VISPAD INSTITUTE OF TECHNOLOGY

Nea onnim no sua a ohu,

Inner Product, Orthogonality, Norm

Fall 2025 💻

(October – December Virtual Internship)

Nea onnim no sua a ohu,

Program Timeline

10th October – 14th December

(10 Weeks)

Lectures

Sundays : 7:00 pm – 9:00 pm

Labs

Saturdays : 7:00 pm – 9:00 pm

At the end of the session the students or candidates should be able to understand and work with:

- Inner Products
- Inequalities
  - The Cauchy-Schwarz Inequality
  - The triangle Inequality
- Orthogonal Vectors and Orthogonal Bases
- Orthogonal Projection and the Closes Point
- The Gram-Schmidt Process
- Orthogonal Subspaces and Complements
- Norms


RECAP

- **Vectors** are ordered list of numbers.
- Each number in this list represent a specific **characteristics** or **feature** of a single data point.

- Imagine you have about houses, and you want to predict their prices.
- For each house, you might record several features:
  1. Size (square feet)
  2. Number of bedrooms
  3. Age (years)
  4. Distance to the nearest school (miles)

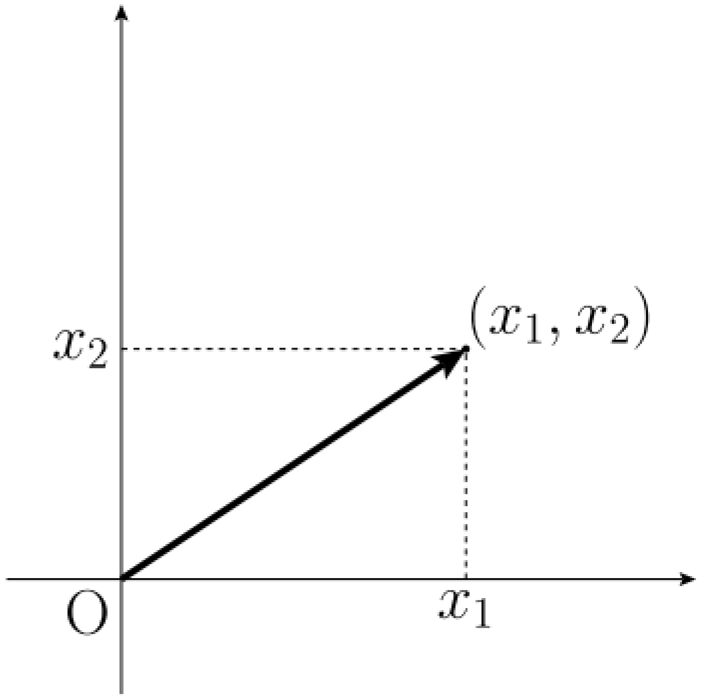
House **A** might have the following characteristics: **1500** sq ft, **3** bedrooms, **20** years old, and **0.5** miles from a school.

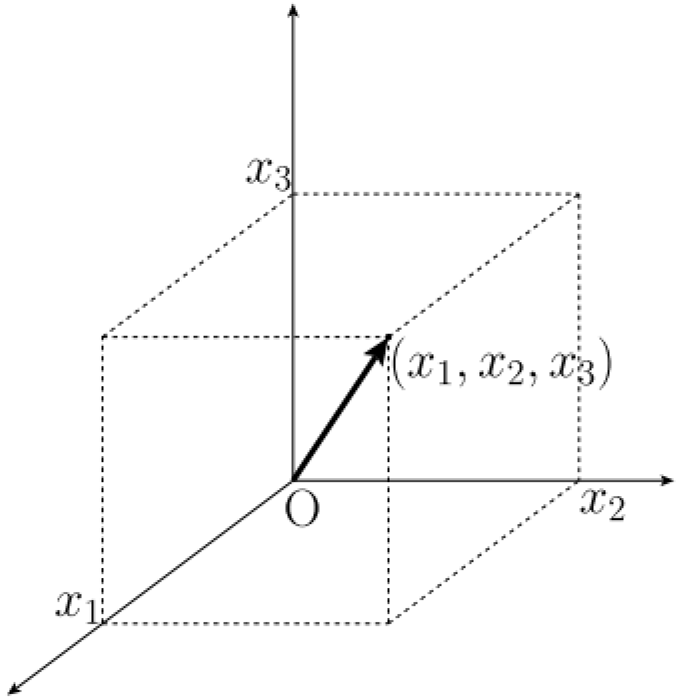
$$v_A = \begin{bmatrix} 1500 \\ 3 \\ 20 \\ 0.5 \end{bmatrix}$$

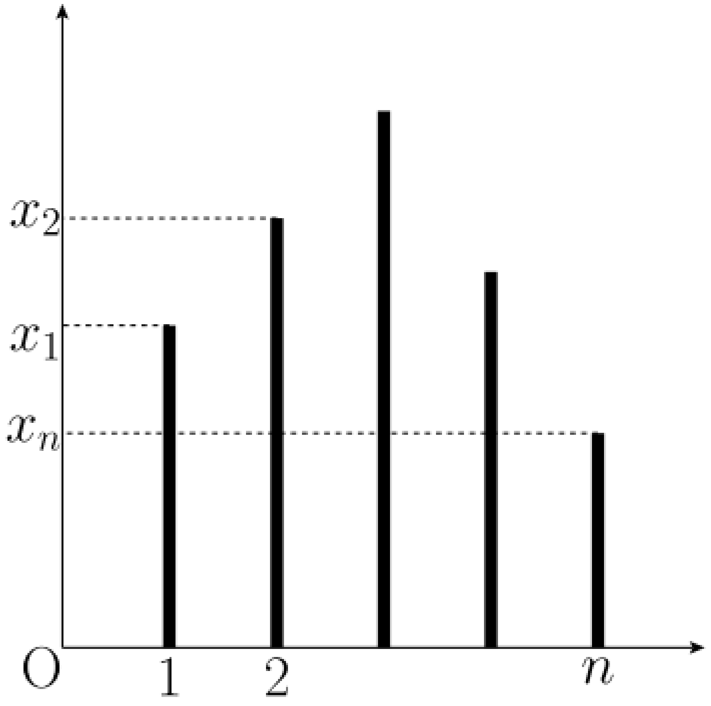
House **A** might have the following characteristics: **2100** sq ft, **4** bedroom, **5** years old, and **1.2** miles from a school.

$$v_B = \begin{bmatrix} 2100 \\ 4 \\ 5 \\ 1.2 \end{bmatrix}$$

- Representing data as vector allows us to think geometrically.







Vector Space

- A vector space is a mathematical structure $(V, F, +, .)$, where:

  $a)$ V is the set of vectors,

  $b)$ F is a field of scalars (Real numbers $\mathbb{R}$)

  $c)$ $+: V \times V \to V$ is the addition operation, satisfying the following properties

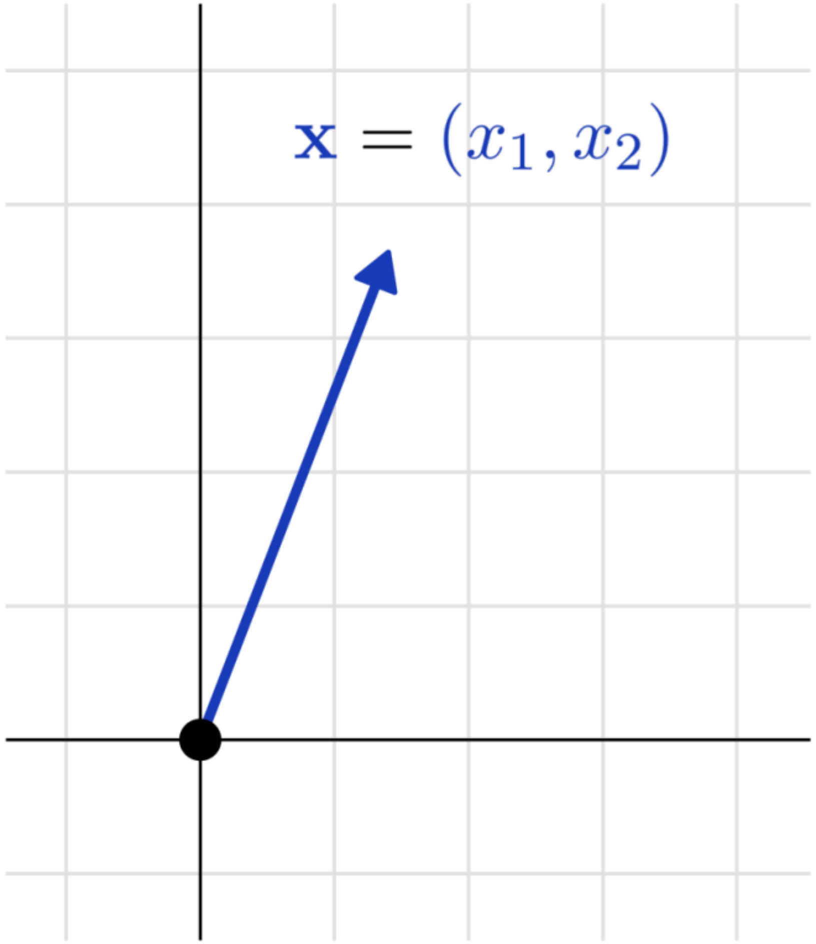
- $x + y = y + x$ (Commutative)
- $x + (y + z) = (x + y) + z$ (Associativity)
- There is an element $0 \in V$ such that $x + 0 = x$ (existence of the null vector)
- And there is an inverse $-x \in V$ for each $x \in V$ such that $x + (-x) = 0$ (existence of additive inverses)

for all vectors $x, y, z \in V$

d) and $\cdot : F \times V \to V$ is the scalar multiplication operation, satisfying

- $a(bx) = (ab)x$ (associativity)
- $a(x + y) = ax + ay$ (distributivity)
- $1x = x$

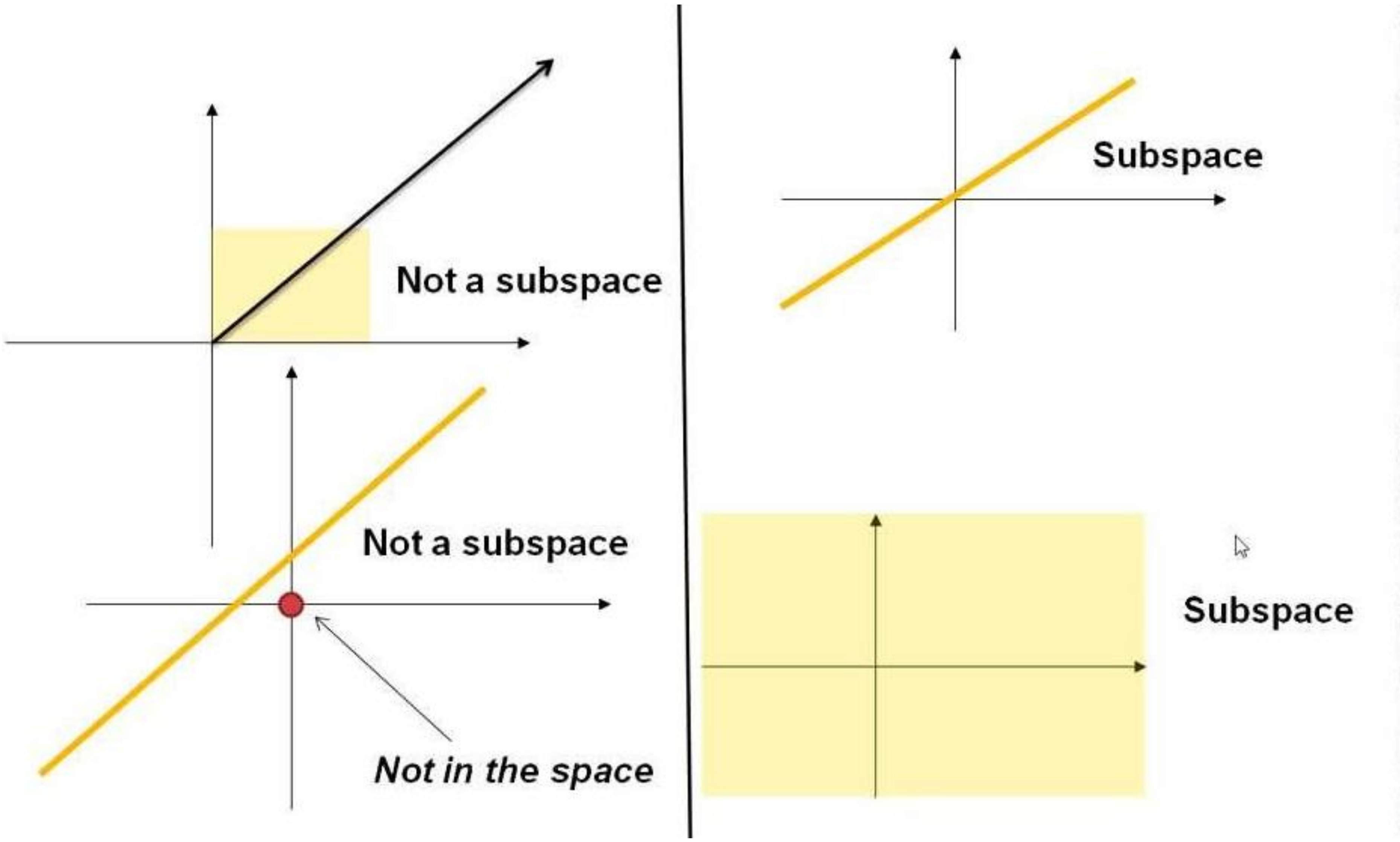
for all scalars $a, b \in F$ and vectors $x, y \in V$.


$\mathbf{x} = (x_1, x_2)$

Subspaces

- For a given vector space V, we are often interested in one of its subsets that is a vector space in its entirety.
- Let V be a vector space. The set $U \subseteq V$ is a *subspace* of V if it is closed under **addition** and scalar **multiplication**.
- U is a proper subspace if it is a subspace and $U \subset V$.

- There are at least two spaces subspaces of each vector space: **itself** and $\{\mathbf{0}\}$.
- For R^2, there are 3 types of subspaces
  - The set containing only the zero vector $\{0\}$ is also a subspace.
  - A line passing through the origin.
  - The entire space R^2 is also a subspace itself.


Not a subspace
Not a subspace
Not in the space
Subspace
Subspace

- For R^3, there are 4 types of subspaces
  - a point – the trivial subspace $V = \{0\}$
  - a line passing through the origin
  - a plane passing through the origin
  - the entire three-dimensional space $V = \mathbb{R}^3$

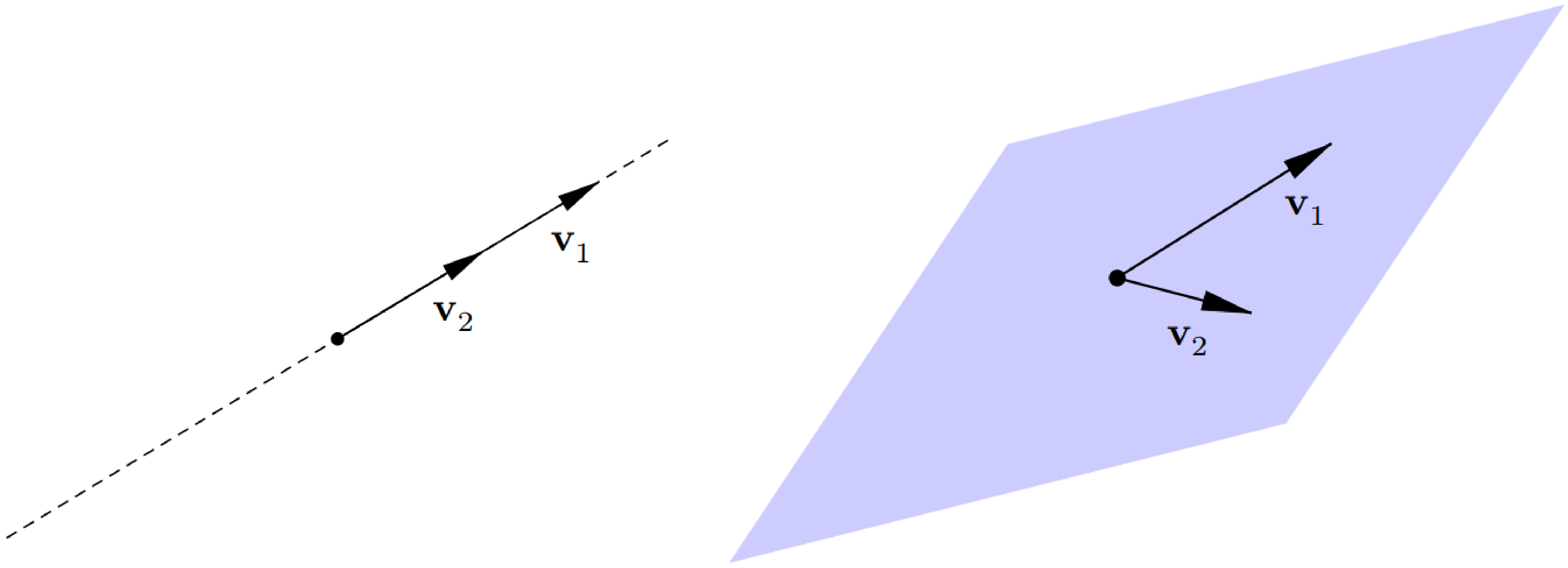
Linear Combinations and Span

- Linear combination is combining vectors to create new ones.
- A linear combination of set of vectors $v_1, v_2, \ldots, v_k$ is any vector w that can be expressed in the form:

$$w = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

where $c_1, c_2, \ldots, c_k$ are scalar constants (real numbers)

- Linear combinations provide a way to take a small set of vectors and generate a whole lot of others from them.
- For a set of vectors S, taking all of its possible linear combinations is called $spanning$, and the generated set is called the **span.**



- The span represents the entire region or space that can be “reached” or generated by combining these vectors.

Linear Independence of Vectors

A set of vectors $\{v_1, v_2, \dots, v_k\}$ in a vector space is said to be linearly independent if the only solution to the vector equation: $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = \vec{0}$ is the trivial solution where all the scalar coefficients are zero: $c_1 = 0, c_2 = 0, \dots, c_k = 0$

- The set of vectors is linearly dependent if there exists at least one non-trivial solution, i.e. at least one coefficient c_i is non-zero.
- In 2D ($\mathbb{R}^2$):
  - Two vectors are linearly dependent if they lie on the same line passing through the origin.

- Two vectors are linearly independent if they point in different directions

❑ In 3D ($\mathbb{R}^3$):

- Three vectors are linearly dependent if they all lie within the same plane passing through the origin.
- Three vectors are linearly independent if they are not coplanar.

Basis

- A basis is the fundamental building blocks or coordinate system for a vector space.
- It's the smallest set of directions you need to be able to reach *any* point in that space, without any of those directions being expressible using the others.

- Let V be a vector space and S be a subset of this vector. S is a **basis** of V if:
  - S is linearly independent
  - $span(S) = V$

- ❑ The most familiar example is the standard basis for $\mathbb{R}^n$.
  - For $\mathbb{R}^2$, the standard basis is $\{e_{1,}e_2\} = \{(1,0),(0,1)\}$
  - For $\mathbb{R}^3$, the standard basis is $\{e_1, e_2, e_3\} = \{(1,0,0),(0,1,0),(0,0,1)\}$

- There can be more than one basis for the same vector space.
- Examples include: $\{(1, 1, 1), (1, 1, 0), (0, 1, 1)\}$
- Wavelet basis: $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$

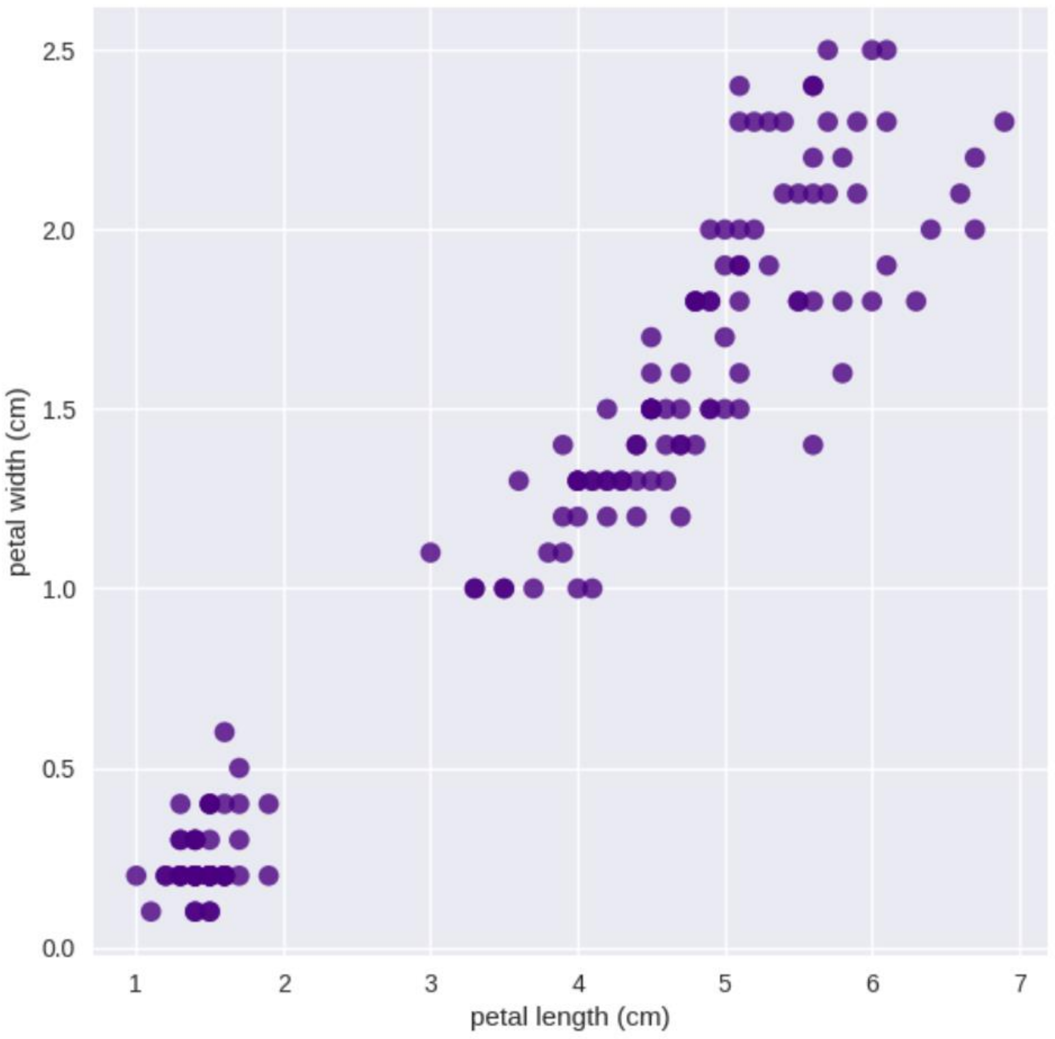
Dimension

- All bases for a given vector space contain the same number of vectors.
- The consistent number is called the dimension of the vector space, denoted as $\dim(V)$.

- The dimension of $\mathbb{R}^2$ is 2, because its bases contain two vectors.
- The dimension of $\mathbb{R}^3$ is 3.
- The dimension of $\mathbb{R}^n$ is n

A plane through the origin in $\mathbb{R}^3$ is a subspace of dimension 2. A line through the origin in $\mathbb{R}^3$ is a subspace of dimension 1. The space containing only the zero vector $\{0\}$ has a dimension 0.


I'm Ready....
...are you READY?
meme-arsenal.ru



- ❑How many classes is shown?
- ❑Can you summarize your reasoning in a single sentence?

Norms and Distances

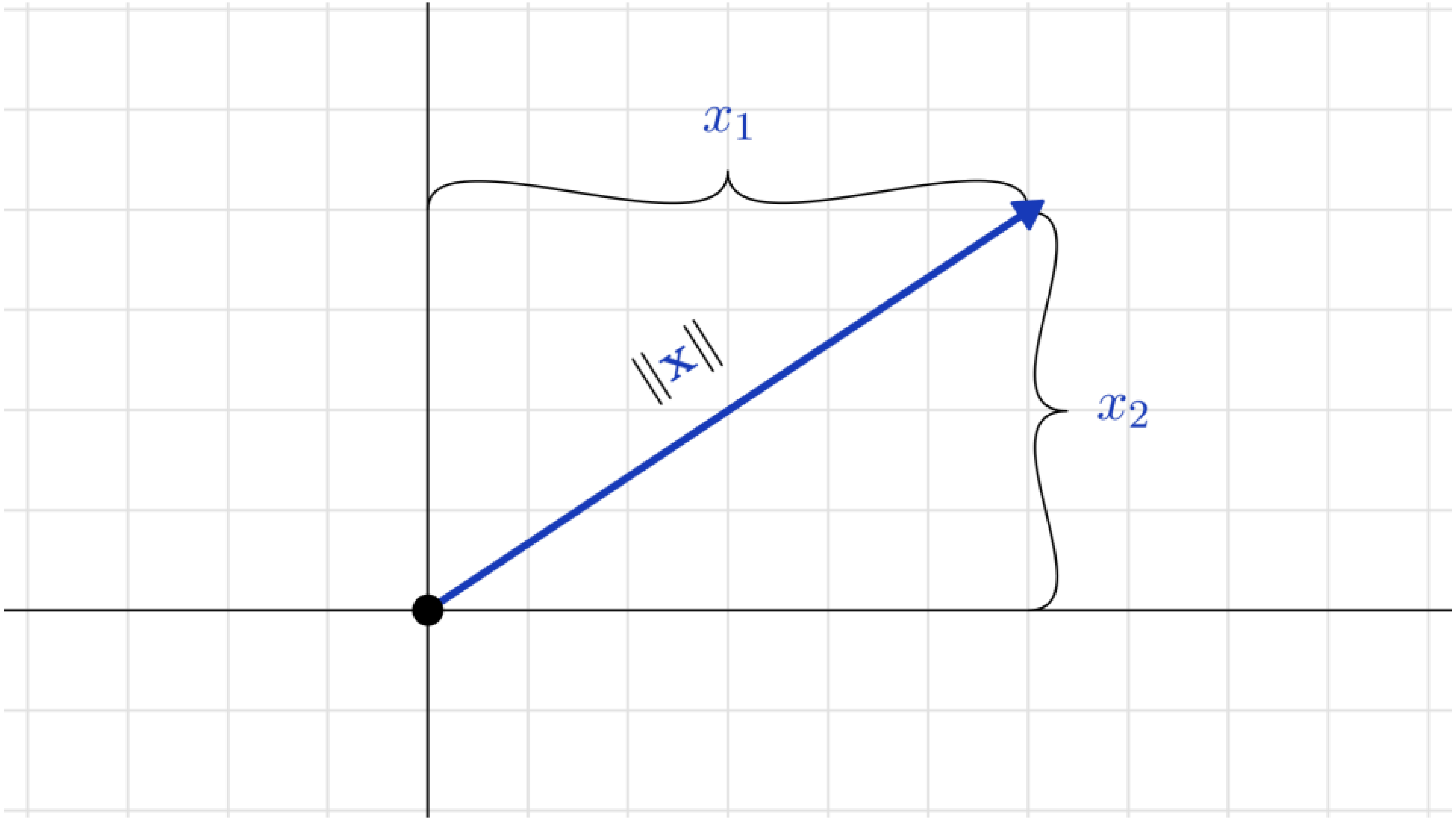
- In addition to their **direction**, vectors also have **magnitude.**
- A norm is a function that takes a vector as input and returns a non-negative scalar value representing its magnitude.

- Different norms measure length in different ways, and the choice of norm can have significant implications in machine learning algorithms, particularly in areas like **regularization** and error **calculation.**

L_2 Norm (Euclidean Norm)

- L_2 norm, also known as Euclidean norm.
- It corresponds to the standard geometric length of a vector, calculated as the square of the sum of the squares of its components.

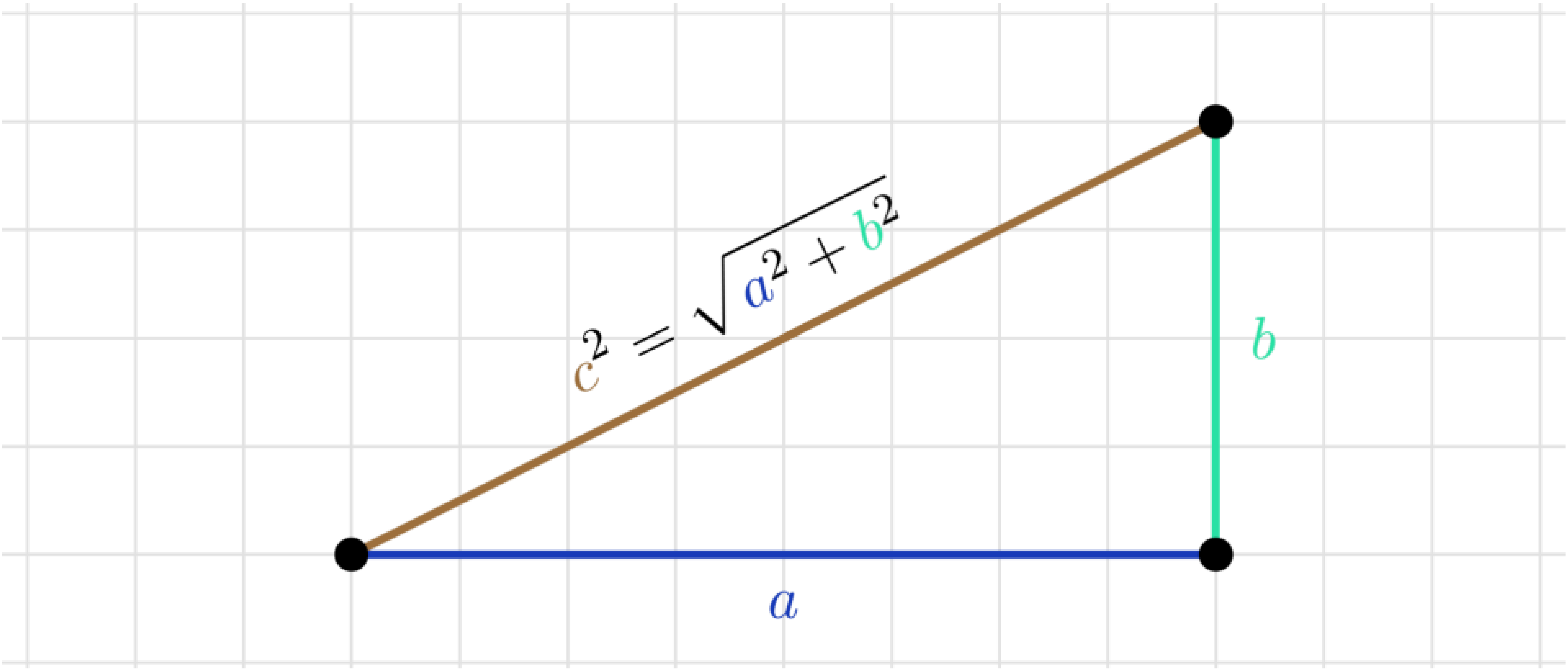
$$\|x\|_2 = \sqrt{x_1^2 + x_2^2}, \qquad x = (x_1, x_2)$$



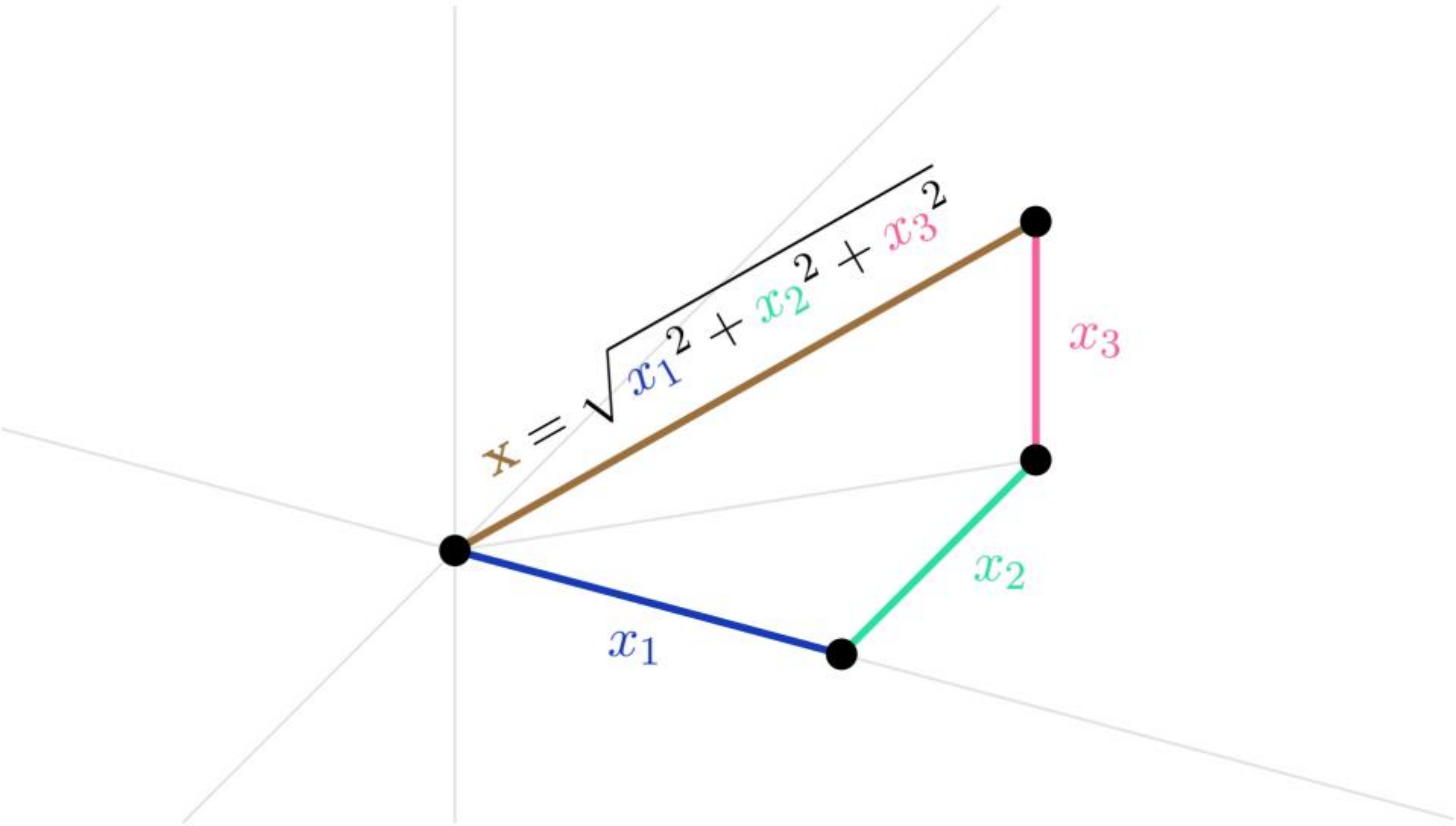
x_1
$\|\mathbf{x}\|$
x_2

The magnitude formula $\sqrt{x_1^2 + x_2^2}$ can be simply generalized to higher dimensions by:

$$\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}, \qquad x = (x_1, \ldots, x_n) \in \mathbb{R}^n.$$



$c^2 = \sqrt{a^2 + b^2}$
b
a



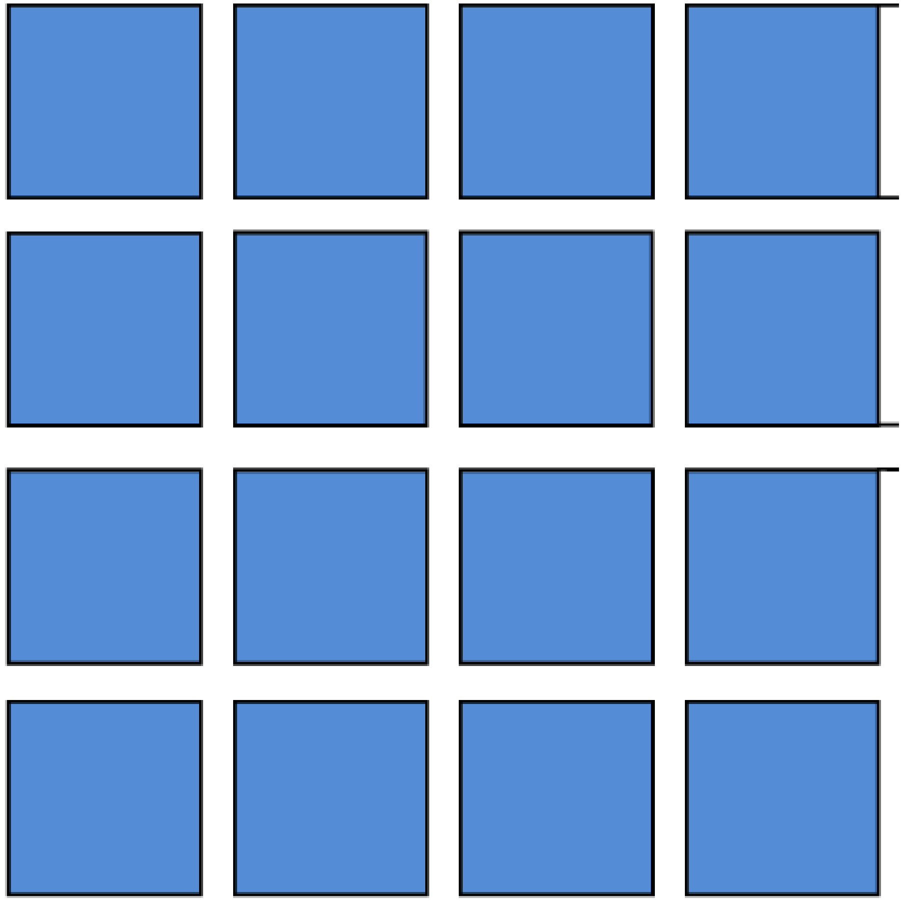
$\mathbf{x} = \sqrt{x_1{}^2 + x_2{}^2 + x_3{}^2}$
x_3
x_2
x_1

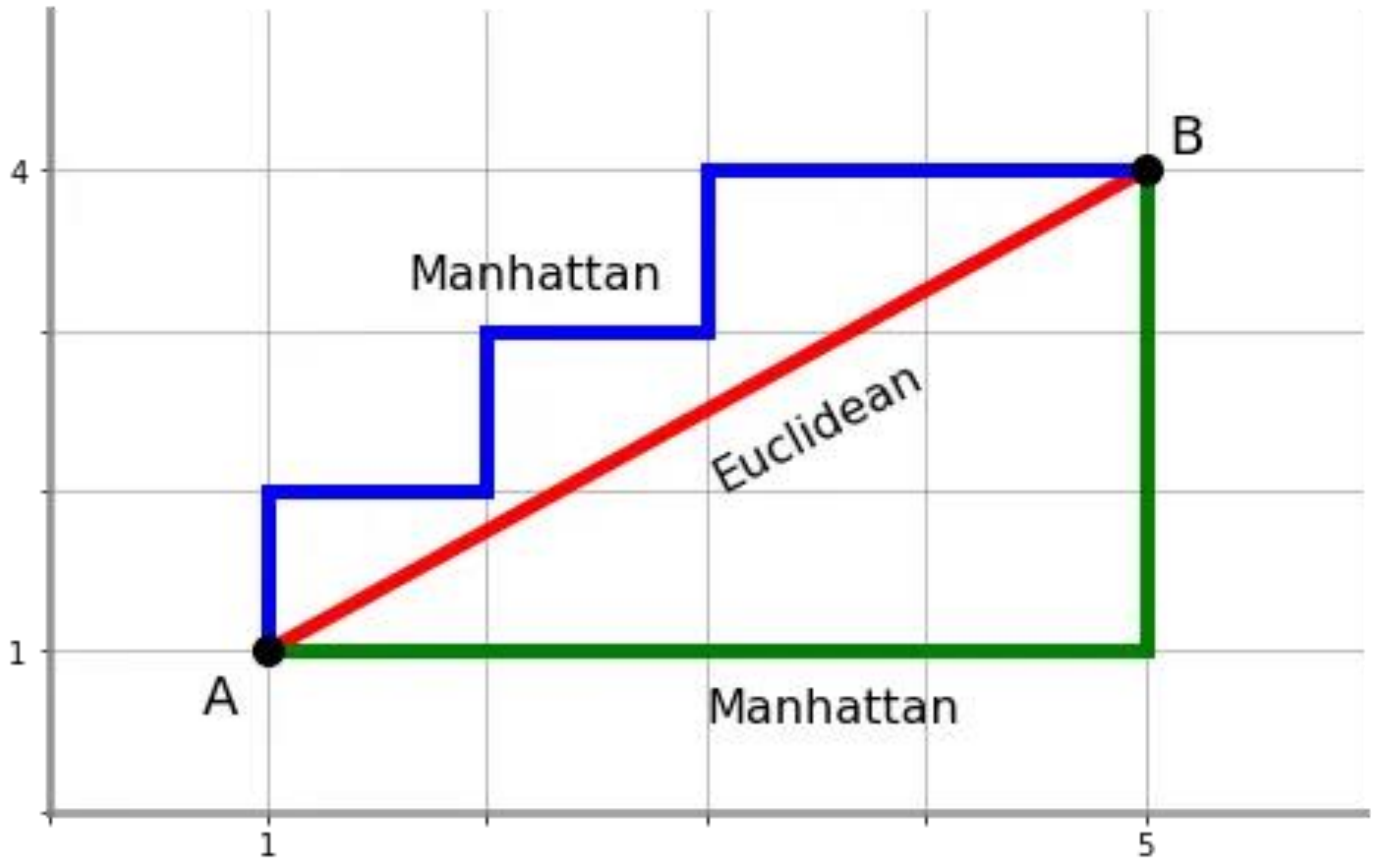
L_1 Norm (Manhattan Norm)

- L_1 norm, also known as Manhattan norm.
- Instead of squaring components, it sums their absolute values.

$$\|x\|_1 = |x_1| + |x_2|, \qquad (x_1, x_2)$$

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n| \, , x = (x_1, \dots, x_n) \in \mathbb{R}^n.$$




B
4
Manhattan
Euclidean
1
A
Manhattan
1
5

The notions of ***magnitude*** and ***distance*** are critical in machine learning, as we can use them to determine the ***similarity*** between data points, ***measure*** and ***control the complexity*** of neural networks, and much more.

Euclidean Distance $(L_2\ Distance)$

- It represents the straight-line distance between the points defined by the vectors in the feature space.
- If $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$, the Euclidean distance:

$$d_2(u, v) = \|u - v\|_2 = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

$$= \sqrt{\sum_{i=1}^{n} (u_i - v_i)^2}$$

Manhattan Distance (L_1 $Distance$)

- It measures the distance by summing the absolute differences of the vector components.
- If $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$, the Euclidean distance:

$$d_1(u, v) = \|u - v\|_1 = \sum_{i=1}^{n} |u_i - v_i|$$

The Manhattan distance is sometimes preferred over Euclidean distance in high-dimensional spaces or when dealing with features that have different units or scales, as it’s less sensitive to large differences in a single dimension compared to the Euclidean distance (which squares the differences).

Inner Products

- ❑ Vector addition, subtraction, and scalar multiplication change a vector’s position or scale.
- ❑ The **inner product** (dot or scalar product) provides a way to “*multipy*” two vectors, yielding a single scalar number.

- ***Inner product*** can be used to measure the similarity of data points.
- The inner product is used to measure the angle between two vectors.

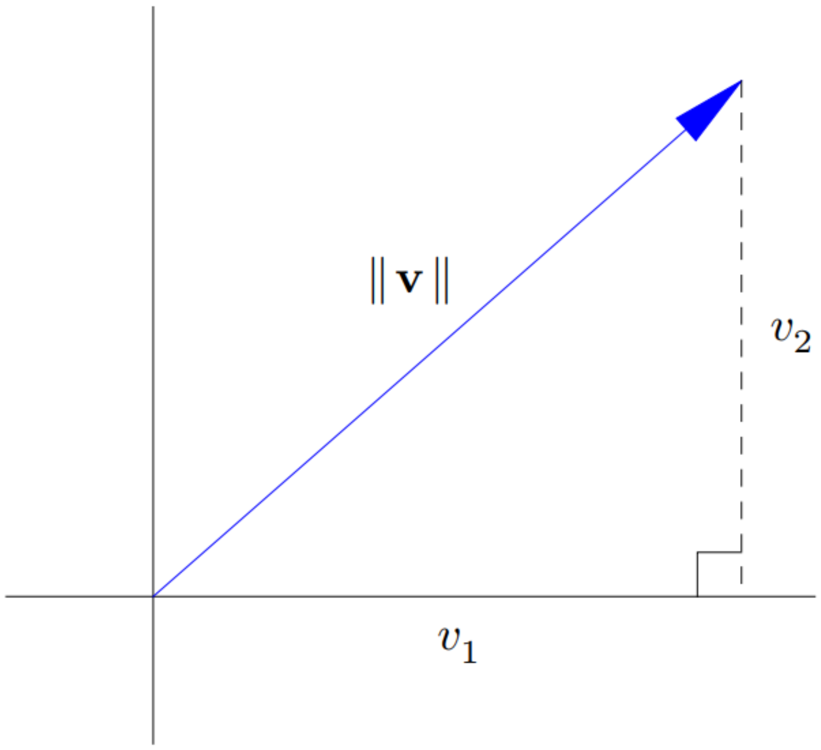
- Given two vectors $v = (u_1, \ldots, u_n)$ and $w = (v_1, \ldots, v_n)$ from the plane, we defined their **dot product** by:

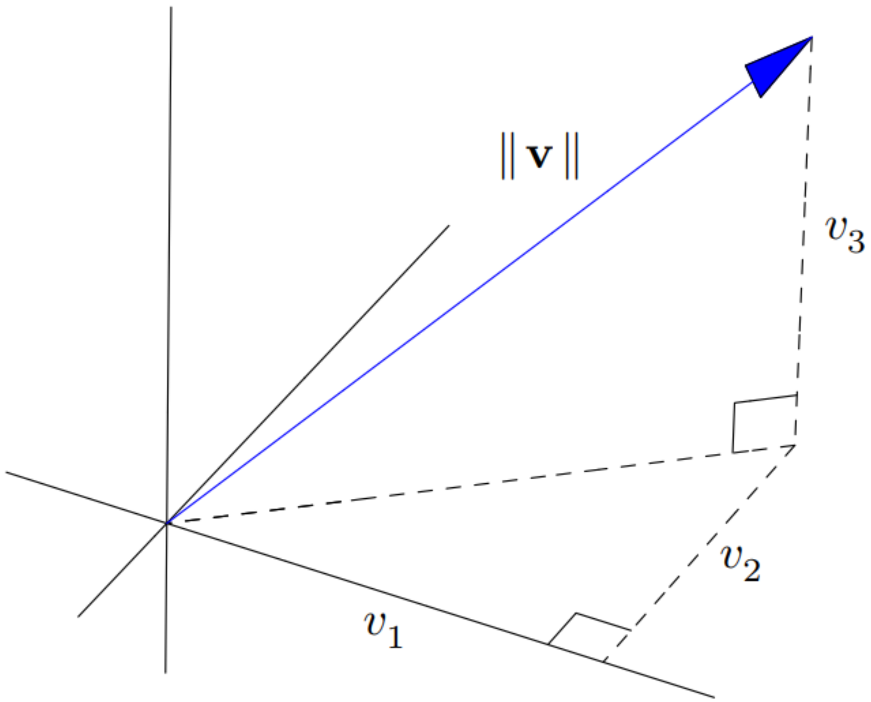
$$v \cdot w = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{i=1}^{n} v_i w_i$$

- The dot product of a vector with itself is: $v.v = v_1^2 + v_2^2 + \cdots + v_n^2$ is the sum of the squares of its entries.
- The *Euclidean norm* or *length* of a vector is found taking the square root:

$$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

- Note that every vector $\boldsymbol{v} \in \mathbb{R}^n$, has nonnegative Euclidean norm: $\|v\| \geq 0$; only the zero vector has norm: $\|v\| = 0$ if and only $\boldsymbol{v} = \mathbf{0}$.





- The elementary properties of ***dot product*** and ***Euclidean norm*** serve to inspire the definition of general **inner products**.
- The dot product is not the only inner product that can be defined on $\mathbb{R}^n$

- An ***inner product*** is a pairing that takes two vectors $v, w \in \mathbb{R}^n$ and produces a real number $\langle v, w \rangle \in \mathbb{R}$.
- The inner product is required to satisfy the following axioms for all $u, v, w \in \mathbb{R}^n$, and scalars $c, d \in \mathbb{R}$

i. Bilinearity:

$$\langle c\boldsymbol{u} + d\boldsymbol{v}, \boldsymbol{w} \rangle = c\langle \boldsymbol{u}, \boldsymbol{w} \rangle + d\langle \boldsymbol{v}, \boldsymbol{w} \rangle$$

$$\langle \boldsymbol{u}, c\boldsymbol{v} + d\boldsymbol{w} \rangle = c\langle \boldsymbol{u}, \boldsymbol{v} \rangle + d\langle \boldsymbol{u}, \boldsymbol{w} \rangle$$

ii. Symmetry:

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{w}, \boldsymbol{v} \rangle$$

iii.Positivity:

$$\langle \boldsymbol{v}, \boldsymbol{v} \rangle > 0 \text{ whenever } \boldsymbol{v} \neq \mathbf{0}, \text{ while } \langle \mathbf{0}, \mathbf{0} \rangle = 0$$

- Given an inner product, the associated norm of a vector $v \in \mathbb{R}^n$ is:

$$\|v\| = \sqrt{\langle v, v \rangle}$$

The positivity axiom implies that $\|v\| \geq 0$ is real and nonnegative, and equals 0 if and only if $v = 0$ is the zero vector.

- If $c \in \mathbb{R}$ is any scalar, then, by bilinearity of the inner product, the norm satisfies the following homogeneity property:

$$\|c\boldsymbol{v}\| = \sqrt{\langle c\boldsymbol{v}, c\boldsymbol{v}\rangle} = \sqrt{c^2\langle \boldsymbol{v}, \boldsymbol{v}\rangle} = |c|\|\boldsymbol{v}\|$$

- Given an inner product, the associated norm of a vector $v \in \mathbb{R}^n$ is: $\|\boldsymbol{v}\| = \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle}$
- The positivity axiom implies that $\|\boldsymbol{v}\| \geq 0$ is real and nonnegative, and equals 0 if and only if $\boldsymbol{v} = 0$ is the zero vector.

- If $c \in \mathbb{R}$ is any scalar, then, by bilinearity of the inner product $\|c\boldsymbol{v}\| = \sqrt{\langle c\boldsymbol{v}, c\boldsymbol{v}\rangle} = \sqrt{c^2\langle \boldsymbol{v}, \boldsymbol{v}\rangle} = |c|\|\boldsymbol{v}\|$

- The most common inner product on $\mathbb{R}^2$, the dot product is defined as:

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle = v \cdot w = v_1 w_1 + v_2 w_2,$$

$$\boldsymbol{v} = \binom{v_1}{v_2}, \boldsymbol{w} = \binom{w_1}{w_2}$$

- A weighted inner norm product can also be defined as: $\langle v, w \rangle = 2v_1w_1 + 5v_2w_2$
- Can we proof that the above formular does indeed define an inner product?
  - Bilinearity
  - Positivity
  - Symmetry

- Does this formula define an inner product? 🤔

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle = v_1 w_1 + 2 v_1 w_2 + 2 v_2 w_1 + v_2 w_2$$

Weighted norms are particularly relevant in statistics and data fitting, when one wants to emphasize the importance of certain measurements and de-emphasize others; this is done by assigning appropriate weights to the different components of the data vector v.

- Given an inner product and associated norm, the vectors $\boldsymbol{u} \in \mathbb{R}^n$, with a unit norm $||\boldsymbol{u}|| = 1$ is known as the ***unit vector.***
- If $\boldsymbol{v} \neq \boldsymbol{0}$ is any nonzero vector, then the vector $\boldsymbol{u} = \boldsymbol{v}/||\boldsymbol{v}||$ obtained by dividing $\boldsymbol{v}$ by its norm is a unit vector parallel to $\boldsymbol{v}$

- The vector $v = (-1, 2)^T$ has length $\|v\| = \sqrt{5}$ with respect to the standard Euclidean norm.
- The unit vector pointing in the same direction is:

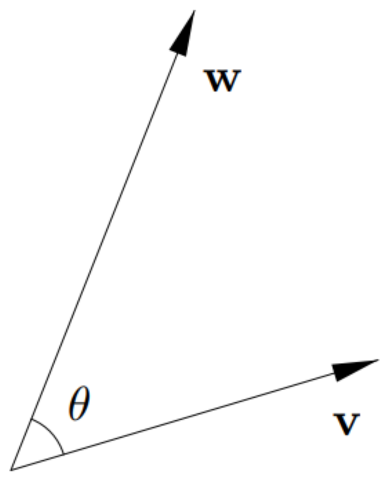
$$u = \frac{v}{\|v\|} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

Inequalities

- ❑There are two fundamental inequalities that are valid for any inner product.
  - Cauchy-Schwarz inequality
  - Triangle inequality

The Cauchy-Schwarz Inequality

- In Euclidean geometry, the dot product between two vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$ can be geometrically characterized by the equation: $\boldsymbol{v}.\boldsymbol{w} = \|\boldsymbol{v}\| \|\boldsymbol{w}\| \cos\theta$

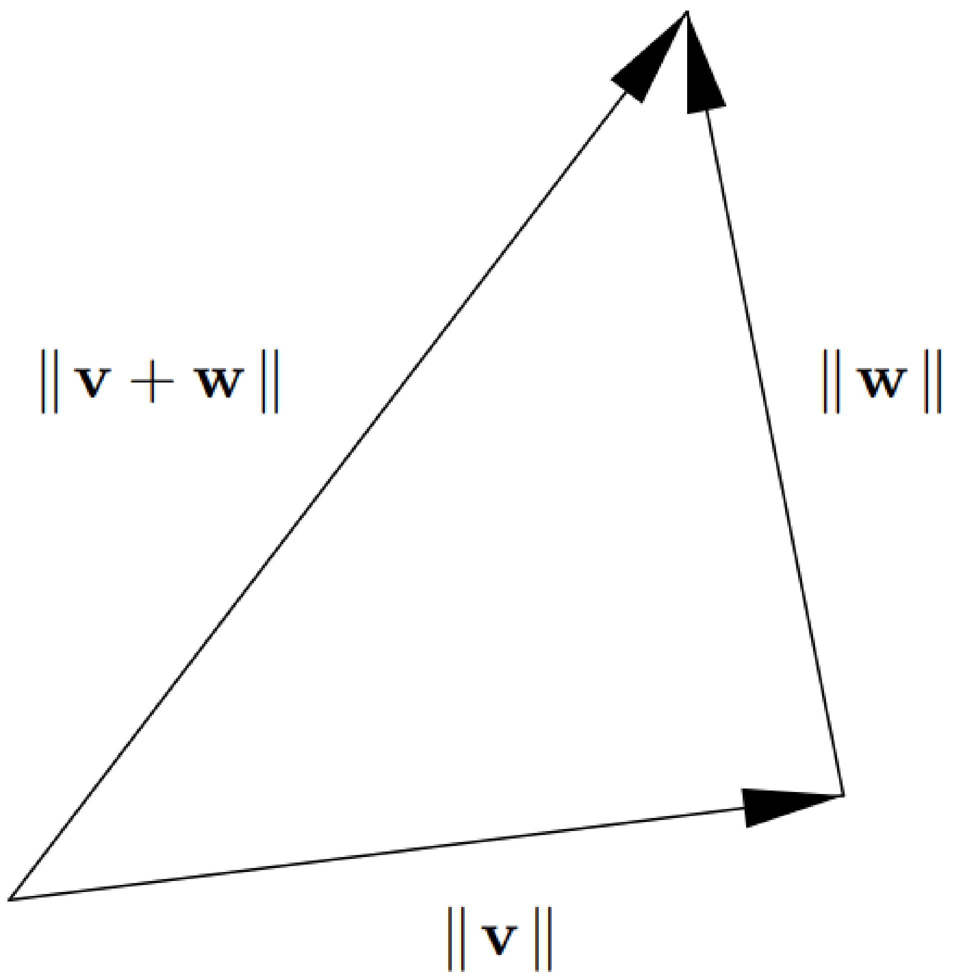


- Given $-1 \leq \cos\theta \leq 1$, the dot product between two vectors is bounded by the product of their lengths: $-\|\boldsymbol{v}\|\|\boldsymbol{w}\| \leq \boldsymbol{v} \cdot \boldsymbol{w} \leq \|\boldsymbol{v}\|\|\boldsymbol{w}\|$ or equivalently, $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq \|\boldsymbol{v}\|\|\boldsymbol{w}\|$
- $|\boldsymbol{v} \cdot \boldsymbol{w}| \leq \|\boldsymbol{v}\|\|\boldsymbol{w}\|$ is the simplest form of the general *Cauchy-Schwarz inequality.*

Every inner product satisfies the Cauchy-Schwarz inequality $|\langle v, w\rangle| \leq \|v\| \|w\|$ for all $v, w \in \mathbb{R}^n$, where $\|\cdot\|$ is the associated norm. Equality holds if and only if v and w are parallel vectors, i.e., $v = \lambda w$ for some scalar.

The Triangle Inequality

- ***Triangle inequality*** states that the length of one side of a triangle is at most equal to the sum of the lengths of the other two sides.
- If the first two sides of a triangle are represented by vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, then the third corresponds to their sum $\boldsymbol{v} + \boldsymbol{w}$


$\|\mathbf{v}+\mathbf{w}\|$
$\|\mathbf{w}\|$
$\|\mathbf{v}\|$

The norm associated with an inner product satisfies the triangle inequality $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in \mathbb{R}^n$. Equality holds if and only if v and w are parallel vectors that point in the same direction, i.e., $v = cw$ for nonnegative scalar $c \geq 0$

Can we prove the triangle inequality: $\|v + w\| \leq \|v\| + \|w\|$

- The vectors $v = \binom{3}{1}$ and $w = \binom{-1}{2}$ sum to $v + w = \binom{2}{3}$. Their Euclidean norms are $\|v\| = \sqrt{10}$ and $\|w\| = \sqrt{5}$, while $\|v + w\| = \sqrt{13}$.
- $\sqrt{13} \leq \sqrt{10} + \sqrt{5}$ is true.

Orthogonal Vectors and Orthogonal Bases

- In Euclidean geometry, a particular noteworthy configuration occurs when two vectors are **perpendicular.**
- This means that they meet at a right angle ($\theta = \measuredangle(v, w) = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$, and hence $\cos\theta = 0$

- From the equation $\|\boldsymbol{v}+\boldsymbol{w}\| \leq \|\boldsymbol{v}\|+\|\boldsymbol{w}\|$, vectors v, w are perpendicular if and only if their dot product vanishes: $\boldsymbol{v} \cdot \boldsymbol{w} = 0$.

Two vectors $v, w \in \mathbb{R}^n$ are called orthogonal if their inner product vanishes: $\langle v, w \rangle = 0$

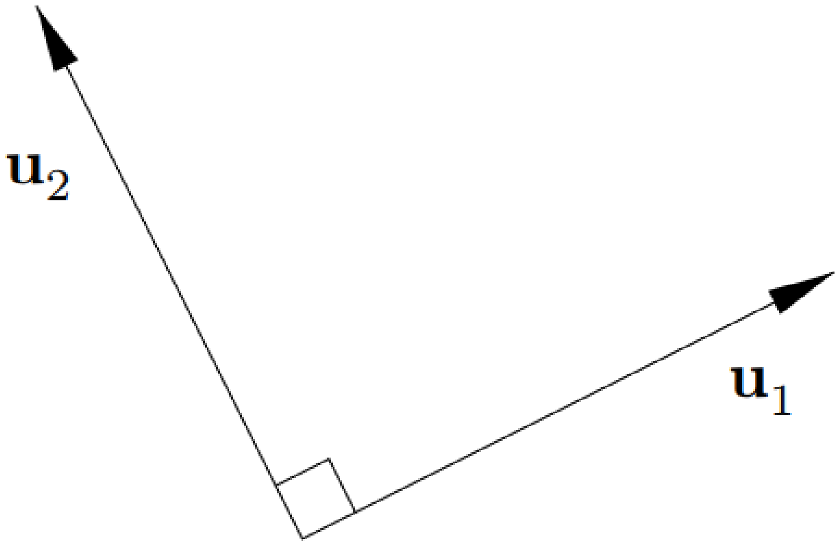
- The zero vector is orthogonal to all other vectors: $\langle \mathbf{0}, \boldsymbol{v} \rangle = 0$ for all $\boldsymbol{v} \in \mathbb{R}^n$, and is the only vector with this property.
- The vectors $\boldsymbol{v} = (1, 2)^T$ and $\boldsymbol{w} = (6, -3)^T$ are orthogonal with respect to the Euclidean dot product in $\mathbb{R}^2$, since $\boldsymbol{v} \cdot \boldsymbol{w} = 1 \cdot 6 + 2 \cdot (-3) = 0$

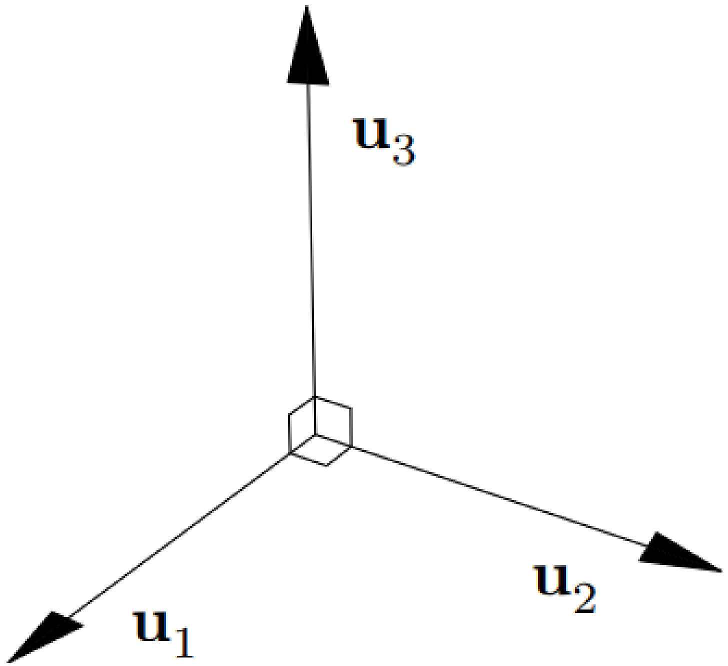
- Orthogonality depends upon which inner product is being used.
- Considering a weighted inner product: $\langle v, w \rangle = 2v_1w_1 + 5v_2w_2$, the vectors are not orthogonal.
- $\langle v, w \rangle = \left\langle \binom{1}{2}, \binom{6}{-3} \right\rangle = 2 \cdot 1 \cdot 6 + 5 \cdot 2 \cdot (-3) = -18$

A basis $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ of a k −dimensional subspace $V \subseteq \mathbb{R}^n$ is called orthogonal if $\langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle = 0$ for all $i \neq j$. The basis is called orthonormal if, in addition, each vector has unit length: $\|\boldsymbol{u}_i\| = 1$, for all $i = 1, \ldots, k$

- The simplest example of an orthonormal basis is the standard basis $e_1, \ldots, e_n$.

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \; e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \; e_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$





$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = 0$, for $i \neq j$, while $\|\boldsymbol{e}_i\| = 1$ implies normality

- The vectors $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$ are mutually orthogonal under the dot product: $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = \boldsymbol{v}_1 \cdot \boldsymbol{v}_3 = \boldsymbol{v}_2 \cdot \boldsymbol{v}_3 = 0$
- When we divide each orthogonal basis vector by its Euclidean length, the result is the orthonormal basis.

- $\boldsymbol{u}_1 \cdot \boldsymbol{u}_2 = \boldsymbol{u}_1 \cdot \boldsymbol{u}_3 = \boldsymbol{u}_2 \cdot \boldsymbol{u}_3 = 0$ and $\|\boldsymbol{u}_1\| = \|\boldsymbol{u}_2\| = \|\boldsymbol{u}_3\| = 1$

If $v_1, \ldots, v_k$ is an orthogonal basis of a subspace V, then the normalized vectors $u_i = \frac{v_i}{\|v_i\|}, i = 1, \ldots, k$, form an orthonormal basis

Every orthogonal collection of nonzero vectors is automatically linearly independent.

What are the advantages of orthogonal and orthonormal bases?

Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ be an orthonormal basis for a k-dimensional subspace $V \subseteq \mathbb{R}^n$. Then one can write any vector $\boldsymbol{v} \in V$ as a linear combination $v = c_1\boldsymbol{u}_1 + \cdots + c_k\boldsymbol{u}_k$ in which its coordinates $c_i = \langle \boldsymbol{u}_i, \boldsymbol{v} \rangle$, $i = 1, \ldots, k$, are explicitly given as inner products.

Moreover, its norm is given by the Pythagorean formula $\|v\| = \sqrt{c_1^2 + \cdots + c_k^2} = \sqrt{\sum_{i=1}^{k} \langle u_i, v \rangle^2}$

The wavelet basis $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$ are orthogonal basis of $\mathbb{R}^4$ under the dot product i.e $\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0$ for $i \neq j$

What are their Euclidean norms?

$\|\boldsymbol{v}_1\| = 2, \quad \|\boldsymbol{v}_2\| = 2, \quad \|\boldsymbol{v}_3\| = \sqrt{2}, \quad \|\boldsymbol{v}_4\| = \sqrt{4}$, the corresponding orthonormal wavelet basis is

$$\boldsymbol{u}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \boldsymbol{u}_2 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}, \boldsymbol{u}_3 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{u}_4 = \begin{pmatrix} 0 \\ 0 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$\boldsymbol{v} = \begin{pmatrix} 4 \\ -2 \\ 1 \\ 5 \end{pmatrix} = 4\boldsymbol{u}_1 - 2\boldsymbol{u}_2 + 3\sqrt{2}\boldsymbol{u}_3 - 2\sqrt{2}\boldsymbol{u}_4$$
$$= 2v_1 - v_2 + 3v_3 - 2v_4$$

where the orthonormal wavelet basis coodinates are computed directly by taking dot products:

$$u_1 \cdot v = 4, \quad u_2 \cdot v = -2, \quad u_3 \cdot v = 3\sqrt{2}, \quad u_4 \cdot v = -2\sqrt{2}$$