Mathematical Logic

INTRODUCTION

Proposition: A proposition or statement is a declarative sentence which is either true or false but not both. The truth or falsity of a proposition is called its truth-value.

These two values ‗true‘ and ‗false‘ are denoted by the symbols T and F respectively. Sometimes these are also denoted by the symbols 1 and 0 respectively.

Example 1: Consider the following sentences:

1. Delhi is the capital of India.

2. Kolkata is a country.

3. 5 is a prime number.

4. 2 + 3 = 4.

These are propositions (or statements) because they are either true of false.

Next consider the following sentences:

5. How beautiful are you?

6. Wish you a happy new year

7. x + y = z

8. Take one book.

These are not propositions as they are not declarative in nature, that is, they do not declare a definite truth value T or F.

Propositional Calculus is also known as **statement calculus**. It is the branch of mathematics that is used to describe a logical system or structure.

A logical system consists of (1) a universe of propositions, (2) truth tables (as axioms) for the logical operators and (3) definitions that explain equivalence and implication of propositions.

Connectives

The words or phrases or symbols which are used to make a proposition by two or more propositions are called **logical connectives or simply connectives**. There are five basic connectives called negation, conjunction, disjunction, conditional and biconditional.

Negation

The **negation** of a statement is generally formed by writing the word 'not' at a proper place in the statement (proposition) or by prefixing the statement with the phrase

'It is not the case that'. If p denotes a statement then the negation of p is written as p and read as 'not p'. If the truth value of p is T then the truth value of p is F. Also if the truth value of p is F then the truth value of p is T.

Table 1. Truth table for negation

p	¬p
T	F
F	T

Example 2: Consider the statement p: Kolkata is a city. Then ¬p: Kolkata is not a city. Although the two statements 'Kolkata is not a city' and 'It is not the case that Kolkata is a city' are not identical, we have translated both of them by p. The reason is that both these statements have the same meaning.

Conjunction

The **conjunction** of two statements (or propositions) p and q is the statement $p \wedge q$ which is read as 'p and q'. The statement $p \wedge q$ has the truth value T whenever both p and q have the truth value T. Otherwise it has truth value F.

Table 2. Truth table for conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 3: Consider the following statements

p : It is raining today.

q : There are 10 chairs in the room.

Then $p \wedge q$: It is raining today and there are 10 chairs in the room.

Note: Usually, in our everyday language the conjunction 'and' is used between two statements which have some kind of relation. Thus a statement 'It is raining today and 1 + 1 = 2' sounds odd, but in logic it is a perfectly acceptable statement formed from the statements 'It is raining today' and '1 + 1 = 2'.

Example 4: Translate the following statement:
'Jack and Jill went up the hill' into symbolic form using conjunction.

Solution: Let p : Jack went up the hill,   q : Jill went up the hill.

Then the given statement can be written in symbolic form as $p \wedge q$.

Disjunction

The **disjunction** of two statements p and q is the statement $p \vee q$ which is read as 'p or q'. The statement $p \vee q$ has the truth value F only when both p and q have the truth value F. Otherwise it has truth value T.

Table 3: Truth table for disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 5: Consider the following statements

p : I shall go to the game.

q : I shall watch the game on television.

Then $p \vee q$: I shall go to the game or watch the game on television.

Conditional proposition

If p and q are any two statements (or propositions) then the statement

$p \to q$ which is read as,

'If p, then q' is called a **conditional statement** (or **proposition**) or **implication** and the connective is the **conditional connective.**

The conditional is defined by the following table:

Table 4. Truth table for conditional

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

In this conditional statement, p is called the **hypothesis** or **premise** or **antecedent** and q is called the **consequence** or **conclusion.**

To understand better, this connective can be looked as a conditional promise. If the promise is violated (broken), the conditional (implication) is false. Otherwise it is true. For this reason, the only circumstances under which the conditional $p \to q$ is false is when p is true and q is false.

Example 6: *Translate the following statement:*

*'**The crop will be destroyed if there is a flood**' into symbolic form using conditional connective.*

Solution: Let c : the crop will be destroyed;

f : there is a flood.

Let us rewrite the given statement as

'**If there is a flood, then the crop will be destroyed'. So, the symbolic form of the given statement is $f \to c$**.

Example 7: Let p and q denote the statements:

p : You drive over 70 km per hour.

q : You get a speeding ticket.

Write the following statements into symbolic forms.

(i) You will get a speeding ticket if you drive over 70 km per hour.

(ii) Driving over 70 km per hour is sufficient for getting a speeding ticket.

(iii) If you do not drive over 70 km per hour then you will not get a speeding ticket.

(iv) Whenever you get a speeding ticket, you drive over 70 km per hour.

Solution: (i) p → q (ii) p → q (iii) p → q (iv) q → p.

Notes: 1. In ordinary language, it is customary to assume some kind of relationship between the antecedent and the consequent in using the conditional. But in logic, the antecedent and the consequent in a conditional statement are not required to refer to the same subject matter. For example, the statement 'If I get sufficient money then I shall purchase a high-speed computer' sounds reasonable.

On the other hand,a statement such as 'If I purchase a computer then this pen is red' does not make sense in our conventional language.

But according to the definition of conditional, this proposition is perfectly acceptable and has a truth-value which depends on the truth-values of the component statements.

2. Some of the **alternative** terminologies used to express $p \to q$ (if p, then q) are the following:

(*i*) *p* implies *q*
(ii) p **only if *q* ('If *p*, then *q'* formulation emphasizes the antecedent, whereas '*p* only if *q'* formulation emphasizes the consequent. The difference is only stylistic.)**
(iii) q **if *p*, or *q* when *p*.**
(iv) q **follows from *p*, or *q* whenever *p*.**
(v) p **is sufficient for *q*, or a sufficient condition for *q* is *p*.**
(vi) *q* is necessary for *p*, or a necessary condition for *p* is *q*.
(*vii*) *q* is consequence of *p*.

Converse, Inverse and Contrapositive

If $P \to Q$ is a conditional statement, then

(1). $Q \to P$ is called its *converse*

(2). $\neg P \to \neg Q$ is called its *inverse*
(3). $\neg Q \to \neg P$ is called its *contrapositive*. Truth table for $Q \to P$ (converse of $P \to Q$)

P	Q	Q → *P*
T	T	T
T	F	T
F	T	F
F	F	T

Truth table for $\neg P \to \neg Q$ (inverse of $P \to Q$)

P	Q	$\neg P$	$\neg Q$	$\neg P \to \neg Q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	F	T

Truth table for $\neg Q \to \neg P$ (contrapositive of $P \to Q$)

P	Q	$\neg Q$	$\neg P$	$\neg Q \to \neg P$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	F	T

Example: Consider the statement

P : It rains.

Q: The crop will grow. The implication $P \to Q$ states that

R: If it rains then the crop will grow.

The **converse** of the implication $P \to Q$, namely $Q \to P$ states that

S: If the crop will grow then there has been rain.

The **inverse** of the implication $P \to Q$, namely $\neg P \to \neg Q$ states that

U: If it does not rain then the crop will not grow.

The **contraposition** of the implication $P \to Q$, namely $\neg Q \to \neg P$ states that

T : If the crop do not grow then there has been no rain.

Example 9: ***Construct the truth table for (p → q) ∧ (q →p)***

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	*T*	*T*	*T*	*T*
T	*F*	*F*	*T*	*F*
F	*T*	*T*	*F*	*F*
F	*F*	*T*	*T*	*T*

Biconditional proposition

If *p* and *q* are any two statements (propositions), then the statement $p \leftrightarrow q$ which is read as **'*p* if and only if *q*'** and abbreviated as **'*p* iff *q*'** is called a **biconditional statement** and the connective is the **biconditional connective.**

The truth table of $p \leftrightarrow q$ is given by the following table:

Table 6. Truth table for biconditional

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

It may be noted that $p \leftrightarrow q$ is true only when both p and q are true or when both p and q are false. Observe that p q is true when both the conditionals $p \to q$ and $q \to p$ are true, *i.e.*, the truth- values of $(p \to q) \wedge (q \to p)$, given in Ex. 9,

TAUTOLOGY AND CONTRADICTION

Tautology: A statement formula which is true regardless of the truth values of the statements which replace the variables in it is called a **universally valid formula** or a **logical truth** or a **tautology.**

Ex: $P \vee \neg P$

Contradiction: A statement formula which is false regardless of the truth values of the statements which replace the variables in it is said to be a **contradiction.**

Ex: $P \wedge \neg P$

Contingency: A statement formula which is neither a tautology nor a contradiction is known as a **contingency.**

Substitution Instance

A formula A is called a substitution instance of another formula B if A can be obtained form B by substituting formulas for some variables of B, with the condition that the same formula is substituted for the same variable each time it occurs.

Example: Let $B : P \to (J \wedge P)$.

Substitute $R \leftrightarrow S$ for P in B, we get

(i): $(R \leftrightarrow S) \to (J \wedge (R \leftrightarrow S))$

Then A is a substitution instance of B.

Note that $(R \leftrightarrow S) \to (J \wedge P)$ is not a substitution instance of B because the variables

P in $J \wedge P$ was not replaced by $R \leftrightarrow S$.

Equivalence of Formulas

Two formulas A and B are said to equivalent to each other if and only if $A \leftrightarrow B$ is a tautology.

If $A \leftrightarrow B$ is a tautology, we write $A \Leftrightarrow B$ which is read as A is equivalent to B.

Note : 1. $\Leftrightarrow$ is only symbol, but not connective.

2. $A \leftrightarrow B$ is a tautology if and only if truth tables of A and B are the same.
3. Equivalence relation is symmetric and transitive.

Method I. Truth Table Method: One method to determine whether any two statement formulas are equivalent is to construct their truth tables.

Example: Prove $P \vee Q \Leftrightarrow \neg(\neg P \wedge \neg Q)$.

Solution:

P	Q	$P \vee Q$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$\neg(\neg P \wedge \neg Q)$	$(P \vee Q) \leftrightarrow \neg(\neg P \wedge \neg Q)$
T	T	T	F	F	F	T	T
T	F	T	F	T	F	T	T
F	T	T	T	F	F	T	T
F	F	F	T	T	T	F	T

As $(P \vee Q) \leftrightarrow \neg(\neg P \wedge \neg Q)$ is a tautology, then $P \vee Q \Leftrightarrow \neg(\neg P \wedge \neg Q)$.

Example: Prove $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$.

Solution:

P	Q	$P \rightarrow Q$	$\neg P$	$\neg PVQ$	$(P \rightarrow Q) \leftrightarrow (\neg PVQ)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

As $(P \rightarrow Q)$ $(\neg P \vee Q)$ is a tautology then $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$.

Equivalence Formulas:

1. Idempotent laws:

(a) $P \vee P \Leftrightarrow P$ (b) $P \wedge P \Leftrightarrow P$

2. Associative laws:

(a) $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ (b) $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$

3. Commutative laws:

(a) $P \vee Q \Leftrightarrow Q \vee P$ (b) $P \wedge Q \Leftrightarrow Q \wedge P$

4. Distributive laws:

$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

5. Identity laws:

(a) (i) $P \vee F \Leftrightarrow P$ (ii) $P \vee T \Leftrightarrow T$

(b) (i) $P \wedge T \Leftrightarrow P$ (ii) $P \wedge F \Leftrightarrow F$

6. Component laws:

(a) (i) $P \vee \neg P \Leftrightarrow T$ (ii) $P \wedge \neg P \Leftrightarrow F$

(b) (i) $\neg\neg P \Leftrightarrow P$ (ii) $\neg T \Leftrightarrow F$, $\neg F \Leftrightarrow T$

7. Absorption laws:

(a) $P \vee (P \wedge Q) \Leftrightarrow P$ (b) $P \wedge (P \vee Q) \Leftrightarrow P$

8. Demorgan's laws:

(a) $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ (b) $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

Method II. Replacement Process: Consider a formula A : $P \to (Q \to R)$. The formula $Q \to R$ is a part of the formula A. If we replace $Q \to R$ by an equivalent formula $\neg Q \vee R$ in A,

we get another formula B : $P \to (\neg Q \vee R)$. One can easily verify that the formulas A and B are equivalent to each other. This process of obtaining B from A as the replacement process.

Example: Prove that $P \to (Q \to R) \Leftrightarrow P \to (\neg Q \vee R) \Leftrightarrow (P \wedge Q) \to R$.

Solution: $P \to (Q \to R) \Leftrightarrow P \to (\neg Q \vee R)$ [$\because Q \to R \Leftrightarrow \neg Q \vee R$]

$\Leftrightarrow \neg P \vee (\neg Q \vee R)$ [$\because P \to Q \Leftrightarrow \neg P \vee Q$]

$\Leftrightarrow (\neg P \vee \neg Q) \vee R$ [by Associative laws]

$\Leftrightarrow \neg(P \wedge Q) \vee R$ [by De Morgan's laws]

$\Leftrightarrow (P \wedge Q) \to R$ [$\because P \to Q \Leftrightarrow \neg P \vee Q$].

Example: Prove that $(P \to Q) \wedge (R \to Q) \Leftrightarrow (P \vee R) \to Q$.

Solution: $(P \to Q) \wedge (R \to Q) \Leftrightarrow (\neg P \vee Q) \wedge (\neg R \vee Q)$

$\Leftrightarrow (\neg P \wedge \neg R) \vee Q$

$\Leftrightarrow \neg(P \vee R) \vee Q$

$\Leftrightarrow P \vee R \to Q$

Example: Prove that $P \to (Q \to P) \Leftrightarrow \neg P \to (P \to Q)$.

Solution: $P \to (Q \to P) \Leftrightarrow \neg P \vee (Q \to P)$

$\Leftrightarrow \neg P \vee (\neg Q \vee P)$

$\Leftrightarrow (\neg P \vee P) \vee \neg Q$

$\Leftrightarrow T \vee \neg Q$

$\Leftrightarrow \boldsymbol{T}$

and $\neg P \to (P \to Q) \Leftrightarrow \neg(\neg P) \vee (P \to Q)$

$\Leftrightarrow P \vee (\neg P \vee Q)$

$\Leftrightarrow (P \vee \neg P) \vee Q$

$\Leftrightarrow T \vee Q$

$\Leftrightarrow \boldsymbol{T}$

So, $P \to (Q \to P) \Leftrightarrow \neg P \to (P \to Q)$.

Example: Prove that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$.

Solution:

$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$

$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R)$ [Associative and Distributive laws]

$\Leftrightarrow (\neg(P \vee Q) \wedge R) \vee ((Q \vee P) \wedge R)$ [De Morgan‘s laws]

$\Leftrightarrow (\neg(P \vee Q) \vee (P \vee Q)) \wedge R$ [Distributive laws]

$\Leftrightarrow T \wedge R$ [$\because \neg P \vee P \Leftrightarrow T$]

$\Leftrightarrow R$

Example: Show $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is tautology.

Solution: By De Morgan‘s laws, we have

$\neg P \wedge \neg Q \Leftrightarrow \neg(P \vee Q)$

$\neg P \vee \neg R \Leftrightarrow \neg(P \wedge R)$

Therefore

Also

$(\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R) \Leftrightarrow \neg(P \vee Q) \vee \neg(P \wedge R)$

$\Leftrightarrow \neg((P \vee Q) \wedge (P \vee R))$

$\neg(\neg P \wedge (\neg Q \vee \neg R)) \Leftrightarrow \neg(\neg P \wedge \neg(Q \wedge R))$

$\Leftrightarrow P \vee (Q \wedge R)$

$$\Leftrightarrow (P \vee Q) \wedge (P \vee R)$$

Hence $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \Leftrightarrow (P \vee Q) \wedge (P \vee Q) \wedge (P \vee R)$

$$\Leftrightarrow (P \vee Q) \wedge (P \vee R)$$

Thus $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$

$$\Leftrightarrow [(P \vee Q) \wedge (P \vee R)] \vee \neg[(P \vee Q) \wedge (P \vee R)]$$

$$\Leftrightarrow T$$

Hence the given formula is a tautology.

Exercise: Show that $(P \wedge Q) \rightarrow (P \vee Q)$ is a tautology.

Example: Write the negation of the following statements.

(a). Jan will take a job in industry or go to graduate school.

(b). James will bicycle or run tomorrow.

(c).If the processor is fast then the printer is slow.

Solution: **(a)**. Let P : Jan will take a job in industry.

Q : Jan will go to graduate school.

The given statement can be written in the symbolic

as $P \vee Q$. The negation of $P \vee Q$ is given by $\neg(P \vee Q)$.

$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q.$$

$\neg P \wedge \neg Q$: Jan will not take a job in industry and he will not go to graduate school.

(b). Let P : James will bicycle.

Q: James will run tomorrow.

The given statement can be written in the symbolic

as $P \vee Q$. The negation of $P \vee Q$ is given by $\neg(P \vee Q)$.

$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q.$$

$\neg P \wedge \neg Q$: James will not bicycle and he will not run tomorrow.

(c). Let P : The processor is fast.

Q: The printer is slow.

The given statement can be written in the symbolic as $P \rightarrow Q$.

The negation of $P \rightarrow Q$ is given by $\neg(P \rightarrow Q)$.

$$\neg(P \rightarrow Q) \Leftrightarrow \neg(\neg P \vee Q) \Leftrightarrow P \wedge \neg Q.$$

$P \wedge \neg Q$: The processor is fast and the printer is fast.

Example: Use Demorgans laws to write the negation of each statement.

(a). I want a car and worth a cycle.
(b). My cat stays outside or it makes a mess.

(c). I've fallen and I can't get up.

(d). You study or you don't get a good grade.

Solution: (a). I don't want a car or not worth a cycle.

(b). My cat not stays outside and it does not make a mess.
(c). I have not fallen or I can get up.
(d). You can not study and you get a good grade.

Exercises: 1. Write the negation of the following statements.
(a). **If it is raining, then the game is canceled.**
(b). ***If he studies then he will pass the examination.***

Duality Law

Two formulas A and A^* are said to be *duals* of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. The connectives $\vee$ and $\wedge$ are called *duals* of each other. If the formula A contains the special variable T or F, then A^*, its dual is obtained by replacing T by F and F by T in addition to the above mentioned interchanges.

Example: Write the dual of the following formulas:

(i). $(P \vee Q) \wedge R$ (ii). $(P \wedge Q) \vee T$ (iii). $(P \wedge Q) \vee (P \vee \neg(Q \wedge \neg S))$

Solution: The duals of the formulas may be written as

(i). $(P \wedge Q) \vee R$ (ii). $(P \vee Q) \wedge F$ (iii). $(P \vee Q) \wedge (P \wedge \neg(Q \vee \neg S))$

Result 1: The negation of the formula is equivalent to its dual in which every variable is replaced by its negation.
We can prove

$$\neg A(P_1, P_2, ..., P_n) \Leftrightarrow A^*(\neg P_1, \neg P_2, ..., \neg P_n)$$

Example: Prove that (a). $\neg(P \wedge Q) \rightarrow (\neg P \vee (\neg P \vee Q)) \quad \Leftrightarrow (\neg P \vee Q)$

(b). $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P \wedge Q)$

Solution: (a).$\neg(P \wedge Q) \rightarrow (\neg P \vee (\neg P \vee Q))$

$$\Leftrightarrow (P \wedge Q) \vee (\neg P \vee (\neg P \vee Q)) \; [\because P \rightarrow Q \Leftrightarrow \neg P \vee Q]$$
$$\Leftrightarrow (P \wedge Q) \vee (\neg P \vee Q)$$
$$\Leftrightarrow (P \wedge Q) \vee \neg P \vee Q$$
$$\Leftrightarrow ((P \wedge Q) \vee \neg P)) \vee Q$$
$$\Leftrightarrow ((P \vee \neg P) \wedge (Q \vee \neg P)) \vee Q$$
$$\Leftrightarrow (T \wedge (Q \vee \neg P)) \vee Q$$
$$\Leftrightarrow (Q \vee \neg P) \vee Q$$
$$\Leftrightarrow Q \vee \neg P$$
$$\Leftrightarrow \neg P \vee Q$$

(b). From (a) Writing the dual $\quad (P \wedge Q) \vee (\neg P \vee (\neg P \vee Q)) \Leftrightarrow \neg P \vee Q$

Writing the dual

$$(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P \wedge Q)$$

Tautological Implications

A statement formula A is said to *tautologically imply* a statement B if and only if $A \rightarrow B$ is a tautology.

In this case we write $A \Rightarrow B$, which is read as 'A implies B'.

Note: $\Rightarrow$ is not a connective, $A \Rightarrow B$ is not a statement formula.

$A \Rightarrow B$ states that $A \rightarrow B$ is tautology.

Clearly $A \Rightarrow B$ guarantees that B has a truth value T whenever A has the truth value T .

One can determine whether $A \Rightarrow B$ by constructing the truth tables of A and B in the same manner as was done in the determination of $A \Leftrightarrow B$.

Example: Prove that $(P \rightarrow Q) \Rightarrow (\neg Q \rightarrow \neg P)$.

P	Q	$\neg P$	$\neg Q$	$P \rightarrow Q$	$\neg Q \rightarrow \neg P$	$(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Since,all the entries in the last column are true, $(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$ is a tautology.

Hence $(P \to Q) \Rightarrow (\neg Q \to \neg P)$.

In order to show any of the given implications, it is sufficient to show that an assignment of the truth value T to the antecedent of the corresponding conditional leads to the truth value T for the consequent. This procedure guarantees that the conditional becomes tautology, thereby proving the implication.

Example: Prove that $\neg Q \wedge (P \to Q) \Rightarrow \neg P$.

Solution: Assume that the antecedent $\neg Q \wedge (P \to Q)$ has the truth value T , then both $\neg Q$ and $P \to Q$ have the truth value T , which means that Q has the truth value F , $P \to Q$ has the truth value T . Hence P must have the truth value F .

Therefore the consequent $\neg P$ must have the truth value T.

$$\neg Q \wedge (P \to Q) \Rightarrow \neg P .$$

Another method to show $A \Rightarrow B$ is to assume that the consequent B has the truth value F and then show that this assumption leads to A having the truth value F . Then $A \to B$ must have the truth value T .

Example: Show that $\neg(P \to Q) \Rightarrow P$.

Solution: Assume that P has the truth value F . When P has F , $P \to Q$ has T , then $\neg(P \to Q)$ has F

. Hence $\neg(P \to Q) \to P$ has T .

$$\neg(P \to Q) \Rightarrow P$$

Other Connectives

We introduce the connectives NAND, NOR which have useful applications in the design of computers.

NAND: The word NAND is a combination of 'NOT' and 'AND' where 'NOT' stands for negation and 'AND' for the conjunction. It is denoted by the symbol $\uparrow$.

If P and Q are two formulas then

$$P \uparrow Q \Leftrightarrow \neg(P \wedge Q)$$

The connective $\uparrow$ has the following equivalence:

$$P \uparrow P \Leftrightarrow \neg(P \wedge P) \Leftrightarrow \neg P \vee \neg P \Leftrightarrow \neg P .$$

$$(P \uparrow Q) \uparrow (P \uparrow Q) \Leftrightarrow \neg(P \uparrow Q) \Leftrightarrow \neg(\neg(P \wedge Q)) \Leftrightarrow P \wedge Q.$$

$$(P \uparrow P) \uparrow (Q \uparrow Q) \Leftrightarrow \neg P \uparrow \neg Q \Leftrightarrow \neg(\neg P \wedge \neg Q) \Leftrightarrow P \vee Q.$$

NAND is Commutative: Let P and Q be any two statement formulas.

$$(P \uparrow Q) \Leftrightarrow \neg(P \wedge Q)$$

$$\Leftrightarrow \neg(Q \wedge P)$$

$$\Leftrightarrow (Q \uparrow P)$$

$\therefore$ NAND is commutative.

NAND is not Associative: Let P , Q and R be any three statement formulas.

Consider $P \uparrow (Q \uparrow R) \Leftrightarrow \neg(P \wedge (Q \uparrow R)) \Leftrightarrow \neg(P \wedge (\neg(Q \wedge R)))$

$\Leftrightarrow \neg P \vee (Q \wedge R))$ $(P \uparrow Q) \uparrow R$

$\Leftrightarrow \neg(P \wedge Q) \uparrow R$

$\Leftrightarrow \neg(\neg(P \wedge Q) \wedge R)$

$\Leftrightarrow (P \wedge Q) \vee \neg R$

Therefore the connective $\uparrow$ is not associative.

NOR: The word NOR is a combination of 'NOT' and 'OR' where 'NOT' stands for negation and 'OR' for the disjunction. It is denoted by the symbol $\downarrow$.

If P and Q are two formulas then

$P \downarrow Q \Leftrightarrow \neg(P \vee Q)$

The connective $\downarrow$ has the following equivalence:

$P \downarrow P \Leftrightarrow \neg(P \vee P) \Leftrightarrow \neg P \wedge \neg P \Leftrightarrow \neg P$.

$(P \downarrow Q) \downarrow (P \downarrow Q) \Leftrightarrow \neg(P \downarrow Q) \Leftrightarrow \neg(\neg(P \vee Q)) \Leftrightarrow P \vee Q$.

$(P \downarrow P) \downarrow (Q \downarrow Q) \Leftrightarrow \neg P \downarrow \neg Q \Leftrightarrow \neg(\neg P \vee \neg Q) \Leftrightarrow P \wedge Q$.

NOR is Commutative: Let P and Q be any two statement formulas.

$(P \downarrow Q) \Leftrightarrow \neg(P \vee Q)$

$\Leftrightarrow \neg(Q \vee P) \Leftrightarrow (Q \downarrow P)$

$\therefore$ NOR is commutative.

NOR is not Associative: Let P , Q and R be any three statement formulas. Consider

$P \downarrow (Q \downarrow R) \Leftrightarrow \neg(P \vee (Q \downarrow R))$

$\Leftrightarrow \neg(P \vee (\neg(Q \vee R)))$

$\Leftrightarrow \neg P \wedge (Q \vee R)$ $(P \downarrow Q) \downarrow R$

$\Leftrightarrow \neg(P \vee Q) \downarrow R$

$\Leftrightarrow \neg(\neg(P \vee Q) \vee R)$

$\Leftrightarrow (P \vee Q) \wedge \neg R$

Therefore the connective $\downarrow$ is not associative.

Evidently, $P \uparrow Q$ and $P \downarrow Q$ are duals of each other.

Since

$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$.

Example: Express $P \downarrow Q$ interms of $\uparrow$ only.

Solution:

$$P \downarrow Q \Leftrightarrow \neg(P \vee Q)$$

$$\Leftrightarrow (P \vee Q) \uparrow (P \vee Q)$$

$$\Leftrightarrow [(P \uparrow P) \uparrow (Q \uparrow Q)] \uparrow [(P \uparrow P) \uparrow (Q \uparrow Q)]$$

Example: Express $P \uparrow Q$ interms of $\downarrow$ only.

Solution:

$$\mathrm{P} \uparrow Q \Leftrightarrow \neg(P \wedge Q)$$

$$\Leftrightarrow (P \wedge Q) \downarrow (P \wedge Q)$$

$$\Leftrightarrow [(P \downarrow P) \downarrow (Q \downarrow Q)] \downarrow [(P \downarrow P) \downarrow (Q \downarrow Q)]$$

Truth Tables

Example: Show that $(A \oplus B) \vee (A \downarrow B) \Leftrightarrow (A \uparrow B)$.

We prove this by constructing truth table.

A	B	$A \oplus B$	$A \downarrow B$	$(A \oplus B) \vee (A \downarrow B)$	$A \uparrow B$
T	T	F	F	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	T	T	T

As columns $(A \oplus B) \vee (A \downarrow B)$ and $(A \uparrow B)$ are identical.

$$\therefore (A \oplus B) \vee (A \downarrow B) \Leftrightarrow (A \uparrow B).$$

Normal Forms

If a given statement formula $A(p_1, p_2, \ldots p_n)$ involves n atomic variables, we have 2^n possible combinations of truth values of statements replacing the variables.

The formula A is a **tautology** if A has the truth value T for all possible assignments of the truth values to the variables $p_1, p_2, \ldots p_n$ and A is called a **contradiction** if A has the truth value F for all possible assignments of the truth values of the n variables. A is said to be ***satisfiable*** if A has the truth value T for atleast one combination of truth values assigned to $p1, p2, \ldots pn$.

The problem of determining whether a given statement formula is a Tautology, or a Contradiction or a satifiable is called a decision problem.

The construction of truth table involves a finite number of steps, but the construction may not be practical. We therefore **reduce the given statement formula** to normal form and find whether a given statement formula is a Tautology or Contradiction or atleast satisfiable.

It will be convenient to use the word “**product**” in place of “**conjunction**” and “**sum**” inplace of “**disjunction**” in our current discussion.

A product of the variables and their negations in a formula is called an ***elementary product***. Similarly, a sum of the variables and their negations in a formula is called an ***elementary sum***.

Let P and Q be any atomic variables.

Then $\boldsymbol{P}$ **,** $\boldsymbol{\neg P \wedge Q}$**,** $\boldsymbol{\neg Q \wedge P \quad \neg P}$ **,** $\boldsymbol{P \quad \neg P}$ **, and** $\boldsymbol{Q \wedge \neg P}$ are some examples of **elementary products**.

On the other hand, $\boldsymbol{P}$ **,** $\boldsymbol{\neg P \vee Q}$**,** $\boldsymbol{\neg Q \vee P \vee \neg P}$ **,** $\boldsymbol{P \vee \neg P}$ **, and** $\boldsymbol{Q \vee \neg P}$ are some examples of **elementary sums**.

Any part of an elementary sum or product which is itself an elementary sum or product is called a *factor* of the original elementary sum or product. Thus $\neg Q, \wedge \neg P$, and $\neg Q \wedge P$ are some of the factors of $\neg Q \wedge P \wedge \neg P$.

Disjunctive Normal Form (DNF)

A formula which is equivalent to a given formula and which consists of a sum of elementary products is called a *disjunctive normal form* of the given formula.

Example: Obtain disjunctive normal forms of

(a) $\boldsymbol{P \wedge (P \rightarrow Q)}$**;** **(b)** $\boldsymbol{\neg(P \vee Q) \leftrightarrow (P \wedge Q)}$**.**

Solution: (a) We have

$$P \wedge (P \rightarrow Q) \Leftrightarrow P \wedge (\neg P \vee Q)$$

$$\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$$

(b) $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$

$$\Leftrightarrow (\neg(P \vee Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge \neg(P \wedge Q))$$

[using $\boldsymbol{R \leftrightarrow S \Leftrightarrow (R \wedge S) \vee (\neg R \wedge \neg S)}$**]**

$$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q))$$

$$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge \neg P) \vee ((P \vee Q) \wedge \neg Q)$$

$$\boldsymbol{\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee (P \wedge \neg P) \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)}$$

which is the required disjunctive normal form.

Note: The DNF of a given formula is not unique.

Conjunctive Normal Form (CNF)

A formula which is equivalent to a given formula and which consists of a product of elementary sums is called a *conjunctive normal form* of the given formula.

The method for obtaining conjunctive normal form of a given formula is similar to the one given for disjunctive normal form. Again, the conjunctive normal form is not unique.

Example: Obtain conjunctive normal forms of

$$\text{(a) } P \wedge (P \to Q); \qquad \text{(b) } \neg(P \vee Q) \leftrightarrow (P \wedge Q).$$

Solution: (a). $P \wedge (P \to Q)$

$$\Leftrightarrow P \wedge (\neg P \vee Q)$$

(b). $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$

$$\Leftrightarrow (\neg(P \vee Q) \to (P \wedge Q)) \wedge ((P \wedge Q) \to \neg(P \vee Q))$$

$$\Leftrightarrow ((P \vee Q) \vee (P \wedge Q)) \wedge (\neg(P \wedge Q) \vee \neg(P \vee Q))$$

$$\Leftrightarrow [(P \vee Q \vee P) \wedge (P \vee Q \vee Q)] \wedge [(\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q)]$$

$$\Leftrightarrow (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge (\neg P \vee \neg Q \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg Q)$$

Note: A given formula is tautology if every elementary sum in CNF is tautology.

Example: Show that the formula $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.

Solution: First we obtain a CNF of the given formula.

$$Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q) \Leftrightarrow Q \vee ((P \vee \neg P) \wedge \neg Q)$$

$$\Leftrightarrow (Q \vee (P \vee \neg P)) \wedge (Q \vee \neg Q)$$

$$\Leftrightarrow (Q \vee P \vee \neg P) \wedge (Q \vee \neg Q)$$

Since each of the elementary sum is a tautology, hence the given formula is tautology.

PDNF and PCNF follow in class notes

Theory of Inference for Statement Calculus

Definition: The main aim of logic is to provide rules of inference to infer a conclusion from certain premises. The theory associated with rules of inference is known as **inference theory** .

Definition: If a conclusion is derived from a set of premises by using the accepted rules of reasoning, then such a process of derivation is called a deduction or a formal proof and the argument is called a ***valid argument*** or conclusion is called a ***valid conclusion*****.**

Note: Premises means set of **assumptions, axioms, hypothesis.**

Definition: Let A and B be two statement formulas. We say that "*B logically follows from A* "or" B is a *valid conclusion (consequence)* of the premise A" iff $A \to B$ is a tautology, that is $A \Rightarrow B$. *We say that from a set of premises* $\{H_1, H_2, \cdots, H_m\}$*, a conclusion C follows logically iff*

$$H_1 \wedge H_2 \wedge \ldots \wedge H_m \Rightarrow C \qquad (1)$$

Note: To determine whether the conclusion logically follows from the given premises, we use the following methods:

- Truth table method
- *Without constructing truth table method.*

Validity Using Truth Tables

Given a set of premises and a conclusion, it is possible to determine whether the conclusion logically follows from the given premises by constructing truth tables as follows.

Let $P_1, P_2, \cdots, P_n$ be all the atomic variables appearing in the premises $H_1, H_2, \cdots, H_m$ and in the conclusion C. If all possible combinations of truth values are assigned to $P_1, P_2, \cdots, P_n$ and if the truth values of $H_1, H_2, \ldots, H_m$ and C are entered in a table. We look for the rows in which all $H_1, H_2, \cdots, H_m$ have the value T. If, for every such row, C also has the value T, then (1) holds. That is, the conclusion follows logically.

Alternatively, we look for the rows on which C has the value F. If, in every such row, at least one of the values of $H_1, H_2, \cdots, H_m$ is F, then (1) also holds. We call such a method a '***truth table technique***' *for the determination of the validity of a conclusion.*

Example: Determine whether the conclusion C follows logically from the premises

H_1 and H_2.

(a) $H_1 : P \to Q$ $\quad H_2 : P$ $\quad C : Q$

(b) $H_1 : P \to Q$ $\quad H_2 : \neg P$ $\quad C : Q$

(c) $H_1 : P \to Q$ $\quad H_2 : \neg(P \wedge Q)$ $\quad C : \neg P$

(d) $H_1 : \neg P$ $\quad H_2 : P \leftrightarrow Q$ $\quad C : \neg(P \wedge Q)$

(e) $H_1 : P \to Q$ $\quad H_2 : Q$ $\quad C : P$

Solution: We first construct the appropriate truth table, as shown in table.

P	Q	$P \to Q$	$\neg P$	$\neg(P \wedge Q)$	$P \leftrightarrow Q$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	T	F
F	F	T	T	T	T

(a) We observe that the first row is the only row in which both the premises have the value T. The conclusion also has the value T in that row. Hence it is valid.

In (**b**) the third and fourth rows, the conclusion Q is true only in the third row, but not in the fourth, and hence the conclusion is not valid.

Similarly, we can show that the conclusions are valid in (**c**) and (**d**) but not in (**e**).

Rules of Inference

The following are two important rules of inferences.

Rule P: A premise may be introduced at any point in the derivation.

Rule T: A formula S may be introduced in a derivation if S is tautologically implied by one or more of the preceding formulas in the derivation.

Implication Formulas

$I_1 : P \wedge Q \Rightarrow P$ (simplification)

$I_2 : P \wedge Q \Rightarrow Q$

$I_3 : P \Rightarrow P \vee Q$

$I_4 : Q \Rightarrow P \vee Q$

$I_5 : \neg P \Rightarrow P \to Q$

$I_6 : Q \Rightarrow P \to Q$

$I_7 : \neg(P \to Q) \Rightarrow P$

$I_8 : \neg(P \to Q) \Rightarrow \neg Q$

$I_9 : P, Q \Rightarrow P \wedge Q$

$I_{10} : \neg P, P \vee Q \Rightarrow Q$ (disjunctive syllogism)

$I_{11} : P, P \to Q \Rightarrow Q$ (modus ponens)

$I_{12} : \neg Q, P \to Q \Rightarrow \neg P$ (modus tollens)

$I_{13} : P \to Q, Q \to R \Rightarrow P \to R$ (hypothetical syllogism)

$I_{14} : P \vee Q, P \to R, Q \to R \Rightarrow R$ (dilemma)

Example: Demonstrate that R is a valid inference from the premises P → Q, Q → R, and P .

Solution:

$\{1\}$	(1)	$P \to Q$	Rule P
$\{2\}$	(2)	P	Rule P,
$\{1, 2\}$	(3)	Q	Rule T, (1), (2), and I_{11}
$\{4\}$	(4)	$Q \to R$	Rule P
$\{1, 2, 4\}$	(5)	R	Rule T, (3), (4), and I_{11}

Hence the result.

Example: Show that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \to \neg H$, $\neg H \to (A \wedge \neg B)$, and $(A \wedge \neg B) \to (R \vee S)$.

Solution:

$\{1\}$	(1)	$(C \vee D) \to \neg H$	Rule P
$\{2\}$	(2)	$\neg H \to (A \wedge \neg B)$	Rule P
$\{1, 2\}$	(3)	$(C \vee D) \to (A \wedge \neg B)$	Rule T, (1), (2), and I_{13}
$\{4\}$	(4)	$(A \wedge \neg B) \to (R \vee S)$	Rule P
$\{1, 2, 4\}$	(5)	$(C \vee D) \to (R \vee S)$	Rule T, (3), (4), and I_{13}
$\{6\}$	(6)	$C \vee D$	Rule P
$\{1, 2, 4, 6\}$	(7)	$R \vee S$	Rule T, (5), (6), and I_{11}

Hence the result.

Example: Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$.

Solution:

$\{1\}$	(1)	$P \vee Q$	Rule P
$\{1\}$	(2)	$\neg P \to Q$	Rule T, (1) $P \to Q \Leftrightarrow \neg P \vee Q$
$\{3\}$	(3)	$Q \to S$	Rule P
$\{1, 3\}$	(4)	$\neg P \to S$	Rule T, (2), (3), and I_{13}
$\{1, 3\}$	(5)	$\neg S \to P$	Rule T, (4), $P \to Q \Leftrightarrow \neg Q \to \neg P$
$\{6\}$	(6)	$P \to R$	Rule P
$\{1, 3, 6\}$	(7)	$\neg S \to R$	Rule T, (5), (6), and I_{13}
$\{1,3,6\}$	(8)	S V R	Rule T,(7) and $P \to Q \Leftrightarrow \neg P \vee Q$

Hence the result.

Example: Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$, and $\neg M$.

Solution:

$\{1\}$	(1)	$P \to M$	Rule P
$\{2\}$	(2)	$\neg M$	Rule P
$\{1, 2\}$	(3)	$\neg P$	Rule T, (1), (2), and I_{12}
$\{4\}$	(4)	$P \vee Q$	Rule P
$\{1, 2, 4\}$	(5)	Q	Rule T, (3), (4), and I_{10}
$\{6\}$	(6)	$Q \to R$	Rule P
$\{1,2,4,6\}$	(7)	R	Rule T,(5),(6),and I_{11}
$\{1,2,4,6\}$	(8)	$R \wedge (P \vee Q)$	Rule T,(4),(7),and I_9

Continue to solve

Example: Test the validity of the following argument:

"If you work hard, you will pass the exam. You did not pass. Therefore, you did not work hard".

Example: Test the validity of the following statements:

"If Sachin hits a century, then he gets a free car. Sachin does not get a free car. Therefore, Sachin has not hit a century".

Rules of Conditional Proof or Deduction Theorem

We shall now introduce a third inference rule, known as CP or rule of conditional proof.

Rule CP: If we can derive S from R and a set of premises, then we can derive $R \rightarrow S$ from the set of premises alone.

Rule CP is not new for our purpose her because it follows from the equivalence

$$(P \wedge R) \rightarrow S \Leftrightarrow P \rightarrow (R \rightarrow S)$$

Let P denote the conjunction of the set of premises and let R be any formula. The above equivalence states that if R is included as an additional premise and S is derived from $P \wedge R$, then $R \rightarrow S$ can be derived from the premises P alone.

Rule CP is also called the *deduction theorem* and is generally used if the conclusion of the form $R \rightarrow S$. In such cases, R is taken as an additional premise and S is derived from the given premises and R.

Example: Show that $R \rightarrow S$ can be derived from the premises $P \rightarrow (Q \rightarrow S)$, $\neg R \vee P$, and Q.

Solution: Instead of deriving $R \rightarrow S$, we shall include R as an additional premise and show S first.

$\{1\}$	(1)	$\neg R \vee P$	Rule P
$\{2\}$	(2)	R	Rule P (assumed premise)
$\{1, 2\}$	(3)	P	Rule T, (1), (2), and I_{10}
$\{4\}$	(4)	$P \rightarrow (Q \rightarrow S)$	Rule P
$\{1, 2, 4\}$	(5)	$Q \rightarrow S$	Rule T, (3), (4), and I_{11}
$\{6\}$	(6)	Q	Rule P
$\{1, 2, 4, 6\}$	(7)	S	Rule T, (5), (6), and I_{11}
$\{1, 2, 4, 6\}$	(8)	$R \rightarrow S$	Rule CP

Example: Show that $P \to S$ can be derived from the premises $\neg P \vee Q$, $\neg Q \vee R$, and $R \to S$.

Solution: We include P as an additional premise and derive S.

$\{1\}$	(1) $\neg P \vee Q$	Rule P
$\{2\}$	(2) P	Rule P (assumed premise)
$\{1, 2\}$	(3) Q	Rule T, (1), (2), and I_{10}
$\{4\}$	(4) $\neg Q \vee R$	Rule P
$\{1, 2, 4\}$	(5) R	Rule T, (3), (4), and I_{10}
$\{6\}$	(6) $R \to S$	Rule P
$\{1, 2, 4, 6\}$	(7) S	Rule T, (5), (6), and I_{11}
$\{1, 2, 4, 6\}$	(8) $P \to S$	Rule CP

Example: "If there was a ball game, then traveling was difficult. If they arrived on time, then traveling was not difficult. They arrived on time. Therefore, there was no ball game". Show that these statements constitute a valid argument.

Solution: Let us indicate the statements as follows:

P : There was a ball game.
Q:Traveling was difficult.
R: They arrived on time.

Hence, the given premises are $P \to Q$, $R \to \neg Q$, and R. The conclusion is $\neg P$.

$\{1\}$	(1) $R \to \neg Q$	Rule P
$\{2\}$	(2) R	Rule P
$\{1, 2\}$	(3) $\neg Q$	Rule T, (1), (2), and I_{11}
$\{4\}$	(4) $P \to Q$	Rule P
$\{4\}$	(5) $\neg Q \to \neg P$	Rule T, (4), and $P \to Q \Leftrightarrow \neg Q \to \neg P$
$\{1, 2, 4\}$	(6) $\neg P$	Rule T, (3), (5), and I_{11}

Example: By using the method of derivation, show that following statements con-stitute a valid argument: "If *A* works hard, then either *B* or *C* will enjoy. If *B* enjoys, then *A* will not work hard. If *D* enjoys, then *C* will not. Therefore, if *A* works hard, *D* will not enjoy".

Solution: Let us indicate statements as follows:

Given premises are $P \to (Q \vee R)$, $Q \to \neg P$, and $S \to \neg R$.

The conclusion is $P \to \neg S$. We include P as an additional premise and derive $\neg S$.

$\{1\}$	(1)	P	Rule P (additional premise)
$\{2\}$	(2)	$P \rightarrow (Q \vee R)$	Rule P
$\{1, 2\}$	(3)	$Q \vee R$	Rule T, (1), (2), and I_{11}
$\{1, 2\}$	(4)	$\neg Q \rightarrow R$	Rule T, (3) and $P \rightarrow Q \Leftrightarrow \sim P \vee Q$
$\{1, 2\}$	(5)	$\neg R \rightarrow Q$	Rule T, (4), and $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$
$\{6\}$	(6)	$Q \rightarrow \neg P$	Rule P
$\{1, 2, 6\}$	(7)	$\neg R \rightarrow \neg P$	Rule T, (5), (6), and I_{13}
$\{1, 2, 6\}$	(8)	$P \rightarrow R$	Rule T, (7) and $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$
$\{9\}$	(9)	$S \rightarrow \neg R$	Rule P
$\{9\}$	(10)	$R \rightarrow \neg S$	Rule T, (9) and $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$
$\{1, 2, 6, 9\}$	(11)	$P \rightarrow \neg S$	Rule T, (8), (10) and I_{13}
$\{1, 2, 6, 9\}$	(12)	$\neg S$	Rule T, (1), (11) and I_{11}
$\{1,2,6,9\}$	(13)	$P \rightarrow \neg S$	Rule CP

Example: Determine the validity of the following arguments using propositional logic:

"Smoking is healthy. If smoking is healthy, then cigarettes are prescribed by physicians. Therefore, cigarettes are prescribed by physicians".

Solution: Let us indicate the statements as follows:

P : Smoking is healthy.

Q: Cigarettes are prescribed by physicians.

Hence, the given premises are P , $P \rightarrow Q$. The conclusion is Q.

$\{1\}$	(1) $P \rightarrow Q$	Rule P
$\{2\}$	(2) P	Rule P
$\{1, 2\}$	(3) Q	Rule T, (1), (2), and I_{11}

Hence, the given statements constitute a valid argument.

Consistency of Premises

A set of formulas $H_1, H_2, \cdots, H_m$ is said to be *consistent* if their conjunction has the truth value T for some assignment of the truth values to the atomic variables appearing in $H_1, H_2, \cdots, H_m$.

If, for every assignment of the truth values to the atomic variables, at least one of the formulas $H_1, H_2, \cdots, H_m$ is **false**, so that their conjunction is identically **false**, then the formulas $H_1, H_2, \cdots, H_m$ are called ***inconsistent***.

Alternatively, a set of formulas $H_1, H_2, \cdots, H_m$ is **inconsistent** if their conjunction implies a **contradiction**, that is,

$$H_1 \wedge H_2 \wedge \cdots \wedge H_m \Rightarrow R \wedge \neg R$$

where R is any formula.

Example: Show that the following premises are inconsistent:

(1). If Jack misses many classes through illness, then he fails high school.

(2). If Jack fails high school, then he is uneducated.

(3). If Jack reads a lot of books, then he is not uneducated.

(4). Jack misses many classes through illness and reads a lot of books.

Solution: Let us indicate the statements as follows:

E: Jack misses many classes through illness.

S: Jack fails high school.

A: Jack reads a lot of books.

H: Jack is uneducated.

The premises are $E \to S$, $S \to H$, $A \to \neg H$, and $E \wedge A$.

$\{1\}$	(1)	$E \to S$	Rule P
$\{2\}$	(2)	$S \to H$	Rule P
$\{1, 2\}$	(3)	$E \to H$	Rule T, (1), (2), and I_{13}
$\{4\}$	(4)	$A \to \neg H$	Rule P
$\{4\}$	(5)	$H \to \neg A$	Rule T, (4), and $P \to Q \Leftrightarrow \neg Q \to \neg P$
$\{1, 2, 4\}$	(6)	$E \to \neg A$	Rule T, (3), (5), and I_{13}
$\{1, 2, 4\}$	(7)	$\neg E \vee \neg A$	Rule T, (6) and $P \to Q \Leftrightarrow \neg P \vee Q$
$\{1, 2, 4\}$	(8)	$\neg(E \wedge A)$	Rule T, (7), and $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
$\{9\}$	(9)	$E \wedge A$	Rule P
$\{1, 2, 4, 9\}$	(10)	$\neg(E \wedge A) \wedge (E \wedge A)$	Rule T, (8), (9) and I_9

Thus, the given set of premises leads to a contradiction and hence it is inconsistent.

Example: Show that the following set of premises is inconsistent: "If the contract is valid, then John is liable for penalty. If John is liable for penalty, he will go bankrupt. If the bank will loan him money, he will not go bankrupt. As a matter of fact, the contract is valid, and the bank will loan him money."

Solution: Let us indicate the statements as follows:

V : The contract is valid.

L: John is liable for penalty.

M: Bank will loan him money.

B: John will go bankrupt.

{1}	(1)	$V \to L$	Rule P
{2}	(2)	$L \to B$	Rule P
{1, 2}	(3)	$V \to B$	Rule T, (1), (2), and I_{13}
{4}	(4)	$M \to \neg B$	Rule P
{4}	(5)	$B \to \neg M$	Rule T, (4), and $P \to Q \Leftrightarrow \neg Q \to \neg P$
{1, 2, 4}	(6)	$V \to \neg M$	Rule T, (3), (5), and I_{13}
{1, 2, 4}	(7)	$\neg V \vee \neg M$	Rule T, (6) and $P \to Q \Leftrightarrow \neg P \vee Q$
{1, 2, 4}	(8)	$\neg(V \wedge M)$	Rule T, (7), and $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
{9}	(9)	$V \wedge M$	Rule P
{1, 2, 4, 9}	(10)	$\neg(V \wedge M) \wedge (V \wedge M)$	Rule T, (8), (9) and I_9

Thus, the given set of premises leads to a contradiction and hence it is inconsistent.

Indirect Method of Proof

The method of using the rule of conditional proof and the notion of an inconsistent set of premises is called the ***indirect method of proof* or *proof by contradiction*.**

In order to show that a conclusion C follows logically from the premises $H_1, H_2, \cdots,$ H_m, we assume that C is false and consider $\neg C$ as an **additional premise**. If the new set of premises is inconsistent, so that they imply a **contradiction**. Therefore, the assumption that $\neg C$ is true does not hold.

Hence, C is true whenever $H_1, H_2, \cdots, H_m$ are true. Thus, C follows logically from the premises $H_1, H_2, \cdots, H_m$.

Example: Show that $\neg(P \wedge Q)$ follows from $\neg P \wedge \neg Q$.

Solution: We introduce $\neg\neg(P \wedge Q)$ as additional premise and show that this additional premise leads to a contradiction.

{1}	(1) $\neg\neg(P \wedge Q)$	Rule P (assumed)
{1}	(2) $P \wedge Q$	Rule T, (1), and $\neg\neg P \Leftrightarrow P$
{1}	(3) P	Rule T, (2), and I_1
{4}	(4) $\neg P \wedge \neg Q$	Rule P

{4}	(5) $\neg P$	Rule T, (4), and $I1$
{1, 4}	(6) $P \wedge \neg P$	Rule T, (3), (5), and $I9$

Hence, our assumption is wrong.

Thus, $\neg(P \wedge Q)$ follows from $\neg P \wedge \neg Q$.

Example: Using the indirect method of proof, show that

$$P \to Q,\ Q \to R,\ \neg(P \wedge R),\ P \vee R \Rightarrow R.$$

Solution: We include $\neg R$ as an additional premise. Then we show that this leads to a contradiction.

{1}	(1) $P \to Q$	Rule P
{2}	(2) $Q \to R$	Rule P
{1, 2}	(3) $P \to R$	Rule T, (1), (2), and $I13$
{4}	(4) $\neg R$	Rule P (assumed)
{1, 2, 4}	(5) $\neg P$	Rule T, (4), and $I12$
{6}	(6) $P \vee R$	Rule P
{1, 2, 4, 6}	(7) R	Rule T, (5), (6) and $I10$
{1, 2, 4, 6}	(8) $R \wedge \neg R$	Rule T, (4), (7), and $I9$

Hence, our assumption is wrong.

Example: Show that the following set of premises are inconsistent, using proof by contradiction

$$P \to (Q \vee R),\ Q \to \neg P,\ S \to \neg R,\ P \Rightarrow P \to \neg S.$$

Solution: We include $\neg(P \to \neg S)$ as an additional premise. Then we show that this leads to a contradiction.

$$\therefore \neg(P \to \neg S) \Leftrightarrow \neg(\neg P \vee \neg S) \Leftrightarrow P \wedge S.$$

{1}	(1) $P \to (Q \vee R)$	Rule P
{2}	(2) P	Rule P
{1, 2}	(3) $Q \vee R$	Rule T, (1), (2), and Modus Ponens
{4}	(4) $P \wedge S$	Rule P (assumed)
{1,2,4}	(5) S	Rule T, (4), and $P \wedge Q \Rightarrow P$
{6}	(6) $S \to \neg R$	Rule P
{1, 2, 4, 6}	(7) $\neg R$	Rule T, (5), (6) and Modus Ponens
{1, 2, 4, 6}	(8) Q	Rule T, (3), (7), and $P \wedge Q, \neg Q \Rightarrow$

P

$\{9\}$	(9) $Q \rightarrow \neg P$	Rule P
$\{1, 2, 4, 6\}$	(10) $\neg P$	Rule T, (8), (9), and $P \wedge Q, \neg Q \Rightarrow P$
$\{1, 2, 4, 6\}$	(11) $P \wedge \neg P$	Rule T, (2), (10), and $P, Q \Rightarrow P \wedge Q$

Hence, it is proved that the given premises are inconsistent.

The Predicate Calculus

Predicate

A part of a declarative sentence describing the properties of an object is called a predicate. The logic based upon the analysis of predicate in any statement is called predicate logic.

Consider two statements:

John is a bachelor

Smith is a bachelor.

In each statement "**is a bachelor**" is a predicate. Both John and Smith have the same property of being a bachelor. In the statement logic, we require two different symbols to express them and these symbols do not reveal the common property of these statements. In predicate calculus these statements can be replaced by a single statement **"*x* is a bachelor**". A predicate is symbolized by a capital letters which is followed by the list of variables. The list of variables is enclosed in parenthesis. If P stands for the predicate "**is a bachelor**", then $\boldsymbol{P(x)}$ stands for "***x* is a bachelor**",where x is a predicate variable.

\`The domain for **P (x) : x is a bachelor, can be taken as the set of all human names.** Note that $P(x)$ is not a statement, but just an expression. Once a value is assigned to x, $P(x)$ becomes a statement and has the truth value. If x is Ram, then $P(x)$ is a statement and its truth value is true.

Note: Remainng Terminology will discuss in class

Quantifiers

Quantifiers: Quantifiers are words that are refer to quantities such as 'some' or 'all'.

Universal Quantifier: The phrase '**for all**' (denoted by ∀) is called the universal quantifier.

For example, consider the sentence "All human beings are mortal".
Let $P(x)$ denote 'x is a mortal'.

Then, the above sentence can be written as $(\forall x \in S)P(x)$ or $\forall x P(x)$ where S denote the set of all human beings.

$\forall x$ represents each of the following phrases, since they have essentially the same for all x

For every *x*
For each *x*.

Existential Quantifier: The phrase 'there exists' (denoted by ∃) is called the existential quantifier.

For example, consider the sentence

"There exists *x* such that $x^2 = 5$".

This sentence can be written as

$$(\exists x \in R)P(x) \text{ or } (\exists x)P(x), \text{ where } P(x) : x^2 = 5.$$

$\exists x$ represents each of the following phrases

There exists an *x*
There is an *x*
For some *x*
There is at least one *x*.

Example: Write the following statements in symbolic form:

(i). Something is good

(ii). Everything is good

(iii). Nothing is good

(iv). Something is not good.

Solution: Statement (i) means **"There is atleast one *x* such that, *x* is good".**

Statement (ii) means **"For all *x*, *x* is good".**

Statement (iii) means, **"For all *x*, *x* is not good".**

Statement (iv) means, **"There is atleast one *x* such that, *x* is not good".**

Thus, if $G(x)$: x is good, then

statement (i) can be denoted by $(\exists x)G(x)$

statement (ii) can be denoted by $(\forall x)G(x)$

statement (iii) can be denoted by $(\forall x)\neg G(x)$

statement (iv) can be denoted by $(\exists x)\neg G(x)$.

Example: Let $K(x)$: x is a man

$L(x)$: x is mortal

$M(x)$: x is an integer

$N(x)$: x either positive or negative

Express the following using quantifiers:

- **All men are mortal**
- **Any integer is either positive or negative.**

Solution: (a) The given statement can be written as

for all *x*, if *x* is a man, then *x* is mortal and this can be expressed as

$(x)(K(x) \rightarrow L(x))$.

(b). The given statement can be written as
for all x, if x is an integer, then x is either positive or negative and this can be expressed as **$(x)(M(x) \rightarrow N(x))$.**

Free and Bound Variables

Given a formula containing a part of the form $(x)P(x)$ or $(\exists x)P(x)$, such a part is called an x-bound part of the formula. Any occurrence of x in an x-bound part of the formula is called a bound occurrence of x, while any occurrence of x or of any variable that is not a bound occurrence is called a free occurrence. The smallest formula immediately following $(\forall x)$ or $(\exists x)$ is called the scope of the quantifier.

Consider the following formulas:

- $(x)P(x, y)$
- $(x)(P(x) \rightarrow Q(x))$
- $(x)(P(x) \rightarrow (\exists y)R(x, y))$
- $(x)(P(x) \rightarrow R(x)) \vee (x)(R(x) \rightarrow Q(x))$
- $(\exists x)(P(x) \wedge Q(x))$
- $(\exists x)P(x) \wedge Q(x)$.

In (1), $P(x, y)$ is the scope of the quantifier, and occurrence of x is bound occurrence, while the occurrence of y is free occurrence.

In (2), the scope of the universal quantifier is $P(x) \rightarrow Q(x)$, and all occurences of x are bound.

In (3), the scope of (x) is $P(x) \rightarrow (\exists y)R(x, y)$, while the scope of $(\exists y)$ is $R(x, y)$. All occurrences of both x and y are bound occurrences.

In (4), the scope of the first quantifier is $P(x) \rightarrow R(x)$ and the scope of the second is $R(x) \rightarrow Q(x)$. All occurrences of x are bound occurrences.

In (5), the scope $(\exists x)$ is $P(x) \wedge Q(x)$.

In (6), the scope of $(\exists x)$ is $P(x)$ and the last of occurrence of x in $Q(x)$ is free.

Negations of Quantified Statements

(i). $\neg(x)P(x) \Leftrightarrow (\exists x)\neg P(x)$

(ii). $\neg(\exists x)P(x) \Leftrightarrow (x)(\neg P(x))$.

Example: Let $P(x)$ denote the statement "x is a professional athlete" and let $Q(x)$ denote the statement "x plays soccer". The domain is the set of all people.

(a). Write each of the following proposition in English.

- **$(x)(P(x) \rightarrow Q(x)$**

- $(\exists x)(P(x) \wedge Q(x))$
- $(x)(P(x) \vee Q(x))$

(b). Write the negation of each of the above propositions, both in symbols and in words.

Solution:

(a). (i). For all x, if x is an professional athlete then x plays soccer.
"All professional athletes plays soccer" or "Every professional athlete plays soccer".

(ii). There exists an x such that x is a professional athlete and x plays soccer.
"Some professional athletes paly soccer".

(iii). For all x, x is a professional athlete or x plays soccer.
"Every person is either professional athlete or plays soccer".

(b). (i). **In symbol: We know that**

$$\neg(x)(P(x) \to Q(x)) \Leftrightarrow (\exists x)\neg(P(x) \to Q(x)) \Leftrightarrow (\exists x)\neg(\neg(P(x)) \vee Q(x))$$

$$\Leftrightarrow (\exists x)(P(x) \wedge \neg Q(x))$$

There exists an x such that, x is a professional athlete and x does not play soccer.
In words: "Some professional athlete do not play soccer".

(ii). $\neg(\exists x)(P(x) \wedge Q(x)) \Leftrightarrow (x)(\neg P(x) \vee \neg Q(x))$

In words: "Every people is neither a professional athlete nor plays soccer "or" All people either not a professional athlete or do not play soccer".

(iii). $\neg(x)(P(x) \vee Q(x)) \Leftrightarrow (\exists x)(\neg P(x) \wedge \neg Q(x))$.

In words: "Some people are not professional athlete or do not play soccer".

Inference Theory of the Predicate Calculus

To understand the inference theory of predicate calculus, it is important to be famil-iar with the following rules:

Rule US: Universal specification or instantiation

$(x)A(x) \Rightarrow A(y)$ From $(x)A(x)$, one can conclude $A(y)$.

Rule ES: Existential specification

$$(\exists x)A(x) \Rightarrow A(y)$$

From $(\exists x)A(x)$, one can conclude $A(y)$.

Rule EG: Existential generalization

$$A(x) \Rightarrow (\exists y)A(y)$$

From $A(x)$, one can conclude $(\exists y)A(y)$.

Rule UG: Universal generalization

$$A(x) \Rightarrow (y)A(y)$$

From $A(x)$, one can conclude $(y)A(y)$.

Equivalence formulas:

E_{31} : $(\exists x)[A(x) \vee B(x)] \Leftrightarrow (\exists x)A(x) \vee (\exists x)B(x)$

E_{32} : $(x)[A(x) \wedge B(x)] \Leftrightarrow (x)A(x) \wedge (x)B(x)$

E_{33} : $\neg(\exists x)A(x) \Leftrightarrow (x)\neg A(x)$

E_{34} : $\neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$

E_{35} : $(x)(A \vee B(x)) \Leftrightarrow A \vee (x)B(x)$

E_{36} : $(\exists x)(A \wedge B(x)) \Leftrightarrow A \wedge (\exists x)B(x)$

E_{37} : $(x)A(x) \to B \Leftrightarrow (x)(A(x) \to B)$

E_{38} : $(\exists x)A(x) \to B \Leftrightarrow (x)(A(x) \to B)$

E_{39} : $A \to (x)B(x) \Leftrightarrow (x)(A \to B(x))$

E_{40} : $A \to (\exists x)B(x) \Leftrightarrow (\exists x)(A \to B(x))$

E_{41} : $(\exists x)(A(x) \to B(x)) \Leftrightarrow (x)A(x) \to (\exists x)B(x)$

E_{42} : $(\exists x)A(x) \to (x)B(X) \Leftrightarrow (x)(A(x) \to B(X))$.

Example: Verify the validity of the following arguments:
"All men are mortal. Socrates is a man. Therefore, Socrates is mortal".

or

Show that $(x)[H(x) \to M(x)] \wedge H(s) \Rightarrow M(s)$.

Solution: Let us represent the statements as follows:

$H(x)$: x is a man
$M(x)$: x is a mortal
s : Socrates
Thus, we have to show that $(x)[H(x) \to M(x)] \wedge H(s) \Rightarrow M(s)$.

$\{1\}$	(1)	$(x)[H(x) \to M(x)]$	Rule P
$\{1\}$	(2)	$H(s) \to M(s)$	Rule US, (1)
$\{3\}$	(3)	$H(s)$	Rule P
$\{1, 3\}$	(4)	$M(s)$	Rule T, (2), (3), and I_{11}

Example: Establish the validity of the following argument:"All integers are rational numbers. Some integers are powers of 2. Therefore, some rational numbers are powers of 2".

Solution: Let $P(x)$: x is an integer

$R(x)$: x is rational number

$S(x)$: x is a power of 2

Hence, the given statements becomes

$(x)(P(x) \to R(x)), (\exists x)(P(x) \wedge S(x)) \Rightarrow (\exists x)(R(x) \wedge S(x))$

{1}	(1)	$(\exists x)(P(x) \wedge S(x))$	Rule P
{1}	(2)	$P(y) \wedge S(y)$	Rule ES, (1)
{1}	(3)	$P(y)$	Rule T, (2) and $P \wedge Q \Rightarrow P$
{1}	(4)	$S(y)$	Rule T, (2) and $P \wedge Q \Rightarrow Q$
{5}	(5)	$(x)(P(x) \to R(x))$	Rule P
{5}	(6)	$P(y) \to R(y)$	Rule US, (5)
{1,5}	(7)	$R(y)$	Rule T, (3), (6) and $P, P \to Q \Rightarrow Q$
{1,5}	(8)	$R(y) \wedge S(y)$	Rule T, (4), (7) and $P, Q \Rightarrow P \wedge Q$
{1,5}	(9)	$(\exists x)(R(x) \wedge S(x))$	Rule EG, (8)

Hence, the given statement is valid.

Example: Show that $(x)(P(x) \to Q(x)) \wedge (x)(Q(x) \to R(x)) \Rightarrow (x)(P(x) \to R(x))$.

Solution:

{1}	(1)	$(x)(P(x) \to Q(x))$	Rule P
{1}	(2)	$P(y) \to Q(y)$	Rule US, (1)
{3}	(3)	$(x)(Q(x) \to R(x))$	Rule P
{3}	(4)	$Q(y) \to R(y)$	Rule US, (3)
{1, 3}	(5)	$P(y) \to R(y)$	Rule T, (2), (4), and $I13$
{1, 3}	(6)	$(x)(P(x) \to R(x))$	Rule UG, (5)

Example: Show that $(\exists x)M(x)$ follows logically from the premises $(x)(H(x) \to M(x))$ and $(\exists x)H(x)$.

Solution:

{1}	(1)	$(\exists x)H(x)$	Rule P
{1}	(2)	$H(y)$	Rule ES, (1)
{3}	(3)	$(x)(H(x) \to M(x))$	Rule P
{3}	(4)	$H(y) \to M(y)$	Rule US, (3)
{1, 3}	(5)	$M(y)$	Rule T, (2), (4), and $I11$
{1, 3}	(6)	$(\exists x)M(x)$	Rule EG, (5)

Hence, the result.

Example: Show that $(\exists x)[P(x) \wedge Q(x)] \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$.

Solution:

{1}	(1) $(\exists x)(P(x) \wedge Q(x))$	Rule P
{1}	(2) $P(y) \wedge Q(y)$	Rule ES, (1)
{1}	(3) $P(y)$	Rule T, (2), and I_1
{1}	(4) $(\exists x)P(x)$	Rule EG, (3)
{1}	(5) $Q(y)$	Rule T, (2), and I_2
{1}	(6) $(\exists x)Q(x)$	Rule EG, (5)
{1}	(7) $(\exists x)P(x) \wedge (\exists x)Q(x)$	Rule T, (4), (5) and I_9

Hence, the result.

Note: Is the converse true?

{1}	(1) $(\exists x)P(x) \wedge (\exists x)Q(x)$	Rule P
{1}	(2) $(\exists x)P(x)$	Rule T, (1) and I_1
{1}	(3) $(\exists x)Q(x)$	Rule T, (1), and I_1
{1}	(4) $P(y)$	Rule ES, (2)
{1}	(5) $Q(s)$	Rule ES, (3)

Here in step (4), y is fixed, and it is not possible to use that variable again in step (5). Hence, the *converse is not true.*

Example: Show that from $(\exists x)[F(x) \wedge S(x)] \rightarrow (y)[M(y) \rightarrow W(y)]$ and $(\exists y)[M(y) \wedge \neg W(y)]$ the conclusion $(x)[F(x) \rightarrow \neg S(x)]$ follows.

{1}	(1)	$(\exists y)[M(y) \wedge \neg W(y)]$	Rule P
{1}	(2)	$[M(z) \wedge \neg W(z)]$	Rule ES, (1)
{1}	(3)	$\neg[M(z) \rightarrow W(z)]$	Rule T, (2), and $\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$
{1}	(4)	$(\exists y)\neg[M(y) \rightarrow W(y)]$	Rule EG, (3)
{1}	(5)	$\neg(y)[M(y) \rightarrow W(y)]$	Rule T, (4), and $\neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$
{1}	(6)	$(\exists x)[F(x) \wedge S(x)] \rightarrow (y)[M(y) \rightarrow W(y)]$	Rule P
{1, 6}	(7)	$\neg(\exists x)[F(x) \wedge S(x)]$	Rule T, (5), (6) and I_{12}
{1, 6}	(8)	$(x)\neg[F(x) \wedge S(x)]$	Rule T, (7), and $\neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$
{1, 6}	(9)	$\neg[F(z) \wedge S(z)]$	Rule US, (8)
{1, 6}	(10)	$\neg F(z) \vee \neg S(z)$	Rule T, (9), and De Morgan‘s laws

{1, 6}	(11)	$F(z) \to \neg S(z)$	Rule T, (10), and $P \to Q \Leftrightarrow \neg P \vee Q$
{1, 6}	(12)	$(x)(F(x) \to \neg S(x))$	Rule UG, (11)

Hence, the result.

Example: Show that $(x)(P(x) \vee Q(x)) \Rightarrow (x)P(x) \vee (\exists x)Q(x)$

Solution: We shall use the indirect method of proof by assuming $\neg((x)P(x) \vee (\exists x)Q(x))$ as an additional premise.

{1}	(1)	$\neg((x)P(x) \vee (\exists x)Q(x))$	Rule P (assumed)
{1}	(2)	$\neg(x)P(x) \wedge \neg(\exists x)Q(x)$	Rule T, (1) $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
{1}	(3)	$\neg(x)P(x)$	Rule T, (2), and I_1
{1}	(4)	$(\exists x)\neg P(x)$	Rule T, (3), and $\neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$
{1}	(5)	$\neg(\exists x)Q(x)$	Rule T, (2), and I_2
{1}	(6)	$(x)\neg Q(x)$	Rule T, (5), and $\neg(\exists x)A(x) \Leftrightarrow (x)\neg A(x)$
{1}	(7)	$\neg P(y)$	Rule ES, (5), (6) and I_{12}
{1}	(8)	$\neg Q(y)$	Rule US, (6)
{1}	(9)	$\neg P(y) \wedge \neg Q(y)$	Rule T, (7), (8)and I_9
{1}	(10)	$\neg(P(y) \vee Q(y))$	Rule T, (9), and $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
{11}	(11)	$(x)(P(x) \vee Q(x))$	Rule P
{11}	(12)	$(P(y) \vee Q(y))$	Rule US
{1,11}	(13)	$\neg(P(y) \vee Q(y)) \wedge (P(y) \vee Q(y))$	Rule T, (10), (11), and I_9
{1,11}	(14)	F	Rule T, and (13)

which is a contradiction.Hence, the statement is valid.

Example: Using predicate logic, prove the validity of the following argument: "Every husband argues with his wife. *x* is a husband. Therefore, *x* argues with his wife".

Solution: Let $P(x)$: x is a husband.

$Q(x)$: x argues with his wife.

Thus, we have to show that $(x)[P(x) \to Q(x)] \wedge P(x) \Rightarrow Q(y)$.

{1}	(1)	$(x)(P(x) \to Q(x))$	Rule P
{1}	(2)	$P(y) \to Q(y)$	Rule US, (1)
{1}	(3)	$P(y)$	Rule P

$\{1\}$ (4) $Q(y)$ Rule T, (2), (3), and I_{11}

Example: Prove using rules of inference

Duke is a Labrador retriever.

All Labrador retriever like to swim.
Therefore Duke likes to swim.

Solution: We denote

$L(x)$: x is a Labrador retriever.

$S(x)$: x likes to swim.

d: Duke.

We need to show that $L(d) \wedge (x)(L(x) \rightarrow S(x)) \Rightarrow S(d)$.

$\{1\}$	(1)	$(x)(L(x) \rightarrow S(x))$	Rule P
$\{1\}$	(2)	$L(d) \rightarrow S(d)$	Rule US, (1)
$\{2\}$	(3)	$L(d)$	Rule P
$\{1, 2\}$	(4)	$S(d)$	Rule T, (2), (3), and I_{11}.

Questions

1. Test the Validity of the Following argument: "All dogs are barking. Some animals are dogs. Therefore, some animals are barking".

2. Test the Validity of the Following argument:
"Some cats are animals. Some dogs are animals. Therefore, some cats are dogs".

3. Symbolizes and prove the validity of the following arguments :
(i) Himalaya is large. Therefore every thing is large.
(ii) Not every thing is edible. Therefore nothing is edible.

4. a) Find the PCNF of $(\sim p \leftrightarrow r) \wedge (q \leftrightarrow p)$?
b) Explain in brief about duality Law?

c) Construct the Truth table for $\sim(\sim p \wedge \sim q)$?
d) Find the disjunctive Normal form of $\sim(p \rightarrow (q \wedge r))$?

5. Define Well Formed Formula? Explain about Tautology with example?
6. Explain in detail about the Logical Connectives with Examples?
7. Obtain the principal conjunctive normal form of the formula $(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$
8. Prove that $(\exists x)P(x) \wedge Q(x) \rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$. Does the converse hold?
9. Show that from i) $(\exists x)(F(x) \wedge S(x)) \rightarrow (y)(M(y) \rightarrow W(y))$
ii) $(\exists y)(M(y) \wedge \neg W(y))$ the conclusion $(x)(F(x) \rightarrow \neg S(x))$ follows.

10. Obtain the principal disjunctive and conjunctive normal forms of $(P \rightarrow (Q \wedge R)) \wedge (\neg P \rightarrow (\neg Q \wedge \neg R))$. Is this formula a tautology?

11. Prove that the following argument is valid: No Mathematicians are fools. No one who is not a fool is an administrator. Sitha is a mathematician. Therefore Sitha is not an administrator.

12. Test the Validity of the Following argument: If you work hard, you will pass the exam. You did not pass. Therefore you did not work hard.

13. Without constructing the Truth Table prove that $(p \rightarrow q) \rightarrow q = p v q$?

14. Using normal forms, show that the formula $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.

15. Show that $(x)(P(x) \vee Q(x)) \rightarrow (x)P(x) \vee (\exists x)Q(x)$

16. Show that $\neg (P \wedge Q) \rightarrow (\neg P \vee (\neg P \vee Q)) \Leftrightarrow (\neg P \vee Q)$

$(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P \wedge Q)$

17. Prove that $(\exists x)(P(x) \wedge Q(x)) \rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$

18. Example: Prove or disprove the validity of the following arguments using the rules of inference. (i) All men are fallible (ii) All kings are men (iii) Therefore, all kings are fallible.

19. Test the Validity of the Following argument:

"Lions are dangerous animals, there are lions, and therefore there are dangerous animals."

MULTIPLE CHOICE QUESTIONS

1: Which of the following propositions is tautology?

A.(p v q)→q B. p v (q→p) C.p v (p→q) D.Both (b) & (c)

Option: C

2: Which of the proposition is p^ (~ p v q) is

A.A tautology B.A contradiction C.Logically equivalent to p ^ q D.All of above

Option: C

3: Which of the following is/are tautology?

A.a v b → b ^ c B.a ^ b → b v c C.a v b → (b → c) D.None of these

Option: B

4: Logical expression (A^ B) → (C' ^ A) → (A ≡ 1) is

A.ContradictionB.Valid C.Well-formed formula D.None of these

Option: D

5: Identify the valid conclusion from the premises Pv Q, Q → R, P → M, ˥M

A.P ^ (R v R) B.P ^ (P ^ R) C.R ^ (P v Q) D.Q ^ (P v R)

Option: D

6: Let a, b, c, d be propositions. Assume that the equivalence a ↔ (b v ˥b) and b ↔ c hold. Then truth value of the formula (a ^ b) → ((a ^ c) v d) is always

A.True B.False C.Same as the truth value of a D.Same as the truth value of b

Option: A

7: Which of the following is a declarative statement?

A. It's right B. He says C.Two may not be an even integer D.I love you

Option: B

8: P → (Q → R) is equivalent to

A. (P ^ Q) → R B.(P v Q) → R C.(P v Q) → ˥R D.None of these

Option: A

9: Which of the following are tautologies?

A.((P v Q) ^ Q) ↔ Q B.((P v Q) ^ ˥P) → Q C.((P v Q) ^ P) → P D.Both (a) & (b)

Option: D

10: If F1, F2 and F3 are propositional formulae such that F1 ^ F2 → F3 and F1 ^ F2→F3 are both tautologies, then which of the following is TRUE?

A.Both F1 and F2 are tautologies B.The conjuction F1 ^ F2 is not satisfiable

C.Neither is tautologies D.None of these

Option: B

11. Consider two well-formed formulas in propositional logic

F1 : P →˥P F2 : (P →˥P) v (˥P →) Which of the following statement is correct? A.F1 is satisfiable, F2 is unsatisfiable B.F1 is unsatisfiable, F2 is satisfiable C.F1 is unsatisfiable, F2 is valid D.F1 & F2 are both satisfiable

Option: C

12: What can we correctly say about proposition P1 : (p v ˥q) ^ (q →r) v (r v p)

A.P1 is tautology B.P1 is satisfiable

C.If p is true and q is false and r is false, the P1 is true

D.If p as true and q is true and r is false, then P1 is true

Option: C

13: (P v Q) ^ (P → R)^ (Q →S) is equivalent to

A.S ^ R B.S → R C.S v R D.All of above

Option: C

14: The functionally complete set is

A.{ ˥, ^, v } B.{↓, ^ }C.{↑} D.None of these

Option: C

15: (P v Q) ^ (P→R) ^ (Q → R) is equivalent to

A.P B.Q C.R D.True = T

Option: C

16: ˥(P → Q) is equivalent to

A.P ^ ˥Q B.P ^ QC.˥P v Q D.None of these

Option: A

17: In propositional logic , which of the following is equivalent to p → q?

A.~p → q B.~p v q C.~p v~ q D.p →q

Option: B

18: Which of the following is FALSE? Read ^ as And, v as OR, ~as NOT, →as one way implication and ↔ as two way implication?

A.((x → y)^ x) →y B.((~x →y)^ (~x ^ ~y))→y C.(x → (x v y)) D.((x v y) ↔(~x v ~y))

Option: D

19: Which of the following well-formed formula(s) are valid?

A.((P → Q)^(Q → R))→ (P → R) B.(P → Q) →(˥P → ˥Q)

C.(P v (˥P v ˥Q)) →P D.((P → R) v (Q → R)) → (P v Q}→R)

Option: A

20: Let p and q be propositions. Using only the truth table decide whether p ↔ q does not imply p → ˥q is

A.True B.False C.None D.Both A and B

Option: A

UNIT-2

Set Theory

Set:A set is collection of well defined objects.

In the above definition the words set and collection for all practical purposes are Synonymous. We have really used the word set to define itself.
Each of the objects in the set is called a member of an element of the set. The objects themselves can be almost anything. Books, cities, numbers, animals, flowers, etc. Elements of a set are usually denoted by lower-case letters. While sets are denoted by capital letters of English larguage.

The symbol $\in$ indicates the membership in a set.

If "*a* is an element of the set *A*", then we write $a \in A$.
The symbol $\in$ is read "is a member of " or " is an element of ".
The symbol $\notin$ is used to indicate that an object is not in the given set.
The symbol $\notin$ is read " is not a member of " or " is not an element of ".
If x is not an element of the set A then we write $x \notin A$.

Subset :

A set A is a subset of the set B if and only if every element of A is also an element of B. We also say that A is contained in B, and use the notation $A \subseteq B$.

Proper Subset*:*

A set A is called proper subset of the set B. If (*i*) A is subset of B and (*ii*) B is not a subset A i.e., A is said to be a proper subset of B if every element of A belongs to the set B, but there is atleast one element of B, which is not in A. If A is a proper subset of B, then we denote it by $A \subset B$.

Super set: If A is subset of B, then B is called a superset of A.

Null set: The set with no elements is called an empty set or null set. A Null set is designated by the symbol ϕ . The null set is a subset of every set, i.e., If A is any set then $\phi \subset A$.

Universal set:
In many discussions all the sets are considered to be subsets of one particular set. This set is called the universal set for that discussion. The Universal set is often designated by the script letter μ . Universal set in not unique and it may change from one discussion to another.

Power set:
The set of all subsets of a set A is called the power set of A.
The power set of A is denoted by $P(A)$. If A has n elements in it, then $P(A)$ has 2^n elements:

Disjoint sets:
Two sets are said to be disjoint if they have no element in common.

Union of two sets:
The union of two sets *A* and *B* is the set whose elements are all of the elements in *A* or in *B* or in both. The union of sets *A* and *B* denoted by $A \cup B$ is read as "*A* union *B*".

Intersection of two sets:
The intersection of two sets *A* and *B* is the set whose elements are all of the elements common to both *A* and *B*. The intersection of the sets of "*A*" and "*B*" is denoted by $A \cap B$ and is read as "*A* intersection *B*"

Difference of sets:
If *A* and *B* are subsets of the universal set *U*, then the relative complement of *B* in *A*is the set of all elements in *A* which are not in *A*. It is denoted by $A - B$ thus: $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

Complement of a set:
If *U* is a universal set containing the set *A*, then $U - A$ is called the complement of *A*. It is denoted by A^1 . Thus
$A1 = \{x: x \notin A\}$

Inclusion-Exclusion Principle:

The inclusion–exclusion principle is a counting technique which generalizes the familiar method of obtaining the number of elements in the unionof two finite sets; symbolically expressed as

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

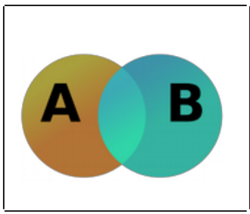


Fig. Venn diagram showing the union of sets A and B

where *A* and *B* are two finite sets and $|S|$ indicates the cardinality of a set *S* (which may be considered as the number of elements of the set, if the set is finite). The formula expresses the fact that the sum of the sizes of the two sets may be too large since some elements may be counted twice. The double-counted elements are those in the intersection of the two sets and the count is corrected by subtracting the size of the intersection.

The principle is more clearly seen in the case of three sets, which for the sets *A*, *B* and *C* is given by

$$|A \cup B \cup BC| = |A| + |B| + |C| - |A \cap B| - |C \cap B| - |A \cap C| + |A \cap B \cap C|.$$

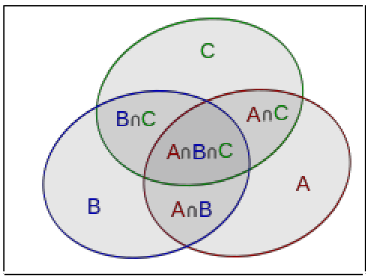


Fig. Inclusion–exclusion illustrated by a Venn diagram for three sets

This formula can be verified by counting how many times each region in the Venn diagram figure is included in the right-hand side of the formula. In this case, when removing the contributions of over-counted elements, the number of elements in the mutual intersection of the three sets has been subtracted too often, so must be added back in to get the correct total.

In general, Let A1, · · · , Ap be finite subsets of a set U. Then,

$$|A_1 \cup A_2 \cup \ldots \cup A_p| = \sum_{1\le i\le p} |A_i| - \sum_{1\le i_1<i_2\le p} |A_{i_1} \cap A_{i_2}| + \sum_{1\le i_1<i_2<i_3\le p} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \ldots + (-1)^{p-1} |A_1 \cap A_2 \cap \cdots \cap A_p|,$$

Example: How many natural numbers n ≤ 1000 are not divisible by any of 2, 3?

Ans: Let $A_2 = \{n \in N \mid n \le 1000, 2|n\}$ and $A_3 = \{n \in N \mid n \le 1000, 3|n\}$.
Then, $|A_2 \cup A_3| = |A_2| + |A_3| - |A_2 \cap A_3| = 500 + 333 - 166 = 667$.
So, the required answer is $1000 - 667 = 333$.

Example: How many integers between 1 and 10000 are divisible by none of 2, 3, 5, 7?

Ans: For $i \in \{2, 3, 5, 7\}$, let $A_i = \{n \in N \mid n \le 10000, i|n\}$.
Therefore, the required answer is $10000 - |A_2 \cup A_3 \cup A_5 \cup A_7| = 2285$.

Relations

Definition: Any set of ordered pairs defines a *binary relation*.

We shall call a binary relation simply a relation. Binary relations represent relationships between elements of two sets. If *R* is a relation, a particular ordered pair, say $(x,y) \in R$ can be written as *xRy* and can be read as "*x* is in relation *R* to *y*".

Example: Give an example of a relation.

Solution: The relation –greater than‖ for real numbers is denoted by ′ >′. If *x* and *y* are any two real numbers such that $x > y$, then we say that $(x, y) \in >$. Thus the relation > is { } >= (*x,y)*: *x* and *y* are real numbers and $x > y$

***Example:* Define a relation between two sets *A* = {5, 6, 7} and *B* = {*x*, *y*}.**

Solution: If *A* = {5, 6, 7} and *B* = {*x*, *y*}, then the subset *R* = {(5, *x*),(5, *y*), (6, *x*), (6, *y*)} is a relation from *A* to *B*.

Definition: Let *S* be any relation. The *domain* of the relation *S* is defined as the set of all first elements of the ordered pairs that belong to *S* and is denoted by *D*(*S*).

$$D(S) = \{ x : (x, y) \in S, \text{ for some } y \}$$

The *range* of the relation *S* is defined as the set of all second elements of the ordered pairs that belong to *S* and is denoted by *R*(*S*).

$$R(S) = \{ y : (x, y) \in S, \text{ for some } x\}$$

Example: $A = \{2, 3, 4\}$ and $B = \{3, 4, 5, 6, 7\}$. Define a relation from A to B by $(a, b) \in R$ if a divides b

Solution: We obtain $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$.

Domain of $R = \{2, 3, 4\}$ and range of $R = \{3, 4, 6\}$.

Properties of Binary Relations in a Set

A relation R on a set X is said to be

- ***Reflexive relation if xRx or $(x, x) \in R$, $\forall x \in X$***
- ***Symmetric relation if xRy then yRx, $\forall x, y \in X$***
- ***Transitive relation if xRy and yRz then xRz, $\forall x, y, z \in X$***
- ***Irreflexive relation if $x \not R x$ or $(x, x) \notin R$, $\forall x \in X$***
- ***Antisymmetric relation if for every x and y in X, whenever xRy and yRx, then x = y.***

Examples: **(i)**. If $R_1 = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\}$ be a relation on $A = \{1, 2, 3\}$, then R_1 is a reflexive relation, since for every $x \in A$, $(x, x) \in R_1$.

(ii). If $R_2 = \{(1, 1), (1, 2), (2, 3), (3, 3)\}$ be a relation on $A = \{1, 2, 3\}$, then R_2 is not a reflexive relation, since for every $2 \in A$, $(2, 2) \notin R_2$.

(iii). If $R_3 = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 1), (3, 1)\}$ be a relation on $A = \{1, 2, 3\}$, then R_3 is a symmetric relation.

(iv). If $R_4 = \{(1, 2), (2, 2), (2, 3)\}$ on $A = \{1, 2, 3\}$ is an antisymmetric.

Example: Given $S = \{1, 2, ..., 10\}$ and a relation R on S, where $R = \{(x, y)| x + y = 10\}$.What are the properties of the relation R?

Solution: Given that

$$S = \{1, 2, ..., 10\}$$
$$R = \{(x, y)|\ x + y = 10\}$$
$$= \{(1, 9), (9, 1), (2, 8), (8, 2), (3, 7), (7, 3), (4, 6), (6, 4), (5, 5)\}.$$

(i). For any $x \in S$ and $(x, x) \notin R$. Here, $1 \in S$ but $(1, 1) \notin R$.

$\Rightarrow$the relation R is not reflexive. It is also not irreflexive,

since $(5, 5) \in R$.

(ii). $(1, 9) \in R \Rightarrow (9, 1) \in R$

$(2, 8) \in R \Rightarrow (8, 2) \in R$.....

$\Rightarrow$the relation is symmetric, but it is not antisymmetric.

(iii). $(1, 9) \in R$ and $(9, 1) \in R$

⇒ (1, 1) $\notin R$

⇒The relation R is not transitive. Hence, R is symmetric.

Relation Matrix and the Graph of a Relation

Relation Matrix: *A relation R from a finite set X to a finite set Y can be represented by a matrix is called* ***the relation matrix of R.***
Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be finite sets containing m and n elements, respectively, and R be the relation from A to B. Then R can be represented by an $m \times n$ matrix

$M_R = [r_{ij}]$, which is defined as follows:

$$r_{ij} = \begin{cases} 1, \text{ if } (x_i, y_j) \in R \\ 0, \text{ if } (x_i, y_j) \; R \end{cases}$$

Example. Let A = {1, 2, 3, 4} and B = {b1, b2, b3}. Consider the relation R = {(1, b2), (1, b3), (3, b2), (4, b1), (4, b3)}. Determine the matrix of the relation.

Solution: $A = \{1, 2, 3, 4\}$, $B = \{b_1, b_2, b_3\}$.

Relation $R = \{(1, b_2), (1, b_3), (3, b_2), (4, b_1), (4, b_3)\}$.

Matrix of the relation R is written as

That is $M_R = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

Example: Let $A = \{1, 2, 3, 4\}$. Find the relation R on A determined by the matrix

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

Solution: The relation $R = \{(1, 1), (1, 3), (2, 3), (3, 1), (4, 1), (4, 2), (4, 4)\}$.

Properties of a relation in a set:
(i). If a relation is reflexive, then all the diagonal entries must be 1.
(ii). If a relation is symmetric, then the relation matrix is symmetric, i.e., $r_{ij} = r_{ji}$ for every i and j.
(iii). If a relation is antisymmetric, then its matrix is such that if $r_{ij} = 1$ then $r_{ji} = 0$ for $i \neq j$.

Graph of a Relation: A relation can also be represented pictorially by drawing its *graph*. Let R be a relation in a set $X = \{x_1, x_2, \ldots, x_m\}$. The elements of X are represented by points or circles called *nodes*. These nodes are called *vertices*. If $(x_i, x_j) \in R$, then we connect the nodes x_i and x_j by means of an arc and put an arrow on the arc in the direction from x_i to x_j . This is called an *edge*. If all the nodes corresponding to the ordered pairs in R are connected by arcs with proper arrows, then we get a graph of the relation R.

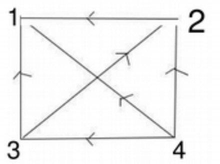
Note: (i). If $x_i R x_j$ and $x_j R x_i$, then we draw two arcs between x_i and x_j with arrows pointing in both directions.
(ii). If $x_i R x_i$, then we get an arc which starts from node x_i and returns to node x_i. This arc is called a *loop*.

Properties of relations:

(i). If a relation is reflexive, then there must be a loop at each node. On the other hand, if the relation is irreflexive, then there is no loop at any node.
(ii). If a relation is symmetric and if one node is connected to another, then there must be a return arc from the second node to the first.
(iii). For antisymmetric relations, no such direct return path should exist.
(iv). If a relation is transitive, the situation is not so simple.

Example: Let X = {1, 2, 3, 4} and R={(x, y)| x > y}. Draw the graph of R and also give its matrix.

Solution: R = {(4, 1), (4, 3), (4, 2), (3, 1), (3, 2), (2, 1)}.
The graph of R and the matrix of R are



Graph of R

$$M_R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Partition and Covering of a Set

Let S be a given set and $A = \{A_1, A_2, \cdots, A_m\}$ where each A_i, $i = 1, 2, \cdots, m$ is a subset of S and $\bigcup_{i=1}^{m} A_i = S$

Then the set *A* is called a *covering* of *S*, and the sets $A_1, A_2, \cdots, A_m$ are said to *cover S*. If, in addition, the elements of *A*, which are subsets of *S*, are mutually disjoint, then *A* is called a *partition* of *S*, and the sets $A_1, A_2, \cdots, A_m$ are called the *blocks* of the partition.

Example: Let $S = \{a, b, c\}$ and consider the following collections of subsets of S. $A = \{\{a, b\}, \{b, c\}\}$, $B = \{\{a\}, \{a, c\}\}$, $C = \{\{a\}, \{b, c\}\}$, $D = \{\{a, b, c\}\}$, $E = \{\{a\}, \{b\}, \{c\}\}$, and $F = \{\{a\}, \{a, b\}, \{a,c\}\}$. Which of the above sets are covering?

Solution: The sets *A, C, D, E, F* are covering of *S*. But, the set *B* is not covering of *S*, since their union is not *S*.

Example: Let $S = \{a, b, c\}$ and consider the following collections of subsets of S. $A = \{\{a, b\}, \{b, c\}\}$, $B = \{\{a\}, \{b, c\}\}$, $C = \{\{a, b, c\}\}$, $D = \{\{a\}, \{b\}, \{c\}\}$, and $E = \{\{a\}, \{a, c\}\}$. Which of the above sets are covering?

Solution: The sets *B, C* and *D* are partitions of *S* and also they are covering. Hence, every partition is a covering.

The set *A* is a covering, but it is not a partition of a set, since the sets *{a, b}* and *{b, c}* are not disjoint. Hence, every covering need not be a partition.

The set *E* is not partition, since the union of the subsets is not *S*. The partition *C* has one block and the partition *D* has three blocks.

Example: List of all ordered partitions $S = \{a, b, c, d\}$ of type (1, 2, 2).

Solution:

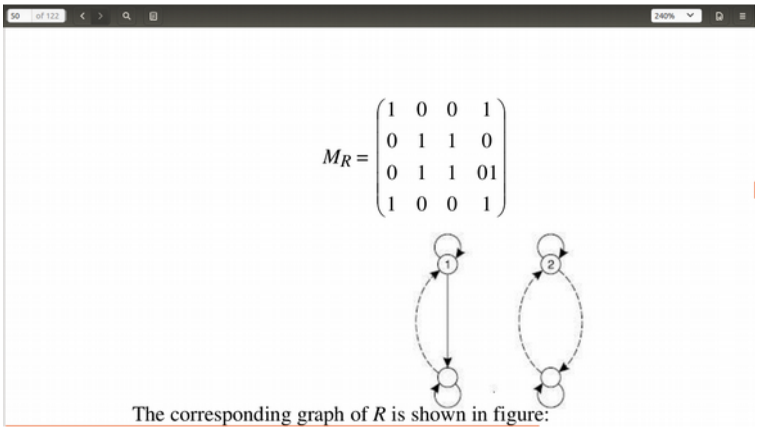
$(\{a\}, \{b\}, \{c, d\})$, $(\{b\}, \{a\}, \{c, d\})$
$(\{a\}, \{c\}, \{b, d\})$, $(\{c\}, \{a\}, \{b, d\})$
$(\{a\}, \{d\}, \{b, c\})$, $(\{d\}, \{a\}, \{b, c\})$
$(\{b\}, \{c\}, \{a, d\})$, $(\{c\}, \{b\}, \{a, d\})$
$(\{b\}, \{d\}, \{a, c\})$, $(\{d\}, \{b\}, \{a, c\})$
$(\{c\}, \{d\}, \{a, b\})$, $(\{d\}, \{c\}, \{a, b\})$.

Equivalence Relations

A relation R in a set X is called an *equivalence relation* if it is reflexive, symmetric and transitive. The following are some examples of equivalence relations:

1. Equality of numbers on a set of real numbers.
2. Equality of subsets of a universal set.

Example: Let X = {1, 2, 3, 4} and R == {(1, 1), (1, 4), (4, 1), (4, 4), (2, 2), (2, 3), (3, 2), (3, 3)}.Prove that R is an equivalence relation.



Clearly, the relation R is reflexive, symmetric and transitive. Hence, R is an equivalence relation.

Example: Let X = {1, 2, 3, ..., 7} and R =(x, y)| $x - y$ is divisible by 3. Show that R is an equivalence relation.

Solution: (i). For any $x \in X$, $x - x = 0$ is divisible by 3.

$\therefore xRx$

$\Rightarrow R$ is reflexive.

- (ii) For any $x, y \in X$, if xRy, then $x - y$ is divisible by 3.

$\Rightarrow -(x - y)$ is divisible by 3.
$\Rightarrow y - x$ is divisible by 3.
$\Rightarrow yRx$

Thus, the relation R is symmetric.

(iii). For any $x, y, z \in X$, let xRy and yRz.

$\Rightarrow (x - y) + (y - z)$ is divisible by 3
$\Rightarrow x - z$ is divisible by 3
$\Rightarrow xRz$

Hence, the relation R is transitive.

Thus, the relation R is an equivalence relation.

Congruence Relation: Let I denote the set of all positive integers, and let m be apositive integer. For $x \in I$ and $y \in I$, define R as $R = \{(x, y)|\ x - y$ is divisible by m $\}$

The statement "$x - y$ is divisible by m" is equivalent to the statement that both x and y have the

same remainder when each is divided by m.

In this case, denote R by $\equiv$ and to write xRy as $x \equiv y \pmod m$, which is read as "x equals to y modulo m". The relation $\equiv$ is called a *congruence relation*.

Example: 83 ≡ 13(mod 5), since 83-13=70 is divisible by 5.

Example: Prove that the relation "congruence modulo *m*" over the set of positive integers is an equivalence relation.

Solution: Let N be the set of all positive integers and m be a positive integer.
We define the relation "congruence modulo m" on N as follows:

Let $x, y \in N$. $x \equiv y \pmod m$ if and only if $x - y$ is divisible by m.

Let $x, y, z \in N$. Then

(i). $x - x = 0.m$

$\Rightarrow x \equiv x \pmod m$ for all $x \in N$

(ii).Let $x \equiv y \pmod m$. Then, $x - y$ is divisible by m.
$\Rightarrow -(x - y) = y - x$ is divisible by m.
i.e., $y \equiv x \pmod m$

∴ The relation ≡ is symmetric.

$\Rightarrow x - y$ and $y - z$ are divisible by m. Now $(x - y) + (y - z)$ is divisible by m.
i.e., $x - z$ is divisible by m.
$\Rightarrow x \equiv z \pmod m$

∴ The relation ≡ is transitive.

Since the relation ≡ is reflexive, symmetric and transitive, the relation *congruence modulo m* is an equivalence relation.

Example: Let R denote a relation on the set of ordered pairs of positive integers such that (x,y)R(u,v) iff xv = yu. Show that R is an equivalence relation.

Solution: Let R denote a relation on the set of ordered pairs of positive integers.

Let x, y, u and v be positive integers. Given (x, y)R(u, v) if and only if xv = yu.

(i). Since xy = yx is true for all positive integers

⇒ (x, y)R(x, y), for all ordered pairs (x, y) of positive integers.

∴ The relation R is reflexive.

(**ii**). Let (x, y)R(u, v)

⇒ xv = yu ⇒ yu

= xv ⇒ uy = vx

⇒ (u, v)R(x, y)

∴ The relation R is symmetric.

(iii). Let x, y, u, v, m and n be positive integers

Let (x, y)R(u, v) and (u, v)R(m, n)

⇒ xv = yu and un = vm

⇒ xvun = yuvm

⇒ xn = ym, by canceling uv

⇒ (x, y)R(m, n)

∴ The relation R is transitive.

Since R is reflexive, symmetric and transitive, hence the relation R is an equivalence relation.

Compatibility Relations

Definition: A relation *R* in *X* is said to be a *compatibility relation* if it is reflexive and symmetric. Clearly, all equivalence relations are compatibility relations. A compatibility relation is sometimes denoted by ≈.

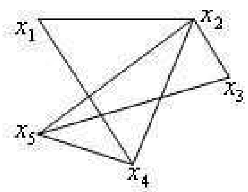
Example:Let $X = \{$ball, bed, dog, let, egg$\}$, and let the relation *R* be given by

$R = \{(x, y) \mid x, y \in X \wedge xRy$ if x and y contain some common letter$\}$.

Then *R* is a compatibility relation, and *x, y* are called compatible if *xRy*.

Note: ball≈bed, bed≈egg. But ball$\not\approx$egg. Thus ≈ is not transitive.

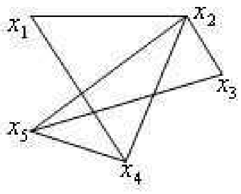
Denoting "ball" by $x1$, "bed" by $x2$, "dog" by $x3$, "let" by $x4$, and "egg" by $x5$, the graph of ≈ is given as follows:



Maximal Compatibility Block:

Let *X* be a set and ≈ a compatibility relation on *X*. A subset $A \subseteq X$ is called a *maximal compatibility block* if any element of *A* is compatible to every other element of *A* and no element of $X - A$ is compatible to all the elements of *A*.

Example: The subsets *{x*1*, x*2*, x*4*}*, *{x*2*, x*3*, x*5*}*, *{x*2*, x*4*, x*5*}*, *{x*1*, x*4*, x*5*}* are maximal compatibility blocks.

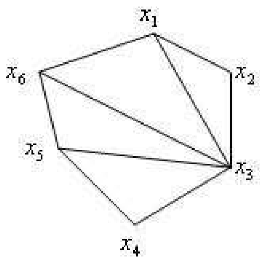


Example: Let the compatibility relation on a set *{x*1*, x*2*, ..., x*6*}* be given by the matrix:

$$\begin{array}{c|ccccc}
x_2 & 1 & & & & \\
x_3 & 1 & 1 & & & \\
x_4 & 0 & 0 & 1 & & \\
x_5 & 0 & 0 & 1 & 1 & \\
x_6 & 1 & 0 & 1 & 0 & 1 \\
\hline
 & x_1 & x_2 & x_3 & x_4 & x_5
\end{array}$$

Draw the graph and find the maximal compatibility blocks of the relation.

Solution:



The maximal compatibility blocks are $\{x_1, x_2, x_3\}$,$\{x_1, x_3, x_6\}$,$\{x_3, x_5, x_6\}$,$\{x_3, x_4, x_5\}$.

Composition of Binary Relations

Let R be a relation from X to Y and S be a relation from Y to Z. Then a relation written as $R \circ S$ is called a *composite relation* of R and S where $R \circ S = \{(x, z) \mid x \in X, z \in Z$, and there exists $y \in Y$ with $(x, y) \in R$ and $(y, z) \in S\}$.

Theorem: If R is relation from A to B, S is a relation from B to C and T is a relation from C to D then $T \circ (S \circ R) = (T \circ S) \circ R$

Example: Let $R = \{(1, 2), (3, 4), (2, 2)\}$ and $S = \{(4, 2), (2, 5), (3, 1), (1, 3)\}$. Find $R \circ S$, $S \circ R$, $R \circ (S \circ R)$, $(R \circ S) \circ R$, $R \circ R$, $S \circ S$, and $(R \circ R) \circ R$.

Solution: Given $R = \{(1, 2), (3, 4), (2, 2)\}$ and $S = \{(4, 2), (2, 5), (3, 1), (1, 3)\}$.

$R \circ S = \{(1, 5), (3, 2), (2, 5)\}$

$S \circ R = \{(4, 2), (3, 2), (1, 4)\} \neq R \circ S$

$(R \circ S) \circ R = \{(3, 2)\}$

$R \circ (S \circ R) = \{(3, 2)\} = (R \circ S) \circ R$

$R \circ R = \{(1, 2), (2, 2)\}$

$R \circ R \circ S = \{(4, 5), (3, 3), (1, 1)\}$

Example: Let A = {a, b, c}, and R and S be relations on A whose matrices are as given below:

Example: Let $A = \{a, b, c\}$, and R and S be relations on A whose matrices are as given below:

$$M_R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } M_S = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Find the composite relations $R \circ S$, $S \circ R$, $R \circ R$, $S \circ S$ and their matrices.

Solution:

$R = \{(a, a), (a, c), (b, a), (b, b), (b, c), (c, b)\}$

$S = \{(a, a), (b, b), (b, c), (c, a), (c, c)\}$. From these, we find that

$R \circ S = \{(a, a), (a, c), b, a), (b, b), (b, c), (c, b), (c, c)\}$

$S \circ R = \{(a, a), (a, c), (b, b), (b, a), (b, c), (c, a), (c, b), (c, c)\}$

$R \circ R = R^2 = \{(a, a), (a, c), (a, b), (b, a), (b, c), (b, b), (c, a), (c, b),$
$(c, c)\}$ $S \circ S = S^2 = \{(a, a), (b, b), (b, c), (b, a), (c, a), (c, c)\}$.

The matrices of the above composite relations are as given below:

$$M_{RO\,S} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}; M_{SO\,R} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}; M_{RO\,R} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix};$$

$$M_{SO\,S} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Transitive Closure

Let X be any finite set and R be a relation in X. The relation $R^+ = R \cup R^2 \cup R^3 \cup \cdots \cup R^n$ in X is called the *transitive closure* of R in X.

Example: Let the relation $R = \{(1, 2), (2, 3), (3, 3)\}$ on the set $\{1, 2, 3\}$. What is the transitive closure of R?

Solution: Given that $R = \{(1, 2), (2, 3), (3, 3)\}$.

The transitive closure of R is $R^+ = R \cup R^2 \cup R^3 \cup \cdots$

$= R = \{(1, 2), (2, 3), (3, 3)\}$

$R^2 = R \circ R = \{(1, 2), (2, 3), (3, 3)\} \circ \{(1, 2), (2, 3), (3, 3)\} = \{(1, 3), (2, 3), (3, 3)\}$

$R^3 = R^2 \circ R = \{(1, 3), (2, 3), (3, 3)\}$
$R^4 = R^3 \circ R = \{(1, 3), (2,3),(3, 3)\}$
$R^+ = R \cup R^2 \cup R^3 \cup R^4 \cup \ldots$

$= \{(1, 2), (2, 3), (3, 3)\} \cup \{(1, 3), (2, 3), (3, 3)\} \cup \{(1, 3), (2, 3), (3, 3)\} \cup \ldots$

$=\{(1, 2), (1, 3), (2, 3), (3, 3)\}$.

Therefore $R^+ = \{(1, 2), (1, 3), (2, 3), (3, 3)\}$.

Example: Let $X = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 3), (3, 4)\}$ be a relation on X. Find R^+.

Solution: Given $R = \{(1, 2), (2, 3), (3, 4)\}$

$R^2 = \{(1, 3), (2, 4)\}$

$R^3 = \{(1, 4)\}$

$R^4 = \{(1, 4)\}$

$R^+ = \{(1, 2), (2, 3), (3, 4), (1, 3), (2, 4), (1, 4)\}$.

Partial Ordering

A binary relation R in a set P is called a ***partial order relation* or a *partial ordering* in P iff R is reflexive, antisymmetric, and transitive.** i.e.,

- aRa for all $a \in P$
- aRb and $bRa \Rightarrow a = b$
- aRb and $bRc \Rightarrow aRc$

A set P together with a partial ordering R is called a *partial ordered set* or *poset*. The relation R is often denoted by the symbol $\leq$ which is diff erent from the usual less than equal to symbol. Thus, if $\leq$ is a partial order in P , then the ordered pair $(P, \leq)$ is called a **poset**.

Example: Show that the relation "greater than or equal to" is a partial ordering on the set of i ntegers.

Solution: Let Z be the set of all integers and the relation $R = ' \geq '$

(i). Since $a \geq a$ for every integer a, the relation $' \geq '$ is reflexive.

(ii). Let a and b be any two integers.

Let aRb and $bRa \Rightarrow a \geq b$ and $b \geq a$

$\Rightarrow a = b$

$\therefore$ The relation $' \geq '$ is antisymmetric. (iii).

Let a, b and c be any three integers.

Let aRb and $bRc \Rightarrow a \geq b$ and $b \geq c$

$\Rightarrow a \geq c$

$\therefore$ The relation $'\geq'$ is transitive.

Since the relation $'\geq'$ is reflexive, antisymmetric and transitive, $'\geq'$ is partial ordering on the set of integers. Therefore, $(Z, \geq)$ is a poset.

Example: Show that the inclusion $\subseteq$ is a partial ordering on the set power set of a set S.

Solution: Since (i). $A \subseteq A$ for all $A \subseteq S$, $\subseteq$ is reflexive.

(ii). $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$, $\subseteq$ is antisymmetric. (iii). $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$, $\subseteq$ is transitive.

Thus, the relation $\subseteq$ is a partial ordering on the power set of S.

Example: Show that the divisibility relation $'/'$ is a partial ordering on the set of positive integers. Solution: Let Z^+ be the set of positive integers.

Since (i). a/a for all $a \in Z^+$, $/$ is reflexive.

(ii). a/b and $b/a \Rightarrow a = b$, $/$ is antisymmetric.

(iii). a/b and $b/c \Rightarrow a/c$, $/$ is transitive.

It follows that $/$ is a partial ordering on Z^+ and $(Z^+, /)$ is a poset.

Note: On the set of all integers, the above relation is not a partial order as a and $-a$ both divide each other, but $a = -a$. i.e., the relation is not antisymmetric. Definition: Let $(P, \leq)$ be a partially ordered set. ***If for every $x, y \in P$ we have either $x \leq y \vee y \leq x$, then $\leq$ is called a simple ordering or linear ordering on P, and $(P, \leq)$ is called a totally ordered or simply ordered set or a chain.***

Note: It is not necessary to have $x \leq y$ or $y \leq x$ for every x and y in a poset P. In fact, x may not be related to y, in which case we say that x and y are incomparable.

Examples:

(i). The poset $(Z, \leq)$ is a totally ordered.
Since $a \leq b$ or $b \leq a$ whenever a and b are integers.

(ii). The divisibility relation $/$ is a partial ordering on the set of positive integers.

Therefore $(Z^+, /)$ is a poset and it is not a totally ordered, since it contain elements that are

incomparable, such as 5 and 7, 3 and 5.

Definition: In a poset $(P, \leq)$, an element $y \in P$ is said to *cover* an element $x \in P$ if $x < y$ and if there does not exist any element $z \in P$ such that $x \leq z$ and $z \leq y$; that is, y covers $x \Leftrightarrow (x < y \wedge (x \leq z \leq y \Rightarrow x = z \vee z = y))$.

Hasse Diagrams

A partial order ≤ on a set *P* can be represented by means of a diagram known as Hasse diagram of (*P*, ≤). In such a diagram,

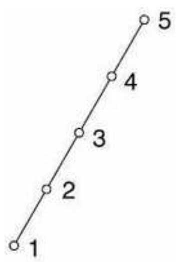
(i). Each element is represented by a small circle or dot.

(ii). The circle for $x \in P$ is drawn below the circle for $y \in P$ if $x < y$, and a line is drawn between *x* and *y* if *y* covers *x*.

(iii). If $x < y$ but *y* does not cover *x*, then *x* and *y* are not connected directly by a single line.

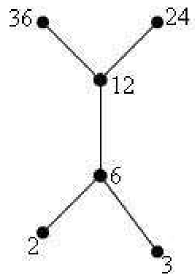
Note: For totally ordered set (*P*, ≤), the Hasse diagram consists of circles one below the other. The poset is called a chain.

Example: Let *P* = {1, 2, 3, 4, 5} and ≤ be the relation "less than or equal to" then the Hasse diagram is:



It is a totally ordered set.

Example: Let *X* = {2, 3, 6, 12, 24, 36}, and the relation ≤ be such that *x* ≤ *y* if *x* divides *y*. Draw the Hasse diagram of (*X*, ≤). Solution: The Hasse diagram is is shown below:



It is not a total order set.

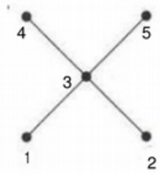
Example: Draw the Hasse diagram for the relation *R* on *A* = {1, 2, 3, 4, 5} whose relation matrix given below:

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

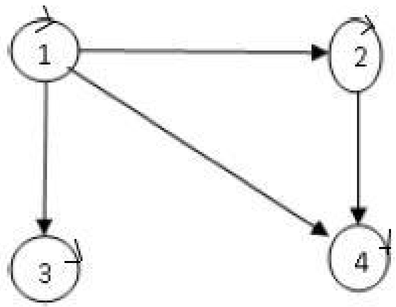
Solution:

R= {(1, 1), (1, 3), (1, 4), (1, 5), (2, 2), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5), (4, 4), (5.5)}.

Hasse diagram for M_R is



Example: A partial order R on the set A = {1, 2, 3, 4} is represented by the following digraph.Draw the Hasse diagram for R.

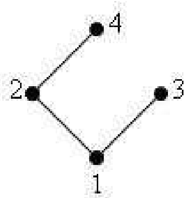


Solution: By examining the given digraph , we find that

R= {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)}.

We check that *R* is reflexive, transitive and antisymmetric. Therefore, *R* is partial order relation on *A*.

The hasse diagram of *R* is shown below:



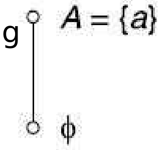
Example: Let *A* be a finite set and $\rho(A)$ be its power set. Let ⊆ be the inclusion relation on the elements of $\rho(A)$. Draw the Hasse diagram of $\rho(A)$, ⊆) for

- $A = \{a\}$
- $A = \{a, b\}$.

Solution: (i). Let $A = \{a\}$

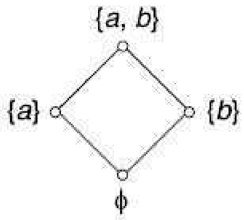
$\rho(A) = \{\phi, a\}$

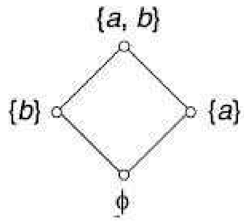
Hasse diagram of ($\rho(A)$, ⊆) is shown in Fig



(ii). Let $A = \{a, b\}$. $\rho(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$.

The Hasse diagram for ($\rho(A)$, ⊆) is shown in fig:



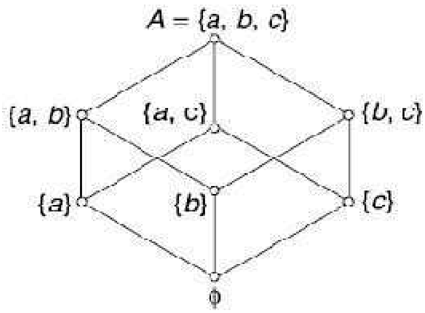


Example: Draw the Hasse diagram for the partial ordering ⊆ on the power set *P* (*S*) where *S* = *{a, b, c}*.

Solution: $S = \{a, b, c\}$.

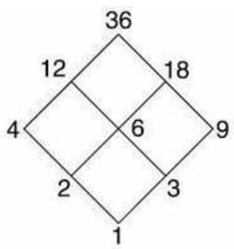
$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Hasse diagram for the partial ordered set is shown in fig:



Example: Draw the Hasse diagram representing the positive divisions of 36 (i.e., *D*36).

Solution: We have $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$ if and only *a* divides *b*.
The Hasse diagram for *R* is shown in Fig.



Minimal and Maximal elements(members): Let $(P, \leq)$ denote a partially or-dered set. An element $y \in P$ is called a ***minimal member*** of P relative to $\leq$ if for no $x \in P$, is $x < y$.

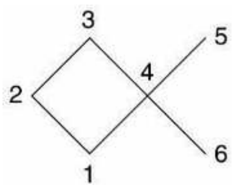
Similarly an element $y \in P$ is called a **maximal member** of P relative to the partial ordering $\leq$ if for no $x \in P$, is $y < x$.

Note:

(i).The minimal and maximal members of a partially ordered set need not unique.

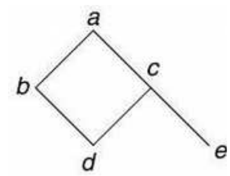
(ii)Maximal and minimal elements are easily calculated from the Hasse diagram. They are the '*top*' and '*bottom*' elements in the diagram.

Example:



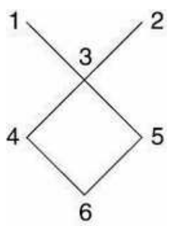
In the Hasse diagram, there are two maximal elements and two minimal elements. The elements 3, 5 are maximal and the elements 1 and 6 are minimal.

Example: Let A = {a, b, c, d, e} and let the partial order on A in the natural way.
The element *a* is maximal.
The elements *d* and *e* are minimal.



Upper and Lower Bounds: Let $(P, \leq)$ be a partially ordered set and let $A \subseteq P$. Any element $x \in P$ is called an ***upper bound*** for A if for all $a \in A$, $a \leq x$. Similarly, any element $x \in P$ is called a ***lower bound*** for A if for all $a \in A$, $x \leq a$.

Example: A = {1, 2, 3, ..., 6} be ordered as pictured in figure.

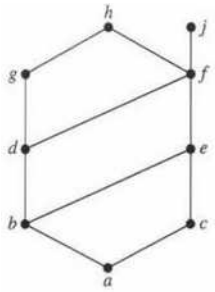


If $B = \{4, 5\}$ then the upper bounds of B are 1, 2, 3. The lower bound of B is 6.

Least Upper Bound and Greatest Lower Bound:

Let $(P, \leq)$ be a partial ordered set and let $A \subseteq P$. An element $x \in P$ is a *least upper bound* or *supremum* for A if x is an upper bound for A and $x \leq y$ where y is any upper bound for A. Similarly, the *the greatest lower bound* or *in* *mum* for A is an element $x \in P$ such that x is a lower bound and $y \leq x$ for all lower bounds y.

Example: Find the great lower bound and the least upper bound of {b, d, g}, if they exist

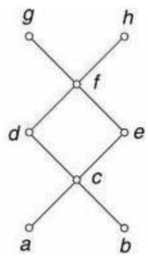


in the **poset shown in fig:**

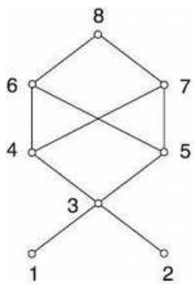
Solution: The upper bounds of *{b, d, g}* are g and h. Since $g < h$, g is the least upper bound. The lower bounds of *{b, d, g}* are a and b. Since $a < b$, b is the greatest lower bound.

Example: Let $A = \{a, b, c, d, e, f, g, h\}$ denote a partially ordered set whose Hasse diagram is shown in Fig:

If B = {c, d, e} then f, g, h are upper bounds of B. The element f is least upper bound.



Example: Consider the poset $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ whose Hasse diagram is shown in Fig and let $B = \{3, 4, 5\}$



The elements 1, 2, 3 are lower bounds of B. 3 is greatest lower bound.

Functions

A function is a special case of relation.

Definition: Let X and Y be any two sets. A relation f from X to Y is called a **function** if for every $x \in X$, there is a unique element $y \in Y$ such that $(x, y) \in f$.

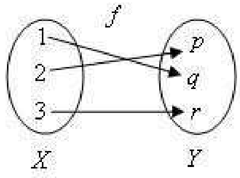
Note: The definition of function requires that a relation must satisfies two additional conditions in order to qualify as a function. These conditions are as follows:

(i) For every $x \in X$ must be related to some $y \in Y$, i.e., the domain of f must be X and nor merely a subset of X.

(ii).Uniqueness, i.e., $(x, y) \in f$ and $(x, z) \in f \Rightarrow y = z$.

The notation $f : X \to Y$, means f is a function from X to Y .

Example: Let $X = \{1, 2, 3\}$, $Y = \{p, q, r\}$ and $f = \{(1, p), (2, q), (3, r)\}$ then $f(1) = p$, $f(2) = q$, $f(3) = r$. Clearly f is a function from X to Y .



Domain and Range of a Function: If $f : X \to Y$ is a function, then X is called the Domain of f and the set Y is called the **codomain** of f. The range of f is defined as the set of all images under f.

It is denoted by $f(X) = \{y|$ for some x in X, $f(x) = y\}$ and is called the **image** of X in Y . The Range f is also denoted by R_f .

Example: If the function f is defined by $f(x)=x^2 + 1$ on the set $\{-2, -1, 0, 1, 2\}$, find the range of f.

Solution: $f(-2) = (-2)^2 + 1 = 5$

$$f(-1) = (-1)^2 + 1 = 2$$

$$f(0) = 0 + 1 = 1$$
$$f(1) = 1 + 1 = 2$$
$$f(2) = 4 + 1 = 5$$

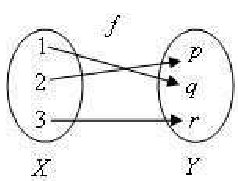
Therefore, the range of $f = \{1, 2, 5\}$.

Types of Functions

One-to-one(Injection): A mapping $f : X \to Y$ is called *one-to-one* if distinct elements of X are mapped into distinct elements of Y , i.e., f is one-to-one if

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

or equivalently $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for $x_1, x_2 \in X$.



Example: $f : R \rightarrow R$ defined by $f(x) = 3x$, $\forall x \in R$ is one-one, since

$$f(x_1)=f(x_2) \Rightarrow 3x_1=3x_2 \Rightarrow x_1=x_2 \text{ , } \forall x_1, x_2 \in R.$$

Example: Determine whether $f : Z \rightarrow Z$ given by $f(x) = x^2$, $x \in Z$ is a one-to-One function.

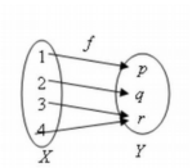
Solution: The function $f : Z \rightarrow Z$ given by $f(x) = x^2$, $x \in Z$ is not a one-to-one function. This is because both 3 and -3 have 9 as their image, which is against the definition of a one-to-one function.

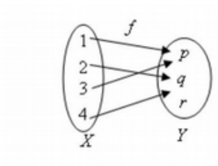
Onto(Surjection): A mapping $f : X \rightarrow Y$ is called onto if the range set $R_f=Y$.

If $f : X \rightarrow Y$ is onto, then each element of Y is f-image of atleast one element of X.

i.e., $\{f(x) : x \in X\} = Y$.

If f is not onto, then it is said to be into.

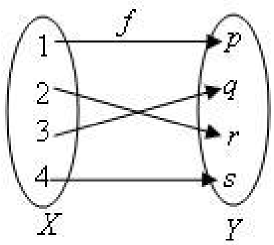




Surjective Not Surjective

Example: $f : R \rightarrow R$, given by $f(x)=2x$, $\forall x \in R$ is onto.

Bijection or One-to-One, Onto: A mapping $f : X \rightarrow Y$ is called one-to-one, onto or bijective if it is both one-to-one and onto. Such a mapping is also called a one-to-one correspondence between X and Y .



Example: Show that a mapping $f : R \rightarrow R$ defined by $f(x) = 2x + 1$ for $x \in R$ is a bijective map from R to R.

Solution: Let $f : R \rightarrow R$ defined by $f(x) = 2x + 1$ for $x \in R$. We need to prove that f is a bijective

map, i.e., it is enough to prove that f is one-one and onto.

- Proof of f being one-to-one

  Let x and y be any two elements in R such that $f(x) = f(y)$

  $\Rightarrow 2x+1=2y+1$

  $\Rightarrow x=y$

Thus, $f(x)=f(y) \Rightarrow x=y$
This implies that f is one-to-one.

Proof of *f* being onto
Let y be any element in the codomain *R*
$\Rightarrow f(x)=y$
$\Rightarrow 2x+1=y$
$\Rightarrow x=(y-1)/2$
Clearly, $x=(y-1)/2 \in R$
Thus, every element in the codomain has pre-image in the domain.
This implies that f is onto
Hence, f is a bijective map.

Identity function: Let X be any set and f be a function such that $f : X \rightarrow X$ is defined by $f(x) = x$ for all $x \in X$. Then, f is called the identity function or identity transformation on X. It can be denoted by *I* or I_x .

Note: The identity function is both one-to-one and onto.

Let $I_x(x)=I_x(y)$
$\Rightarrow x=y$
$\Rightarrow I_x$ is one-to-one
I_x is onto since $x=I_x(x)$ for all x.

Composition of Functions

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two functions. Then the composition of f and g denoted by **g ◦ f**,is the function from X to Z defined as

$$(g \circ f)(x) = g(f(x)), \text{ for all } x \in X.$$

Note. In the above definition it is assumed that the range of the function f is a subset of Y (the Domain of g), i.e., $R_f \subseteq D_g$. $g \circ f$ is called the ***left composition g with f***.

Example: Let X = {1, 2, 3}, Y = {p, q} and Z = {a, b}. Also let $f:X \rightarrow Y$ be f = {(1, p), (2, q), (3,q)} and $g : Y \rightarrow Z$ be given by g = {(p, b), (q, b)}. Find $g \circ f$.

Solution: $g \circ f$ = {(1, b), (2, b), (3, b)}.

Example: Let X = {1, 2, 3} and f, g, h and s be the functions from X to X given by

f = {(1, 2), (2, 3), (3, 1)} g = {(1, 2), (2, 1), (3, 3)}
h = {(1, 1), (2, 2), (3, 1)} s = {(1, 1), (2, 2), (3, 3)}

Find $f \circ f$; $g \circ f$; $f \circ h \circ g$; $s \circ g$; $g \circ s$; $s \circ s$; and $f \circ s$.

Solution:

$f \circ g$ = {(1, 3), (2, 2), (3, 1)}
$g \circ f$ = {(1, 1), (2, 3), (3, 2)} $\neq f \circ g$

$f \circ h \circ g = f \circ (h \circ g) = f \circ \{(1, 2), (2, 1), (3, 1)\}$

$= \{(1, 3), (2, 2), (3, 2)\}$

$s \circ g = \{(1, 2), (2, 1), (3, 3)\} = g$

$g \circ s = \{(1, 2), (2, 1), (3, 3)\}$

$\therefore s \circ g = g \circ s = g$

$s \circ s = \{(1, 1), (2, 2), (3, 3)\} = s$

$f \circ s = \{(1, 2), (2, 3), (3, 1)\}$

Thus, $s \circ s = s$, $f \circ g \neq g \circ f$, $s \circ g = g \circ s = g$ and $h \circ s = s \circ h = h$.

Example: Let f(x) = x + 2, g(x) = x − 2 and h(x) = 3x for x ∈ R, where R is the set of real numbers. Find $g \circ f$; $f \circ g$; $f \circ f$; $g \circ g$; $f \circ h$; $h \circ g$; $h \circ f$; and $f \circ h \circ g$.

Solution: $f : R \to R$ is defined by $f(x) = x + 2$

f: $R \to R$ is defined by $g(x) = x - 2$

$h : R \to R$ is defined by $h(x) = 3x$

- $g \circ f : R \to R$

  Let $x \in R$. Thus, we can write

  $(g \circ f)(x) = g(f(x)) = g(x + 2) = x + 2 - 2 = x$

  $\therefore (g \circ f)(x) = \{(x, x)|\ x \in R\}$
- $(f \circ g)(x) = f(g(x)) = f(x - 2) = (x - 2) + 2 = x$

  $\therefore f \circ g = \{(x, x)|\ x \in R\}$
- $(f \circ f)(x) = f(f(x)) = f(x + 2) = x + 2 + 2 = x + 4$

  $\therefore f \circ f = \{(x, x + 4)|\ x \in R\}$
- $(g \circ g)(x) = g(g(x)) = g(x - 2) = x - 2 - 2 = x - 4$

  $\Rightarrow g \circ g = \{(x, x - 4)|\ x \in R\}$
- $(f \circ h)(x) = f(h(x)) = f(3x) = 3x + 2$

  $\therefore f \circ h = \{(x, 3x + 2)|\ x \in R\}$
- $(h \circ g)(x) = h(g(x)) = h(x - 2) = 3(x - 2) = 3x - 6$

  $\therefore h \circ g = \{(x, 3x - 6)|\ x \in R\}$
- $(h \circ f)(x) = h(f(x)) = h(x + 2) = 3(x + 2) = 3x + 6$ $h \circ f =$

  $\{(x, 3x + 6)|\ x \in R\}$
- $(f \circ h \circ g)(x) = [f \circ (h \circ g)](x)$

  $f(h \circ g(x)) = f(3x - 6) = 3x - 6 + 2 = 3x - 4$

  $\therefore f \circ h \circ g = \{(x, 3x - 4)|\ x \in R\}$.

Exercise: What is composition of functions? Let f and g be functions from R to R, where R is a set of real numbers defined by $f(x) = x^2 + 3x + 1$ and $g(x) = 2x - 3$. Find the composition of functions: i) $f \circ f$ ii) $f \circ g$ iii) $g \circ f$.

Inverse Functions

A function $f : X \to Y$ is said to be ***invertible*** of its inverse function f^{-1} is also function from the range of f into X.

Theorem: A function $f : X \to Y$ is invertible $\Leftrightarrow$ f is one-to-one and onto.

Example: Let $X = \{a, b, c, d\}$ and $Y = \{(1, 2, 3, 4\}$ and let $f : X \to Y$ be given by f = $\{(a, 1), (b, 2), (c, 2), (d, 3)\}$. Is f^{-1} a function?

Solution:

$f^{-1} = \{(1, a), (2, b), (2, c), (3, d)\}$. Here, 2 has two distinct images b and c.

Therefore, f^{-1} is not a function.

Example: Let R be the set of real numbers and $f : R \to R$ be given by $f = \{(x, x^2)| \ x \in R\}$. Is f^{-1} a function?

Solution: The inverse of the given function is defined as $f^{-1} = \{(x^2, x)| \ x \in R\}$. Therefore, it is not a function.

Theorem: If $f : X \to Y$ and $g : Y \to X$ be such that $g \circ f = I_X$ and $f \circ g = I_Y$, then f and g are both invertible. Furthermore, $f^{-1} = g$ and $g^{-1} = f$.

Example: Let $X = \{1, 2, 3, 4\}$ and f and g be functions from X to X given by $f = \{(1, 4), (2, 1), (3,2), (4, 3)\}$ and $g = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$. Prove that f and g are inverses of each other.

Solution: We check that

$(g \circ f)(1) = g(f(1)) = g(4) = 1 \quad = I_X(1), \quad (f \circ g)(1) \quad = f(g(1)) = f(2) = 1 = I_X(1).$

$(g \circ f)(2) = g(f(2)) = g(1) = 2 \quad = I_X(2), \quad (f \circ g)(2) \quad = f(g(2)) = f(3) = 2 = I_X(2).$

$(g \circ f)(3) = g(f(3)) = g(2) \quad = 3 = I_X(3), \quad (f \circ g)(3) \quad = f(g(3)) = f(4) = 3 = I_X(3).$

$(g \circ f)(4) = g(f(4)) = g(3) \quad = 4 = I_X(4), \quad (f \circ g)(4) \quad = f(g(4)) = f(1) = 4 = I_X(4).$

Thus, for all $x \in X$, $(g \circ f)(x) = I_X(x)$ and $(f \circ g)(x) = I_X(x)$. Therefore g is inverse of f and f is inverse of g.

Example: Show that the functions $f(x) = x^3$ and $g(x) = x^{1/3}$ for $x \in R$ are inverses of one another.

Solution: $f : R \to R$ is defined by $f(x) = x^3$; f: $R \to R$ is defined by $g(x) = x^{1/3}$

$(f \circ g)(x) = f(g(x)) = f(x^{1/3}) = x^{3(1/3)} = x = I_X(x)$

i.e., $(f \circ g)(x) = I_X(x)$

and $(g \circ f)(x) = g(f(x)) = g(x^3) = x^{3(1/3)} = x = I_X(x)$

i.e., $(g \circ f)(x) = I_X(x)$

Thus, $f = g^{-1}$ or $g = f^{-1}$

i.e ., f and g are inverses of one other.

****Example: f : $R \to R$ is defined by $f(x)=ax+b$, for $a, b \in R$ and $a \neq 0$. Show that f is invertible and find the inverse of f.**

(i) First we shall show that f is one-to-one

ii Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

iii $ax_1 + b = ax_2 + b$

$ax_1 = ax_2$

$\Rightarrow x_1 = x_2$

$\therefore f$ is one-to-one.

To show that f is onto.

Let $y \in R$(codomain) such that $y = f(x)$ for some $x \in R$.

$\Rightarrow y=ax + b$

$\Rightarrow ax = y - b$

$\Rightarrow x = (y-b)/a$

Given $y \in R$(codomain), there exists an element $x = (y-b)/a \in R$ s~~uch~~ that $f(x) = y$.

$\therefore f$ is onto

$\Rightarrow f$ is invertible and $f^{-1}(x)= (x-b)/a$

Example: Let $f : R \to R$ be given by $f(x) = x^3 - 2$. Find f^{-1}.

(i) First we shall show that f is one-to-one

Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow x_1^3 - 2 = x_2^3 -$

$2 \Rightarrow x_1^3 = x_2^3$

$\Rightarrow x_1 = x_2$

$\therefore f$ is one-to-one.

- To show that f is onto.

$\Rightarrow y = x^3 - 2$

$\Rightarrow x^3 = y+2$

$\Rightarrow x= \sqrt[3]{y+2}$

Given $y \in R$(codomain), there exists an element $x = \sqrt[3]{y+2} \in R$ such that $f(x) = y$.

$\therefore f$ is onto

$\Rightarrow f$ is invertible and $f^{-1}(x) = \sqrt[3]{x+2}$

Floor and Ceiling functions:

Let x be a real number, then the least integer that is not less than x is called the CEILING of x.
The CEILING of x is denoted by $\lceil x \rceil$.

Examples: $\lceil 2.15 \rceil = 3$, $\lceil \sqrt{5} \rceil = 3$, $\lceil -7.4 \rceil = -7$, $\lceil -2 \rceil = -2$
Let x be any real number, then the greatest integer that does not exceed x is called the Floor of x.
The FLOOR of x is denoted by $\lfloor x \rfloor$.
Examples: $\lfloor 5.14 \rfloor = 5$, $\lfloor \sqrt{5} \rfloor = 2$, $\lfloor -7.6 \rfloor = -8$, $\lfloor 6 \rfloor = 6$, $\lfloor -3 \rfloor = -3$

Example: Let f and g abe functions from the positive real numbers to positive real numbers defined by $f(x) = \lfloor 2x \rfloor$, $g(x)=x^2$. Calculate $f \circ g$ and $g \circ f$.
Solution: $f \circ g(x) = f(g(x)) = f(x^2) = \lfloor 2x^2 \rfloor$
$g \circ f(x) = g(f(x)) = g(\lfloor 2x \rfloor) = (\lfloor 2x \rfloor)^2$

Recursive Function

Total function: Any function $f : N^n \rightarrow N$ is called total if it is defined for every n-tuple in N^n.
Example: $f(x, y) = x + y$, which is defined for all $x, y \in N$ and hence it is a total function.

Partial function: If $f : D \rightarrow N$ where $D \subseteq N^n$, then f is called a partial function.
Example: $g(x, y) = x - y$, which is defined for only $x, y \in N$ which satisfy $x \geq y$.
Hence $g(x, y)$ is partial.

Initial functions:
The initial functions over the set of natural numbers is given by
Zero function Z: $Z(x) = 0$, for all x.
Successor function S: $S(x) = x + 1$, for all x.
Projection function $U_i^n : U_i^n (x_1 , x_2 , \ldots, x_n) = x_i$ for all n tuples
$(x_1 , x_2 , \ldots, x_n), 1 \leq i \leq n$.
Projection function is also called ***generalized identity function.***
For example, $U_1^1(x)=x$ *for every* $x \in N$ *is the identity function.*$_1$

Composition of functions of more than one variable:
The operation of composition will be used to generate the other function.
Let $f_1(x, y)$, $f_2(x, y)$ and $g(x, y)$ be any three functions. Then the composition of g with f_1 and f_2 is defined as a function $h(x, y)$ given by

$$h(x, y) = g(f_1(x,y), f_2(x, y)).$$

In general, let f_1 , f_2 , ..., f_n each be partial function of m variables and g be a partial function of n variables. Then the composition of g with f 1 , f 2 , ..., f n produces a partial function *h* given by

$$h(x_1, x_2, ..., x_m) = g(f_1(x_1,x_2, ..., x_m), ..., f_n(x_1, x_2, ...x_m)).$$

Note: The function h is total *iff* f_1 , f_2 , ..., f_n and g are total.

Example: *Let f1(x, y)=x+y,* $f_2(x, y)=xy+y^2$ *and g(x,y)=xy. Then*

$h(x, y) = g(f_1(x, y), f_2(x, y))$

$= g(x + y, xy + y^2)$

$= (x + y)(xy + y^2)$

Recursion: The following operation which defines a function $f(x_1, x_2, ..., x_n, y)$ *of* $n + 1$ variables by using other functions $g(x_1, x_2, .., x_n)$ *and* $h(x_1, x_2, ..., x_n, y, z)$ of n and n + 2 variables, respectively, is called **recursion**.

$f(x_1, x_2, ..., x_n, 0) = g(x_1, x_2, ..., x_n)$

$f(x_1, x_2, ..., x_n, y+1)=h(x_1, x_2, ...,x_n,y,f(x_1,x_2, ..., x_n,y))$

where y is the inductive variable.

Primitive Recursive: A function f is said to be Primitive recursive iff it can be obtained from the initial functions by a finite number of operations of composition and recursion.

Example: Show that the function f(x, y) = x + y is primitive recursive. Hence compute the value of f(2, 4).

Solution: Given that f(x, y) = x + y.

Here, f(x, y) is a function of two variables. If we want f to be defined by recursion, we need a function g of single variable and a function h of three variables. Now,

$$\begin{aligned} f(x, y + 1) &= x + (y + 1) \\ &= (x + y) + 1 \\ &= f(x, y) + 1. \end{aligned}$$

Also, f(x, 0) = x.

We define f(x, 0) as

$$\begin{aligned} f(x,0)=x&=U_1^1(x) \\ &= S(f(x, y)) \\ &=S(U_3^3(x, y,f(x,y))) \end{aligned}$$

If we take $g(x)=U_1^1(x)$ *and* $h(x,y,z)=S(U_3^3(x,y,z))$, *we get* $f(x,0)=g(x)$ *and* $f(x, y + 1) =h(x, y, z)$.

Thus, f is obtained from the initial functions U_1^1, U_3^3 , and S by applying composition once and recursion once.

Hence f is **primitive recursive.**

Here,

$$f(2,0)=2$$
$$\begin{aligned} f(2,4) &= S(f(2,3)) \\ &= S(S(f(2,2))) \\ &= S(S(S(f(2,1)))) \\ &= S(S(S(S(f(2,0))))) \\ &= S(S(S(S(2)))) \\ &= S(S(S(3))) \\ &= S(S(4)) \\ &= S(5) \\ &= 6 \end{aligned}$$

Example: Show that f(x, y) = x ∗ y is primitive recursion.

Solution: Given that $f(x, y) = x*y$.

Here, f(x, y) is a function of two variables. If we want f to be defined by recursion, we need a function g of single variable and a function h of three variables. Now, f(x, 0) = 0 and $f(x, y + 1) = x * (y + 1) = x * y$

$$f(x, y) + x$$

We can write

$$f(x,0)=0=Z(x) \text{ and}$$
$$f(x,y+1)=f1(U_3^3(x,y,f(x,y)), U_1^3(x,y,f(x,y)))$$

where $f_1(x, y) = x + y$, which is primitive recursive. By taking g(x) = Z(x) = 0 and h defined by h(x, y, z) = f1(U33(x,y,z),U13(x,y,z))=f(x,y+1), we see that f defined by recursion. Since g and h are primitive recursive, f is primitive recursive.

Example: Show that $f(x, y)=x^y$ is primitive recursive function.

Solution: Note that $x^0= 1$ for $x \neq 0$ and we put $x^0=0$ for $x = 0$.

Also, $x^{y+1}=x^y*x$

Here $f(x, y)=x^y$ is defined as

$$f(x,0)=1=S(0)=S(Z(x))$$
$$f(x, y + 1) = x * f(x, y)$$
$$U\ 1\ (x, y, f(x, y)) * U\ 3\ (x, y, f(x, y))$$

h(x, y, f(x, y) = f 1 (U 1 (x, y, f(x, y)), U 3 (x, y, f(x, y))) where f 1 (x, y) = x ∗ y, which is primitive recursive.

∴ *f(x, y) is a* ***primitive recursive function.***