Schedule

1. **What are models?**
   Exercise: Python Skills

2. **Energy budget of the Earth**
   Exercise: Simple Energy Balance Model

3. **Nonlinearity, Feedback, and Predictability**
   Exercise: Nonlinearity and Feedbacks
   Exercise: Revised Energy Balance Model

4. **Parametrization and Sensitivity**

5. **Radiative budget**
   Exercise: 1-layer greenhouse model
   Exercise: 2-layer greenhouse model

6. **Introduction to fluid dynamics**
   Exercise: Analytical katabatic flow model

7. **Finite difference method**
   Exercise: Heat Equation
   Exercise: Advection-Diffusion Equation
   Exercise: Boundary layer Evolution
   Exercise: Numerical katabatic flow model

8. **Implicit finite difference methods**
   Exercise: Boundary layer evolution

9. **Optimization problem**
   Exercise: Surface energy balance
   Exercise: Sublimation

10. **COSIPY snow model**
    Exercise: Simulations with COSIPY

11. **Introduction to PALM**
    Exercise: Simulations with PALM

12. **How to write an article**

Five steps of model development

Step 1: Mathematical model

$$\frac{\partial u}{\partial t} = -\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) - g_x - \frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\partial \overline{u'u'}}{\partial x} - \frac{\partial \overline{u'v'}}{\partial y} - \frac{\partial \overline{u'w'}}{\partial z}$$

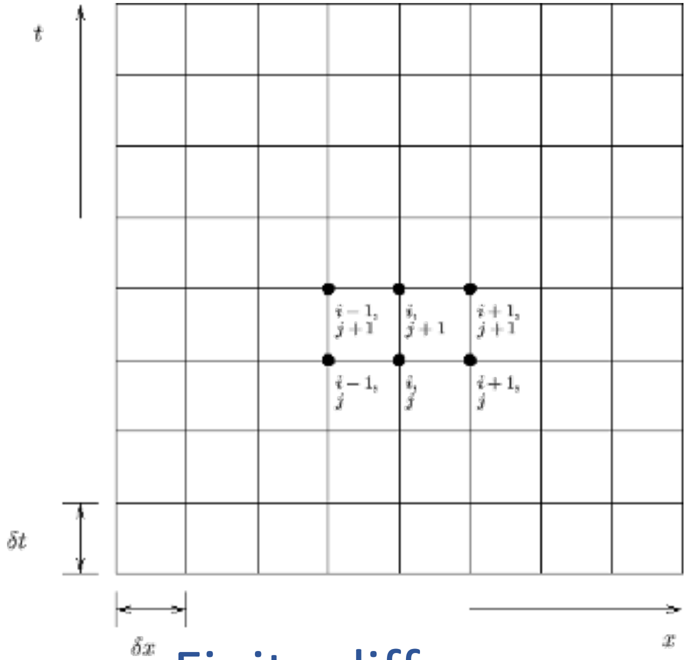
- Partial differential equation (PDE)
- Boundary conditions (BC)
- Does the model fit the problem?
- Does it introduce errors?

Step 2: Discretization → Approximation of the PDEs → system of algebraic equations

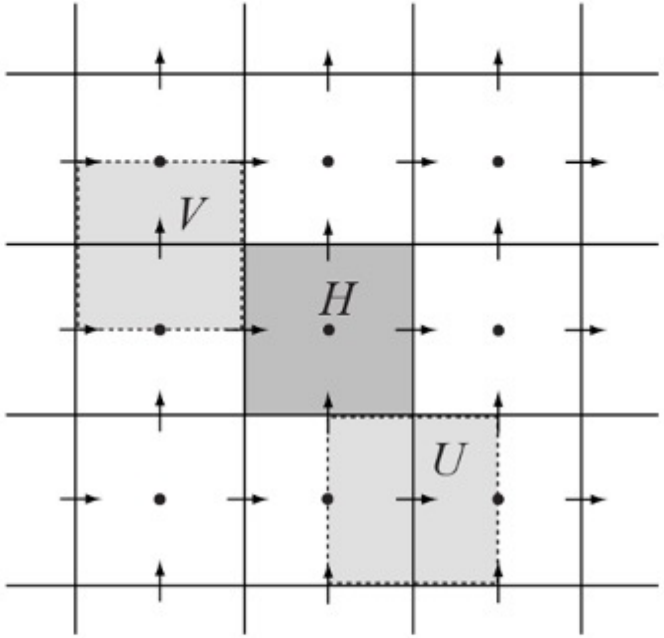
$$\left.\begin{array}{c}\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla p - \mathrm{g} - \nabla\cdot\tau \\ \nabla\cdot\vec{u} = 0\end{array}\right\} \quad Ax = b = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn}\end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_m\end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m\end{pmatrix}$$

Five steps of model development

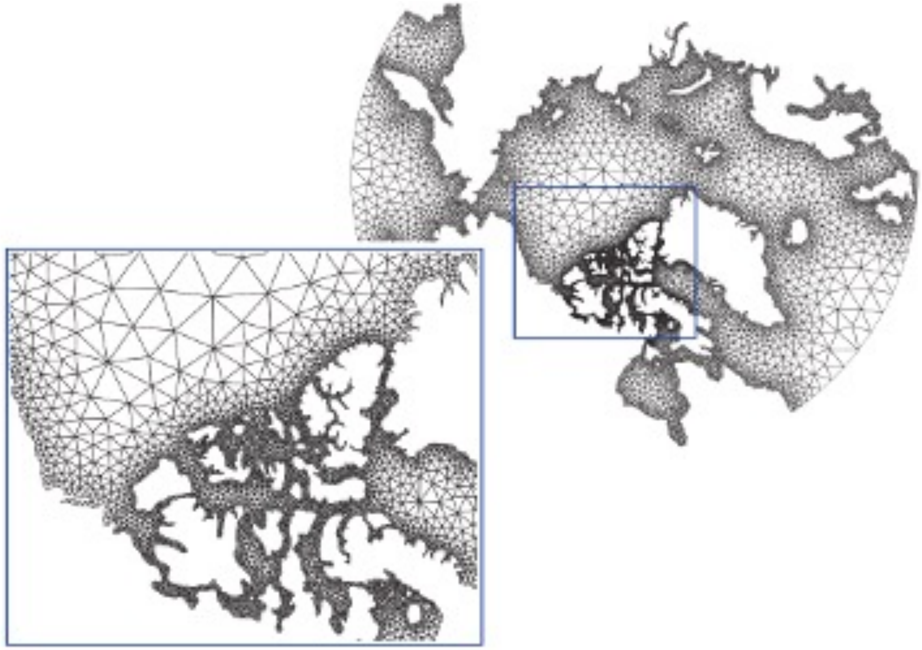
Step 2: Discretization



Finite differences



Finite Volume



Finite Elements

- Geometry: Create mesh or grid
- Model $\frac{\partial}{\partial x} \rightarrow$ arithmetic operations

Five steps of model development

Step 3: Analyse the model

Numerical Scheme must satisfy …

consistency
stability
convergence
conversation
boundedness
accuracy and error behaviour

Step 4: Solve the model

'Steady state' model

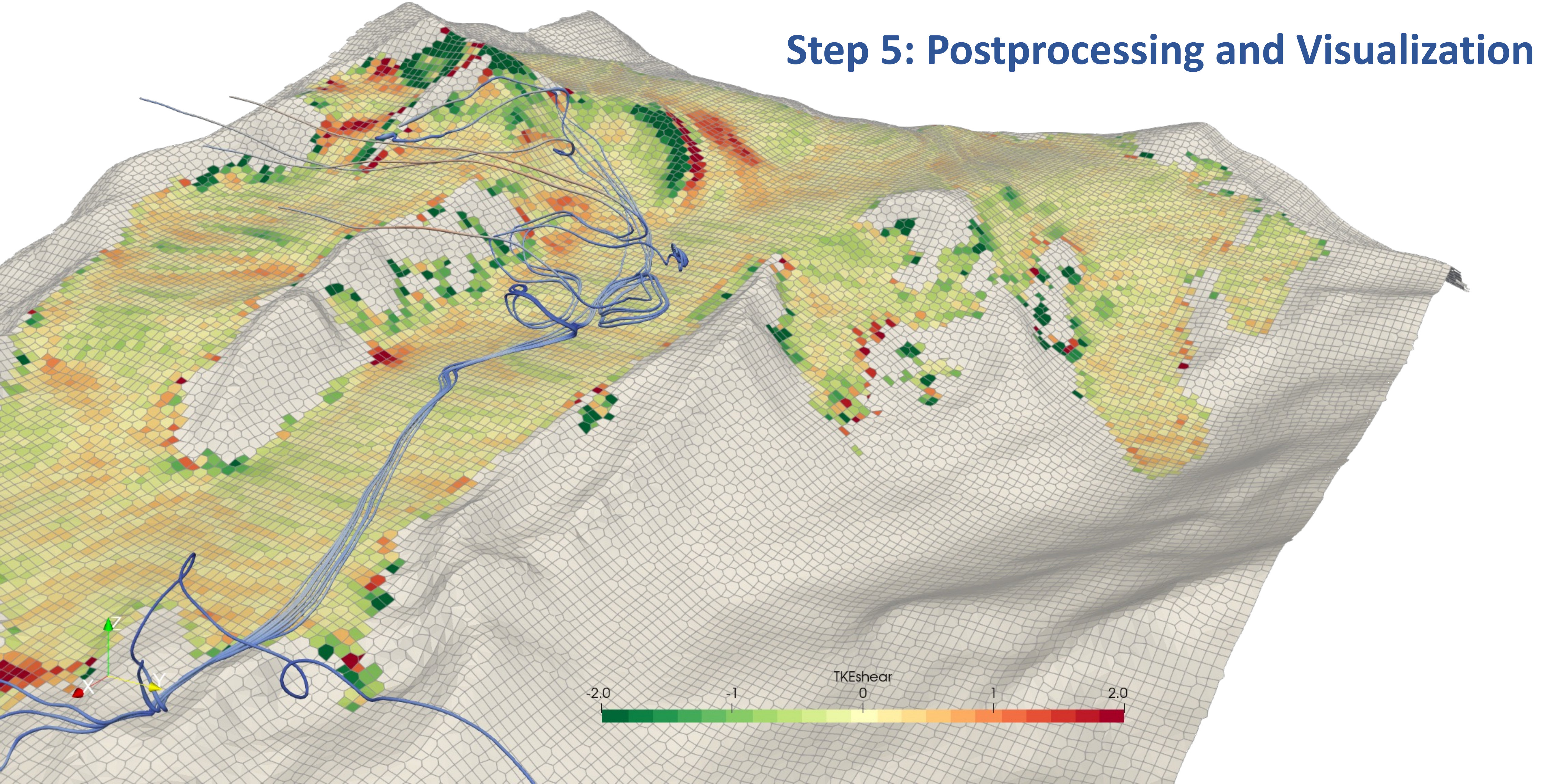
$Ax = b$
Linear solver

'Time dependent' model

Ordinary differential equations (ODE)
Time integration solver

Five steps of model development

Step 5: Postprocessing and Visualization




Leonhard Euler (www.wikipedia.com)

Finite Differences

Step 1: Discretization of domain (grid/mesh)

Step 2: Substitution of derivations by differences

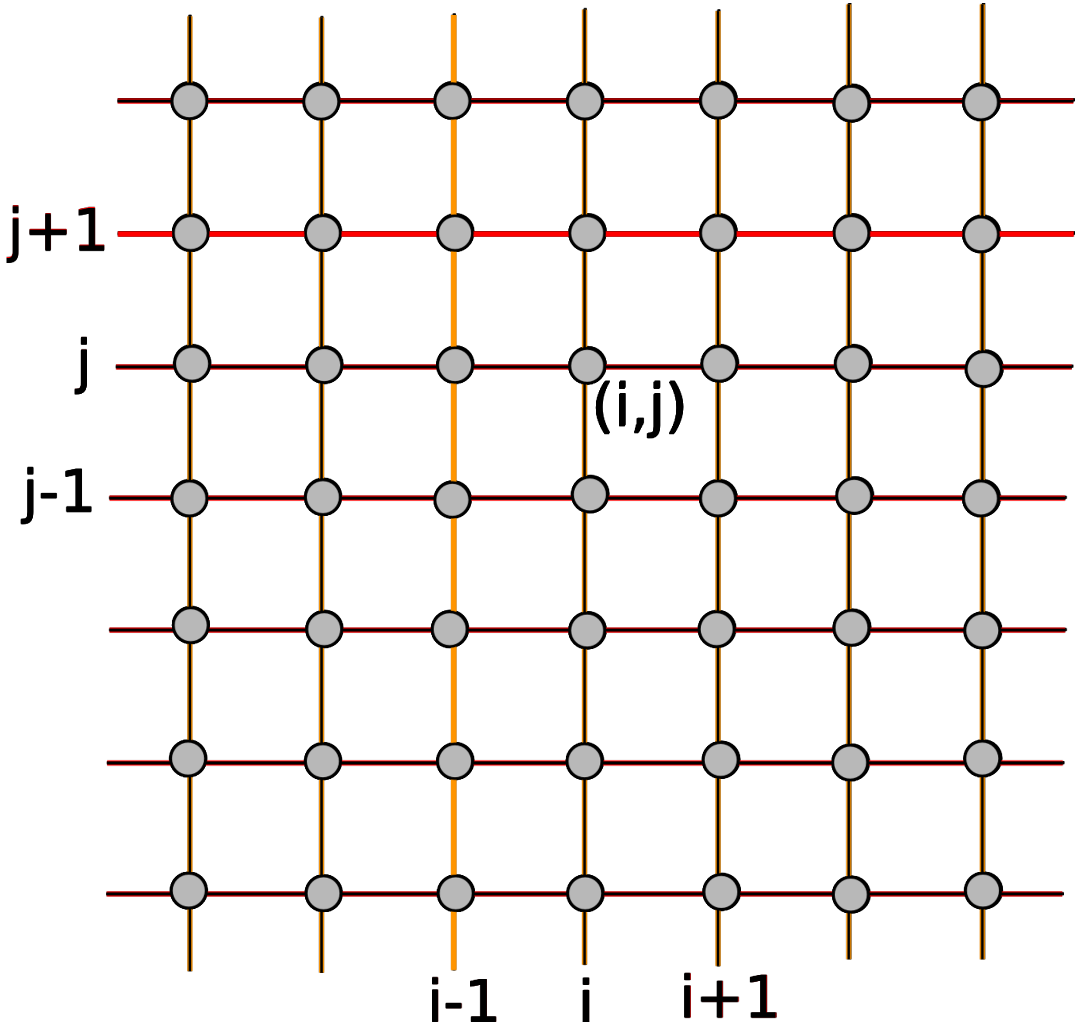
Step 3: Sort the differences equations

Step 4: Solve the difference equations, linear system

Discretization of the domain

Approximate derivatives $\frac{\partial}{\partial x}, \frac{\partial}{\partial t}$ ... numerically

Let's define a grid:



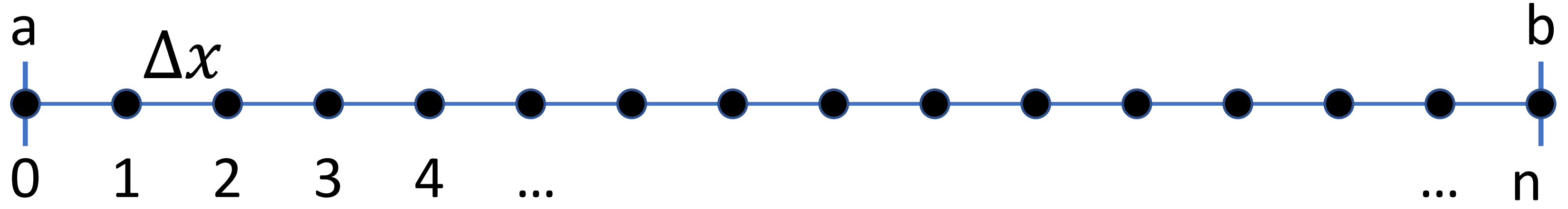
- v
- contains field variables dependent on neighbouring nodes
- provides one algebraic equation
- grid lines from same family don't intersect

Discretization of the domain

Domain: Interval: $I = [a, b]$

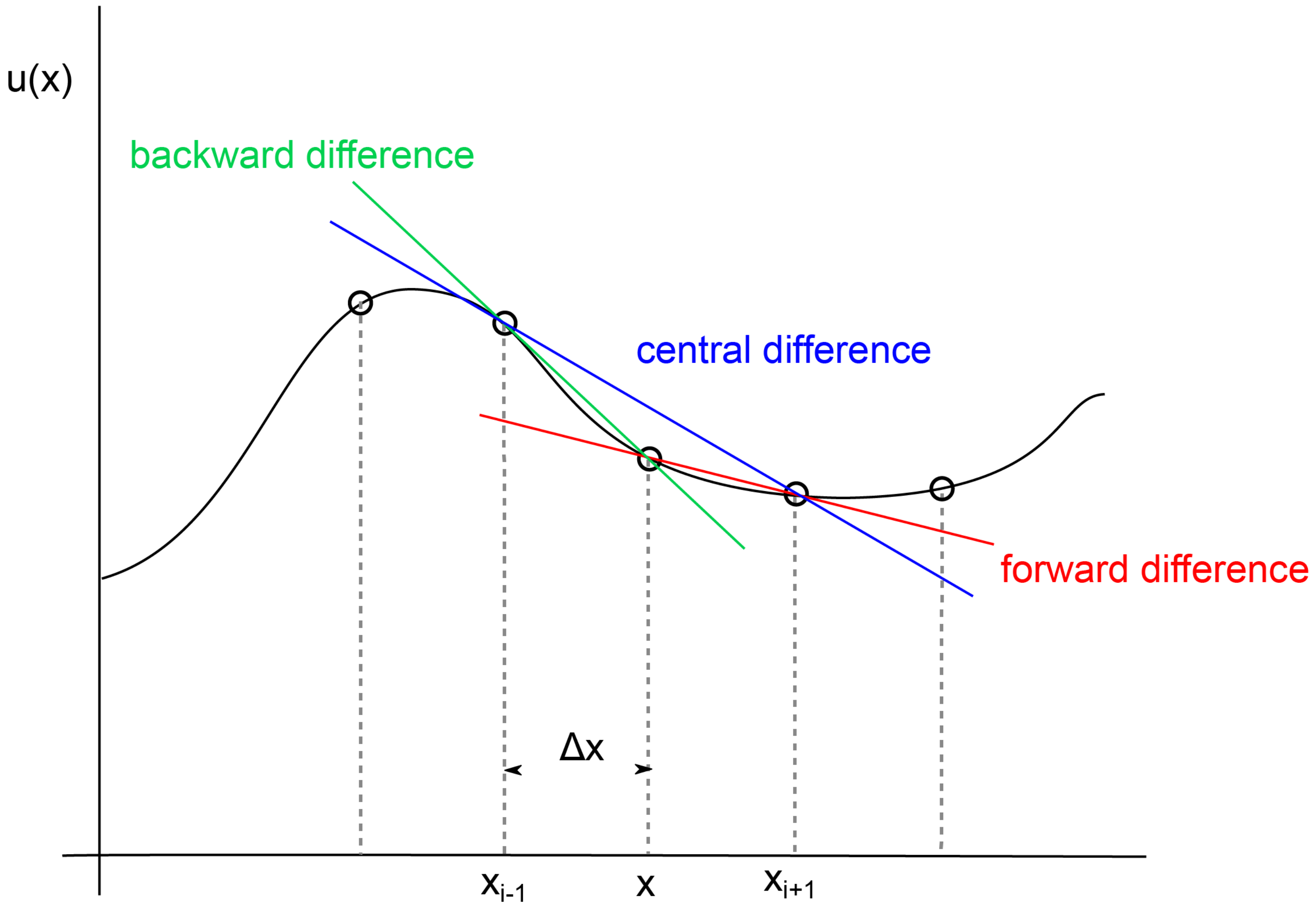
Discretization: Grid points: $x_i = a + i\Delta x, \quad i = 0, \dots, n$

Equidistant grid: Grid step: $\Delta x = \frac{b-a}{n}$



Discretization of the domain

$$\left.\frac{\partial u}{\partial x}\right|_{x_i} = \lim_{\Delta x \to 0} \frac{u(x_i + \Delta x) - u(x_i)}{\Delta x}$$



Forward difference

$$\left.\frac{\partial u}{\partial x}\right|_{x_i} \simeq \lim_{\Delta x \to 0} \frac{u(x_{i+1}) - u(x_i)}{\Delta x}$$

Backward difference

$$\left.\frac{\partial u}{\partial x}\right|_{x_i} \simeq \lim_{\Delta x \to 0} \frac{u(x_i) - u(x_{i-1})}{\Delta x}$$

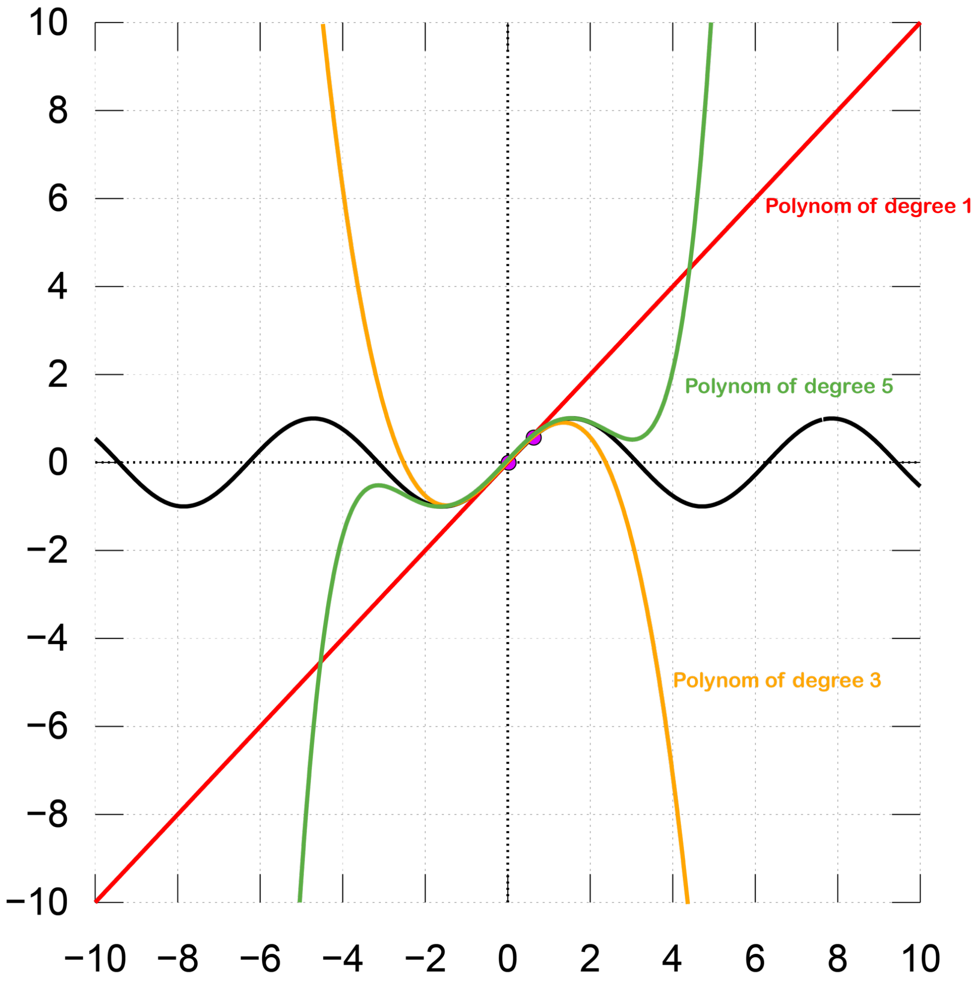
Central difference

$$\left.\frac{\partial u}{\partial x}\right|_{x_i} \simeq \lim_{\Delta x \to 0} \frac{u(x_{i+1}) - u(x_{i-1})}{2\Delta x}$$

Accuracy of finite difference approximation

Taylor-Series:

$$(1) \quad u(x_{i-1}) = u(x_i) - \Delta x u_x(x_i) + \frac{\Delta x^2}{2} u_{xx}(x_i) - \frac{\Delta x^3}{6} u_{xxx}(x_i) + \cdots$$



Accuracy of finite difference approximation

Taylor-Series:

$$(1) \quad u(x_{i-1}) = u(x_i) - \Delta x u_x(x_i) + \frac{\Delta x^2}{2} u_{xx}(x_i) - \frac{\Delta x^3}{6} u_{xxx}(x_i) + \cdots$$

$$(2) \quad u(x_{i+1}) = u(x_i) + \Delta x u_x(x_i) + \frac{\Delta x^2}{2} u_{xx}(x_i) + \frac{\Delta x^3}{6} u_{xxx}(x_i) + \cdots$$

Difference quotient of derivation: (2) − (1)

$$u(x_{i+1}) - u(x_{i-1}) = 2\Delta x u_x(x_i) - \frac{2\Delta x^3}{6} u_{xxx}(x_i) + \mathcal{O}(\Delta x^5)$$

$$\frac{u(x_{i+1}) - u(x_{i-1})}{2\Delta x} = u_x(x_i) - \underbrace{\frac{\Delta x^2}{6} u_{xxx}(x_i) + \mathcal{O}(\Delta x^4)}_{\text{We neglect all h.o.t.}}$$

Accuracy of finite difference approximation

$$\frac{u(x_{i+1}) - u(x_i)}{\Delta x} \simeq \left.\frac{\partial u}{\partial x}\right|_{x_i} + \mathcal{O}(\Delta x)$$

Forward difference

$$\frac{u(x_i) - u(x_{i-1})}{\Delta x} \simeq \left.\frac{\partial u}{\partial x}\right|_{x_i} + \mathcal{O}(\Delta x)$$

Backward difference

$$\frac{u(x_{i+1}) - u(x_{i-1})}{2\Delta x} \simeq \left.\frac{\partial u}{\partial x}\right|_{x_i} + \mathcal{O}(\Delta x^2)$$

Central difference

$$\frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{\Delta x^2} \simeq \left.\frac{\partial^2 u}{\partial x^2}\right|_{x_i} + \mathcal{O}(\Delta x^2)$$

2. Derivative

Grid spacing Δx	Method	Error
0.1	FD, BD	$\mathcal{O}(\Delta x)$ = 0.1 (10%)
0.1	CD	$\mathcal{O}(\Delta x^2)$ = 0.01 (1%)
0.01	FD, BD	$\mathcal{O}(\Delta x)$ = 0.01 (10%)