VISPAD INSTITUTE OF TECHNOLOGY

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Eigenvalues and Singular Values

Fall 2025

(October – December Virtual Internship)

At the end of the session the students or candidates should be able to understand and work with:

- Eigenvalues of matrices
- Finding eigenvalue-eigenvector pairs
- Eigenvectors, eigenspaces, and their bases

- Each square matrix possesses a collection of one or more distinguished scalars, called eigenvalues, each associated with certain distinguished vectors known as eigenvectors.
- When a matrix acts on vectors via matrix multiplication, the eigenvector specify the directions of pure scaling and the eigenvalues the extent the

eigenvector is scaled.

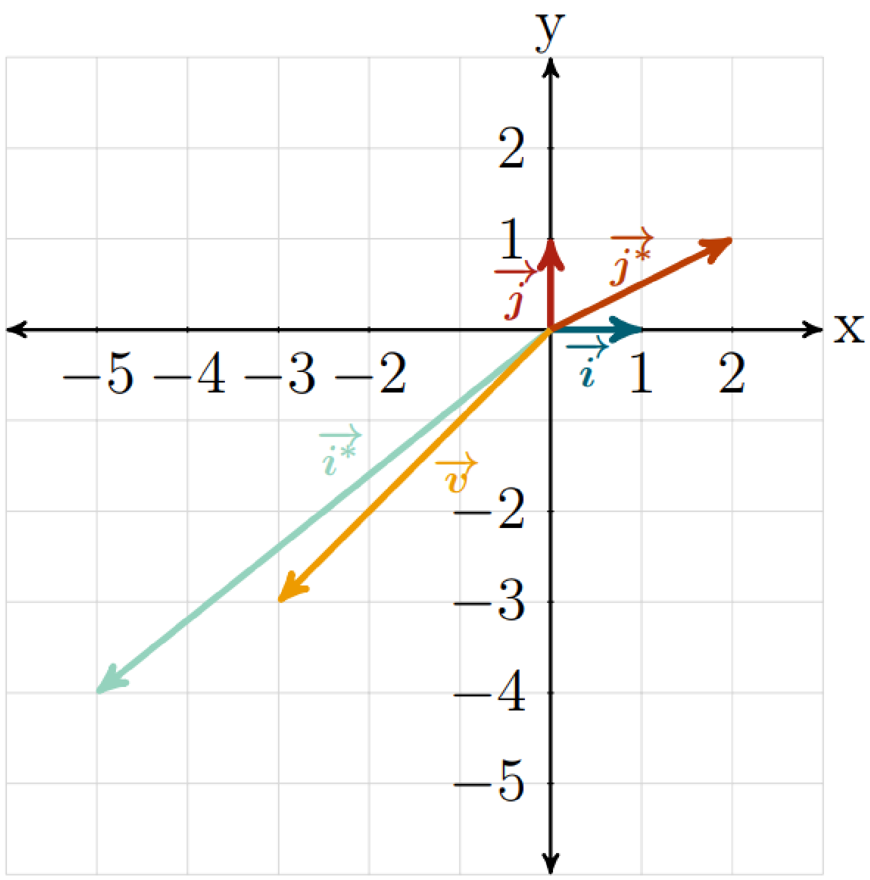
- Eigenvalues and eigenvectors are of absolutely fundamental importance.
- They have a broad range of applications, including machine learning and data analysis, dynamical systems, both continuous and discrete, statistics, and many more.

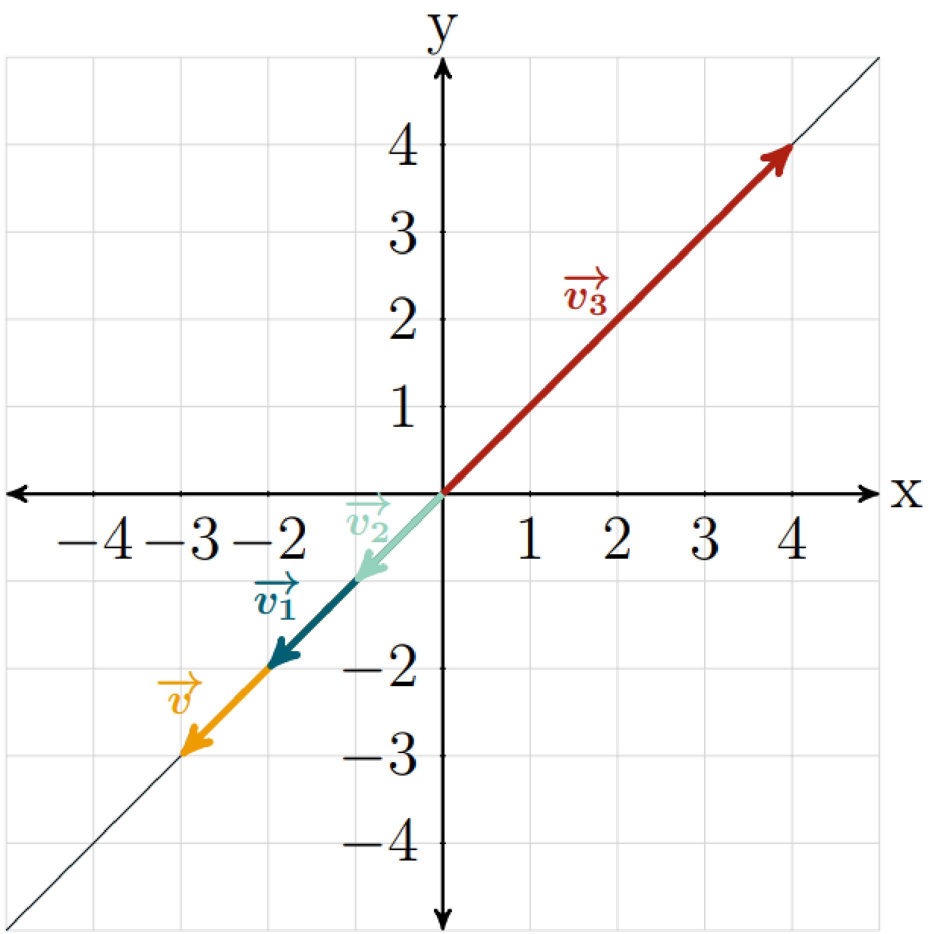
Eigenvalues and Eigenvectors

Let A be a square matrix. A scalar λ is called an *eigenvalue* of A if there is a nonzero vector $v \neq 0$, called an *eigenvector*, such that

$$Av = \lambda v$$

- In geometric terms, the matrix A scales (stretches) the eigenvector v by an amount specified by the eigenvalue λ.
- An eigenvector is a vector that, when transformed by a matrix, only get stretched or shrunk by a certain amount but does not get rotated.





$$i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad A = \begin{pmatrix} -5 & 2 \\ -4 & 1 \end{pmatrix}$$

"*Eigenvalue*" and "*eigenvector*" are hybrid German-English, which can be fully translated as "*proper value*" and "*proper vector*".

The alternative English terms *characteristic value* and *characteristic vector* can be found in some texts.

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} = \lambda I\vec{v}$$

$$A\vec{v} - \lambda I\vec{v} = \overrightarrow{0}$$

$$(A - \lambda I)\vec{v} = \overrightarrow{0}$$

- There are two ways to solve the above the equation.
- The first one is the trivial solution where $\vec{v} = 0$

- The second one is when $\vec{v} \neq 0$.
- A homogenous linear system has a nonzero solution $\vec{v} = \vec{0}$ if and only if its coefficient matrix, which in this case is $A - \lambda I$, is singular.
- This observation is the key to resolving the eigenvector equation.

- A matrix is **singular** if and only if it has a zero eigenvalue.
- Let’s consider the simplest case in detail. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be general 2×2 real matrix with indicated entries $a, b, c, d \in \mathbb{R}$.

A scalar λ will be an eigenvalue if and only if the matrix

$$A - \lambda I = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix}$$

is singular.

$$\det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc$$

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

- The eigenvalues are the solutions to a certain quadratic polynomial equation called the *characteristic equation* associated with the matrix.
- The *characteristic equation* can be immediately solved using the quadratic equation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- There are three possibilities, which can be characterized by the sign of the *discriminant* $\Delta = (b^2 - 4ac)$ of the quadratic equation.
  a. $\Delta > 0$: The characteristic equation has two different real roots $\lambda_1 \neq \lambda_2$. A has two distinct eigenvalues.
  b. $\Delta = 0$: The characteristic equation has a single

real root λ_1, and so A has only one eigenvalue.

c. $\Delta < 0$: The characteristics equation has complex conjugate root $\lambda_{\pm} = \mu \pm iv$, where $i = \sqrt{-1}$ is the imaginary unit.

$$\left|\begin{pmatrix} -5 & 2 \\ -4 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| = 0$$

$$\begin{vmatrix} -5-\lambda & 2 \\ -4 & 1-\lambda \end{vmatrix} = 0$$

$$(-5-\lambda)(1-\lambda) - 2\cdot(-4) = 0$$

$$\lambda^2 + 4\lambda + 3 = 0$$

$$\lambda = -3 \; and \; \lambda = -1$$

$$\lambda = -3$$

$$\begin{pmatrix} -5-(-3) & 2 \\ -4 & 1-(-3) \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} -2 & 2 \\ -4 & 4 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{0}$$

$$M = \begin{pmatrix} -2 & 2 \\ -4 & 4 \end{pmatrix}$$

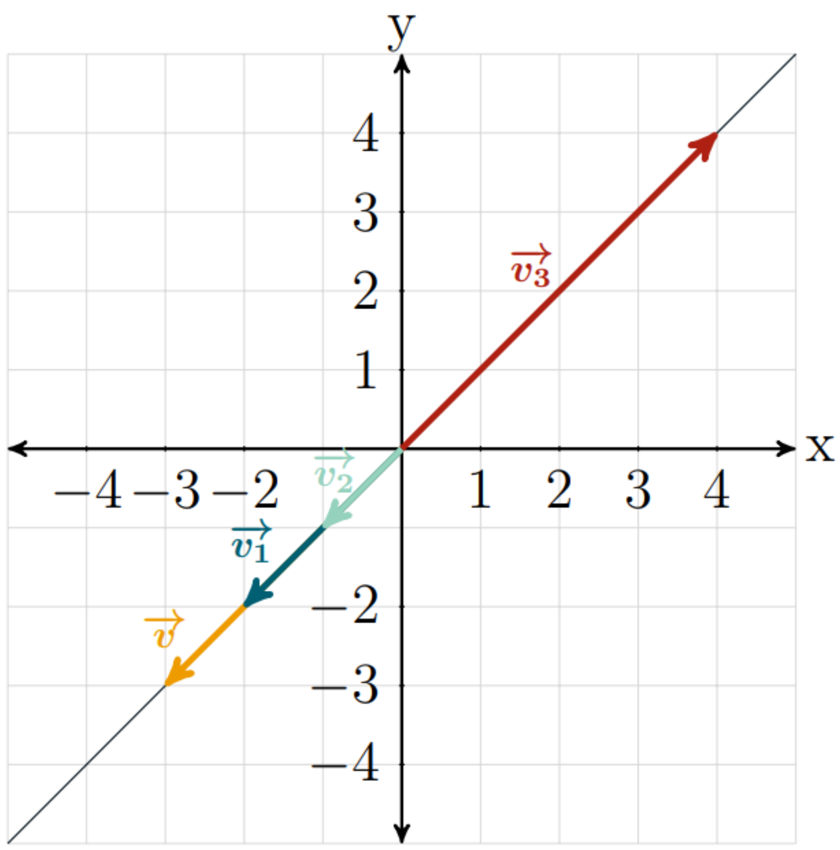
- The determinant of the matrix M is zero.
- This means that we either have no solutions, or a lot them.

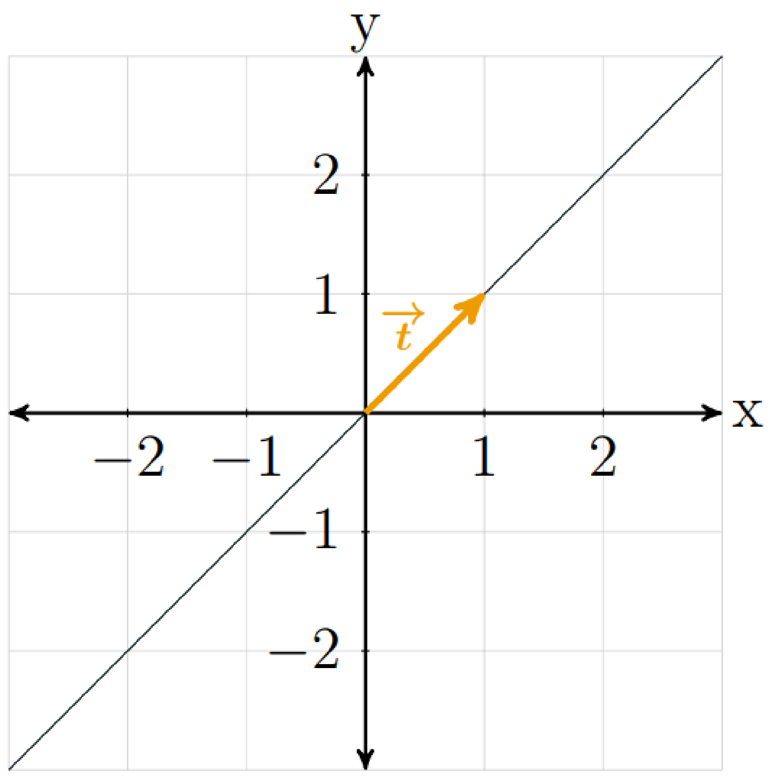
$$\begin{cases} -2v_1 + 2v_2 = 0 \\ -4v_1 + 4v_2 = 0 \end{cases} \Leftrightarrow \begin{cases} -2v_1 = -2v_2 \\ -4v_1 + 4v_2 = 0 \end{cases}$$

$$\begin{cases} v_1 = v_2 \\ -4v_1 + 4v_2 = 0 \end{cases} \Leftrightarrow \begin{cases} v_1 = 1 \\ v_2 = 1 \end{cases}$$

The eigen vector corresponding to the eigenvector $\lambda =$

$- 3$ is the vector $(1, 1)^T$





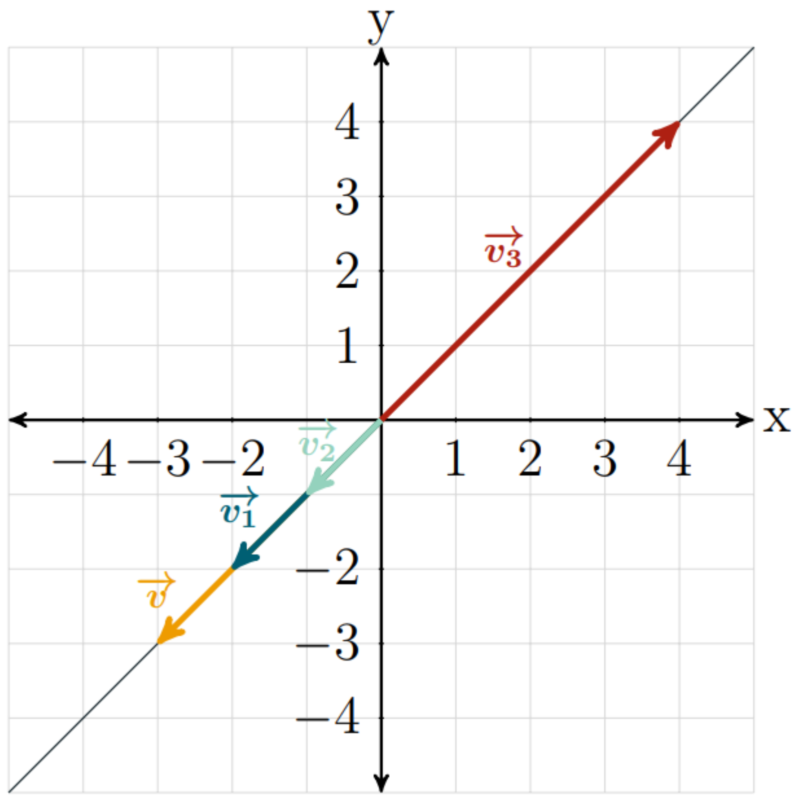
$$\begin{pmatrix} -5 & 2 \\ -4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \end{pmatrix}$$

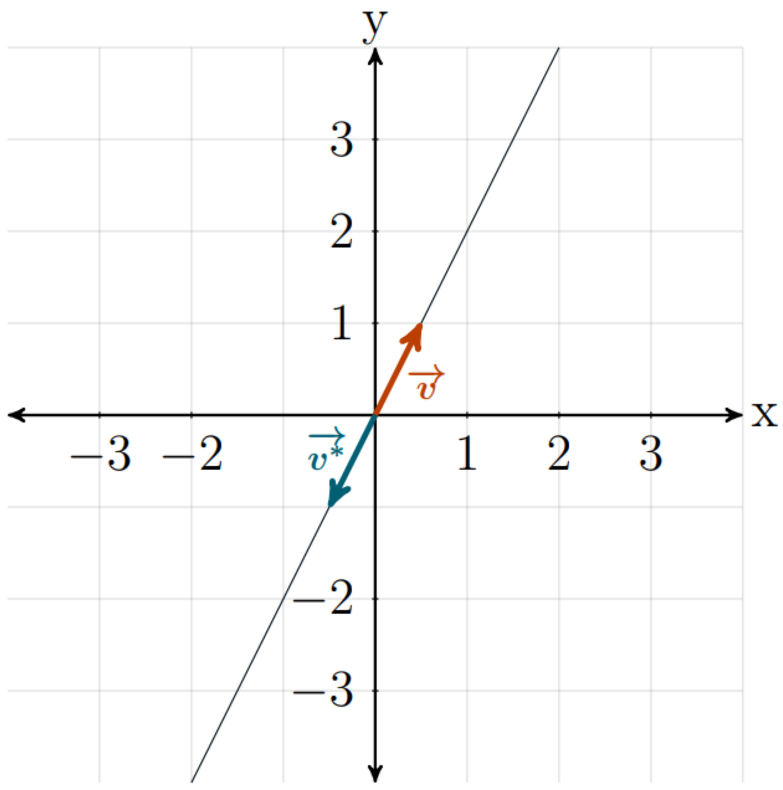
$$\lambda = -1$$

$$\begin{pmatrix} -5-(-1) & 2 \\ -4 & 1-(-1) \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{0}$$

$$\begin{pmatrix} -4 & 2 \\ -4 & 2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{0}$$

$$\vec{v} = \left(\frac{1}{2}, 1\right)^T$$





$$\begin{pmatrix} -5 & 2 \\ -4 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -1 \end{pmatrix}$$

Example

1. Find the eigenvalues and vectors of the matrix: $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$.
2. Find the eigenvalues and vectors of the matrix: $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

$$A\vec{v} = \lambda\vec{v}$$

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = -3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} = -1\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

$$A\overbrace{\begin{pmatrix}1 & \frac{1}{2}\\ 1 & 1\end{pmatrix}}^{P}=\overbrace{\begin{pmatrix}1 & \frac{1}{2}\\ 1 & 1\end{pmatrix}}^{P}\overbrace{\begin{pmatrix}-3 & 0\\ 0 & -1\end{pmatrix}}^{\Lambda}$$

$$AP = P\Lambda$$

$$APP^{-1} = P\Lambda P^{-1}$$

$$A = P\Lambda P^{-1}$$

A square matrix A is complete if and only if there exists a nonsingular matrix P and a diagonal matrix Λ such that $P^{-1}AP = \Lambda$ or equivalently, $A = P\Lambda P^{-1}$

If A is a complete matrix of size $n \times n$, then the sum of its eigenvalues equals its trace, e.i., the sum of its diagonal entries:

$$\sum_{i=1}^{n} \lambda_i = trA = \sum_{i=1}^{n} a_{ii}$$

Matrix Diagonalization

matrix

invertible matrix
(eigenvector columns)

negative 1

$$A = PDP^{-1}$$

equals

diagonal matrix
(eigenvalue diagonals)

inverse of P

- A can be defined as a product of three matrices:
  - the eigen base P and its inverse P^{-1} which represent two rotations.
  - the matrix Λ is a diagonal matrix with eigenvalues for elements.
- In simple terms, we are representing a matrix by two rotations and one scaling term.

Diagonal matrix

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- In calculating the determinant of a diagonal matrix, one only needs to multiply the elements that forms the diagonal.

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = a \cdot b - 0 \cdot 0 = a \cdot b$$

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

- The inverse of the above matrix can be found using Laplace expansion.
- The Laplace expansion is defined as: $\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$

- The expansion can be simplified as:

$$\det(A) = (-1)^{2i} a_{ii} M_{ij}$$

- In simple terms, we are multiplying all of the elements on the diagonal together.
- The inverse of a matrix can be calculated as:

$$A^{-1} = \frac{1}{\det(a)}(adj(A))$$

- The adjoint can be calculated as: $adj(A) =$

$$\begin{pmatrix} M_{11} & 0 & 0 \\ 0 & M_{22} & 0 \\ 0 & 0 & M_{33} \end{pmatrix}$$

- This is equivalent to: $adj(A) = \begin{pmatrix} bc & 0 & 0 \\ 0 & ac & 0 \\ 0 & 0 & ab \end{pmatrix}$

$$A^{-1} = \frac{\begin{pmatrix} bc & 0 & 0 \\ 0 & ac & 0 \\ 0 & 0 & ab \end{pmatrix}}{abc} = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{pmatrix}$$

- ❑ Simply, we have replaced each non-zero element with its inverse.

- Matrix multiplication is not commutative; however, if one of the matrices in the operation is diagonal, we can easily get the results.
- Let’s define a generic 2×2 matrix X as:

$$X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

- If X is multiplied by a diagonal on the left, it will change the matrix row-wise.

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} ax_{11} & bx_{12} \\ ax_{21} & bx_{22} \end{pmatrix}$$

- If X is multiplied by a diagonal matrix on the right, it will change the matrix column-wise.

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} ax_{11} & bx_{12} \\ ax_{21} & bx_{22} \end{pmatrix}$$

- Powering of matrix comes with comfort when diagonalization technique is used.
- Multiplying 2×2 by itself is baby talk 😂 😂.
- In machine learning, we can deal with very large matrices 😒 😒.

- We know that $A = P\Lambda P^{-1}$
- Let A be a $n \times n$ matrix where n is giant.
- This giant matrix can be:
  - information about the weather (states of observation climate conditions)
- Powering A to the t can be computed by multiplying A by itself t times which is extremely expensiver in

of computation time.

- Matrix diagonalization can be used instead of ….

$$A^t = A.A.A \dots A \quad t \text{ times}$$

- Let's start squaring A:

$$A^2 = P\Lambda P^{-1}P\Lambda P^{-1}$$

$$A^2 = P\Lambda\Lambda P^{-1}$$

$$A^2 = P\Lambda^2 P^{-1}$$

- Let's compute A^3

$$A^3 = P\Lambda^2 P^{-1} \cdot P\Lambda P^{-1}$$

$$A^3 = P\Lambda^3 P^{-1}$$

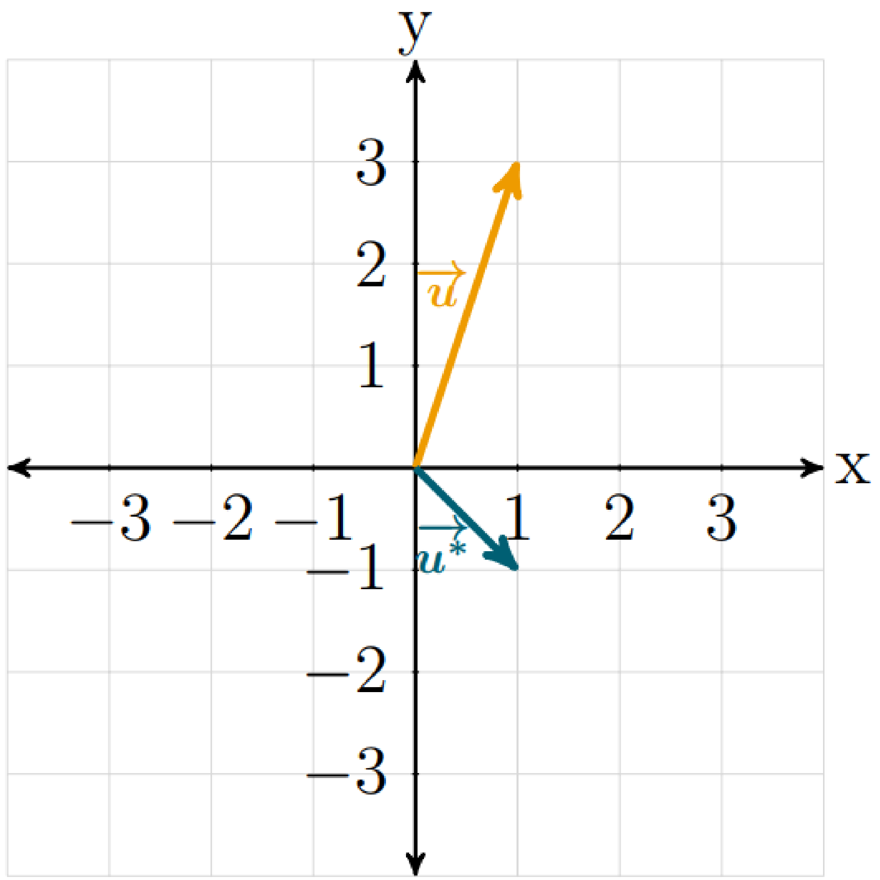
- If you want to power A to t, the formula is:

$$A^t = P\Lambda^t P^{-1}$$

- In geometric interpretation, we are representing a linear transformation as one rotation, followed by a stretch, and then another rotation.

- ❑ Let $A = \begin{pmatrix} -5 & 2 \\ -4 & 1 \end{pmatrix}, \vec{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

- ❑ $A \cdot \vec{u} = \begin{pmatrix} -5 & 2 \\ -4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



- Let's rotate, stretch and rotate the vector $\vec{u}$ with $P\Lambda P^{-1}$.
- The result has to be a vector with coordinates $(1,-1)^T$.
- P is the matrix with the eigenvectors of A for columns.

- Λ is a matrix whose diagonal has for entries the eigenvalues of A, and the remaining entries are 0.
- P^{-1} is the inverse of P
- The first rotation:

$$P^{-1} \cdot \vec{u} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \overrightarrow{u_1^*}$$

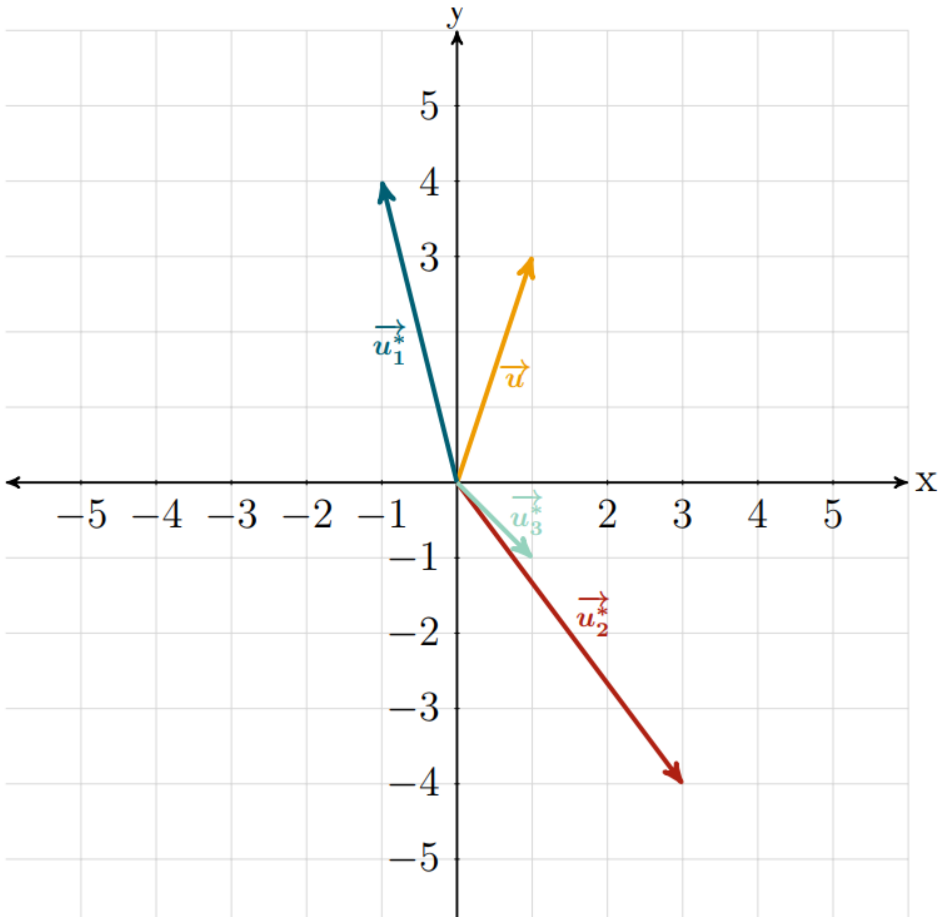
- Now, let's scale it:

$$\Lambda \cdot \overrightarrow{u_1^*} = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \overrightarrow{u_2^*}$$

- Finally, let's rotate it again:

$$P \cdot \overrightarrow{u_2^*} = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \overrightarrow{u_3^*}$$

- This process is known as *eigendecomposition*, which is a special case of matrix decomposition.

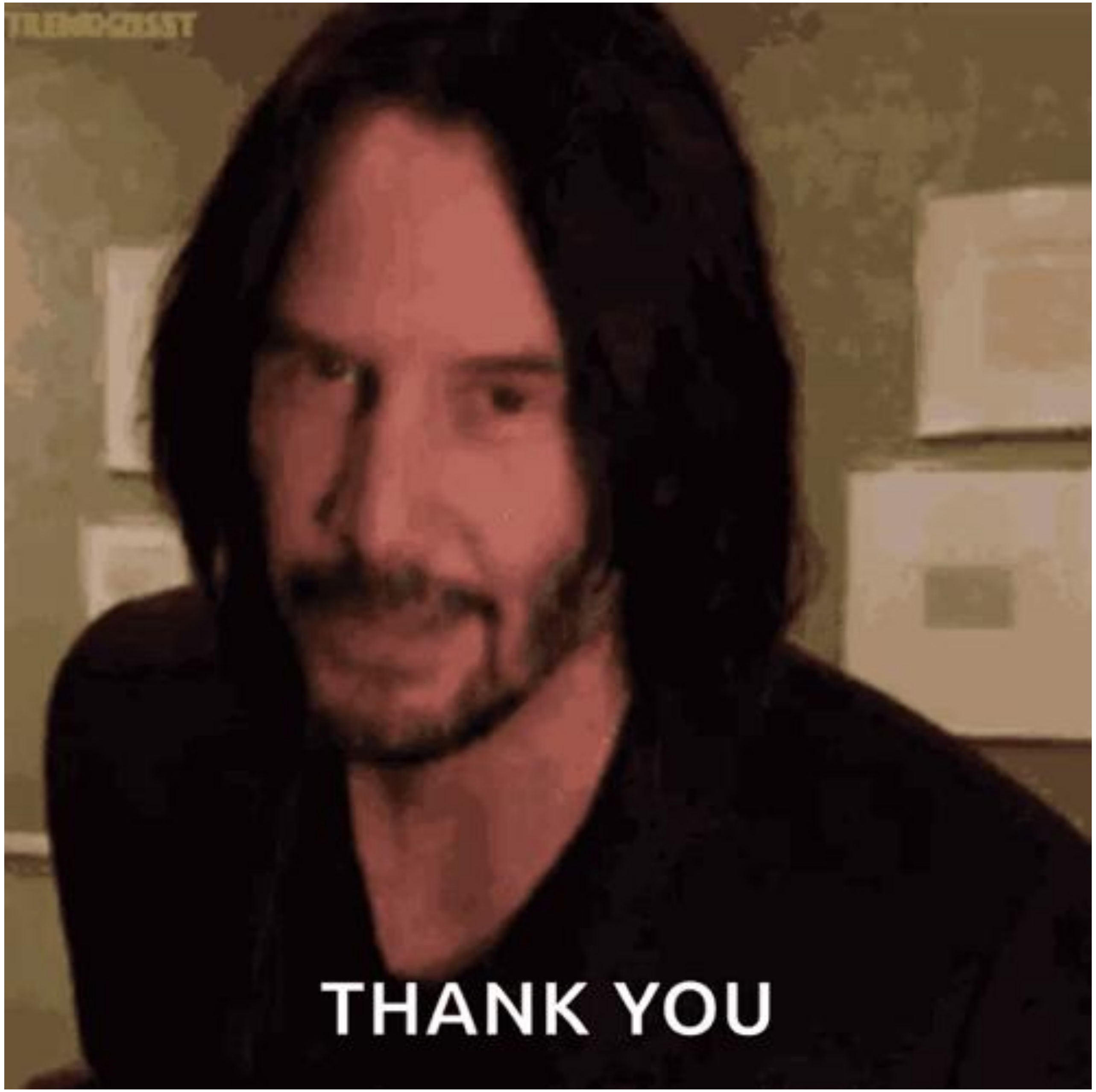


- ❑ Given the matrix $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$, find the eigen values and the corresponding eigen vectors.
- ❑ Determine the values of P, Λ and P^{-1}
- ❑ Find the value of $P^{-1}AP$
- ❑ Find the trA

- The only catch is that the matrix that we decompose has to be *square.*
- Square matrices are hard to come by in real world application.
- If square matrices are not that common, there's got to be a way to decompose any matrix shape.
- *Single value decomposition* can used to achieve this.

- Understanding eigen-decomposition is fundamental for grasping several important machine learning algorithms.
- *Principal Component Analysis (PCA)* relies directly on the eigen-decomposition of the data’s covariance matrix.

- The eigenvectors indicate the directions of maximum variance (the principal components), and the eigenvalues quantify this variance.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