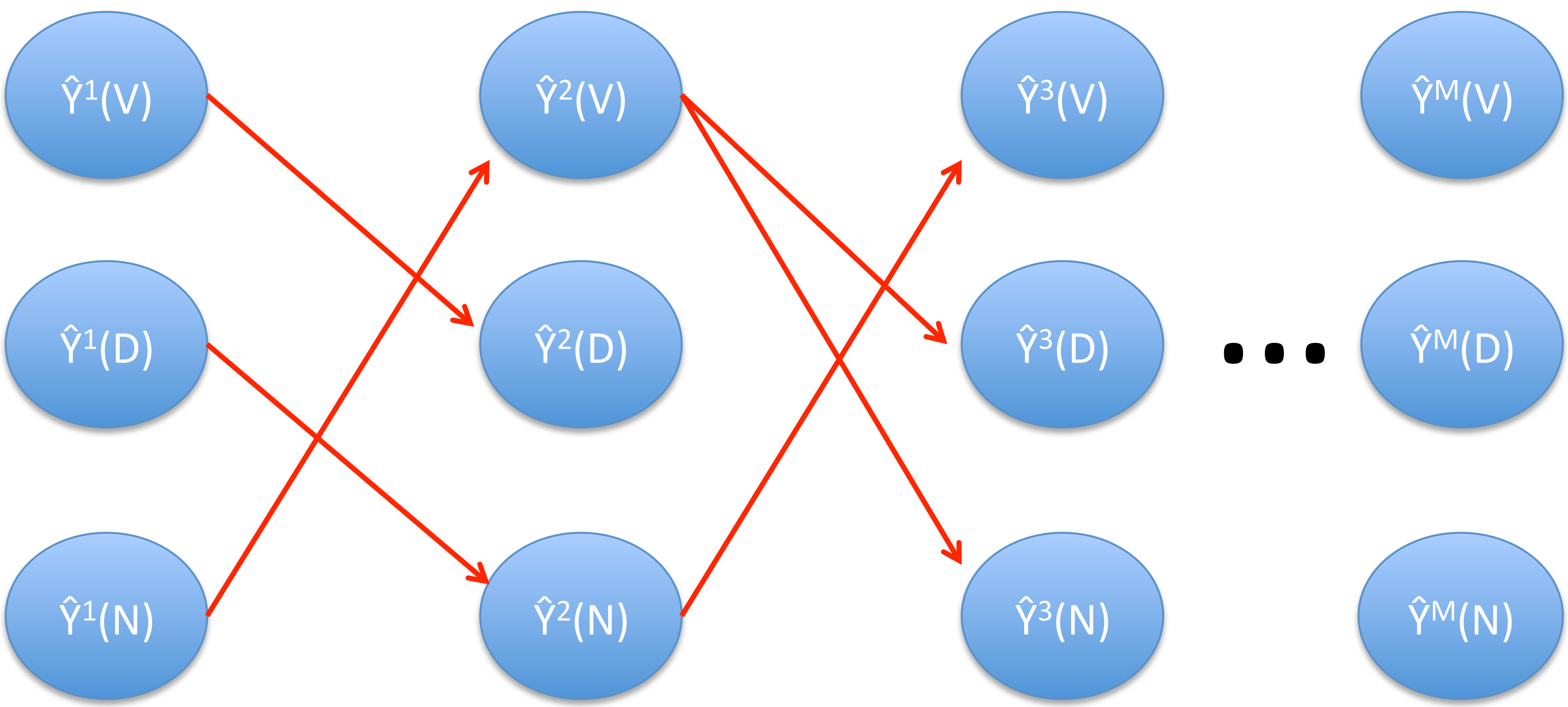
- Prob. of (V,V,V) & (N,V,V) differ in 3 terms
- $P(y^1|y^0)$, $P(x^1|y^1)$, $P(y^2|y^1)$
- **None of these depend on y^3!**

$$\hat{Y}^M(V) = \left(\underset{y^{1:M-1} \in \left\{\hat{Y}^{M-1}(T)\right\}_T}{\operatorname{argmax}} P(y^{1:M-1}, x^{1:M-1}) P(y^M = V \mid y^{M-1}) P(x^M \mid y^M = V) \underbrace{P(End \mid y^M = V)}_{\text{Optional}} \right) \oplus V$$

Store each
$\hat{Y}^1(Z)$ & $P(\hat{Y}^1(Z), x^1)$

Store each
$\hat{Y}^2(Z)$ & $P(\hat{Y}^2(Z), x^{1:2})$

Store each
$\hat{Y}^3(Z)$ & $P(\hat{Y}^3(Z), x^{1:3})$



Ex: $\hat{Y}^2(V) = (N, V)$

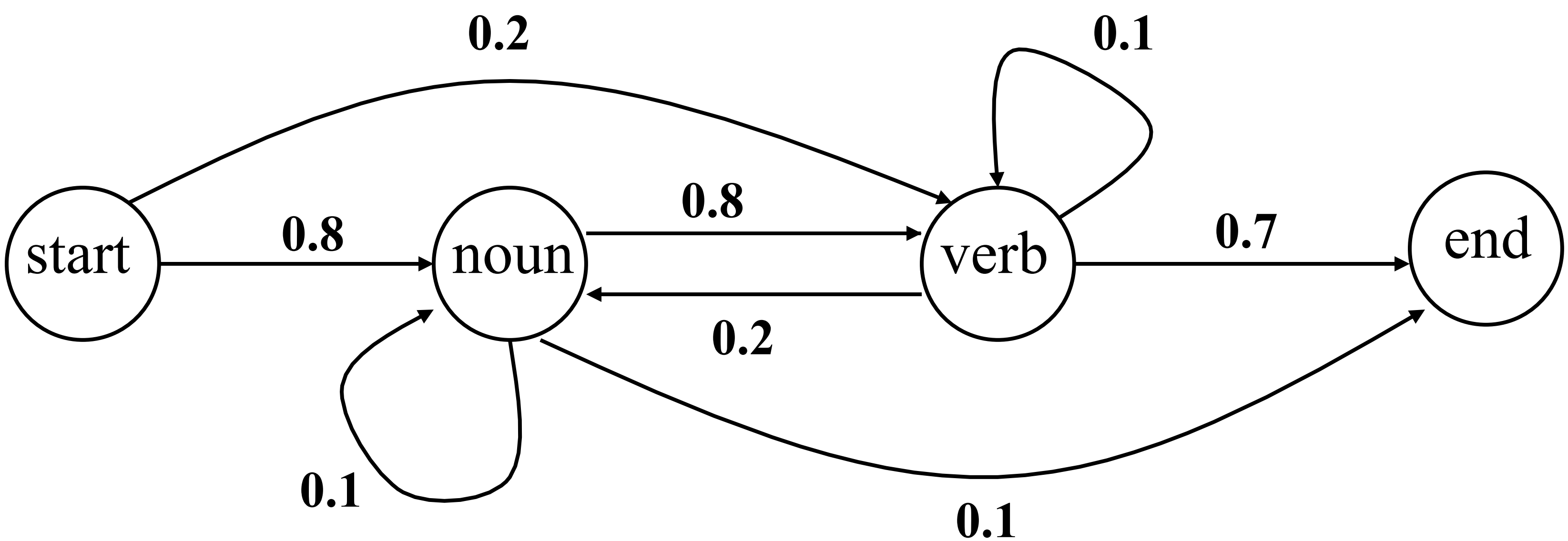
Ex: $\hat{Y}^3(V) = (D,N,V)$

Viterbi Algorithm

- Solve: $$\underset{y}{\operatorname{argmax}} P\left(y \mid x\right) = \underset{y}{\operatorname{argmax}} \frac{P(y,x)}{P(x)}$$
$$= \underset{y}{\operatorname{argmax}} P(y,x)$$
$$= \underset{y}{\operatorname{argmax}} P(x \mid y)P(y)$$
- For k=1..M
  - Iteratively solve for each $\hat{Y}^k(Z)$
    - Z looping over every POS tag.
- Predict best $\hat{Y}^M(Z)$
- Also known as Mean A Posteriori (MAP) inference

Numerical Example

x= (Fish Sleep)



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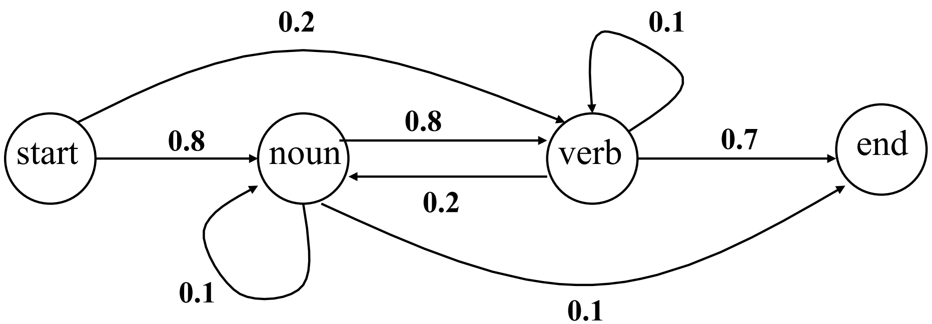
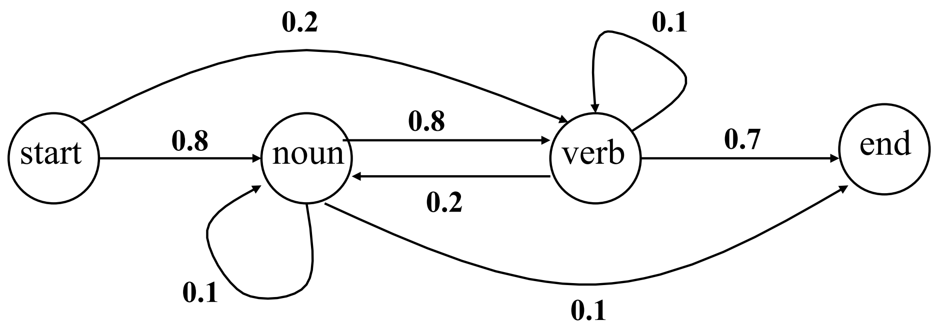


P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5
x="sleep"	0.2	0.5

	0	1	2	3
start	1			
verb	0			
noun	0			
end	0			

Slides borrowed from Ralph Grishman

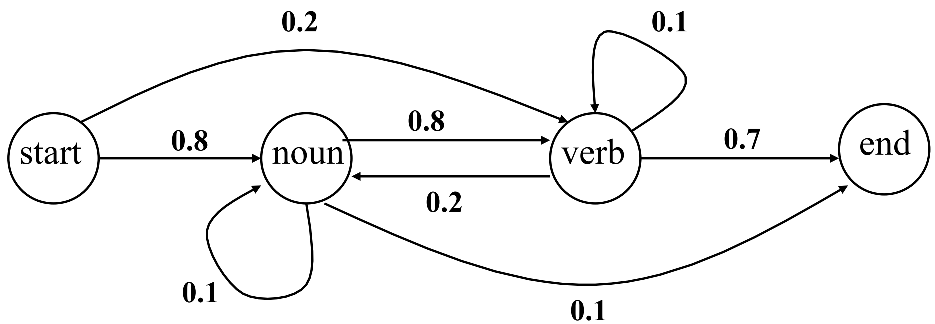


P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5
x="sleep"	0.2	0.5

Token 1: fish

	0	1	2	3
start	1	0		
verb	0	.2 * .5		
noun	0	.8 * .8		
end	0	0		

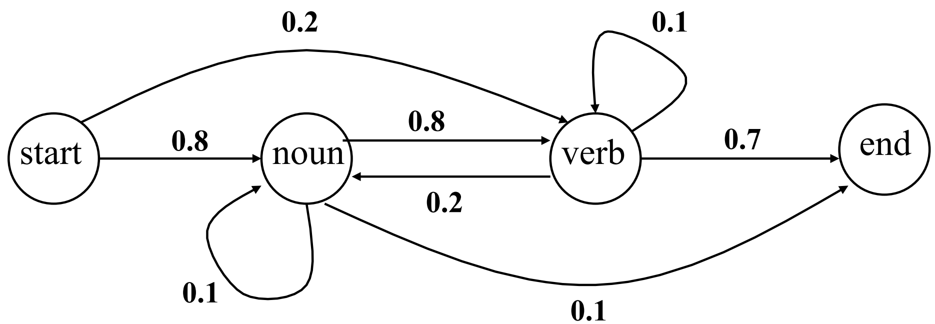
Slides borrowed from Ralph Grishman



P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5
x="sleep"	0.2	0.5

Token 1: fish

	0	1	2	3
start	1	0		
verb	0	.1		
noun	0	.64		
end	0	0		

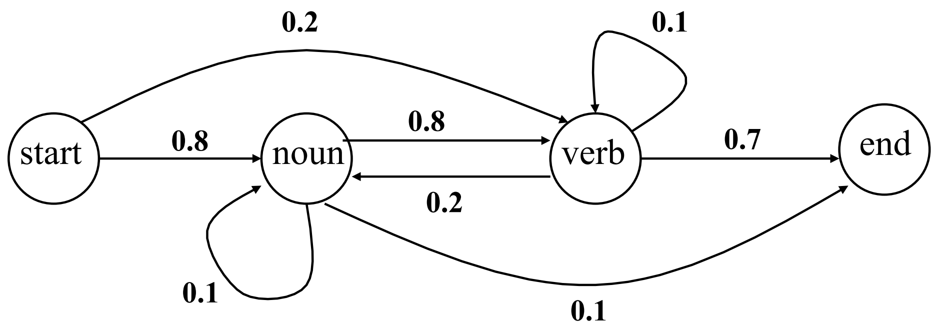


P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5
x="sleep"	0.2	0.5

Token 2: sleep
(if 'fish' is verb)

	0	1	2	3
start	1	0	0	
verb	0	.1	.1*.1*.5	
noun	0	.64	.1*.2*.2	
end	0	0	-	

Slides borrowed from Ralph Grishman



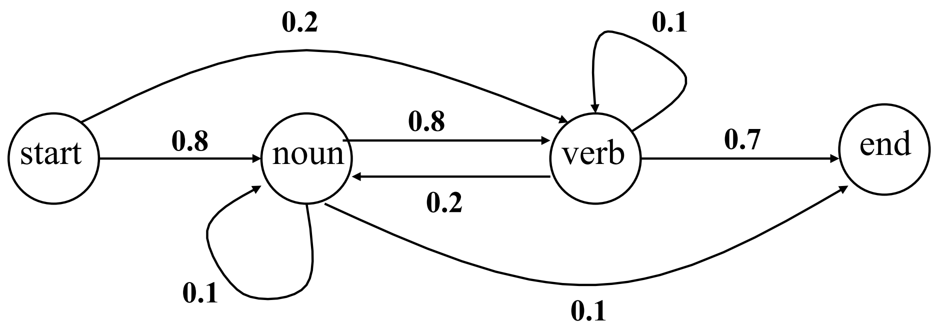
P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5
x="sleep"	0.2	0.5

Token 2: sleep

(if 'fish' is verb)

	0	1	2	3
start	1	0	0	
verb	0	.1	.005	
noun	0	.64	.004	
end	0	0	-	

Slides borrowed from Ralph Grishman



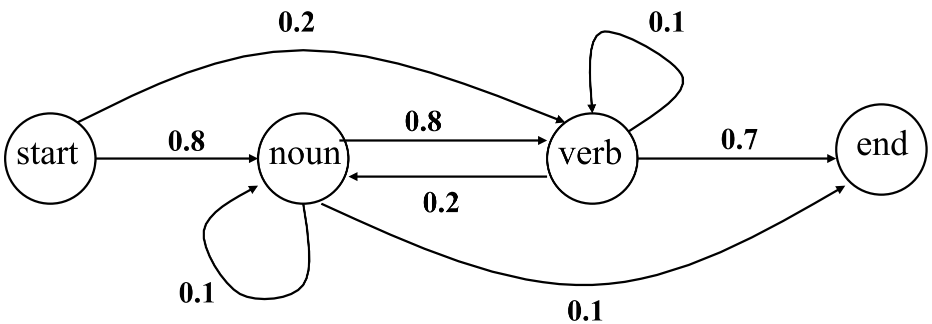
P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5
x="sleep"	0.2	0.5

Token 2: sleep

(if 'fish' is a noun)

	0	1	2	3
start	1	0	0	
verb	0	.1	.005 .64*.8*.5	
noun	0	.64	.004 .64*.1*.2	
end	0	0	-	

Slides borrowed from Ralph Grishman



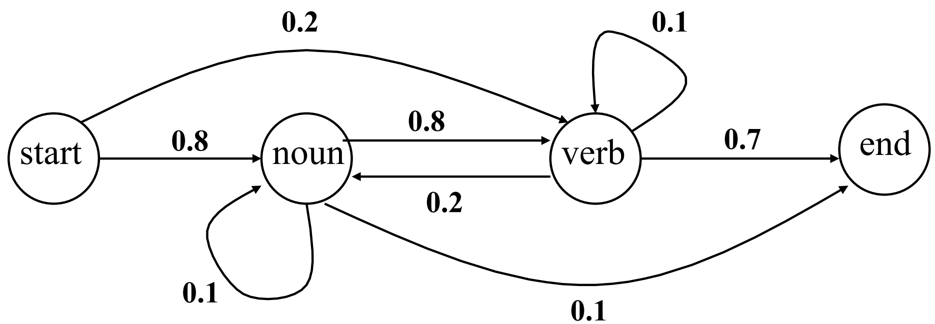
P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5
x="sleep"	0.2	0.5

Token 2: sleep

(if 'fish' is a noun)

	0	1	2	3
start	1	0	0	
verb	0	.1	.005 .256	
noun	0	.64	.004 .0128	
end	0	0	-	

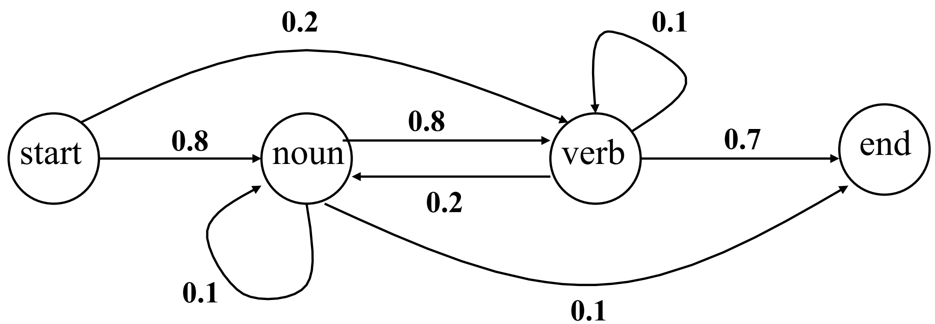
Slides borrowed from Ralph Grishman



P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5
x="sleep"	0.2	0.5

Token 2: sleep
take maximum,
set back pointers

	0	1	2	3
start	1	0	0	
verb	0	.1	~~.005~~ .256	
noun	0	.64	~~.004~~ .0128	
end	0	0	-	

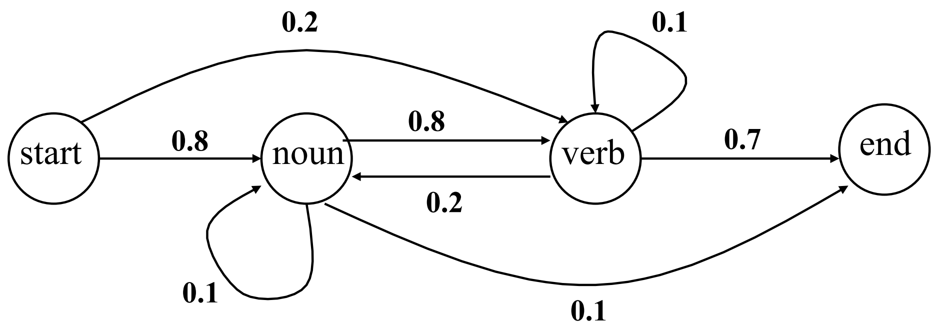


P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5
x="sleep"	0.2	0.5

Token 2: sleep
take maximum,
set back pointers

	0	1	2	3
start	1	0	0	
verb	0	.1	.256	
noun	0	.64	.0128	
end	0	0	-	

Slides borrowed from Ralph Grishman

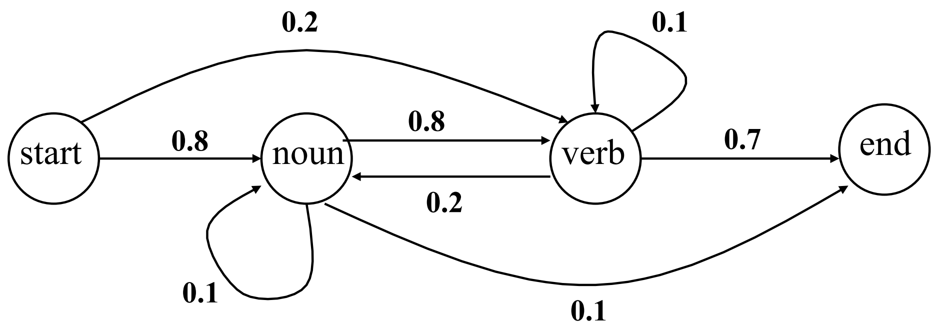


P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5
x="sleep"	0.2	0.5

Token 3: end

	0	1	2	3
start	1	0	0	0
verb	0	.1	.256	-
noun	0	.64	.0128	-
end	0	0	-	.256*.7 .0128*.1

Slides borrowed from Ralph Grishman

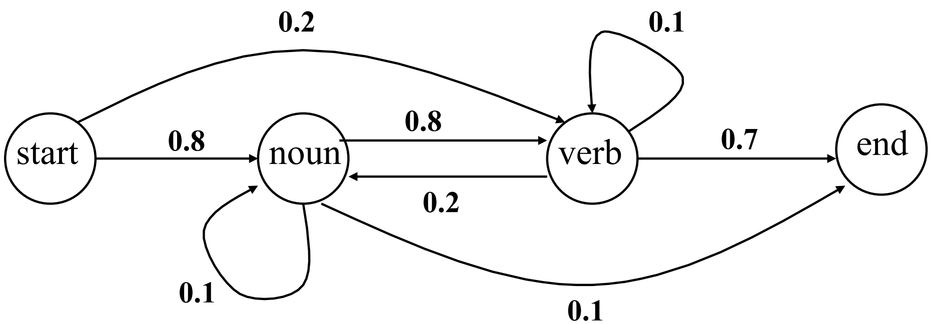


P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5
x="sleep"	0.2	0.5

Token 3: end
take maximum,
set back pointers

	0	1	2	3
start	1	0	0	0
verb	0	.1	.256	-
noun	0	.64	.0128	-
end	0	0	-	.256*.7 ~~.0128*.1~~

Slides borrowed from Ralph Grishman

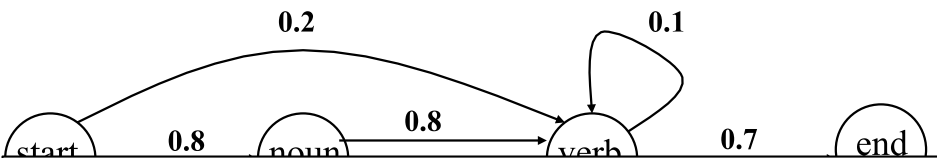


P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5
x="sleep"	0.2	0.5

Decode:
fish = noun
sleep = verb

	0	1	2	3
start	1	0	0	0
verb	0	.1	.256	-
noun	0	.64	.0128	-
end	0	0	-	.256*.7

Slides borrowed from Ralph Grishman



P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5

What might go wrong for long sequences?

sleep = verb

start	1	0	0	0

Underflow!

Small numbers get repeatedly multiplied together – exponentially small!

Slides borrowed from Ralph Grishman

Viterbi Algorithm

(w/ Log Probabilities)

- Solve: $\underset{y}{\operatorname{argmax}} P\left(y \mid x\right) = \underset{y}{\operatorname{argmax}} \frac{P(y,x)}{P(x)}$

$$= \underset{y}{\operatorname{argmax}} P(y,x)$$

$$= \underset{y}{\operatorname{argmax}} \log P(x \mid y) + \log P(y)$$

- For k=1..M
  - Iteratively solve for each log(Ŷ^k(Z))
    - Z looping over every POS tag.
- Predict best log(Ŷ^M(Z))
  - **Log(Ŷ^M(Z)) accumulates additively, not multiplicatively**

Recap: Independent Classification

x="I fish often"

POS Tags:
Det, Noun, Verb, Adj, Adv, Prep

- Treat each word independently
  - Independent multiclass prediction per word

P(y|x)	x="I"	x="fish"	x="often"
y="Det"	0.0	0.0	0.0
y="Noun"	1.0	0.75	0.0
y="Verb"	0.0	0.25	0.0
y="Adj"	0.0	0.0	0.4
y="Adv"	0.0	0.0	0.6
y="Prep"	0.0	0.0	0.0

Prediction: (N, N, Adv)

Correct: (N, V, Adv)

Mistake due to not modeling multiple words.

Assume pronouns are nouns for simplicity.

Recap: Viterbi

- Models pairwise transitions between states
  - Pairwise transitions between POS Tags
  - "1st order" model

$$P\left(x,y\right) = P(End \mid y^M)\prod_{i=1}^{M} P(y^i \mid y^{i-1})\prod_{i=1}^{M} P(x^i \mid y^i)$$

x="I fish often"

Independent: (N, N, Adv)

HMM Viterbi: (N, V, Adv)

*Assuming we defined P(x,y) properly

Training HMMs

Supervised Training

- **Given:** $S = \{(x_i, y_i)\}_{i=1}^{N}$

Word Sequence (Sentence)

POS Tag Sequence

- **Goal:** Estimate P(x,y) using S

$$P(x,y) = P(End \mid y^M) \prod_{i=1}^{M} P(y^i \mid y^{i-1}) \prod_{i=1}^{M} P(x^i \mid y^i)$$

- **Maximum Likelihood!**

Aside: Matrix Formulation

- Define Transition Matrix: A
  - $A_{ab} = P(y^{i+1}=a|y^i=b)$ or $-\text{Log}(P(y^{i+1}=a|y^i=b))$

$P(y^{next}|y)$	y="Noun"	y="Verb"
y^{next}="Noun"	0.09	0.667
y^{next}="Verb"	0.91	0.333

- Observation Matrix: O
  - $O_{wz} = P(x^i=w|y^i=z)$ or $-\text{Log}(P(x^i=w|y^i=z))$

P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5
x="sleep"	0.2	0.5

Aside: Matrix Formulation

$$P\left(x,y\right) = P(End \mid y^M)\prod_{i=1}^{M} P(y^i \mid y^{i-1})\prod_{i=1}^{M} P(x^i \mid y^i)$$

$$P\left(x,y\right) = P(End \mid y^M)\prod_{i=1}^{M} P(y^i \mid y^{i-1})\prod_{i=1}^{M} P(x^i \mid y^i)$$

$$= A_{End,y^M}\prod_{i=1}^{M} A_{y^i,y^{i-1}}\prod_{i=1}^{M} O_{x^i,y^i}$$

$$-\log(P(x,y)) = \tilde{A}_{End,y^M} + \sum_{i=1}^{M} \tilde{A}_{y^i,y^{i-1}} + \sum_{i=1}^{M} \tilde{O}_{x^i,y^i}$$

Log prob. formulation

Each entry of Ã is define as –log(A)

Maximum Likelihood

$$\underset{A,O}{\operatorname{argmax}} \prod_{(x,y)\in S} P\left(x,y\right) = \underset{A,O}{\operatorname{argmax}} \prod_{(x,y)\in S} P(End \mid y^M)\prod_{i=1}^{M} P(y^i \mid y^{i-1})\prod_{i=1}^{M} P(x^i \mid y^i)$$

- Estimate each component separately:

$$A_{ab} = \frac{\sum_{j=1}^{N}\sum_{i=0}^{M_j} 1_{\left[\left(y_j^{i+1}=a\right)\wedge\left(y_j^i=b\right)\right]}}{\sum_{j=1}^{N}\sum_{i=0}^{M_j} 1_{\left[y_j^i=b\right]}} \qquad O_{wz} = \frac{\sum_{j=1}^{N}\sum_{i=1}^{M_j} 1_{\left[\left(x_j^i=w\right)\wedge\left(y_j^i=z\right)\right]}}{\sum_{j=1}^{N}\sum_{i=1}^{M_j} 1_{\left[y_j^i=z\right]}}$$

- (Derived via minimizing neg. log likelihood)

Recap: Supervised Training

$$\underset{A,O}{\operatorname{argmax}} \prod_{(x,y)\in S} P\left(x,y\right) = \underset{A,O}{\operatorname{argmax}} \prod_{(x,y)\in S} P(End \mid y^M)\prod_{i=1}^{M} P(y^i \mid y^{i-1})\prod_{i=1}^{M} P(x^i \mid y^i)$$

- Maximum Likelihood Training
  - Counting statistics
  - Super easy!
  - Why?

- What about unsupervised case?

Recap: Supervised Training

$$\operatorname*{argmax}_{A,O} \prod_{(x,y)\in S} P\left(x,y\right) = \operatorname*{argmax}_{A,O} \prod_{(x,y)\in S} P(End \mid y^M) \prod_{i=1}^{M} P(y^i \mid y^{i-1}) \prod_{i=1}^{M} P(x^i \mid y^i)$$

- Maximum Likelihood Training
  - Counting statistics
  - Super easy!
  - **Why?**

- What about unsupervised case?

Conditional Independence Assumptions

$$\underset{A,O}{\operatorname{argmax}} \prod_{(x,y)\in S} P\left(x,y\right) = \underset{A,O}{\operatorname{argmax}} \prod_{(x,y)\in S} P(End \mid y^M)\prod_{i=1}^{M} P(y^i \mid y^{i-1})\prod_{i=1}^{M} P(x^i \mid y^i)$$

- Everything decomposes to products of pairs
  - I.e., $P(y^{i+1}=a|y^i=b)$ doesn't depend on anything else

- Can just estimate frequencies:
  - How often $y^{i+1}=a$ when $y^i=b$ over training set
  - Note that $P(y^{i+1}=a|y^i=b)$ is a common model across all locations of all sequences.

Conditional Independence Assumptions

$$\underset{A,O}{\operatorname{argmax}} \prod_{(x,y)\in S} P(x,y) = \underset{A,O}{\operatorname{argmax}} \prod_{(x,y)\in S} P(End \mid y^M) \prod_{i=1}^{M} P(y^i \mid y^{i-1}) \prod_{i=1}^{M} P(x^i \mid y^i)$$

\# Parameters:

Transitions A: $\#Tags^2$

Observations O: #Words x #Tags

Avoids directly model word/word pairings

#Tags = 10s

#Words = 10000s

Unsupervised Training

- What about if no y's?
  - Just a training set of sentences

$$S = \{x_i\}_{i=1}^{N}$$

Word Sequence
(Sentence)

- Still want to estimate P(x,y)
  - How?
  - Why?

$$\arg\max \prod_i P(x_i) = \arg\max \prod_i \sum_y P(x_i, y)$$

Unsupervised Training

- What about if no y's?
  - Just a training set of sentences

$$S = \{x_i\}_{i=1}^{N}$$

Word Sequence
(Sentence)

- Still want to estimate P(x,y)
  - How?
  - **Why?**

$$\arg\max \prod_i P(x_i) = \operatorname{argmax} \prod_i \sum_y P(x_i, y)$$

Why Unsupervised Training?

- Supervised Data hard to acquire
  - Require annotating POS tags
- Unsupervised Data plentiful
  - Just grab some text!
- Might just work for POS Tagging!
  - Learn y's that correspond to POS Tags
- Can be used for other tasks
  - Detect outlier sentences (sentences with low prob.)
  - Sampling new sentences.

EM Algorithm (Baum-Welch)

- If we had y's ➔ max likelihood.
- If we had (A,O) ➔ predict y's

Chicken vs Egg!

1. Initialize A and O arbitrarily

Expectation Step

2. Predict prob. of y's for each training x

3. Use y's to estimate new (A,O)

Maximization Step

4. Repeat back to Step 1 until convergence

http://en.wikipedia.org/wiki/Baum%E2%80%93Welch_algorithm

Expectation Step

- Given (A,O)
- For training $x=(x^1,\ldots,x^M)$
  - Predict $P(y^i)$ for each $y=(y^1,\ldots y^M)$

	x^1	x^2	…	x^L
$P(y^i\text{=Noun})$	0.5	0.4	…	0.05
$P(y^i\text{=Det})$	0.4	0.6	…	0.25
$P(y^i\text{=Verb})$	0.1	0.0	…	0.7

  - Encodes current model's beliefs about y
  - "Marginal Distribution" of each y^i

Recall: Matrix Formulation

- Define Transition Matrix: A
  - $A_{ab} = P(y^{i+1}=a|y^i=b)$ or $-\text{Log}(\, P(y^{i+1}=a|y^i=b)\,)$

$P(y^{next}|y)$	y="Noun"	y="Verb"
y^{next}="Noun"	0.09	0.667
y^{next}="Verb"	0.91	0.333

- Observation Matrix: O
  - $O_{wz} = P(x^i=w|y^i=z)$ or $-\text{Log}(P(x^i=w|y^i=z)\,)$

P(x|y)	y="Noun"	y="Verb"
x="fish"	0.8	0.5
x="sleep"	0.2	0.5

Maximization Step

- Max. Likelihood over Marginal Distribution

Supervised:

$$A_{ab} = \frac{\sum_{j=1}^{N}\sum_{i=0}^{M_j} 1_{\left[\left(y_j^{i+1}=a\right)\wedge\left(y_j^i=b\right)\right]}}{\sum_{j=1}^{N}\sum_{i=0}^{M_j} 1_{\left[y_j^i=b\right]}} \qquad O_{wz} = \frac{\sum_{j=1}^{N}\sum_{i=1}^{M_j} 1_{\left[\left(x_j^i=w\right)\wedge\left(y_j^i=z\right)\right]}}{\sum_{j=1}^{N}\sum_{i=1}^{M_j} 1_{\left[y_j^i=z\right]}}$$

Marginals

Marginals

Unsupervised:

$$A_{ab} = \frac{\sum_{j=1}^{N}\sum_{i=0}^{M_j} P(y_j^i = b, y_j^{i+1} = a)}{\sum_{j=1}^{N}\sum_{i=0}^{M_j} P(y_j^i = b)} \qquad O_{wz} = \frac{\sum_{j=1}^{N}\sum_{i=1}^{M_j} 1_{\left[x_j^i=w\right]} P(y_j^i = z)}{\sum_{j=1}^{N}\sum_{i=1}^{M_j} P(y_j^i = z)}$$

Marginals

Computing Marginals
(Forward-Backward Algorithm)

- Solving E-Step, requires compute marginals

	x^1	x^2	…	x^L
$P(y^i=\text{Noun})$	0.5	0.4	…	0.05
$P(y^i=\text{Det})$	0.4	0.6	…	0.25
$P(y^i=\text{Verb})$	0.1	0.0	…	0.7

- Can solve using Dynamic Programming!
  – Similar to Viterbi

Notation

Probability of observing prefix $x^{1:i}$ and having the i-th state be y^i=Z

$$\alpha_z(i) = P(x^{1:i}, y^i = Z \mid A, O)$$

Probability of observing suffix $x^{i+1:M}$ given the i-th state being y^i=Z

$$\beta_z(i) = P(x^{i+1:M} \mid y^i = Z, A, O)$$

Computing Marginals = Combining the Two Terms

$$P(y^i = z \mid x) = \frac{a_z(i)\beta_z(i)}{\sum_{z'} a_{z'}(i)\beta_{z'}(i)}$$

http://en.wikipedia.org/wiki/Baum%E2%80%93Welch_algorithm

Notation

Probability of observing prefix $x^{1:i}$ and having the i-th state be y^i=Z

$$\alpha_z(i) = P(x^{1:i}, y^i = Z \mid A, O)$$

Probability of observing suffix $x^{i+1:M}$ given the i-th state being y^i=Z

$$\beta_z(i) = P(x^{i+1:M} \mid y^i = Z, A, O)$$

Computing Marginals = Combining the Two Terms

$$P(y^i = b, y^{i-1} = a \mid x) = \frac{a_a(i-1)P(y^i = b \mid y^{i-1} = a)P(x^i \mid y^i = b)\beta_b(i)}{\sum_{a',b'} a_{a'}(i-1)P(y^i = b' \mid y^{i-1} = a')P(x^i \mid y^i = b')\beta_{b'}(i)}$$

http://en.wikipedia.org/wiki/Baum%E2%80%93Welch_algorithm

Forward (sub-)Algorithm

- Solve for every: $\alpha_z(i) = P(x^{1:i}, y^i = Z \mid A, O)$
- Naively:

Exponential Time!

$$\alpha_z(i) = P(x^{1:i}, y^i = Z \mid A, O) = \sum_{y^{1:i-1}} P(x^{1:i}, y^i = Z, y^{1:i-1} \mid A, O)$$

- Can be computed recursively (like Viterbi)

$$\alpha_z(1) = P(y^1 = z \mid y^0) P(x^1 \mid y^1 = z) = O_{x^1,z} A_{z,start}$$

$$\alpha_z(i+1) = O_{x^{i+1},z} \sum_{j=1}^{L} \alpha_j(i) A_{z,j}$$

Viterbi effectively replaces sum with max

Backward (sub-)Algorithm

- Solve for every: $\beta_z(i) = P(x^{i+1:M} \mid y^i = Z, A, O)$
- Naively:

Exponential Time!

$$\beta_z(i) = P(x^{i+1:M} \mid y^i = Z, A, O) = \sum_{y^{i+1:L}} P(x^{i+1:M}, y^{i+1:M} \mid y^i = Z, A, O)$$

- Can be computed recursively (like Viterbi)

$$\beta_z(L) = 1$$

$$\beta_z(i) = \sum_{j=1}^{L} \beta_j(i+1) A_{j,z} O_{x^{i+1},j}$$

Forward-Backward Algorithm

- Runs Forward $\alpha_z(i) = P(x^{1:i}, y^i = Z \mid A, O)$
- Runs Backward $\beta_z(i) = P(x^{i+1:M} \mid y^i = Z, A, O)$
- For each training x=(x^1,…,x^M)
  - Computes each P(y^i) for y=(y^1,…,y^M)

$$P(y^i = z \mid x) = \frac{a_z(i)\beta_z(i)}{\sum_{z'} a_{z'}(i)\beta_{z'}(i)}$$

Recap: Unsupervised Training

- Train using only word sequences: $S = \left\{ x_i \right\}_{i=1}^{N}$

Word Sequence
(Sentence)

- y's are "hidden states"
  - All pairwise transitions are through y's
  - Hence hidden Markov Model
- Train using EM algorithm
  - Converge to local optimum

Initialization

- How to choose #hidden states?
  - By hand
  - Cross Validation
    - P(x) on validation data
    - Can compute P(x) via forward algorithm:

$$P(x) = \sum_y P(x,y) = \sum_z \alpha_z(M) P(End \mid y^M = z)$$

Recap: Sequence Prediction & HMMs

- Models pairwise dependences in sequences

x="I fish often"

POS Tags:
Det, Noun, Verb, Adj, Adv, Prep

Independent: (N, N, Adv)

HMM Viterbi: (N, V, Adv)

- Compact: only model pairwise between y's
- **Main Limitation:** Lots of independence assumptions
  – Poor predictive accuracy

Next Week

- Conditional Random Fields
  - Sequential version of logistic regression
  - Removes many independence assumptions
  - More accurate in practice
  - Can only be trained in supervised setting

- Recitation Tonight:
  - Recap of Viterbi and Forward/Backward