Climate Modelling and Data Analysis

Prof. Dr. Tobias Sauter

Date 02.05.2022

Schedule

1. **What are models?**
   Exercise: Python Skills

2. **Energy budget of the Earth**
   Exercise: Simple Energy Balance Model

3. **Nonlinearity, Feedback, and Predictability**
   Exercise: Nonlinearity and Feedbacks
   Exercise: Revised Energy Balance Model

4. **Parametrization and Sensitivity**

5. **Radiative budget**
   Exercise: 1-layer greenhouse model
   Exercise: 2-layer greenhouse model

6. **Introduction to fluid dynamics**
   Exercises: The use of the governing equations
   Exercise: Analytical katabatic flow model

7. **Finite difference method**
   Exercise: Advection-Diffusion Equation
   Exercise: Boundary layer Evolution
   Exercise: Numerical katabatic flow model
   Exercise: Heat Equation

8. **Implicit finite difference methods**
   Exercise: Boundary layer evolution

9. **Optimization problem**
   Exercise: Surface energy balance
   Exercise: Sublimation

10. **COSIPY snow model**
    Exercise: Simulations with COSIPY

11. **Introduction to PALM**
    Exercise: Simulations with PALM

12. **How to write an article**

Learning objective

Learning objectives

- Fundamental concepts of fluid dynamics
- Governing equations
- Discretization
- Finite-difference method

Literature

Recommended textbooks

- Young et. al, 2011: **Introduction to Fluid Mechanics**. Wiley.
- Jacobson M., 2005: **Fundamentals of atmospheric modelling**. Cambridge University Press.
- Kundu P.K., Cohen I.M., and Dowling D.R., 2016: **Fluid Mechanics**. Elsevier.
- Stull, 1988: **An Introduction to Boundary Layer Meteorology**. Springer.

Governing equations for atmospheric flow

- Conservation of mass (continuity equation)
- Conservation of momentum (Newton’s second law)
- Conservation of heat (First law of thermodynamics
- Conservation of moisture
- Conservation of s scalar quantities
- Equation of state

Basics concept of fluid dynamics

Assumptions

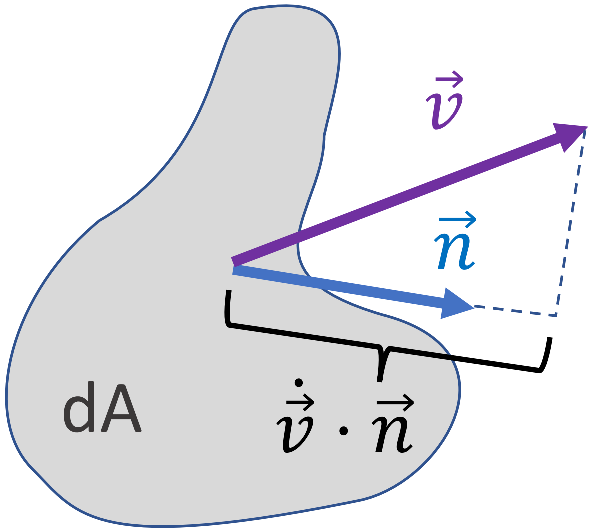
- No molecular structure (no resistance to external shear forces
- Fluid regarded as continuum
- Flow by external forces
- Fluids are similar under action of forces

Conservation of mass (continuity equation)

Conservation law → Basic idea

$$\left\{\text{Rate of inflow to a system}\right\} - \left\{\text{Rate of outflow from a system}\right\} = \left\{\text{Rate of accumulation}\right\}$$

What is the rate of mass flow through a surface dA with normal vector $\vec{n}$?



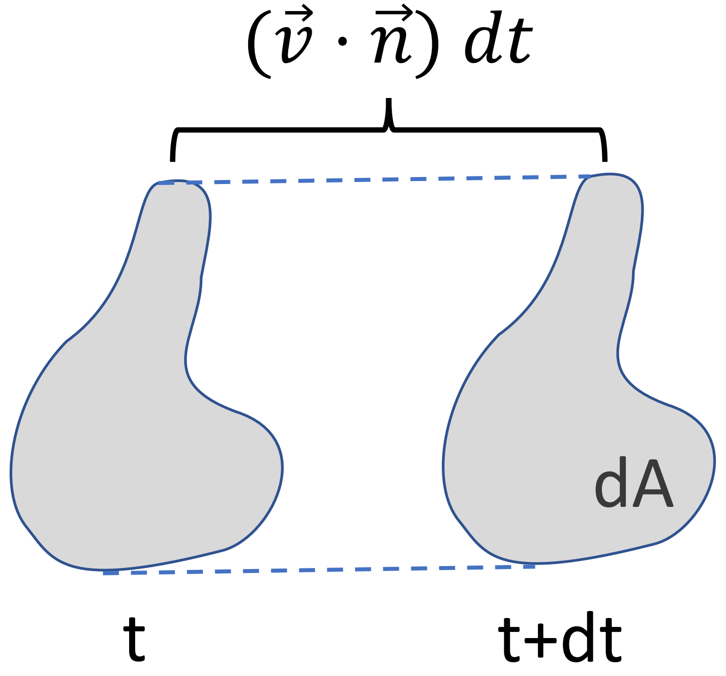
$\vec{v} \cdot \vec{n}$ is the projection of $\vec{v}$ normal to the surface

What **volume** of fluid flows through dA in time dt?

$$(\vec{v} \cdot \vec{n})\, dt\, dA$$

Flow through Surface — Time — Area

$$\left[\frac{m}{s}\right] \quad [s] \quad [m^2] \quad = \quad [m^3]$$



Conservation of mass

What **mass** of fluid flows through dA in time dt?

Density $\rho = \frac{mass}{volume}$ ⟹ mass = (density)(volume) $= \rho(\vec{v} \cdot \vec{n})\, dt\, dA$

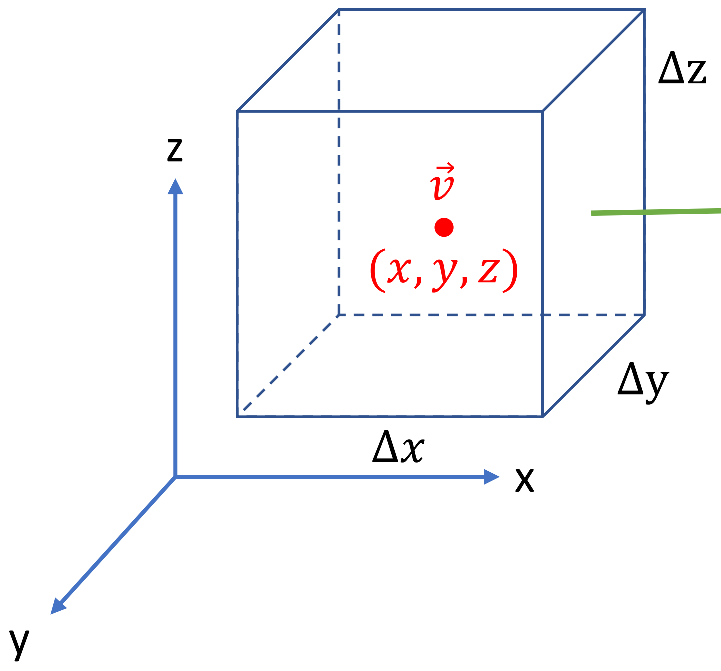
$$[kg] = \left[\frac{kg}{m^3}\right] \quad [m^3]$$

$$\frac{mass}{time} = \dot{m} = \rho(\vec{v} \cdot \vec{n})\, dA$$

Mass flux (mass per unit time per unit area) = $\rho(\vec{v} \cdot \vec{n})$

Conservation of mass

Let's write a conservation law …



$$\vec{n} = 1\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$
$$= \hat{\imath}$$
$$\vec{v} \cdot \vec{n} = u(\widehat{\imath}) + \vec{v}(\widehat{\jmath}) + \vec{w}(\hat{k})$$
$$= u \longleftarrow \text{Scalar}$$

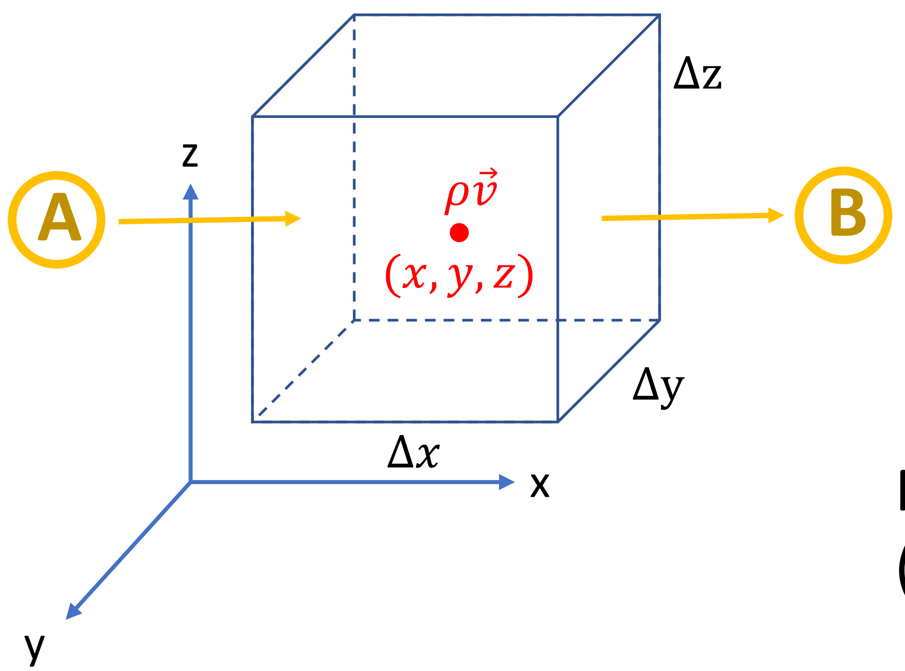
We know the following:

1. $\vec{v} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$

2. $mass\ flux = \dfrac{mass}{(time)(area)} = \rho(\vec{v} \cdot \vec{n})$

Conservation of mass

Let's write a conservation law …



We know the following:

1. $\vec{v} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$

2. $mass\ flux = \dfrac{mass}{(time)(area)} = \rho(\vec{v} \cdot \vec{n})$

Mass flux IN (x-face A)

$$\rho u + \underbrace{\frac{\partial(\rho u)}{\partial x}\left(-\frac{\Delta x}{2}\right)}_{\text{Change in } \rho u \text{ from center to face A}}$$

Mass flux OUT (x-face B)

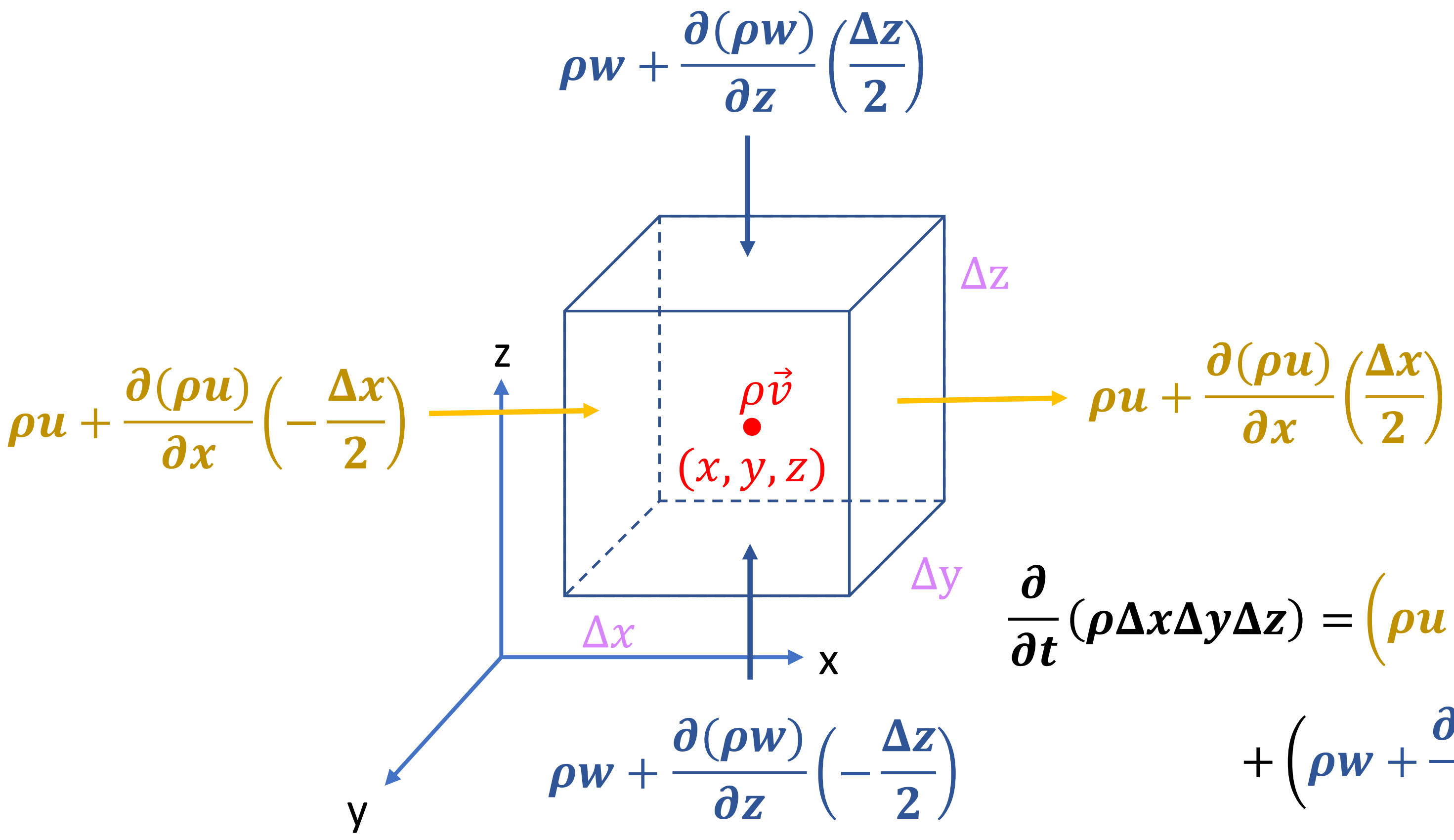
$$\rho u + \frac{\partial(\rho u)}{\partial x}\left(\frac{\Delta x}{2}\right)$$

$\dfrac{mass}{(time)(area)}$ at center of cube (x,y,z)

Change in ρu from center to face A

Conservation of mass

Let’s write a conservation law …



Remember:

(Rate of inflow to a system) - (Rate of outflow from a system) = (Rate of accumulation)

$$\frac{\partial}{\partial t}(\rho\Delta x\Delta y\Delta z) = \left(\rho u + \frac{\partial(\rho u)}{\partial x}\left(-\frac{\Delta x}{2}\right)\Delta y\Delta z\right) - \left(\rho u + \frac{\partial(\rho u)}{\partial x}\left(\frac{\Delta x}{2}\right)\Delta y\Delta z\right)$$

$$+\left(\rho w + \frac{\partial(\rho w)}{\partial z}\left(-\frac{\Delta z}{2}\right)\Delta x\Delta y\right) - \left(\rho w + \frac{\partial(\rho w)}{\partial z}\left(\frac{\Delta z}{2}\right)\Delta x\Delta y\right)$$

$$+\left(\rho v + \frac{\partial(\rho v)}{\partial y}\left(-\frac{\Delta y}{2}\right)\Delta x\Delta z\right) - \left(\rho v + \frac{\partial(\rho v)}{\partial y}\left(\frac{\Delta y}{2}\right)\Delta x\Delta z\right)$$

Conservation of mass

$$\frac{\partial}{\partial t}(\rho\cancel{\Delta x\Delta y\Delta z}) = \left(\cancel{\rho u} + \boxed{\frac{\partial(\rho u)}{\partial x}\left(-\frac{\Delta x}{2}\right)}\Delta y\Delta z\right) - \left(\cancel{\rho u} + \boxed{\frac{\partial(\rho u)}{\partial x}\left(\frac{\Delta x}{2}\right)\Delta y\Delta z}\right)$$

$$+\left(\cancel{\rho w} + \frac{\partial(\rho w)}{\partial z}\left(-\frac{\Delta z}{2}\right)\Delta x\Delta y\right) - \left(\cancel{\rho w} + \frac{\partial(\rho w)}{\partial z}\left(\frac{\Delta z}{2}\right)\Delta x\Delta y\right)$$

$$+\left(\cancel{\rho v} + \frac{\partial(\rho v)}{\partial y}\left(-\frac{\Delta y}{2}\right)\Delta x\Delta z\right) - \left(\cancel{\rho v} + \frac{\partial(\rho v)}{\partial y}\left(\frac{\Delta y}{2}\right)\Delta x\Delta z\right)$$

$$= \left[-\frac{\partial(\rho u)}{\partial x} - \frac{\partial(\rho v)}{\partial y} - \frac{\partial(\rho w)}{\partial z}\right]\cancel{\Delta x\Delta y\Delta z}$$

$$\frac{\partial\rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x} - \frac{\partial(\rho v)}{\partial y} - \frac{\partial(\rho w)}{\partial z}$$

$$\boxed{\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0}$$

Continuity equation or conservation of mass

Conservation of mass

$$\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{t}} + \frac{\partial(\boldsymbol{\rho u})}{\partial \boldsymbol{x}} + \frac{\partial(\boldsymbol{\rho v})}{\partial \boldsymbol{y}} + \frac{\partial(\boldsymbol{\rho w})}{\partial \boldsymbol{z}} = \mathbf{0}$$

Recall the gradient or "DEL" operator

$$\nabla(\cdot) = \frac{\partial(\cdot)}{\partial x}\hat{\imath} + \frac{\partial(\cdot)}{\partial y}\hat{\jmath} + \frac{\partial(\cdot)}{\partial z}\hat{k}$$

So, if $\vec{v} = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$, then

$$\underset{\text{Dot product}}{\nabla \cdot} \vec{v} = \underbrace{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}}_{\text{Scalar}}$$

$$\boxed{\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{t}} + \boldsymbol{\nabla} \cdot (\boldsymbol{\rho v}) = \mathbf{0}}$$

Incompressible flow

ρ = constant (in some cases)

$$\cancel{\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{t}}} + \boldsymbol{\nabla} \cdot (\boldsymbol{\rho v}) = \mathbf{0}$$

$$\cancel{\boldsymbol{\rho}}(\boldsymbol{\nabla} \cdot \boldsymbol{v}) = \mathbf{0}$$

$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{v} = \mathbf{0}}$$

Steady-state

$$\frac{\partial \rho}{\partial t} = 0 \text{ (no changes with time)}$$

$$\boxed{\boldsymbol{\nabla} \cdot (\boldsymbol{\rho v}) = \mathbf{0}}$$

Conservation of a scalar quantity

$$\underbrace{\boxed{\text{Time rate of change of tracer}}} = \underbrace{\boxed{\text{Rate of tracer inflow to CV}} - \boxed{\text{Rate of tracer outflow to CV}}} + \underbrace{\boxed{\text{source term}}} - \underbrace{\boxed{\text{Turbulent tracer flux}}}$$

$$\underbrace{\frac{\partial c}{\partial t} =}_{\text{Accumulation}} \underbrace{-\left(u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z}\right)}_{\text{Advection}} \underbrace{+S_c}_{\text{Source term}} \underbrace{-\frac{\partial \overline{u'c'}}{\partial x} - \frac{\partial \overline{v'c'}}{\partial y} - \frac{\partial \overline{w'c'}}{\partial z}}_{\text{Turbulent mixing}}$$

Conservation of heat

Time rate of change of heat = Rate of heat inflow to CV - Rate of heat outflow to CV + source term + Turbulent heat flux

$$\underbrace{\frac{\partial\theta}{\partial t}}_{\text{Accumulation}} = \underbrace{-\left(u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + w\frac{\partial\theta}{\partial z}\right)}_{\text{Advection}} \underbrace{-\frac{1}{\rho c_p}\left(\frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} + \frac{\partial Q_z^*}{\partial z}\right)}_{\text{Radiation divergence}} \underbrace{-\frac{L_v E}{\rho c_p}}_{\text{Latent heat release}}$$

$$\underbrace{-\frac{\partial\overline{u'\theta'}}{\partial x} - \frac{\partial\overline{v'\theta'}}{\partial y} - \frac{\partial\overline{w'\theta'}}{\partial z}}_{\text{Turbulent heat flux}}$$

θ: potential temperature
L: latent heat
c_p: specific heat
E: phase change of moisture
Q^*: net radiation flux

Conservation of momentum

Netwon's lex segunda (law of motion)

(Example: zonal momentum equation)

Time rate of change of momentum = Rate of momentum inflow to CV - Rate of momentum outflow to CV + Sum of forces acting on CV

$$\underbrace{\frac{\partial u}{\partial t}}_{\text{Accumulation}} = \underbrace{-\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)}_{\text{Advection}} \underbrace{-g_x}_{\text{Gravitational force}} \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial x}}_{\text{Pressure force}} \underbrace{-\frac{\partial \overline{u'u'}}{\partial x} - \frac{\partial \overline{u'v'}}{\partial y} - \frac{\partial \overline{u'w'}}{\partial z}}_{\text{Turbulent flux divergence}}$$

"Navier-Stokes Equation"

Conservation of momentum

$$\underbrace{\rho\frac{\partial u}{\partial t}=}_{\text{Accumulation}}\quad \underbrace{-\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)}_{\text{Flow}}\quad \underbrace{-\rho g_x}_{\text{Gravitational force}}\quad \underbrace{-\frac{\partial p}{\partial x}}_{\text{Pressure force}}\quad \underbrace{-\frac{\partial \overline{u'u'}}{\partial x}+\frac{\partial \overline{u'v'}}{\partial y}+\frac{\partial \overline{u'w'}}{\partial z}}_{\text{Turbulent flux divergence}}$$

We can also write the Navier-Stokes equation in VECTOR FORM

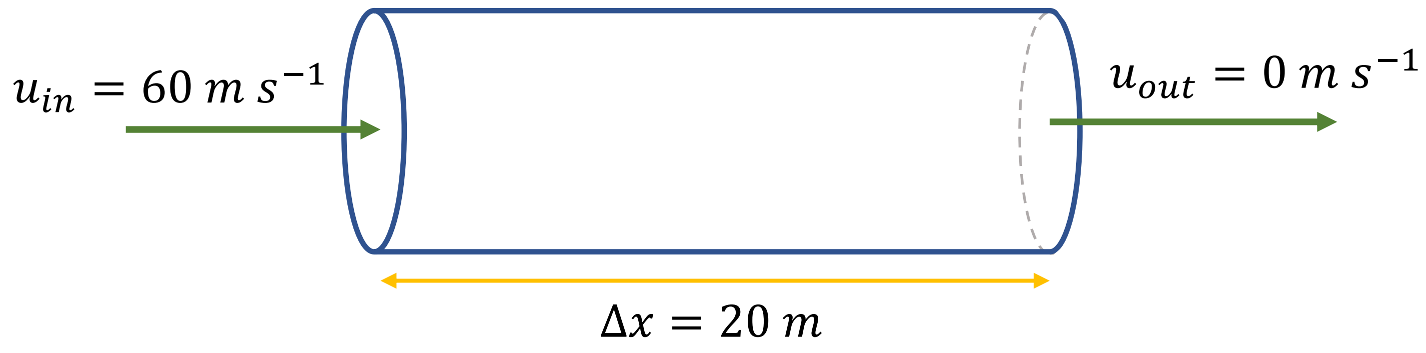
$$\boxed{\rho\frac{D\boldsymbol{u}}{Dt}=\rho\left(\frac{\partial \boldsymbol{u}}{\partial t}+\mathbf{u}(\nabla\cdot\boldsymbol{u})\right)=-\nabla p+\rho\boldsymbol{g}+\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})}$$

where $\boldsymbol{u}$ is the velocity vector (not the u component)

Example: Conservation of air mass

Hurricane-force winds of 60 m s^{-1} blow into an west-facing entrance of a 20 m long pedestrian tunnel. The door at the other end of the tunnel is closed. The initial air density in the tunnel is 1.2 kg m^{-3}. Find the rate of air density increase in the tunnel.

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x} - \cancel{\frac{\partial(\rho v)}{\partial y}} - \cancel{\frac{\partial(\rho w)}{\partial z}}$$



Solution:

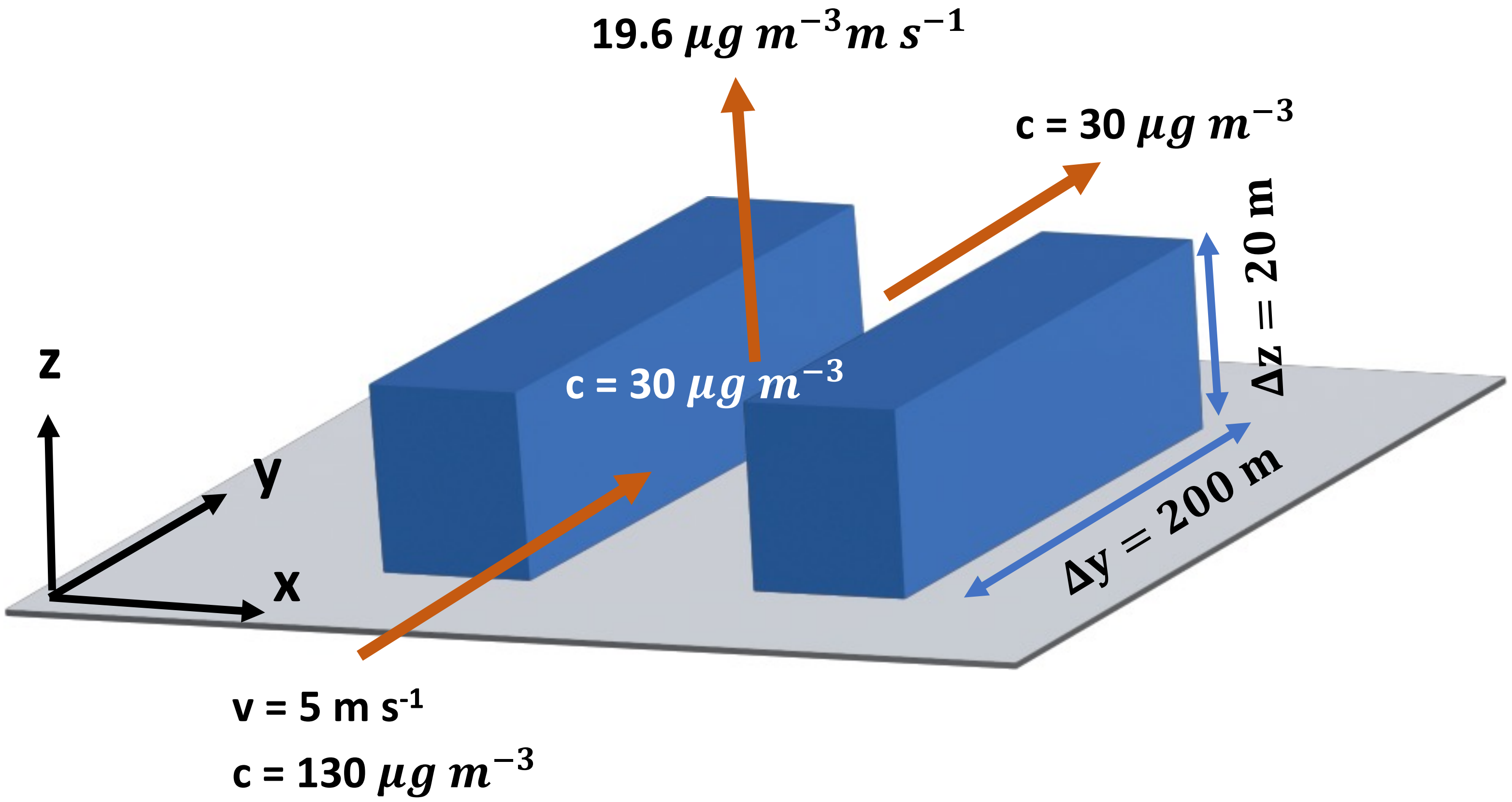
$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}$$

$$= -(1.2\ kg\ m^{-3})\frac{0 - 60\ m\ s^{-1}}{20\ m}$$

$$= +3.6\ kg\ m^{-3}\ s^{-1}$$

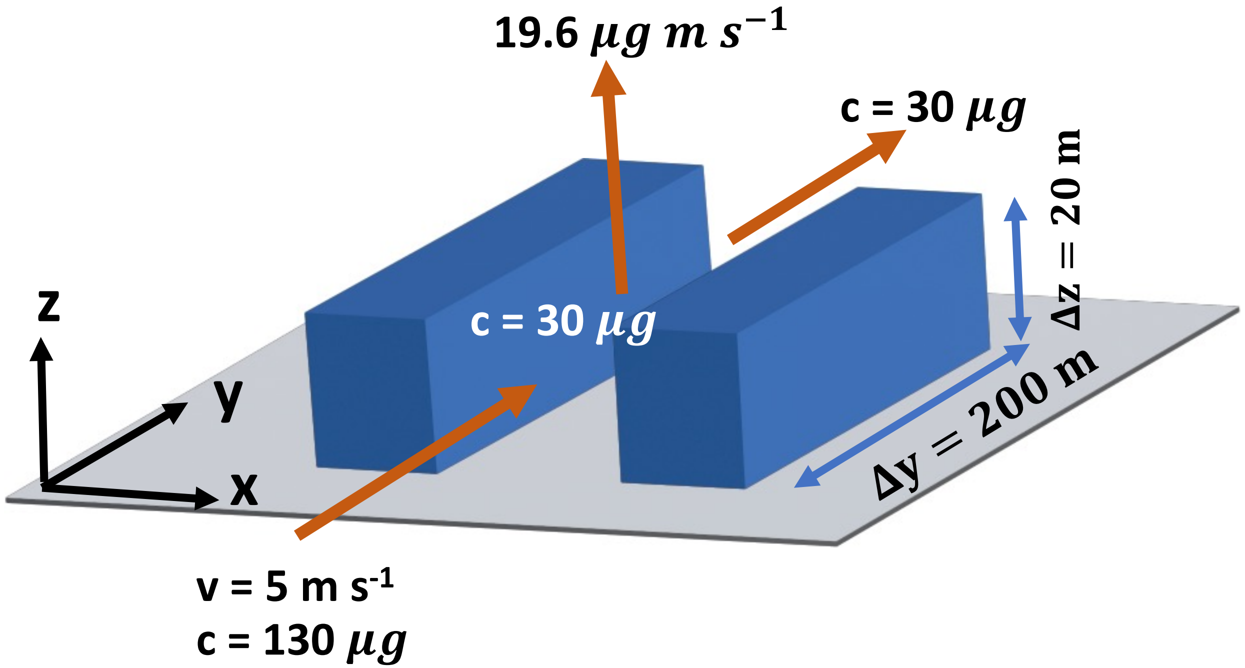
Example: Conservation of particle matter

In a narrow street canyon, a fine particle matter load of 30 $\mu g\ m^{-3}$ is measured. A moderate wind blows from the south, advecting further particles from a busy road. The concentration at the road is 130 $\mu g\ m^{-3}$. Turbulent mixing causes 19.6 $\mu g \cdot m \cdot s^{-1}$ to be entrained upwards. Does the concentration exceed the critical threshold value of 150 $\mu g\ m^{-3}$ after one hour?



$$\frac{\partial c}{\partial t} = \underbrace{-\left(\cancel{u\frac{\partial c}{\partial x}} + v\frac{\partial c}{\partial y} + \cancel{w\frac{\partial c}{\partial z}}\right)}_{\text{Advection}} \underbrace{+\cancel{S_c}}_{\text{Source term}} \underbrace{-\cancel{\frac{\partial \overline{u'c'}}{\partial x}} - \cancel{\frac{\partial \overline{v'c'}}{\partial y}} - \frac{\partial \overline{w'c'}}{\partial z}}_{\text{Turbulent mixing}}$$

Example: Conservation of particle matter



Solution:

a) Calculate the change of particle matter

$$\frac{\partial c}{\partial t} = -v\frac{\partial c}{\partial y} - \frac{\partial \overline{w'c'}}{\partial z}$$

$$= -2\ (m\ s^{-1})\frac{(30-130)\mu g}{200\ m} - \frac{19.6\ \mu g\ m\ s^{-1}}{20\ m}$$

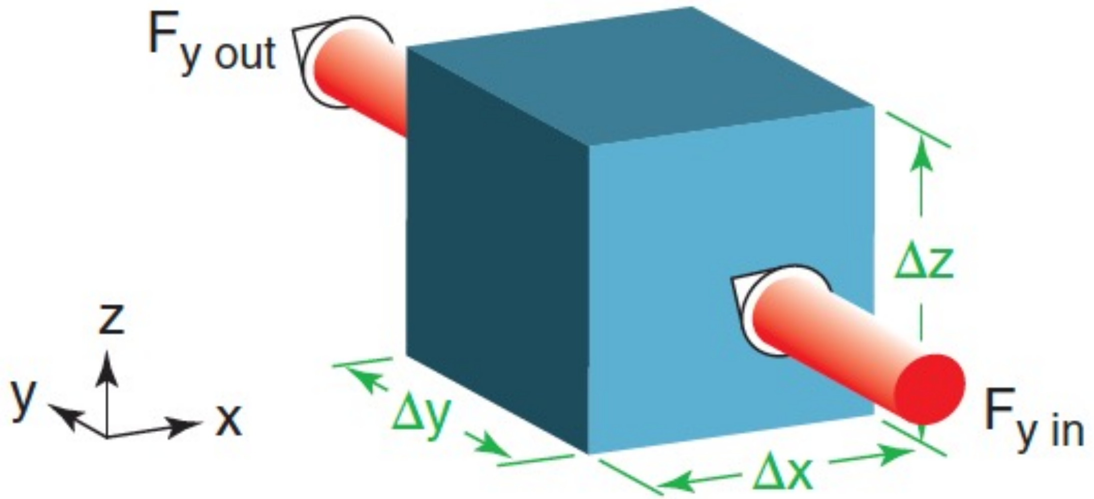
$$= \mathbf{0.02\ \mu g\ s^{-1}}$$

b) Calculate the change of particle matter in one hour.

$$\frac{\partial c}{\partial t} = \frac{c_t - c_0}{dt} = 0.02\ \mu g\ s^{-1} \qquad c_t = c_0 + (0.02\ \mu g\ s^{-1}) \cdot dt = \mathbf{102\ \mu g}$$

Example: Heat advection

The cube of air below has θ = 12°C along its south side, but smoothly increases in temperature to 15°C on the north side. This 100 km square cube is advecting toward the north at 25 km/hour. What warming rate at a fixed thermometer can be attributed to temperature advection?



$$\frac{\partial \theta}{\partial t} = \underbrace{-\left(u\cancel{\frac{\partial \theta}{\partial x}} + v\frac{\partial \theta}{\partial y} + w\cancel{\frac{\partial \theta}{\partial z}}\right)}_{\text{Advection}}$$

$$\underbrace{\cancel{-\frac{1}{\rho c_p}\left(\frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} + \frac{\partial Q_z^*}{\partial z}\right)}}_{\text{Radiation divergence}}$$

$$\underbrace{-\cancel{\frac{L_v E}{\rho c_p}}}_{\text{Latent heat release}}$$

$$\underbrace{\cancel{-\frac{\partial \overline{u'\theta'}}{\partial x} - \frac{\partial \overline{v'\theta'}}{\partial y} - \frac{\partial \overline{w'\theta'}}{\partial z}}}_{\text{Turbulent heat flux}}$$

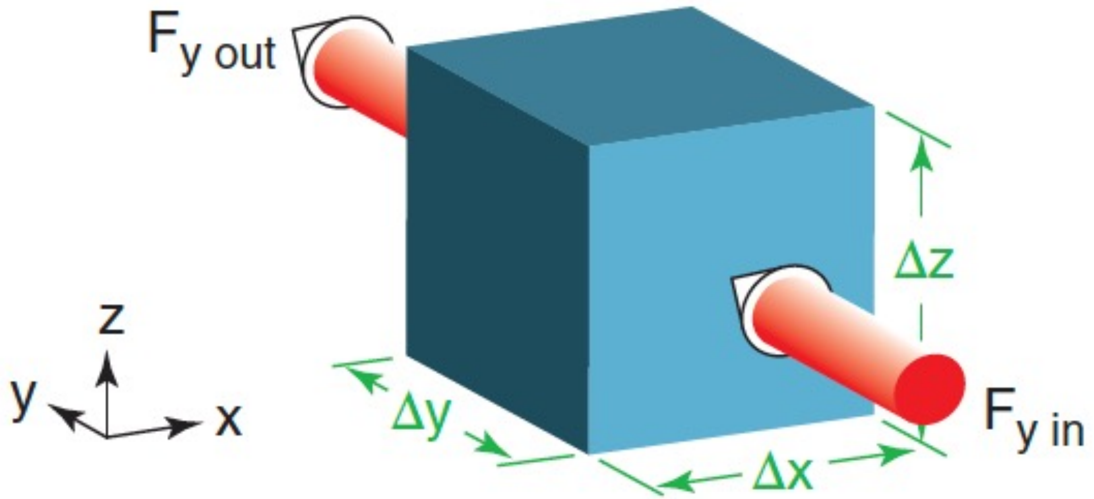
Example: Heat advection

The cube of air below has θ = 12°C along its south side, but smoothly increases in temperature to 15°C on the north side. This 100 km square cube is advecting toward the north at 25 km/hour. What warming rate at a fixed thermometer can be attributed to temperature advection?

$$\frac{\partial\theta}{\partial t} = -v\frac{\partial\theta}{\partial y}$$

$$= -(25\ km\ h^{-1})\cdot\frac{15ºC - 12ºC}{100\ km}$$

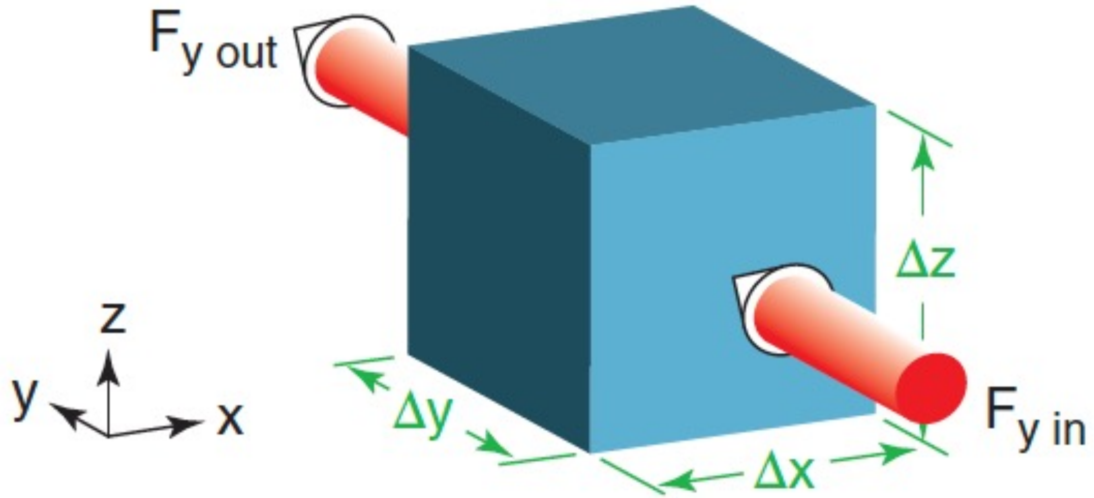
$$= \mathbf{-0.75\ ºC\ h^{-1}}$$



Example: Turbulent heat flux

In the figure below, suppose that the incoming heat flux from the south is 5 W m^{-2}, and the outgoing on the north face of the cube is 7 W m^{-2}.

(a) Convert these fluxes to kinematic units.
(b) What is the value of the kinematic flux gradient?
(c) Calculate the warming rate of air in the cube, assuming the cube has zero humidity and is at a fixed altitude where air density is 1 kg m^{-3}. The cube of air is 10 m on each side.



$$\frac{\partial \theta}{\partial t} = \underbrace{\cancel{-\left(u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z}\right)}}_{\text{Advection}}$$

$$\underbrace{\cancel{-\frac{1}{\rho c_p}\left(\frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} + \frac{\partial Q_z^*}{\partial z}\right)}}_{\text{Radiation divergence}}$$

$$\underbrace{\cancel{-\frac{L_v E}{\rho c_p}}}_{\text{Latent heat release}}$$

$$\underbrace{\cancel{-\frac{\partial \overline{u'\theta'}}{\partial x}} - \frac{\partial \overline{v'\theta'}}{\partial y} \cancel{-\frac{\partial \overline{w'\theta'}}{\partial z}}}_{\text{Turbulent heat flux}}$$

Example: Turbulent heat flux

In the figure below, suppose that the incoming heat flux from the south is 5 W m^{-2}, and the outgoing on the north face of the cube is 7 W m^{-2}.

(a) Convert these fluxes to kinematic units.
(b) What is the value of the kinematic flux gradient?
(c) Calculate the warming rate of air in the cube, assuming the cube has zero humidity and is at a fixed altitude where air density is 1 kg m^{-3}. The cube of air is 10 m on each side.

[K] [K m s^{-1}]

$$\frac{\partial \theta}{\partial t} = -\frac{\partial \overline{v'\theta'}}{\partial y}$$

[s] [m]

(a) Convert heat flux [W m^{-2}] to the kinetic heat flux [K m s^{-1}]

$$\overline{v'\theta'} = \frac{5\, W\, m^{-2}}{\rho \cdot c_p}$$

$$= \frac{5\, W\, m^{-2}}{1\, kg\, m^{-3} \cdot 1004\, J\, kg^{-1} K^{-1}}$$

$$\underline{= \mathbf{4.98 \cdot 10^{-3}\, K \cdot m \cdot s^{-1}}}$$

$$\overline{v'\theta'} = \frac{7\, W\, m^{-2}}{1\, kg\, m^{-3} \cdot 1004\, J\, kg^{-1} K^{-1}}$$

$$\underline{= \mathbf{6.97 \cdot 10^{-3}\, K \cdot m \cdot s^{-1}}}$$

Example: Turbulent heat flux

In the figure below, suppose that the incoming heat flux from the south is 5 W m^{-2}, and the outgoing on the north face of the cube is 7 W m^{-2}.

(a) Convert these fluxes to kinematic units.
(b) What is the value of the kinematic flux gradient?
(c) Calculate the warming rate of air in the cube, assuming the cube has zero humidity and is at a fixed altitude where air density is 1 kg m^{-3}. The cube of air is 10 m on each side.

[K] [K m s^{-1}]

$$\frac{\partial \theta}{\partial t} = -\frac{\partial \overline{v'\theta'}}{\partial y}$$

[s] [m]

(b) What is the kinematic flux gradient?

$$\frac{\partial \overline{v'\theta'}}{\partial y} = \frac{[6.97 \cdot 10^{-3} - 4.98 \cdot 10^{-3}]}{[10 - 0]}$$

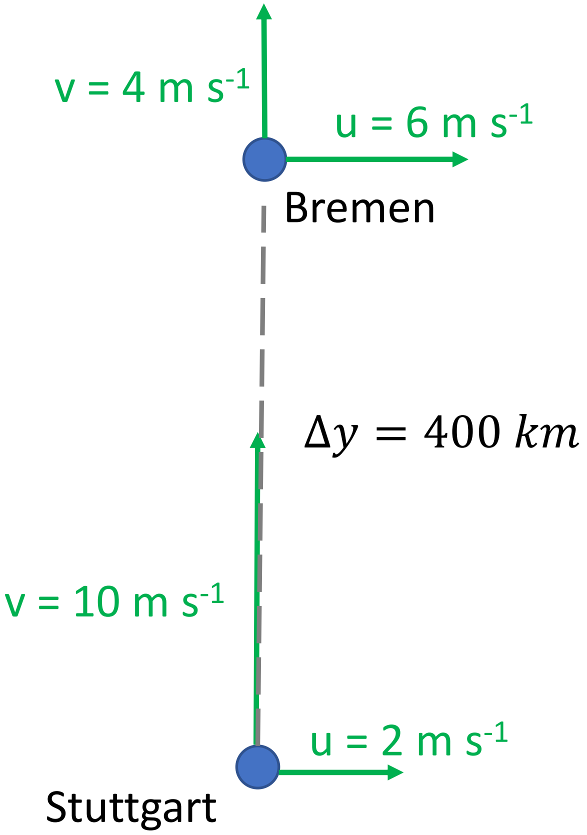
$$= \mathbf{-1.99 \cdot 10^{-4}\, \boldsymbol{K} \cdot \boldsymbol{s}^{-1}}$$

(c) Calculate the warming rate.

$$\frac{\partial \theta}{\partial t} = \frac{\partial \overline{v'\theta'}}{\partial y} = \mathbf{-1.99 \cdot 10^{-4}\, \boldsymbol{K} \cdot \boldsymbol{s}^{-1}}$$

Example: Advection of momentum

Bremen is about 400 km north of Stuttgart. In Bremen the wind components (U, V) are (6, 4) m s^{-1} , while in Stuttgart they are (2, 10) m s^{-1}. What is the value of the advective force per mass?



$$\frac{\partial u}{\partial t} = \underbrace{-\left(\cancel{u\frac{\partial u}{\partial x}} + v\frac{\partial u}{\partial y} + \cancel{w\frac{\partial u}{\partial z}}\right)}_{\text{Advection}}$$

$$\underbrace{\cancel{+g_x}}$$

Gravitational force

$$\underbrace{\cancel{-\frac{1}{\rho}\frac{\partial p}{\partial x}}}$$

Pressure force

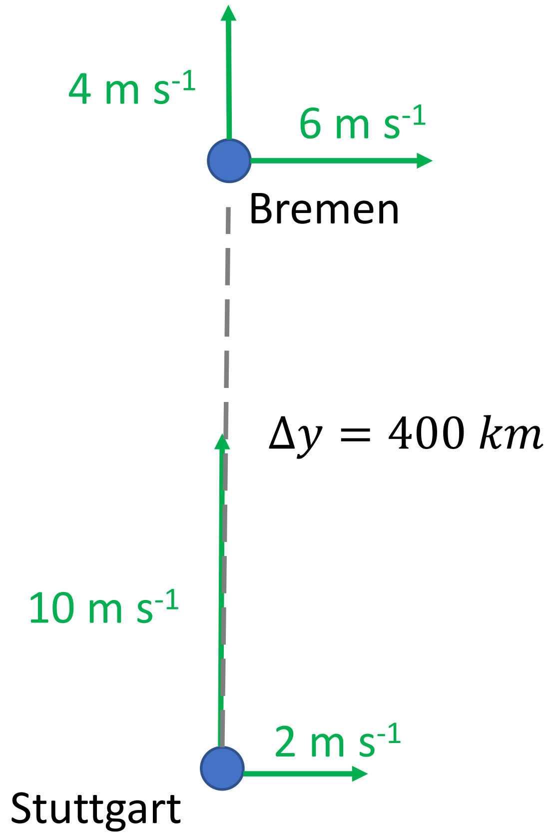
$$\underbrace{\cancel{-\frac{\partial \overline{u'u'}}{\partial x} - \frac{\partial \overline{u'v'}}{\partial y} - \frac{\partial \overline{u'w'}}{\partial z}}}$$

Turbulent flux divergence

Example: Advection of momentum

Bremen is about 400 km north of Stuttgart. In Bremen the wind components (U, V) are (6, 4) m s^{-1} , while in Stuttgart they are (2, 10) m s^{-1}. What is the value of the advective force per mass?

$$\frac{\partial u}{\partial t} = -v\frac{\partial u}{\partial y} \qquad \frac{\partial v}{\partial t} = -v\frac{\partial v}{\partial y}$$



Gradients:

$$\frac{\partial u}{\partial y} = \frac{(6 - 2\, m\, s^{-1})}{400{,}000\, m} = 1.0 \cdot 10^{-5}\, s^{-1}$$

$$\frac{\partial v}{\partial y} = \frac{(4 - 10\, m\, s^{-1})}{400{,}000\, m} = -1.5 \cdot 10^{-5}\, s^{-1}$$

Average wind speed:

$$u = \frac{(6 + 2\, m\, s^{-1})}{2} = 4\, m\, s^{-1} \qquad v = \frac{(4 + 10\, m\, s^{-1})}{2} = 7\, m\, s^{-1}$$

$$\frac{\partial u}{\partial t} = -(7\, m\, s^{-1}) \cdot (1.0 \cdot 10^{-5}\, s^{-1}) = -7 \cdot 10^{-5}\, m \cdot s^{-2}$$

$$\frac{\partial v}{\partial t} = -(7\, m\, s^{-1}) \cdot (-1.5 \cdot 10^{-5}\, s^{-1}) = 1.05 \cdot 10^{-4}\, m \cdot s^{-2}$$

Example: Advection of momentum

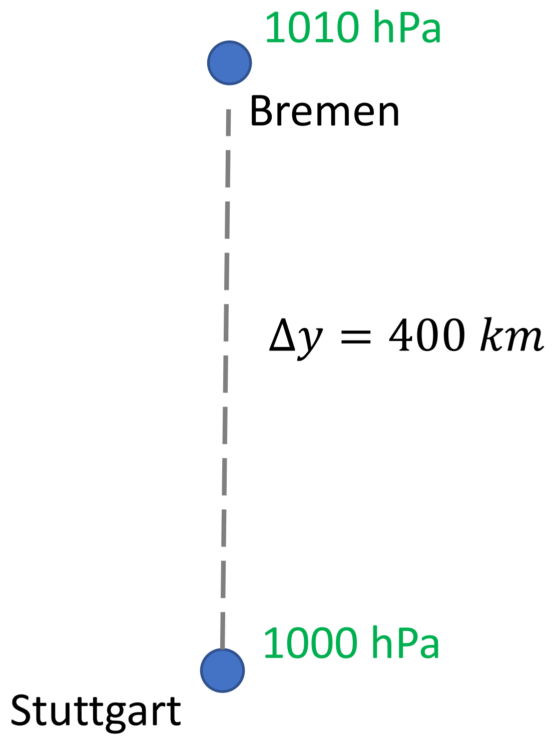
Bremen is about 400 km north of Stuttgart. In Bremen the pressure is 1010 hPa, while in Stuttgart the pressure is 1000 hPa. Find the pressure gradient force? (let $\rho = 1.1\ \mathrm{kg\,m^{-3}}$)

$$\frac{\partial v}{\partial t} = \underbrace{\cancel{-\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)}}_{\text{Advection}}$$

$$\underbrace{\cancel{-g_y}}_{\text{Gravitational force}}$$

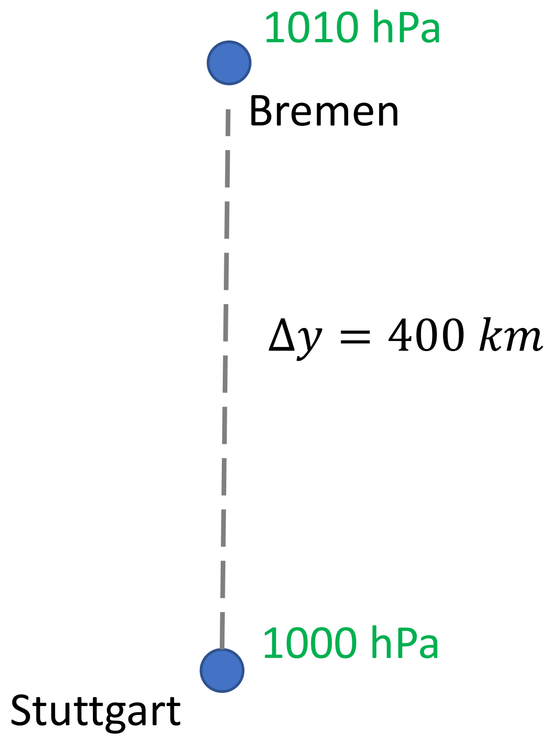
$$\underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial y}}_{\text{Pressure force}}$$

$$\underbrace{\cancel{-\frac{\partial \overline{v'u'}}{\partial x} - \frac{\partial \overline{v'v'}}{\partial y} - \frac{\partial \overline{v'w'}}{\partial z}}}_{\text{Turbulent flux divergence}}$$



Example: Advection of momentum

Bremen is about 400 km north of Stuttgart. In Bremen the pressure is 1010 hPa, while in Stuttgart the pressure is 1000 hPa. Find the pressure gradient force? (let $\rho = 1.1\ \mathrm{kg\ m^{-3}}$)



Solution:

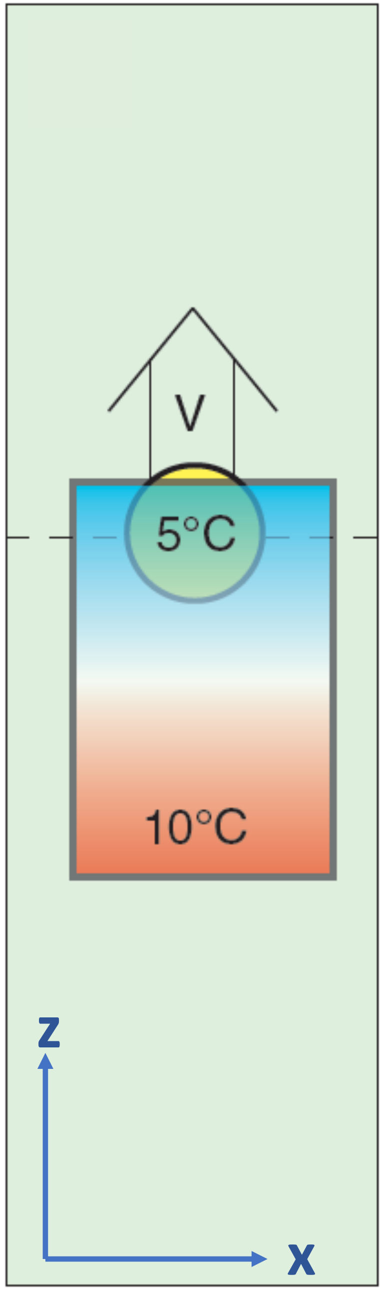
$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$= -\frac{1}{1.1\ kg\ m^{-3}}\frac{(101{,}000 - 100{,}000)Pa}{(400{,}000 - 0)\ m}$$

$$= \mathbf{-2.27 \cdot 10^{-3}\ m \cdot s^{-2}}$$

(hint: 1 Pa = 1 kg m^{-1} s^{-2})

Homework: Heat advection



Homework 1: Given the left figure, assume that higher in the figure corresponds to higher in the atmosphere. Suppose that the 5°C air is at a relative altitude that is 500 m higher than that of the 10°C air. If the updraft is 500 m/(10 hours), what is the temperature at the thermometer (yellow circle) after 10 hours?

Homework 2: The potential temperature of the air increases 5°C per 100 km distance east. If an east wind of 20 m s^{-1} is blowing, find the advective flux gradient, and the temperature change associated with this advection.

Homework 3: Suppose that the turbulent heat flux decreases linearly with height according to $\overline{w'\theta'} = a - b \cdot z$, where a = 0.3 (K ms^{-1}) and b = $3 \cdot 10^{-4}$ (K s^{-1}). If the initial temperature profile is an arbitrary shape, then what will be the shape of the final profile on hour later? Neglecting subsidence, radiation, latent heating, and assume horizontal heterogeneity.

Homework 4: If a horizontal wind of 10 m/s is advecting drier air into a region, where the horizontal moisture gradient is (5 g_{water}/kg_{air})/100 km, then what vertical gradient of turbulent moisture flux in the boundary layer is required to maintain a steady-state specific humidity? Assume there is no body source of moisture.

Homework: Heat conservation

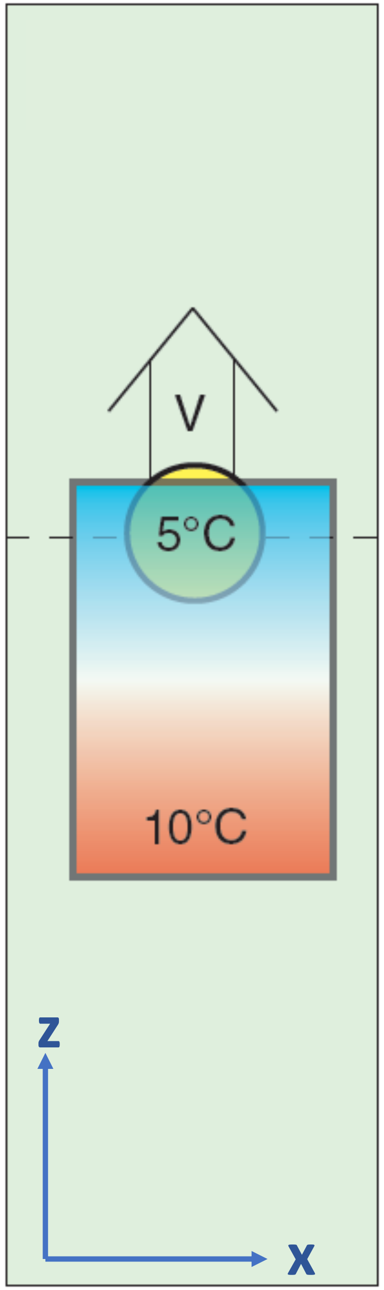
Homework 5: Find the rate of temperature change (°C h^{-1}) with no internal heat source, given the kinematic flux divergence values below. Assume Δx = Δy = Δz = 1 km.

	$\overline{u'\theta'}$ ($K{\cdot}m\ s^{-1}$)	$\overline{v'\theta'}$ ($K{\cdot}m\ s^{-1}$)	$\overline{w'\theta'}$ ($K{\cdot}m\ s^{-1}$)
a)	1	2	3
b)	1	2	-3
c)	1	-2	3
d)	1	-2	-3

Homework 6: Given the wind and temperature gradient, find the value of the kinematic advective flux gradient (°C h^{-1}).

	w ($m\ s^{-1}$)	$\partial\theta/\partial z$ (ºC·km^{-1})
a)	5	-2
b)	5	2
c)	10	-5
d)	10	-10

Homework: Heat advection



Homework 1: Given the left figure, assume that higher in the figure corresponds to higher in the atmosphere. Suppose that the 5°C air is at a relative altitude that is 500 m higher than that of the 10°C air. If the updraft is 500 m/(10 hours), what is the temperature at the thermometer (yellow circle) after 10 hours?

$$\frac{\partial \theta}{\partial t} = -\left(u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z}\right) - \frac{1}{\rho c_p}\left(\frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} + \frac{\partial Q_z^*}{\partial z}\right) - \frac{L_v E}{\rho c_p} - \frac{\partial \overline{u'\theta'}}{\partial x} - \frac{\partial \overline{v'\theta'}}{\partial y} - \frac{\partial \overline{w'\theta'}}{\partial z}$$

$$\frac{\partial \theta}{\partial t} = \frac{\theta_{t+1} - \theta_t}{dt} \Longrightarrow \theta_{n+1} = \theta_t - w\frac{\partial \theta}{\partial z}dt$$

$$= 5\,[ºC] - \;500\,[m\;10h^{-1}]\frac{5 - 10\,[ºC]}{500\,[m]} \cdot 10\,[h] = \mathbf{10}\,[º\boldsymbol{C}]$$

BUT: The air that is initially 10°C will adiabatically cool 9.8°C/km of rise. Here, it rises only 0.5 km in the 10 h, so it cools 9.8°C/2 = 4.9°C.

Its final temperature is 10 ºC - 4.9ºC = **5.1 ºC**

$$\Longrightarrow \frac{\partial \theta}{\partial t} = -w\left(\frac{\partial \theta}{\partial z} + \Gamma_d\right)$$

Homework: Heat advection

Homework 2: The potential temperature of the air increases 5°C per 100 km distance east. If an east wind of 20 m s^{-1} is blowing, find the advective flux gradient, and the temperature change associated with this advection.

$$\frac{\partial \theta}{\partial t} = -\left(u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z}\right) - \frac{1}{\rho c_p}\left(\frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} + \frac{\partial Q_z^*}{\partial z}\right) - \frac{L_v E}{\rho c_p} - \frac{\partial \overline{u'\theta'}}{\partial x} - \frac{\partial \overline{v'\theta'}}{\partial y} - \frac{\partial \overline{w'\theta'}}{\partial z}$$

$$\frac{\partial \theta}{\partial t} = -v\frac{\partial \theta}{\partial y}$$

$$= -(-20\ [m\ s^{-1}])\frac{5[ºC]}{100000\ [m]} = -20\ [m\ s^{-1}] \cdot \mathbf{0.00005}\ [\boldsymbol{ºC\ m^{-1}}]$$

$$= \underline{\mathbf{+0.001}\ [\boldsymbol{ºC\ s^{-1}}]}$$

Homework: Heat advection

Homework 3: Suppose that the turbulent heat flux decreases linearly with height according to $\overline{w'\theta'} = a - b \cdot z$, where a = 0.3 (K ms^{-1}) and b = $3 \cdot 10^{-4}$ (K s^{-1}). If the initial temperature profile is an arbitrary shape, then what will be the shape of the final profile on hour later? Neglecting subsidence, radiation, latent heating, and assume horizontal heterogeneity.

$$\frac{\partial \theta}{\partial t} = -\left(u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z}\right) - \frac{1}{\rho c_p}\left(\frac{\partial Q_x^*}{\partial x} + \frac{\partial Q_y^*}{\partial y} + \frac{\partial Q_z^*}{\partial z}\right) - \frac{L_v E}{\rho c_p} - \frac{\partial \overline{u'\theta'}}{\partial x} - \frac{\partial \overline{v'\theta'}}{\partial y} - \frac{\partial \overline{w'\theta'}}{\partial z}$$

$$\frac{\partial \theta}{\partial t} = -\frac{\partial \overline{w'\theta'}}{\partial z}$$

Pluggin in the expression for $\overline{w'\theta'}$ gives: $$\frac{\partial \theta}{\partial t} = -\frac{\partial \overline{w'\theta'}}{\partial z} = -\frac{\partial (a - b \cdot z)}{\partial z} = +\boldsymbol{b}$$

<u>Answer</u>: The answer is not a function of z; hence, air at each height in the sounding warms at the same rate.

Integrating over time from $t = t_0$ to t gives: $\theta_t = \theta_0 + b(t - t_0)$

$$b(t - t_0) = 3 \cdot 10^{-4}\ [K\ s^{-1}] \cdot 3600\ [s] = \mathbf{1.08}\ \boldsymbol{K}$$

Homework: Heat advection

Homework 4: If a horizontal wind of 10 m/s is advecting drier air into a region, where the horizontal moisture gradient is (5 g_{water}/kg_{air})/100 km, then what vertical gradient of turbulent moisture flux in the boundary layer is required to maintain a steady-state specific humidity? Assume there is no body source of moisture.

$$\frac{\partial q}{\partial t} = \quad -\left(u\frac{\partial q}{\partial x} + v\frac{\partial q}{\partial y} + w\frac{\partial q}{\partial z}\right) \quad +S_q \quad -\frac{\partial \overline{u'q'}}{\partial x} - \frac{\partial \overline{v'q'}}{\partial y} - \frac{\partial \overline{w'q'}}{\partial z}$$

Answer: Stready-state is defined as one where there are no local changes of a variable with time (i.e. $\frac{\partial(\cdot)}{\partial t} = 0$). Choose the x-axis to be aligned with the mean wind direction for simplicity gives

$$u\frac{\partial q}{\partial x} = -\frac{\partial \overline{w'q'}}{\partial z}$$

$$10\ [m\ s^{-1}] \cdot 5 \cdot 10^{-5}\ [g_{water}\ kg_{air}^{-1}\ m^{-1}] = -\frac{\partial \overline{w'q'}}{\partial z}$$

$$\boldsymbol{-\frac{\partial \overline{w'q'}}{\partial z} = -5 \cdot 10^{-4} \left[\frac{g_{water}}{kg_{air} \cdot s}\right]}$$