Poisson Processes

a) Question: how do we mathematically describe a point process:
   i) Binomial description
   ii) Rate based description
b) Sample path of a point process (e.g. a spike train)
   Let (0,T] denote the times where we are recording the spike train.

   Let $0 < u_1 < u_2 < \ldots < u_J \leq T$ be the observed spike times.

   For $t \in (0,T]$ let $N_0^t = \{0 < u_1 < u_2 < \ldots < u_j \cap N(t) = j\}, j \leq J$.

   Where J is the total number of spikes and

   N(t) is the total number of spike observed up to time t.

   Thus, N(t) = j means than at time t we have observed j spikes.

   N_0^t is called the sample path of the point process and is a function that starts at 0 and jumps 1 at the time of each spike.
c) Firing rate
   $$\lambda = \lim_{\Delta \to 0} \frac{\Pr\left(N_0^{t+\Delta} - N_0^t = 1\right)}{\Delta} ,$$
   i) The probability that our sample path jumped by one (eg. that we saw a spike) in a short interval Δ.
d) Homogeneous Poisson process
   i) Description based on rate of spiking as a function of time.
   $$\Pr\left(\text{k spikes in time T} \mid \text{firing rate } \lambda\right) = \frac{e^{-\lambda t}\left(\lambda T\right)^k}{k!} \text{ for k = 0,1, ... or}$$
   $$\Pr\left(\mathrm{N}_0^{\mathrm{T}} = \mathrm{k} \mid \lambda\right) = \frac{e^{-\lambda t}\left(\lambda T\right)^k}{k!} \text{ for k = 0,1, ...}$$
   ii) Given fixed total time T, rewrite as $\Pr\left(\text{k spikes} \mid \text{m}\right) = \frac{e^{-m}\left(m\right)^k}{k!}$

e) Properties of the Poisson distribution

i) the mean of a Poisson distribution with rate parameter m is m.'

by definition $E[k]=\sum_{k=0}^{\infty} k\frac{e^{-m}(m)^k}{k!}$.

$$\sum_{k=0}^{\infty} k\frac{e^{-m}(m)^k}{k!}=me^{-m}\sum_{k=1}^{\infty}\frac{(m)^{k-1}}{(k-1)!}=me^{-m}e^{m}=m, \text{ as } \sum_{k=1}^{\infty}\frac{(m)^{k-1}}{(k-1)!}=e^{m}$$

ii) the variance of a Poisson distribution is equal to its mean.

$E[(k-m)^2]=E[k^2]-E[k]^2=E[k^2]-m^2$.

$$E[k^2]=\sum_{k=0}^{\infty} k^2\frac{e^{-m}(m)^k}{k!}=\sum_{k=0}^{\infty}(k(k-1)+k)\frac{e^{-m}(m)^k}{k!}=$$

$$\sum_{k=0}^{\infty} k\frac{e^{-m}(m)^k}{k!}+\sum_{k=0}^{\infty}(k(k-1))\frac{e^{-m}(m)^k}{k!}=$$

$$m+e^{-m}m^2\sum_{k=2}^{\infty}\frac{(m)^{k-2}}{(k-2)!}=m+e^{-m}m^2e^{m}=m+m^2$$

Thus, $E[k^2]-m^2=m+m^2-m^2=m$.

iii) The interspike interval (ISI) distribution P(t < ISI ≤ t+Δ) for a Poisson process with rate parameter λ is $\lambda e^{-\lambda t}$.

Proof: Pr(t < ISI ≤ t+Δ) is equivalent to saying that we had no spikes up to time t and then one spike in the t+Δ interval. Using the Poisson distribution, we know the probability of that happening is

$$\lim_{\Delta\to 0}\Pr(\text{0 spikes in (0,t), 1 spike in [t,t+}\Delta\text{)})=\lim_{\Delta\to 0}\frac{e^{-\lambda t}(\lambda t)^0}{0!}\cdot\frac{e^{-\lambda\Delta}(\lambda\Delta)^1}{1!}=\lim_{\Delta\to 0}\lambda e^{-\lambda(t+\Delta)}\Delta.$$

Technically, this limit is zero, because the probability of any specific ISI is very small. Nonetheless, we can use this expression to derive the probability density for the ISI by dividing by Δ, leaving us with

$$f(t)=\lambda e^{-\lambda t}.$$

This is known as the exponential distribution.

Properties of the exponential distribution:

$$E[t]=\frac{1}{\lambda}$$

$$Var[t]=\frac{1}{\lambda^2}$$

- iv) The Poisson process is history-less.
  - (1) The time of the last spike has no influence on the time of the next spike.
  - (2) Result of the exponential waiting time distribution.

- f) Measures of "Poisson-ness"
  - i) Coefficient of Variation

    $$\frac{\sigma}{\mu}$$

    - (1) Equal to 1 for a Poission ISI function
      - (a) For exponential distribution $E\left[x\right]=\frac{1}{\lambda}$, $\text{Var}[x]=\frac{1}{\lambda^2}$
  - ii) Fano factor
    - (1) $\frac{\sigma^2}{\mu}$
    - (2) Equal to 1 for Poisson distribution.
- g) Inhomogeneous Poisson Process
  - i) Replace λ with λ(t) and λT with $m(T)=\int_0^T \lambda(u)du$
  - ii) $\Pr\left(\mathrm{N}_0^\mathrm{T}\text{=k} \mid \lambda(t)\right)=\frac{e^{-m(t)}\left(m(t)\right)^k}{k!}=\frac{e^{-\int_0^T \lambda(u)du}\left(\int_0^T \lambda(u)du\right)^k}{k!}$ for k = 0,1, ...
  - iii) Can be used to describe any point process with finite rate, but … may not be a parsimonious description
- h) Example: Quantal Analysis
  - i) Record responses to low level stimulation or spontaneous activity (minis)
  - ii) Identify events, graph histogram of amplitude
  - iii) Fit to poission
    - (1) Compute mean number of events m
    - (2) Test to determine if distribution is Possion with mean m
  - iv) Issues
    - (1) Signal attenuation in dendrites
    - (2) Independence of events
    - (3) Depression / potentiation
    - (4) Etc.

- i) Example: PSTH
  - i) Poisson assumption behind most analyses

(1) Turns out to be wrong is virtually all cases
(2) Refractory periods, bursting, etc.
(3) Issue: when does this matter?

ii) Other issues
(1) Bin sizes
(2) Stationarity
(3) Number of variables (one for each bin)