VISPAD INSTITUTE OF
TECHNOLOGY
Nea onnim no sua a ohu,

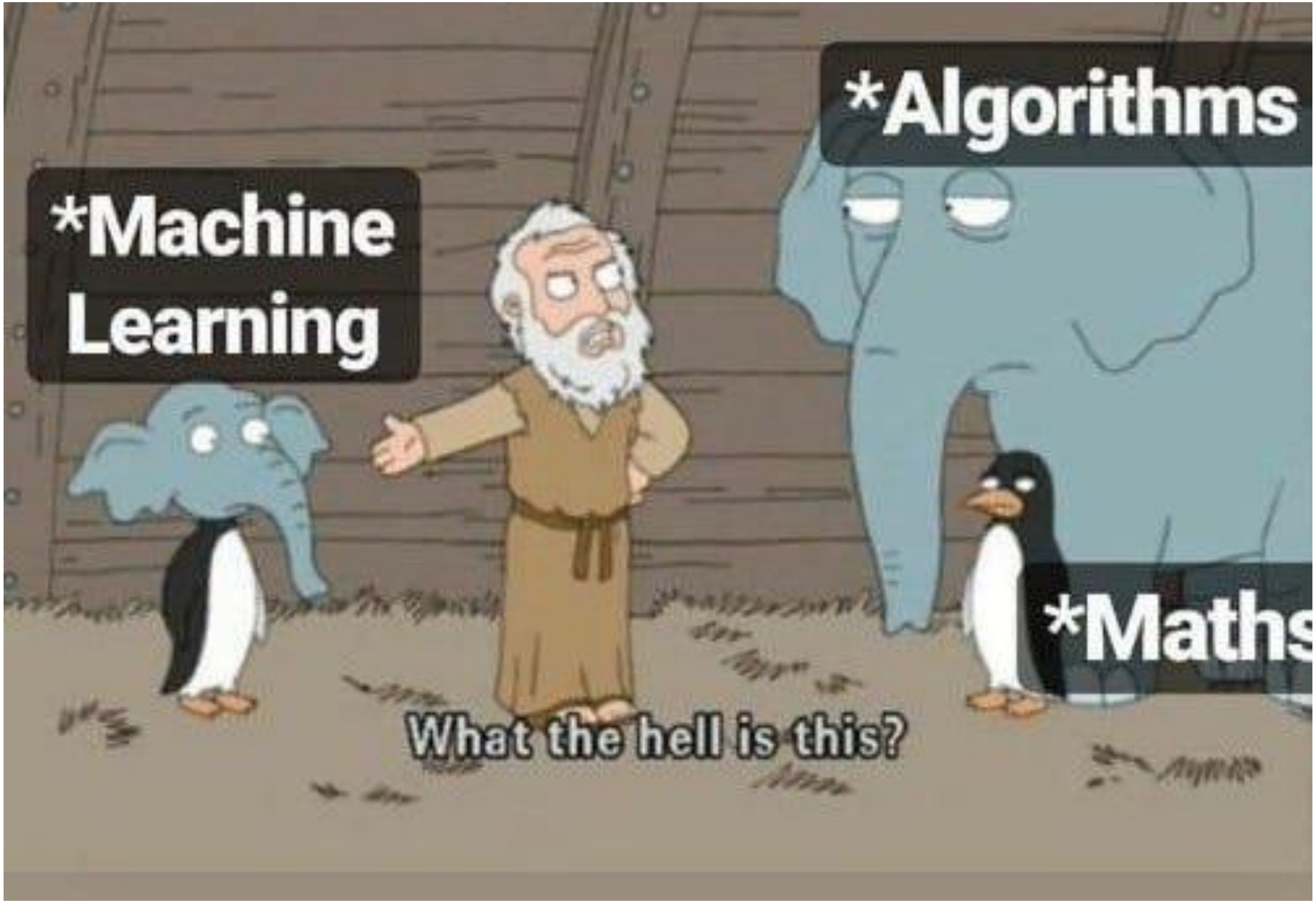
Vectors

Fall 2025 💻

(October – December Virtual Internship)

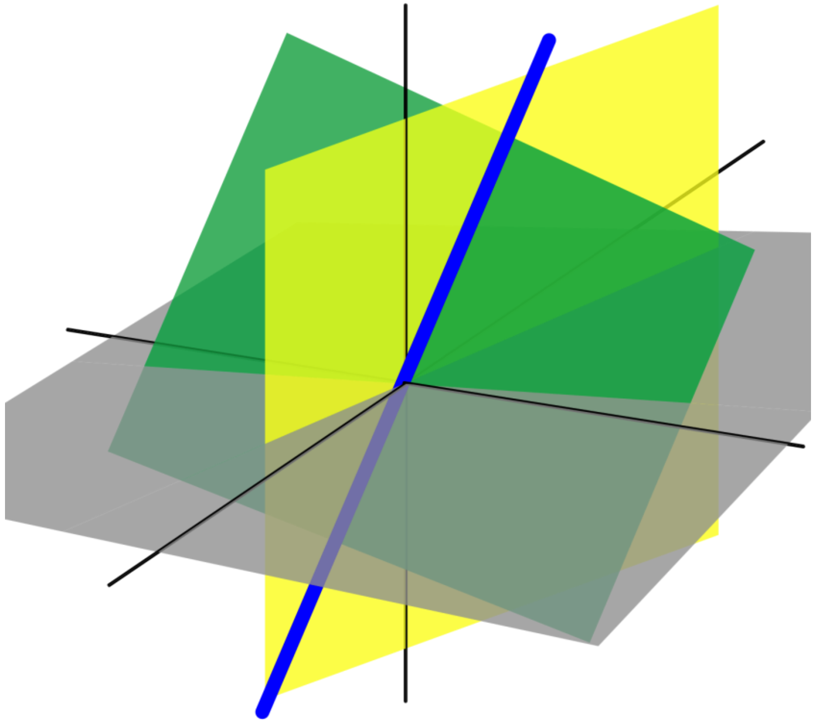
At the end of the session the students or candidates should be able to understand and work with:

- Vector spaces
- Basis
  - Linear combination and independence
  - Spans of vectors
  - Bases
  - Finite dimensional vectors spaces
  - Why bases so important
  - The existences of bases
  - Sub spaces


*Machine
Learning
*Algorithms
*Maths
What the hell is this?

- Mathematics of machine learning rests upon three pillars:
  - linear algebra
  - calculus
  - probability theory
- Linear algebra describes how to **represent and manipulate data**; calculus helps us **fit the models**; while **probability theory helps interpret them.**

Why Linear Algebra



- Linear algebra is essential to forming a **complete** understanding of machine learning.

- The techniques from this discipline belong to a shared collection of algorithm widely used in **artificial intelligence.**

- Its properties and methods allow for faster computation of complex systems and extraction of hidden relationships in sets of data.

Notation

- $\mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ denote, respectively, the **integers**, the **natural numbers (nonnegative integers)**, the **rational numbers**, the **real numbers**, and the **complex numbers**.

- $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$, $\prod_{i=1}^{n} a_i = a_1 a_2 \ldots a_2$, are used for the sum and product of the quantities $a_1, \ldots, a_n$

What is a Vector

- In a simple term, *a vector is a list of numbers where the position of each item in this structure matters.*
- A ***column vector*** consists of a finite number of real numbers, known as its entries, arranged in a vertical column.

- Given a position integer $n = 1,2,3, \ldots,$ the set of all vectors with n entries is denoted by $\mathbb{R}^n$.
- Examples of vectors in $\mathbb{R}^3$:

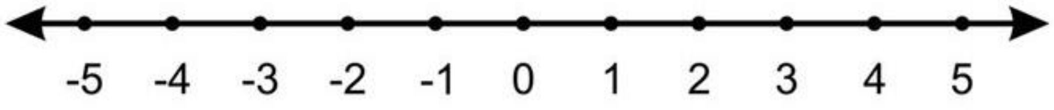
$$\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} \pi \\ \sqrt{2} \\ -\frac{4}{7} \end{pmatrix}, \begin{pmatrix} 3.14 \\ 1.41 \\ -.57 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- In general, a vector $\boldsymbol{v} \in \mathbb{R}^n$ has the form: $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_2 \end{pmatrix}$,

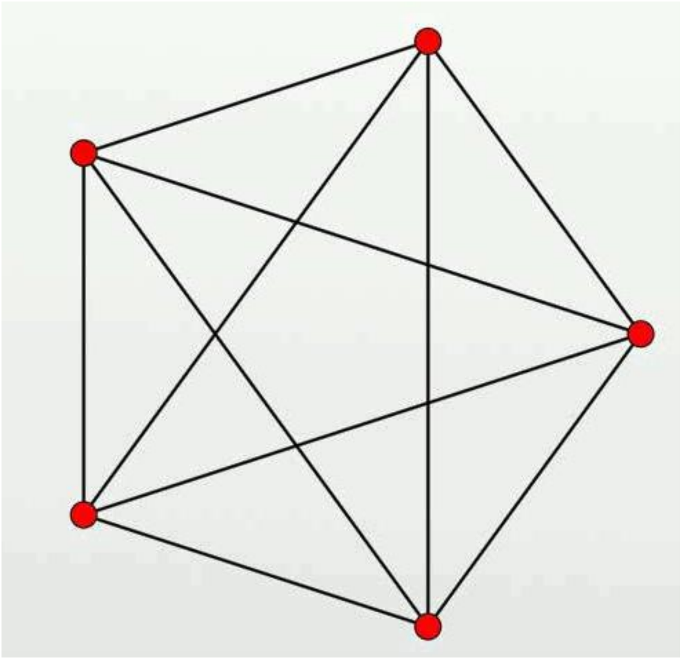
  where $v_1, \ldots, v_n \in \mathbb{R}$.

- Two vectors are equal, $\boldsymbol{v} = \boldsymbol{w}$, if and only if they have the same of entries, so $v, w \in \mathbb{R}^n$ for some $0 < n \in \mathbb{N}$, and all their entries are equal: $v_i = w_i, i = 1, \ldots, n$.

- The set $\mathbb{R}^n$ is known as $n-$ $dimensional\ Euclidean\ space$, which forms the basic setting for Euclidean geometry
- $\mathbb{R}^1 \simeq \mathbb{R}$ can be identified as the real line;
- $\mathbb{R}^2$ is the two-dimensional Euclidean plane;
- $\mathbb{R}^3$ can be identified with three-dimensional space



(A)



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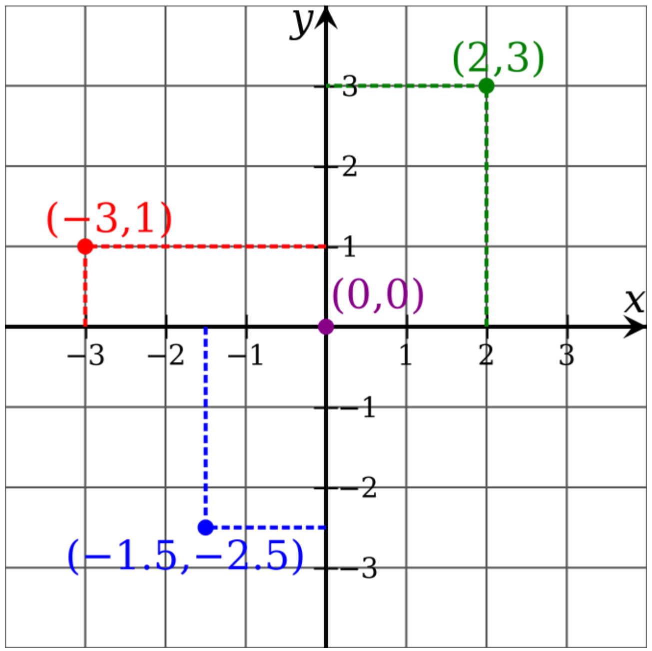

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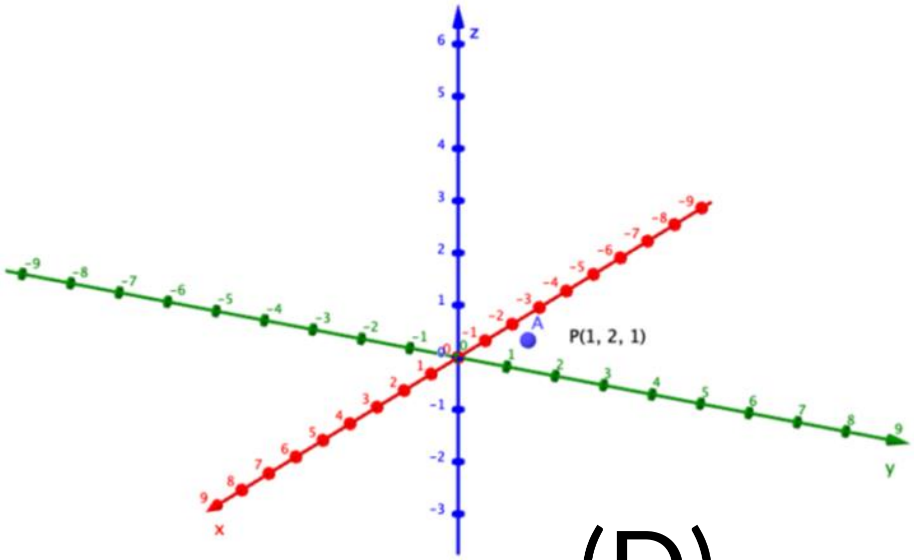


(D)

- A *row vector* contains a finite number of real numbers arranged in a horizontal row.
- ***Column vectors*** are the more important of the two.
- The operation of converting a column vector into a row vector, and vice versa, is known as the ***transpose***.
- It is denoted with a T superscript.

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}^T = (v_1, v_2, \ldots, v_n)$$

$$(v_1, v_2, \ldots, v_n)^T = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

Transposing twice takes you back to where you started:

$(\boldsymbol{v}^T)^T$

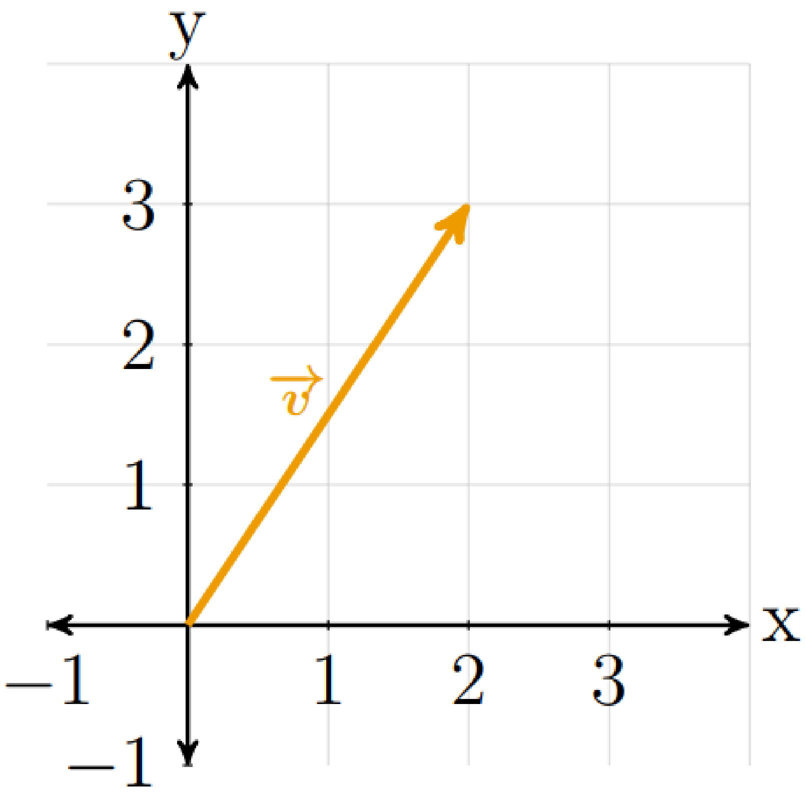
Example

If you are analysing the **height** and **weight** of a class of students, in this domain, a **two-dimensional vector** will represent each student.

$$\vec{v} = \binom{v_1}{v_2} = \binom{1.64}{64}$$

v_1 represent height, v_2 represents weight;

- In linear algebra, a vector is an arrow with a **direction** dictated by its coordinates.

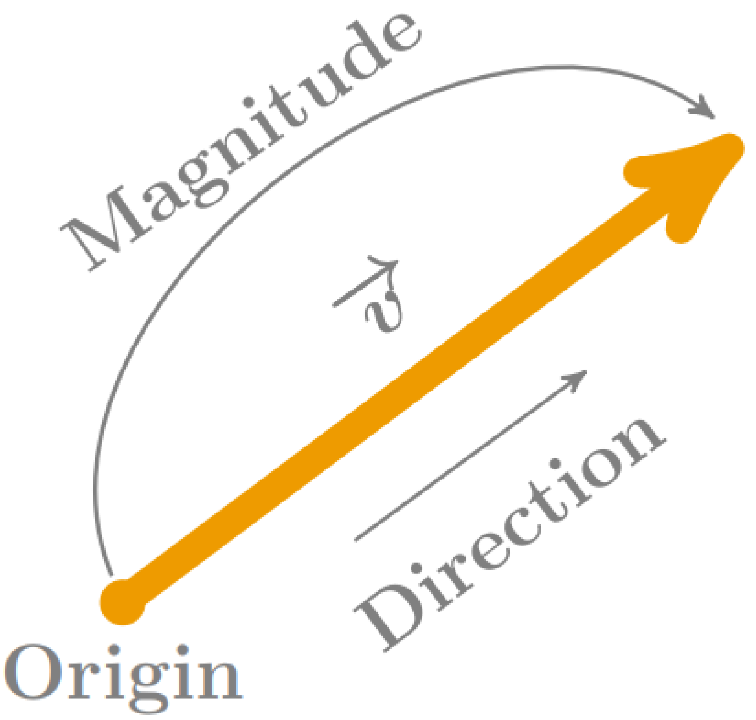


$$\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

- There is no limitation to the dimensions a vector can have:
- A **three-dimensional** vector, t can be represented as:

$$\vec{t} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

- Using abstract concept, a vector is an object that has both a direction and a magnitude.
- The starting point of a vector is called the **BASE** or **ORIGIN.**



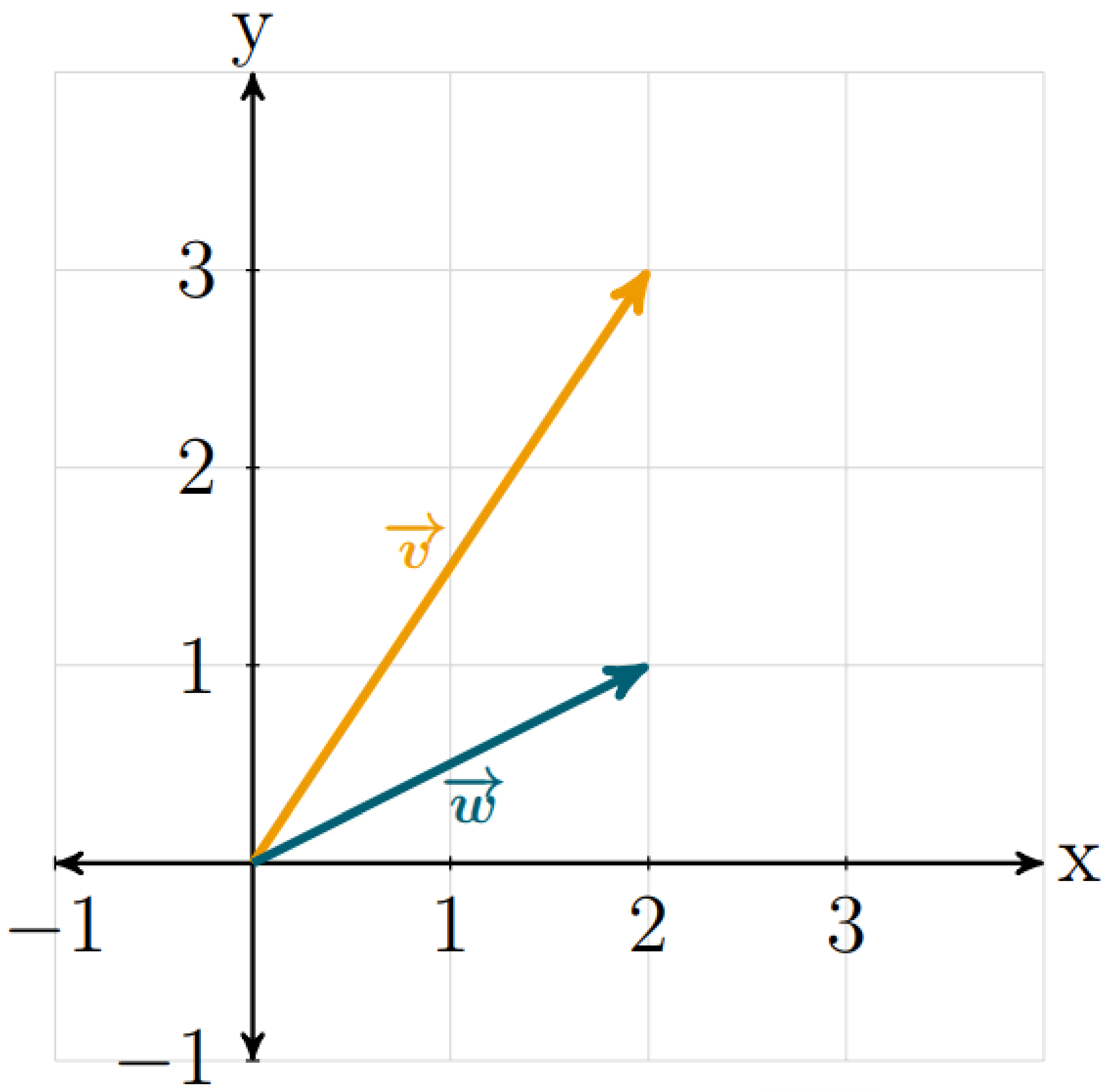
- **Magnitude** is the size of the vector, and it is also a function of where it lands.
- For two vectors to be the same, their **directions** and **magnitudes** must be equal.

- Vectors alone are instrumental mathematical elements.
- They can also represent many things, such as **gravity**, **velocity**, **acceleration**, and **paths**.

- When $n = 1$, a column vector $\boldsymbol{v} = (v_1) \in \mathbb{R}^1$ has a single entry.
- Such a vector can be uniquely identified with the corresponding real number $v_1 \in \mathbb{R}$, and so $\mathbb{R}^1 \simeq \mathbb{R}$

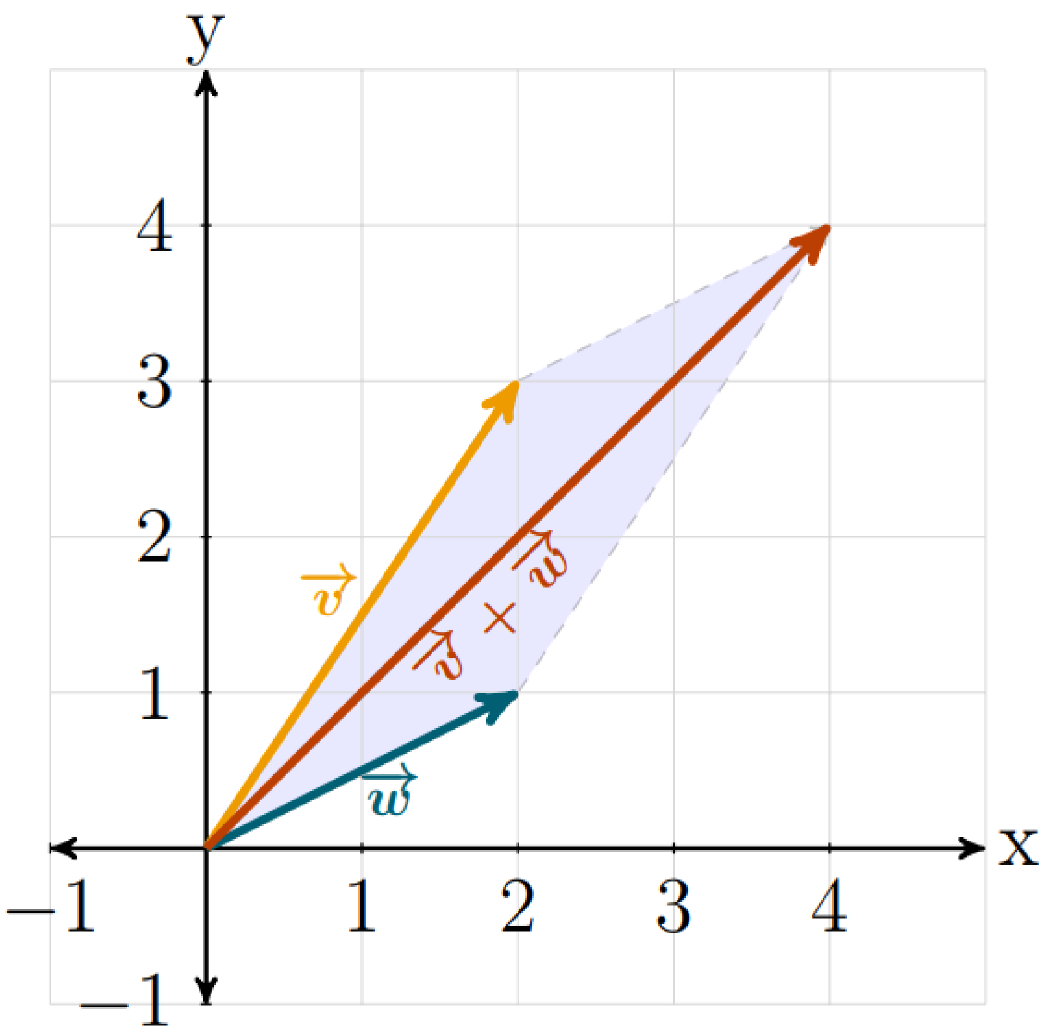
Vector Addition

- Given two vector $\vec{v}$ and $\overrightarrow{w}$, their sum, $\vec{v} + \overrightarrow{w}$ is translation from the edge of $\vec{v}$ with the magnitude and direction of $\overrightarrow{w}$.



$$\vec{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\vec{v} + \vec{w}$$



$$\vec{z} = \vec{v} + \vec{w}$$

$$\vec{z} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\vec{z} = \begin{pmatrix} 2+2 \\ 3+1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

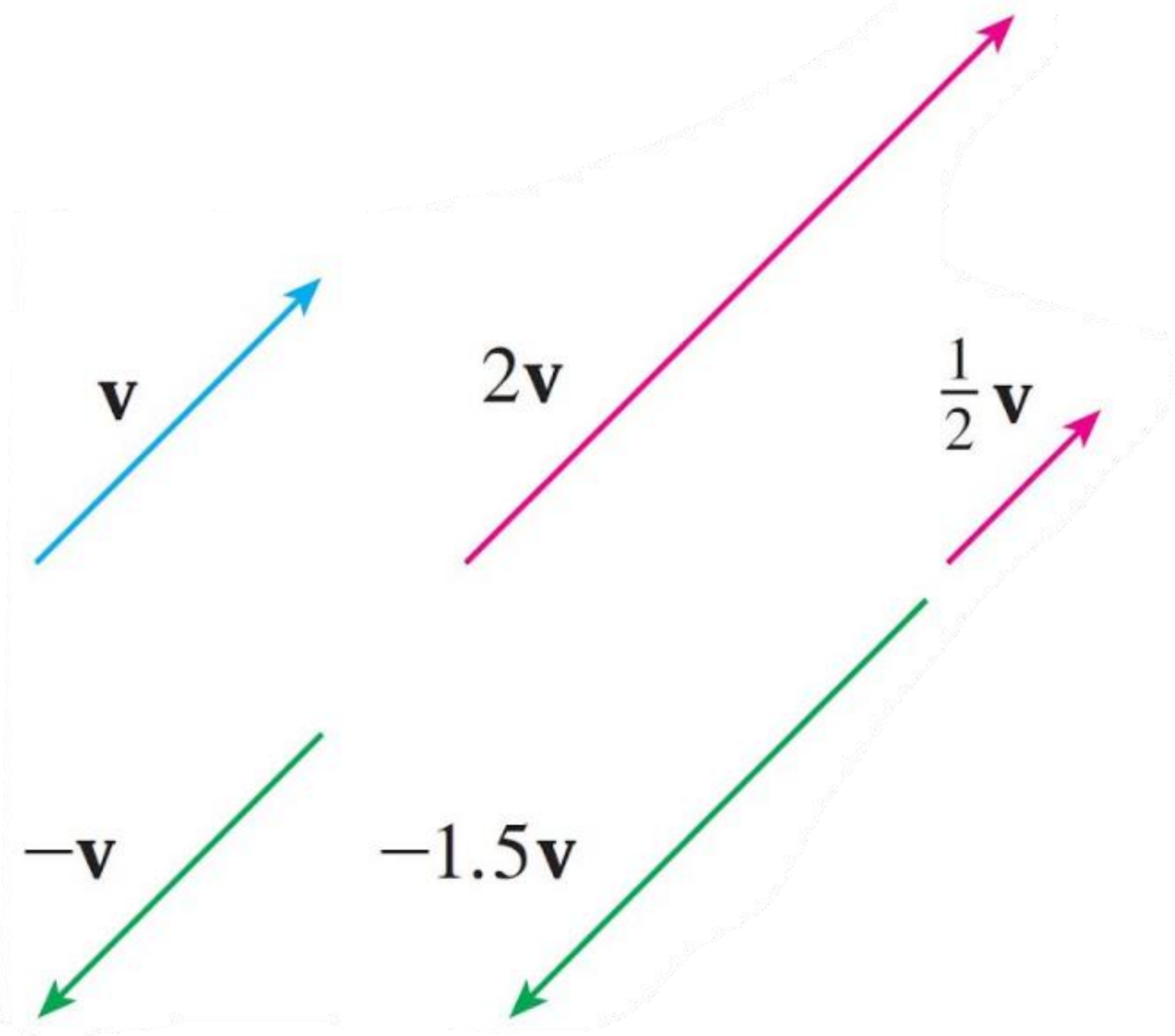
Scalar Multiplication

- A scalar can be any real number.
- A real number is any number you can think of.

$$\lambda \in \mathbb{R}$$

There are four different outcomes when multiplying a vector by a scalar.

1. If $\boldsymbol{\lambda} > \mathbf{1}$ the vector will keep the same direction but stretch
2. If $\mathbf{1} > \boldsymbol{\lambda} > \mathbf{0}$ the vector will keep the same direction but shrink.
3. If $\boldsymbol{\lambda} < -\mathbf{1}$ the vector will change direction and stretch.
4. If $\mathbf{0} > \boldsymbol{\lambda} > -\mathbf{1}$ the vector will change direction and shrink.



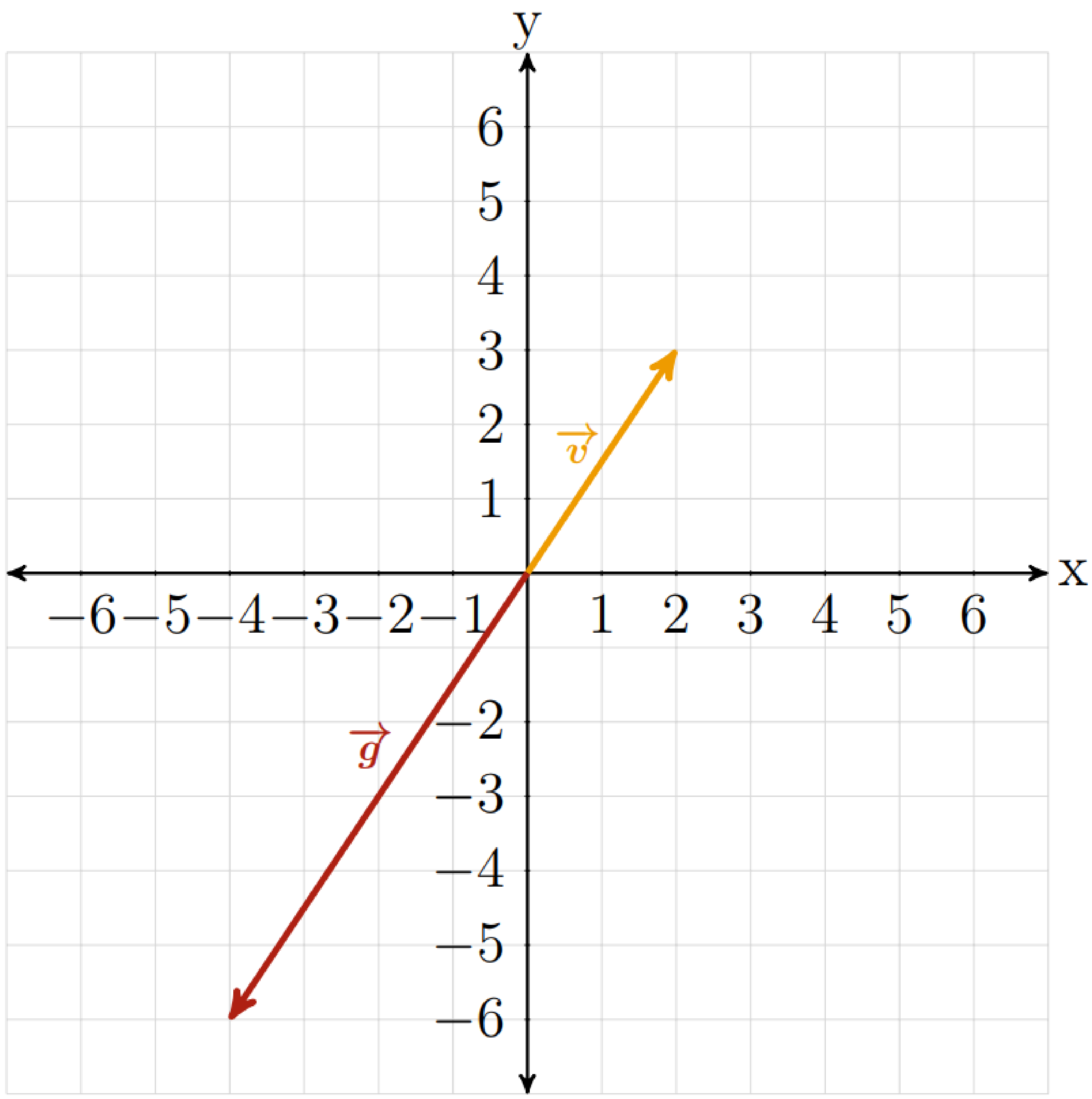
v
2v
1/2 v
−v
−1.5v

- Multiplying a vector $\vec{v}$ by a scalar λ can be defined by: $\lambda\vec{v}$
- If we define a new vector $\vec{g}$ such that:

$$\vec{g} = \lambda\vec{v} \text{ and } \lambda = -2$$

$$\vec{g} = -2 \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

$$\vec{g} = \begin{pmatrix} -4 \\ -6 \end{pmatrix}$$



y
x
6
5
4
3
2
1
−6 −5 −4 −3 −2 −1
1 2 3 4 5 6
−2
−3
−4
−5
−6
$\vec{v}$
$\vec{g}$

The Vector Space

- A **vector space** is a fundamental concept that provides a formal framework for the study and manipulation of vectors.
- Vectors are entities can be **added** together and **multiplied** by scalars.

- A **vector space** is a structured set consisting of a carefully defined group of entities, known as vectors, which adhere to specific rules.
- Vector space is describe using: $(O, +, \times)$

- $\boldsymbol{O}$ is an non-empty set whose elements are called vectors.

- $+$ is a binary operation (vector addition) that takes two vectors from O and produces another vector in O.

- $\times$ is an operation (scalar multiplication) that takes a scalar and a vector from O and produces another vector in O

- If $\vec{v}$ and $\vec{w}$ are two vectors $\in O$ then $\vec{v} + \vec{w}$ must be in O
- $\lambda \in \mathbb{R}$ and $\vec{v} \in O$ then $\lambda\vec{v}$ also needs to be in O.

Axioms of Vector Space

- Commutative property of addition:

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$

- Associative property of addition:

$$\vec{v} + \left(\vec{w} + \vec{t}\right) = (\vec{v} + \vec{w}) + \vec{t}$$

- Associative property of addition:

$$\vec{v} + 0 = \vec{v}$$

- An inverse element exists:

$$\vec{v} + (\overrightarrow{-v}) = \vec{0}$$

- Scalars can be distributed across the members of an addition:

$$c(\vec{v} + \overrightarrow{w}) = c\vec{v} + c\overrightarrow{w}$$

- Vector can be distributed to an addition of two scalars:

$$(c + d)\vec{v} = c\vec{v} + d\vec{v}$$

- The product of two scalars and an element is equivalent to one of the scalars being multiplied by the product of the other scalar and the element:

$$(cd)\vec{v} = c(d\vec{v})$$

- Multiplying an element by 1 just returns the same element:

$$1 \cdot \vec{v} = \vec{v}$$

A Two-dimension Vector Space $(\mathbb{R}^2)$

- This is the space formed by all of the vectors with two dimensions, whose elements are real numbers.
- Let's define two vectors $\vec{v}$ and $\vec{w}$ as being in $\mathbb{R}^2$.
- Let's also define a scalar $\lambda \in \mathbb{R}$

- If we multiply $\vec{v}$ by λ we have:

$$\lambda \cdot \vec{v} = \begin{pmatrix} \lambda \cdot v_1 \\ \lambda \cdot v_2 \end{pmatrix}$$

- Multiplying a real number by a real number results in a new real number.

- If we add $\vec{v}$ and $\vec{w}$:

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$$

Subspaces

- The most important subsets of $\mathbb{R}^n$ are those that **closed** under the operations of vector addition and scalar multiplication.
- They serve to generalize the geometric notions of points, line, and plane in two- and three-dimensional space.

- A subspace of $\mathbb{R}^n$ is a nonempty subset $\emptyset \neq V \subseteq \mathbb{R}^n$ that satisfies.

  a) for every $\boldsymbol{v}, \boldsymbol{w} \in V$, the sum $\boldsymbol{v} + \boldsymbol{w} \in V$ and

  b) for every $\boldsymbol{v} \in V$ and every $c \in \mathbb{R}$, the scalar product $c\boldsymbol{v} \in V$

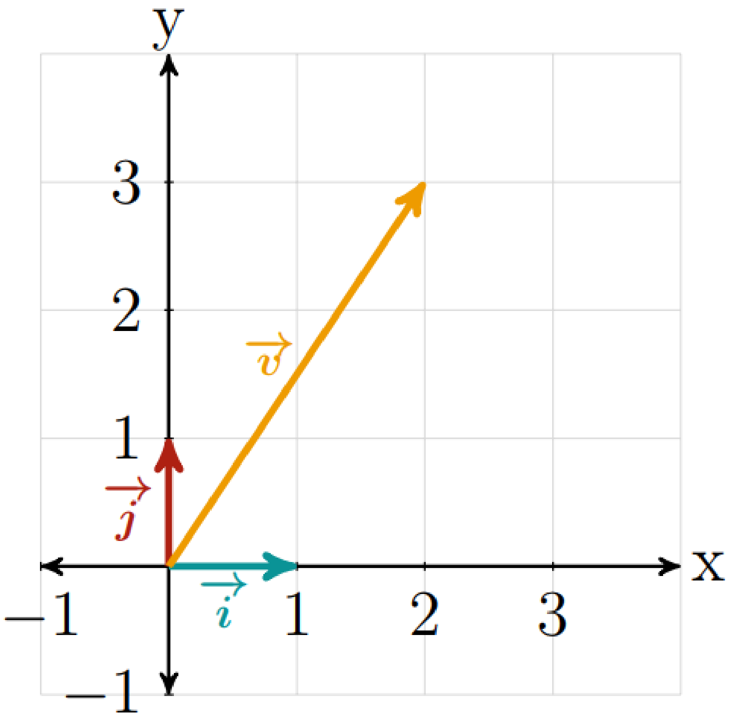
Span and Linear Independence

- Linear means there are no curves, just *lines*, *planes*, or *hyper-planes*, depending on the dimensions that we are working in.

Example of linear combination

$$\alpha \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{where} \quad \alpha, \beta \in \mathbb{R}$$

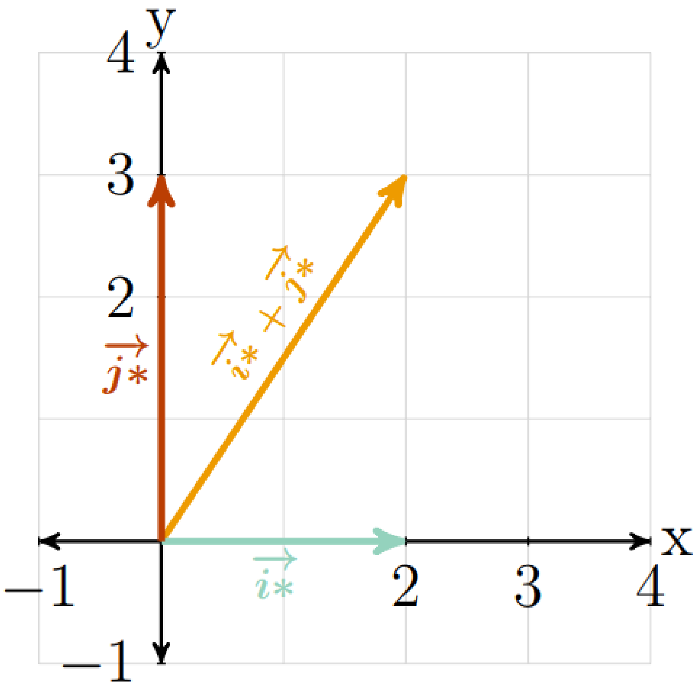
We name $(1 \quad 0)^T$ as $\vec{\imath}$ and $(0 \quad 1)^T$ as $\vec{\jmath}$



$$\vec{v} = (2,3)^T$$

We can stretch $\vec{i}$ by **two** units and then sum it to a **three** units stretched version of $\vec{j}$, the result will be equal to $\vec{v}$.

$$\vec{v} = 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



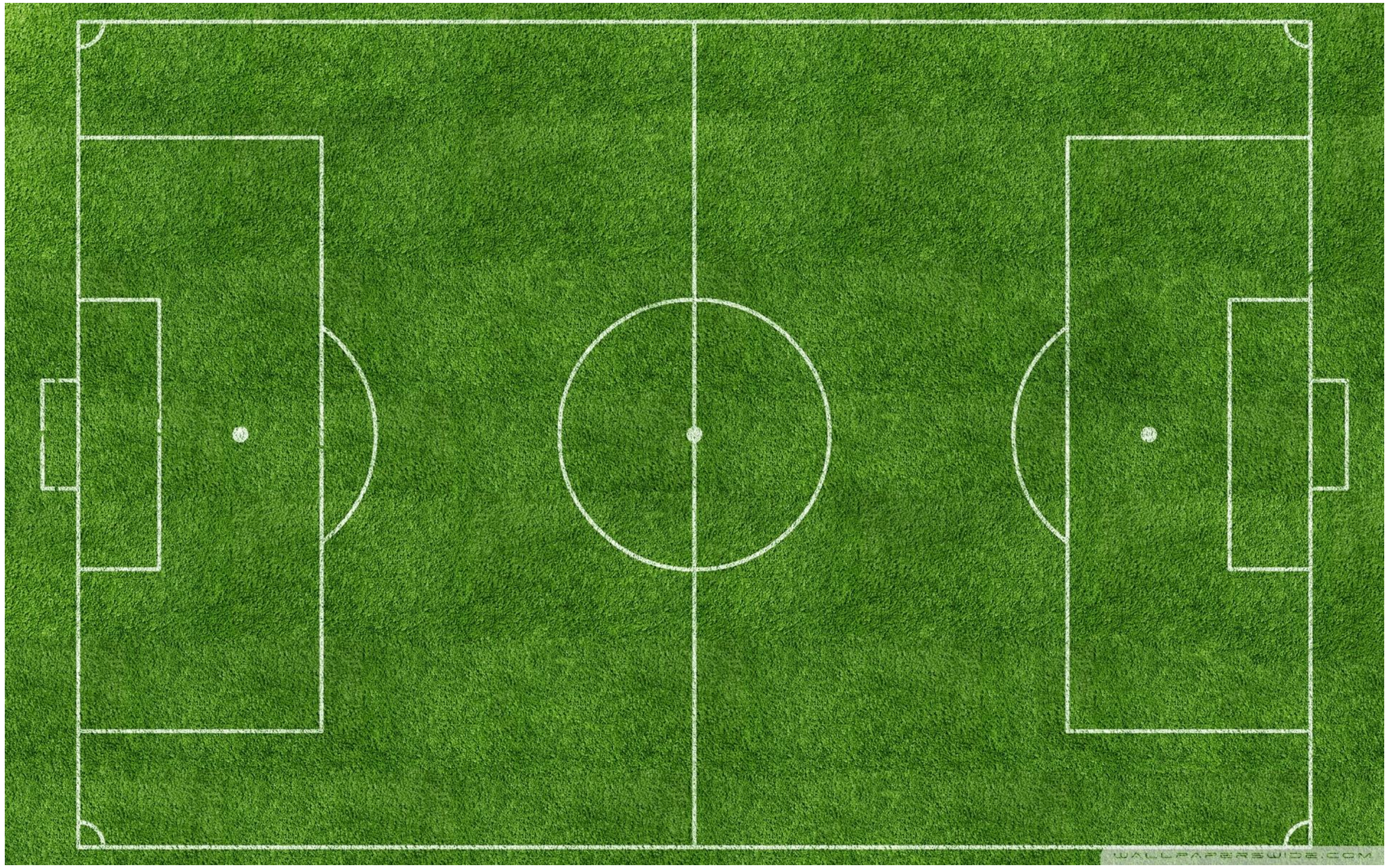
$$\vec{v} = 2 \cdot \vec{\imath} + 3 \cdot \vec{\jmath}$$

The scalars 2 and 3 can be replaced with $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, where both of them are in $\mathbb{R}$

$$\boldsymbol{\alpha} \cdot \vec{\imath} + \boldsymbol{\beta} \cdot \vec{\jmath}$$

- We can display all the vectors of the vector space $\mathbb{R}^2$ using $\boldsymbol{\alpha} \cdot \vec{\imath} + \boldsymbol{\beta} \cdot \vec{\jmath}$
- If I have the entire set of real numbers assigned to the scalars α and β, it means that if I add up the scaled version of $\vec{\imath}$ and $\vec{\jmath}$, I can get any vector within $\mathbb{R}^2$

- The vectors $\vec{i}$ and $\vec{j}$ are linearly independent.
- You can't get to $\vec{j}$ via $\vec{i}$ and vice versa.



- Linear independent can be defined using the equation: $c_1 \cdot \overrightarrow{v_1} + c_2 \cdot \overrightarrow{v_3} + ... + c_n \cdot \overrightarrow{v_n} = 0$
- The factors $c_1, c_2, \ldots, c_n$ are scalars or real numbers.
- The $v`s$ are set of vectors that belong to the space are **linearly independent** if, and only if, the values for the $c`s$ that satisfy that equality are 0.

$$\alpha \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

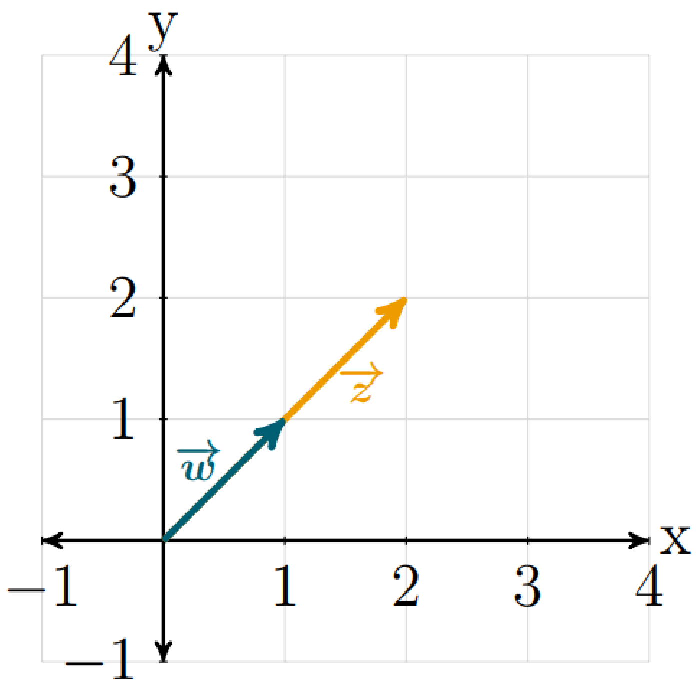
- The only way for the equality to be true is if both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are equal to zero.
- Therefore, $\vec{\imath}$ and $\vec{\jmath}$ are **linearly independent**.

- The vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ are called linearly dependent if there exist scalars $c_1, \ldots, c_k \in \mathbb{R}$, *not all zero*, such that:

$$c_1\boldsymbol{v_1} + \cdots + c_k\boldsymbol{v_k} = \boldsymbol{0}$$

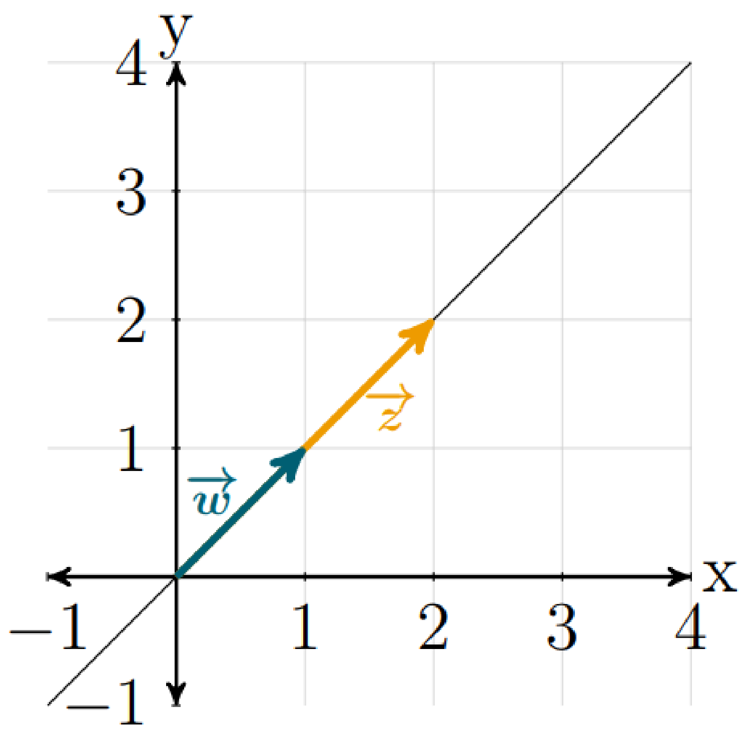
- To check **linear independence**, one needs to show that the only linear combination that produces the zero vector $c_1 = \cdots = c_k = 0$ is the *one and only* solution to the vector equation.

- Let's consider $\vec{w} = (1 \quad 1)^T$ and $\vec{z} = (2 \quad 2)^T$



We can define a linear combination of these two vectors

$$\alpha \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \beta \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$



$$\vec{z} = 2 \cdot \vec{w}$$

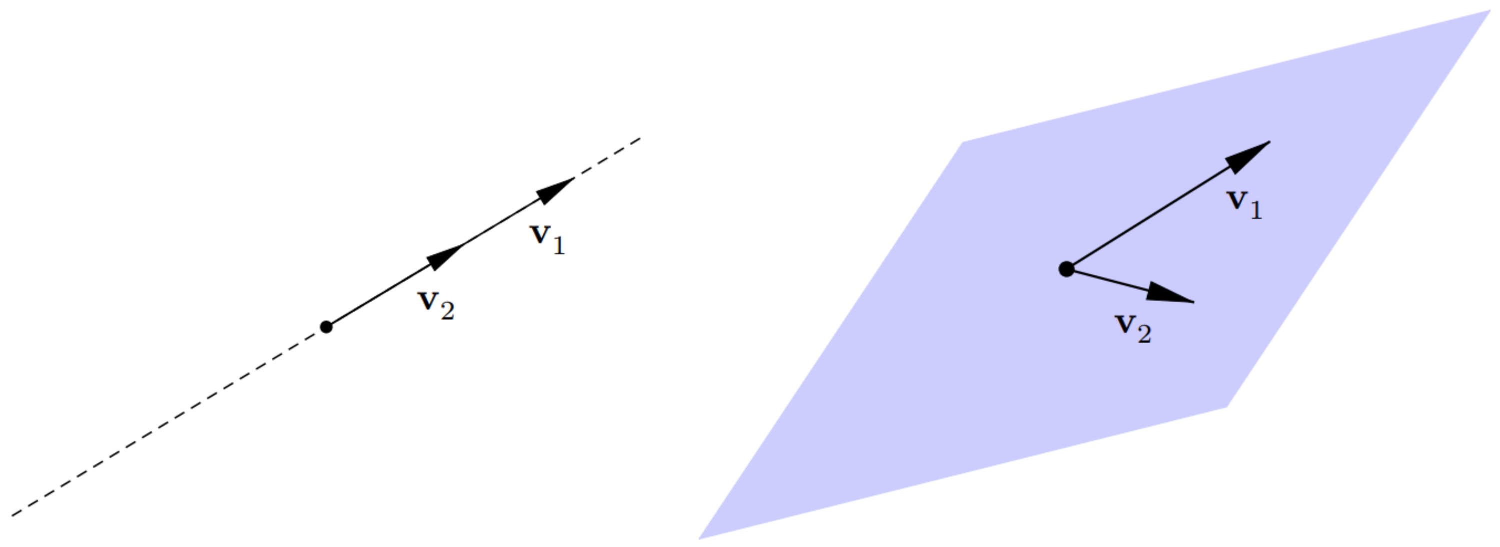
We are not able to represent all of the vectors in the space using the two vectors, $\vec{w}$ and $\vec{z}$

$\vec{z}$ *and* $\vec{w}$ **linearly dependence**

- Two vectors $v, w \in V$ are **linearly dependent** if and only if they are parallel, meaning that one is a scalar multiple of the other.

- All the vectors that result from a linear combination define the **span**.
- The *span* of a finite collection of vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ is the subset $V = span\{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ consisting of all possible linear combinations for $c_1, \ldots, c_k \in \mathbb{R}$

- $\vec{i}$ and $\vec{j}$ span the entire vector space (**plane**) because we can get all the vectors.
- The span of $\vec{w}$ and $\vec{z}$ is a **line**.



Basis and Dimension

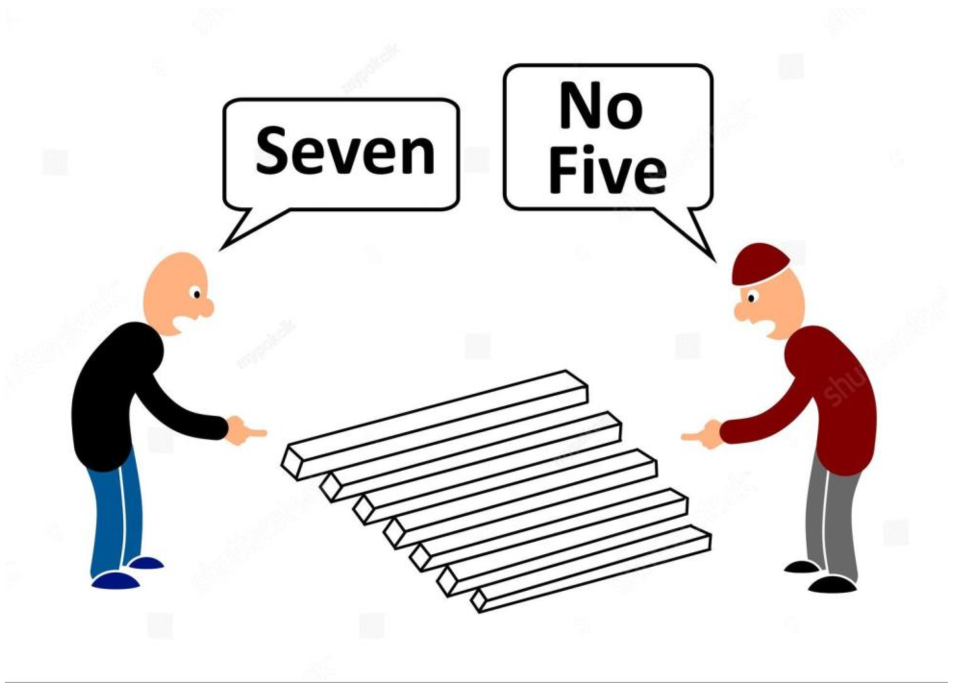
- A ***basis*** of a subspace $V \subseteq \mathbb{R}^n$ is a finite set of vectors $v_1, \ldots, v_k \in V$ that span

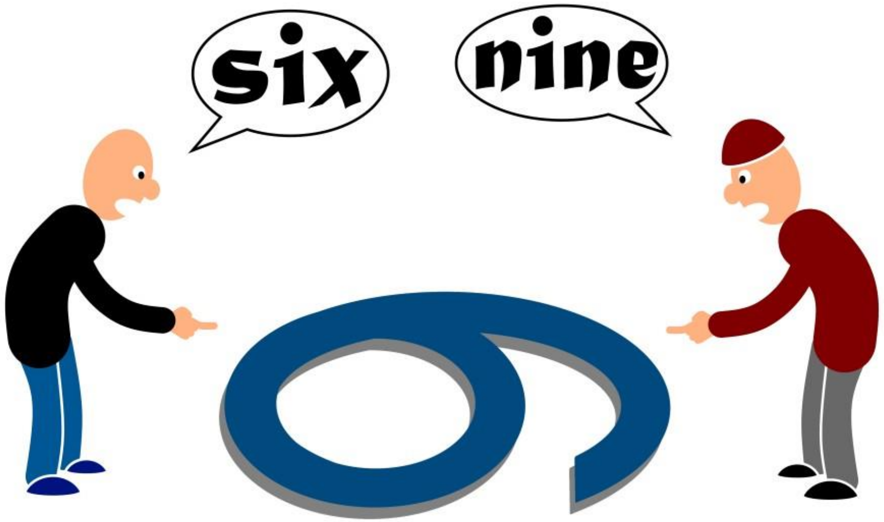
  (a) spans V

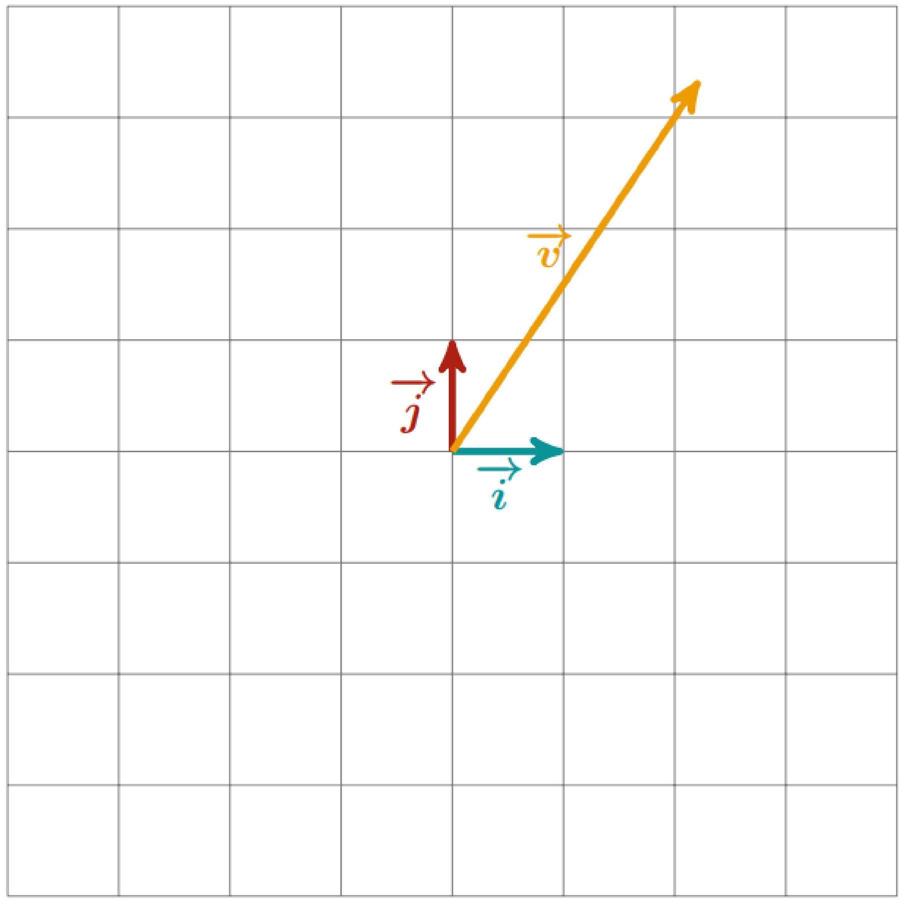
  (b) Is linearly independent

- For a set of vectors to be considered a basis of a vector space, these vectors need to be linearly independent, and their span has to be equal to the entire space.
- $\vec{i}$ and $\vec{j}$ form a basis of $\mathbb{R}^2$.

- A vector space can have **more** than one basis.
- Different basis is the **perspective** from which we observe the same vector in different ways.





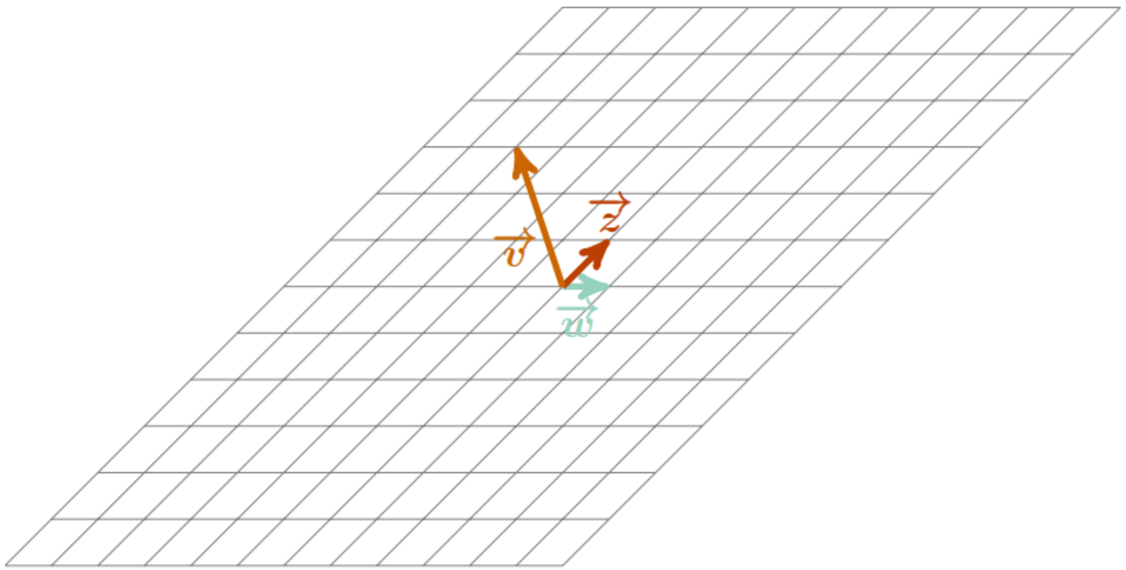


$$\vec{\imath} = (1 \quad 0)^T \qquad \vec{\jmath} = (0 \quad 1)^T$$

$$\vec{v} = (2 \quad 3)^T$$

$$\vec{v} = 2 \cdot \vec{\imath} + 3 \cdot \vec{\jmath}$$

The grids represent out perspective or the basis, which is the way we observe $\vec{v}$ on the basis formed by $\vec{\imath}$ and $\vec{\jmath}$.



$$\vec{w} = (1 \quad 0)^T \quad \vec{z} = (1 \quad 1)^T$$

$$\vec{v} = (2 \quad 3)^T$$

$$\vec{v} = v_1^* \cdot \vec{w} + v_2^* \cdot \vec{z}$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = v_1^* \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_2^* \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Let's calculate the new coordinates of $\vec{v}$ using the new basis above.

Find v_1^* and v_2^*

- The coordinates of $\vec{v}$ in the new basis formed by $\vec{w}$ and $\vec{z}$ are $(-1, 3)^T$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix},$$

Wavelet basis plays an increasingly central role in modern signal and digital image processing.

Change of basis is very useful because you can find **properties** of vectors or data using a different basis or spaces, which allows for **faster** computation or even better results when dealing with machine learning models.

Application

We wish to predict the price of a house

- The dataset consists of two measurements: the **number of bedrooms** and the **number of bathrooms.**
- The vector $\vec{i} = (1, 0)^T$ will point in the directions where the number of bedrooms increases.
- The vector $\vec{j} = (0, 1)^T$ will point in the directions where the number of bathrooms increases.

Let’s assume that:

- Houses with more rooms in total tend to have higher prices.
- When the number of bathrooms is the same or close to the number of bedrooms, there is increase in price.

Let's define new basis:

- ❑ $\overrightarrow{i^*} = (1,1)^T$ and $\overrightarrow{j^*} = (1,-1)^T$
- ❑ The vector $\overrightarrow{i^*}$ will represent the total number of rooms.
- ❑ The vector $\overrightarrow{j^*}$ displays the difference in number between bedrooms and bathrooms.

- We have a house with three bedrooms and two bathrooms.

$$\vec{w} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

- Let's check it from a new perspective, the one described by the new basis, $\overrightarrow{i^*}$ and $\overrightarrow{j^*}$

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = w_1^* \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + w_2^* \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Find the values of w_1^* and w_2^*


Thank you
Thank you very much