Solution

- Remove any edge that has a probability 0
- Construct a graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ representing the problem
- Let s be the initial position of *Rami* and t be the initial position of *Yessine*. And we suppose without a loss of generality that $s \neq t$
- Let $\mathbb{E}[W_{i,j}]$ be the expected length of the walk if *Rami* is at i and *Yessine* is at j
- Let $\mathcal{A}$ be the condensation graph of $\mathcal{G}$ and $\{C_1,\dots,C_m\}$ be its nodes. That is $C_1,\dots,C_m$ are the strongly connected components of $\mathcal{G}$
- Let $\mathcal{S}(\mathcal{A})=\{C_i / \quad C_i \nrightarrow C_j \quad \forall j\}$, that is $\mathcal{S}(\mathcal{A})$ is the set of sinks of $\mathcal{A}$. This set is not empty as $\mathcal{A}$ is a directed acyclic graph.
- Let $\mathcal{G}^2=(\mathcal{V}^2,\mathcal{E}^2)$ be the product graph of $\mathcal{G}$ and $\mathcal{G}$. It is the graph of the simultaneous walk of both *Rami* and *Yessine*

1. Infinite Walk

- First of all, we will observe that eventually, both persons will get stuck at $(C_i,C_j)\in\mathcal{S}(\mathcal{A})^2$
- Note that the expected walk length is infinite if and only if both persons can go with k steps to (u,v) without a crossover such that $\mathbb{E}[W_{u,v}]$ is infinite and the path has a **non-zero** probability (conditioning on the number of steps k). The latter is guaranteed as we removed any edge with 0 probability.
- Now, we can reduce the problem to finding such u,v with the requirement that each of u,v is on some **sink** connected component:
  - If u,v are in the same connected component $C\in\mathcal{S}(\mathcal{A})$, then the $\mathbb{E}[W_{u,v}]=+\infty$ if and only if (u,v) cannot go simultaneously to a common node (i,i)
  - Otherwise if $u\in C, v\in C'$ and $C,C'\in\mathcal{S}(\mathcal{A})$ and $C\neq C'$, It is trivial that $\mathbb{E}[W_{u,v}]=+\infty$
  - If no u,v exists, each person cannot go to some **sink** connected component without a crossover. It can be shown that in that case $\mathbb{E}[W_{s,t}]<+\infty$

With that the expected walk length is infinite if and only if:

1. $\exists i,j\in\{1,\dots,m\}/ \quad s\to^* u\in C_i, t\to^* v\in C_j$ and $C_i,C_j\in\mathcal{S}(\mathcal{A})$
2. There exists a path $(s,t)\to^*(u,v)$ in the product graph that does not cross $(k,k) \quad \forall k\in\{0,\dots,n-1\}$
3. Every path that goes from (u,v) does not cross $(k,k) \quad \forall k\in\{0,\dots,n-1\}$

Now with that, we will do a BFS starting from (s,t) on the graph $\mathcal{G}'$ that is equal to $\mathcal{G}^2$ after removing the nodes $(i,i) \quad \forall i\in\{0,\dots,n-1\}$, and we mark all reachable nodes on the boolean matrix `visited`. Such BFS can be implemented in $\mathcal{O}(n^2+m^2)$

Now for each reachable node (i,j), let $C(i)$ the connected component of i, and $C(j)$ the connected component of j

1. If $C(i)\notin\mathcal{S}(\mathcal{A})$ or $C(j)\notin\mathcal{S}(\mathcal{A})$, skip
2. Otherwise, if $C(i)\neq C(j)$, set $\mathbb{E}[W_{s,t}]\leftarrow+\infty$ and exit the loop

3. Otherwise we have $C(i) = C(j)$, we do a BFS on the product graph $\mathcal{G}^2$ starting from (i, j) (without removing the nodes (k, k)), and we save the reachable nodes on `marked`
   - If there is some node (i, i) such that `marked[i][i]` is True, skip
   - Otherwise, set $\mathbb{E}[W_{s,t}] \leftarrow +\infty$ and exit the loop

After that if there is no assignation $\mathbb{E}[W_{s,t}] \leftarrow +\infty$, then we are sure that $\mathbb{E}[W_{s,t}]$ is finite

With that the time complexity of such brute forcing BFS is $\mathcal{O}(n^2(n^2 + m^2))$.

The overall time complexity of such check is:

$$\mathcal{O}(n^4 + n^2m^2) = \mathcal{O}(n^6)$$

2. Finite Walk

We have:

$$\mathbb{E}[W_{u,v}] = \begin{cases} 0 & \text{if } u = v \\ 1 + \sum_{0 \le i,j < n} p_{u,i} p_{v,j} \mathbb{E}[W_{i,j}] & \text{otherwise} \end{cases}$$

This is a system of n^2 equations on n^2 variables.

It can be solved using Gaussian elimination on:

$$\mathcal{O}(n^6)$$

3. Time Complexity

The time complexity of the overall algorithm is:

$$\mathcal{O}(n^6)$$