Función logaritmo

formula:

$\text{Log } z = \text{Log}|z| + i\arg z$

1. Probar que

a) $\text{Log}(-ei) = 1 - \frac{\pi}{2} i$

//sustituimos valores

$|z| = \sqrt{x^2 + y^2}$ ... 1

$\arg z = -\frac{\pi}{2}$ (H) ... 2

$|z| = \sqrt{0 + (-e)^2}$

$|z| = \sqrt{e^2} = e$

$\text{Log}(-ei) = \text{Log } e + i\left(\frac{-\pi}{2}\right)$

$\text{Log}(-ei) = \text{Log } e - i\frac{\pi}{2}$

$\text{Log}(-ei) = 1 - i\frac{\pi}{2}$

b. $\text{Log}(1-i) = \frac{1}{2}\text{Log } 2 - \frac{\pi}{4} i$

$|z| = \sqrt{1^2 + i^2}$

$= \sqrt{1+1}$

$|z| = \sqrt{2}$

$\text{Log}(1-i) = \text{Log}\sqrt{2} + i\left(-\frac{\pi}{4}\right)$

$\log(1-i) = \log 2^{\frac{1}{2}} - \frac{\pi}{4} i$ //por propiedades de algoritmos

$\text{Log}(1-i) = \frac{1}{2}\log 2 - \frac{\pi}{4} i$

2. probar que cuando $n = 0, \pm 1 \pm 2 \ldots$

a) $\log e = 1 + 2n\pi i$ //función multivaluada

$\log e = \log|e| + i(\theta + 2n\pi)$ donde $n = 0, \pm 1, \pm 2 \pm \ldots$

$\log e = \log e + 2n\pi i$ donde $n = 0 \pm 1 \pm 2 \pm \ldots$

$\log e = 1 + 2n\pi i$ donde $n = 0 \pm 1 \pm 2 \pm \ldots$

b) $\log i = (2n + \frac{1}{2}\pi i)$

Donde

$|z| = \sqrt{0 + i^2}$

$|z| = i$

$\arg z = \frac{\pi}{2}$

Donde $n = 0 \pm 1 \pm 2 \ldots$

Tenemos

$\log i = \log|i| + i\left(\frac{\pi}{2} + 2n\pi\right)$

$\log i = \log 1 + \pi i\left(\frac{1}{2} + 2n\right)$ $n = 0 \pm 1 \pm 2 \ldots$

$\log i = 0 + \left(\frac{1}{2} + 2n\right)\pi i$

Entonces

$$\log i = \left(\frac{1}{2} + 2n\right)\pi i$$

c) $\log(-1+\sqrt{3}\,i) = \operatorname{Log} 2 + 2\left(n+\frac{1}{3}\right)\pi i$

número imaginario número real

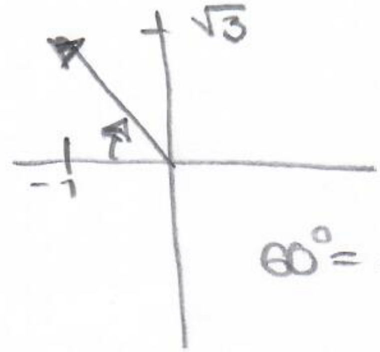
Tenemos

$$\log(-1+\sqrt{3}i) = \operatorname{Log}|-1+\sqrt{3}\,i| + i\left(\frac{2\pi}{3}+2n\pi\right)$$

donde $n = 0 \pm 1 \pm 2 \ldots$

$$\log(-1+\sqrt{3}\,i) = \log 2 + 2\left(n+\frac{1}{3}\right)\pi i$$

Donde $n = 0 \pm 1 \pm 2 \ldots$

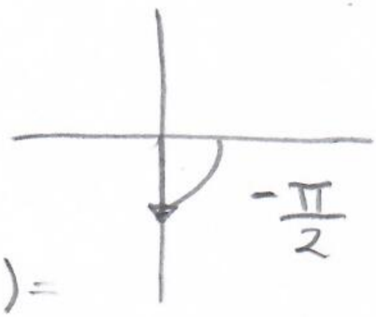


$60^\circ = \frac{2\pi}{3}$

$\arg = \frac{2\pi}{3}$

3. Probar que $[(1+i)^2] = 2\operatorname{Log}(1+i)$ pero $\operatorname{Log}[(-1+i)^2 \neq 2\operatorname{Log}(-1+i)$

Tenemos



$$\operatorname{Log}(1+i)^2 = 2\operatorname{Log}(1+i)$$

$$\operatorname{Log}(1+i)^2 = \operatorname{Log}(1+i)(1+i) = \operatorname{Log}(2i) =$$

donde

$$\underbrace{\log 2i = \log 2}\quad \arg = \frac{\pi}{2}$$

$$\operatorname{Log}(1+i)^2 = \operatorname{Log} 2 + \frac{\pi}{2} i$$

Donde

$$2\operatorname{Log}(1+i) = 2\left(\operatorname{Log}\sqrt{2} + \frac{\pi}{4}\right)i$$ //propiedades de los logaritmos $\sqrt{2} = 2^{1/2}$

Entonces

$$\underbrace{2\operatorname{Log}(1+i) = 2\left(\operatorname{Log}\sqrt{2} + \frac{\pi}{4}\right)i = \operatorname{Log} 2 + \frac{\pi}{2} i}$$

Pero $\operatorname{Log}[(-1+i]^2 \neq 2\operatorname{Log}(-1+i)$

$$\operatorname{Log}(-1+i)^2 = \operatorname{Log}(-2i) = \operatorname{Log} 2 - \frac{\pi}{2} i$$

Entonces

$$2\operatorname{Log}(-1+i) = 2\left(\operatorname{Log}\sqrt{2} + i\frac{3}{4}\pi\right)$$

$$2\operatorname{Log}(-1+i) = \operatorname{Log} 2 + \frac{3}{2} i$$

$$\underbrace{\operatorname{Log}(-1+i)^2 \neq 2\operatorname{Log}(-1+i)}$$

4/ Probar que

a) Si $\log z = \mathrm{Log}\, r + i\Theta \;(r>0,\ \pi/4 < \Theta < 9\pi/4)$,
Entonces $\log(i^2) = 2\log i$

Tenemos

$$\log z = \mathrm{Log}\, r + i\Theta$$

Entonces

$$\log (i)^2 = -1$$

$$\underbrace{\log(-1) = \cancel{\log 1} + i\pi = i\pi}$$

Del otro lado tenemos

$$2\log i = \cancel{2}\left(\cancel{\log 1} + \frac{\pi}{\cancel{2}} i\right) =$$

$$\underbrace{2\log i = i\pi}$$

∴ Se prueba que son iguales

6. Si $\log z = \operatorname{Log} r + i\theta \ (r>0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4})$,
entonces $\log(i^2) \neq 2\log i$

Tenemos

$$\log z = \operatorname{Log} r + i\theta$$

Consideramos

$$\log(i^2) = -i$$

Donde

$$\log(i^2) = \log(-1) = \log 1 + i\pi$$

$$\operatorname{Log}(i^2) = \cancel{\log}(1)^{0} + i\pi$$

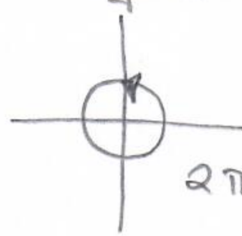
$$\underbrace{\log(i^2) = i\pi}$$

Ahora

$$2\log i = \cancel{2}(\cancel{\log} 1^{0} + \frac{5}{\cancel{2}}\pi i)$$

$$2\log i = 5\pi i$$

Donde $\frac{3}{4}\pi < \theta < \frac{11}{4}\pi$



$$2\pi + \frac{\pi}{2} = \underline{\underline{\frac{5}{2}\pi}}$$

$\therefore \log(i^2) \neq 2\log i$

Nota: solo cuando θ toma los valores $\frac{3}{4}\pi < \theta < \frac{11}{4}\pi$

5. Probar que

a) el conjunto de valores de $\log(i^{1/2})$ es $\left(n+\frac{1}{4}\right)\pi i$

Donde $n = 0 \pm 1 \pm 2 \ldots$

Donde

$$\log(i^{1/2}) = \left(n+\frac{1}{4}\right)\pi i \quad \text{Donde } n = 0 \pm 1 \pm 2 \pm \ldots$$

// por exponentes tenemos

$$\frac{1}{2}\log i = \frac{1}{2}\left[\cancel{\ln 1} + i\left(\frac{\pi}{2} + 2n\pi\right)\right]$$

De forma general

// multiplicando

$$\frac{1}{2}\log i = \left(\frac{\pi}{4} + n\pi\right) i$$

Nos queda

$$\ldots \frac{1}{2}\log i = \left(n+\frac{1}{4}\right)\pi i \ldots$$

y considere con el de los valores de $\left(\frac{1}{2}\right)\log i$

los valores de i son

$$e^{i\frac{\pi}{4}} \text{ y } e^{i\frac{5}{4}\pi}$$

Tenemos

$$\log(e^{i\frac{\pi}{4}}) = \log 1 + i\left(\frac{\pi}{4} + 2n\pi\right)$$

$$\log(e^{i\frac{\pi}{4}}) = \left(2n+\frac{1}{4}\right)\pi i$$

Para

$$\ldots \log(e^{i\frac{5}{4}\pi}) = \cancel{\log 1} + i\left(\frac{5}{4}\pi + 2n\pi\right) \ldots$$

$$\ldots \log(e^{i\frac{5}{4}\pi}) = \left((2n+1) + \frac{1}{4}\right)\pi i \ldots$$

juntando los dos valores tenemos

$$\ldots \left(n+\frac{1}{4}\right)\pi i \ldots$$

∴ Comprobamos que los valores de $\log(i^{1/2})$ son los mismos de $\frac{1}{2}\log i$.

9. Probar que si $\operatorname{Re} z_1 > 0$ y $\operatorname{Re} z_2 > 0$, entonces:
$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$

$z_1 = r_1 \exp i\Theta$ y $z_2 = r_2 \exp i\Theta_2$

$-\frac{\pi}{2} < \Theta < \frac{\pi}{2}$ y $-\frac{\pi}{2} < \Theta_2 < \frac{\pi}{2}$

Entonces -

$-\pi < \Theta_1 + \Theta_2 < \pi$

donde

$\operatorname{Log}(z_1 z_2) = \operatorname{Log}[(r_1 r_2) \exp i(\Theta_1 + \Theta_2]$

Nos queda

$\operatorname{Log}(z_1 z_2) = \log(r_1 r_2) + i(\Theta_1 + \Theta_2)$

$\operatorname{Log}(z_1 z_2) = (\log r_1 + i\Theta_1) + (\log r_2 + i\Theta_2)$

Se puede escribir como

$\operatorname{Log}(z_1 z_2) = \log r_1 e^{i\Theta_1} + \log r_2 e^{i\Theta_2}$

$\operatorname{Log}(z_1 z_2) = \log z_1 + \log z_2$

Se muestra entonces que

$\operatorname{Log}(z_1 z_2) = \log z_1 + \log z_2$

10. Probar que para todo par de números complejos no nulos z_1 y z_2

$$\text{Log}(z_1 z_2) = \text{Log}\, z_1 + \text{Log}\, z_2 + 2N\pi i$$

siendo N uno de los valores 0 ± 1

Tenemos

$$z_1 = r_1 e^{i\Theta_1} \quad y \quad z_2 = r_2 e^{i\Theta_2}$$

//multiplicamos

$$z_1 \cdot z_2 = r_1 r_2 e^{i(\Theta_1 + \Theta_2)}$$

//se pueden sumar los exponentes porque tienen la misma base

//utilizando el argumento

$$\Theta_1 = \arg z_1 \qquad \Theta_1 + \Theta_2 = \arg(\Theta_1 \Theta_2)$$

$$\Theta_2 = \arg z_2 \qquad \arg(z_1 z_2) = \Theta_1 + \Theta + 2n\pi \ (n = 0, \pm 1)$$

Entonces

$$\log(z_1 z_2) = \log r_1 r_2 e^{i(\arg(\Theta_1 \Theta_2))}$$

Tenemos

$$\log(z_1 z_2) = \log r_1 r_2 + i(\arg(z_1 z_2)) + 2 \quad \text{Donde } n = 0 \pm 1 \pm 2 \ldots$$

Nos queda

$$\log(z_1 z_2) = \log r_1 r_2 + i\arg(z_1) + i\arg(z_2) + 2n\pi$$

Agrupamos

$$\log(z_1 z_2) = (\log r_1 + i\arg(z_1)) + (\log r_2 + i\arg(z_2)) + 2n\pi$$

Nos queda

$$\log(z_1 z_2) = \text{Log}(z_1) + \log(z_2) + 2n\pi \ldots$$

Donde : $n = 0 \pm 1 \pm 2 \ldots$