Potencias y raíces

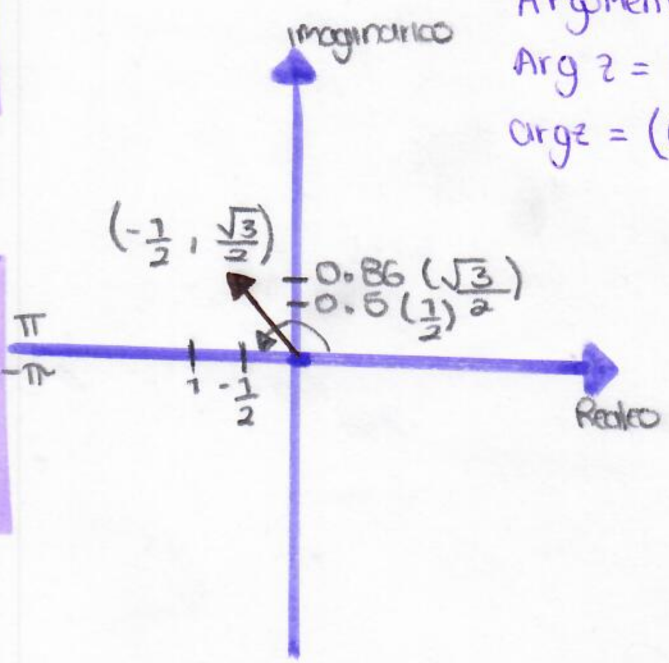
1) $z = \dfrac{-2}{1+\sqrt{3}\,i}$,

Argumento $\Theta \in (-\pi, \pi)$

Argumento $(\Theta + 2n\pi)$

$\text{Arg } z = \Theta$

$\arg z = (\Theta + 2n\pi)$



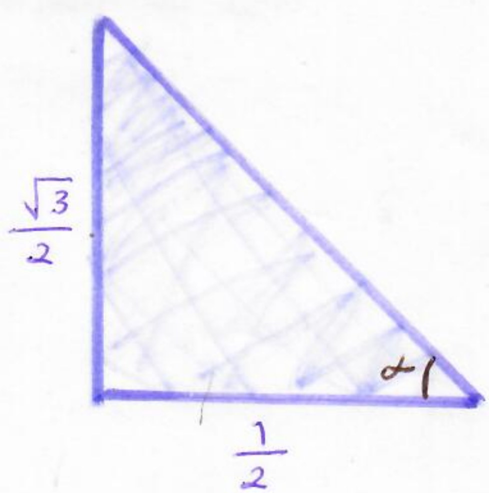
//Separar reales e imaginarios
//multiplicamos por el conjugado

$$z = \frac{-2}{1+\sqrt{3}\,i}\left(\frac{1+\sqrt{3}\,i}{1-\sqrt{3}\,i}\right) = \frac{-2+2\sqrt{3}\,i}{1+3}$$

$$= -\frac{2}{4} + \frac{2\sqrt{3}i}{4}$$

$$= -\frac{1}{2} + \frac{\sqrt{3}i}{2}$$

//Calcular ángulo



Calcular la hipotenuza

$$\Theta = \tan^{-1}\left(\frac{co}{ca}\right)$$

$$\Theta = \left(\frac{\frac{\sqrt{3}}{\cancel{2}}}{\frac{1}{\cancel{2}}}\right) = \Theta = \sqrt{3}$$

$$\tan^{-1} = \sqrt{3} = \underline{\underline{60^\circ}}$$

Pasar a radianes

$$\frac{60^\circ}{180^\circ}(\pi) = 180^\circ - 60^\circ = \Theta$$

$$\Theta = 120$$

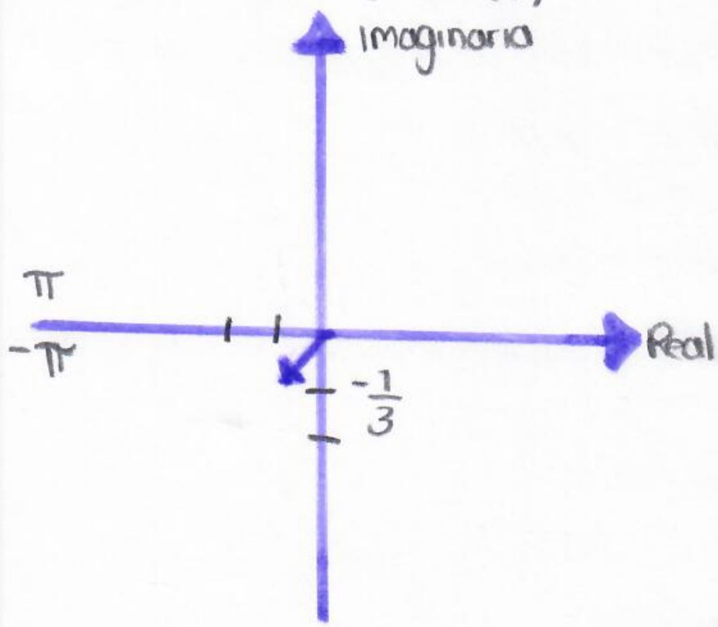
$$\underline{\arg z = \left(\frac{2}{3}\pi + 2n\pi\right)}$$

b) Hallar un valor del argumento para los números complejos.

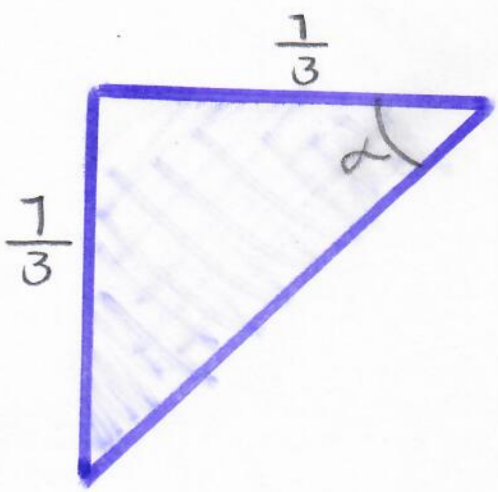
$$z = \frac{i}{-2-2i}$$

//Multiplicamos por el conjugado

$$\frac{i}{-2-2i}\left(\frac{-2+2i}{-2+2i}\right) = \frac{-2i-2}{4+2} = -\frac{2}{6} - \frac{2}{6}i = -\frac{1}{3} - \frac{1}{3}i$$



//Calcular la hipotenusa



$$\alpha = \tan^{-1}\left(\frac{\frac{1}{3}}{\frac{1}{3}}\right)$$

$$\alpha = 45^\circ$$

$$180^\circ - 45^\circ = 135^\circ$$

$$rad = \frac{135^\circ}{180^\circ}\pi = -\frac{3}{4}\pi$$

$$\arg z = \left(-\frac{3}{4}\pi + 2n\pi\right)$$

c) Hallar un valor del argumento para los números complejos

$z = (\sqrt{3} - i)^6$

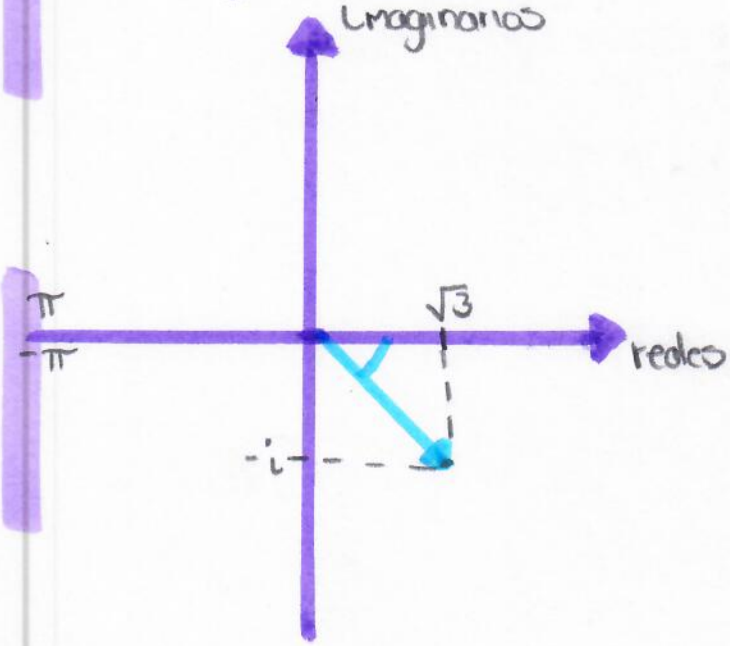
$z = z_0^c$

$z_0 = (\sqrt{3} - i)$

$c = 6$

$z_0 = \begin{cases} |z_0| = 2 \\ \Theta_0 = -\frac{\pi}{6} \end{cases}$

Coordenadas para el número $z_0 = \sqrt{3} - i$



$\Theta \in (-\pi, \pi]$

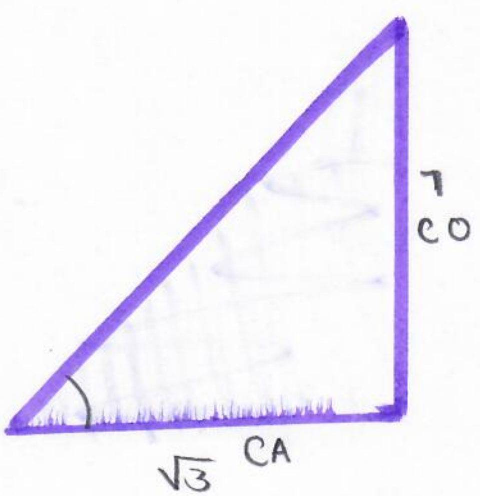
$z_0 = 2^6 [\cos(-\pi) + i\,\mathrm{sen}(-\pi)]$

$= \cos(-\pi) = \cos(\pi)$

$\underline{\Theta = \pi}$ // Pertenece al rango

Argumento principal

$\boxed{\Theta = \pi}$



Calcular hipotenuza

$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30 = \underline{-\frac{\pi}{6}}$

Magnitud del número complejo

$|z_0| = \sqrt{(\sqrt{3})^2 + (-i)^2}$

$|z_0| = \sqrt{3+1} = \sqrt{4}$

$|z_0| = \underline{2}$

Encontrar el valor del argumento

$z_0 = |z_0| \cdot e^{i\Theta}$ ⇒ forma exponencial //

$z_0 = 2 \cdot e^{i\left(-\frac{\pi}{6}\right)} = \left[2 \cdot e^{i\left(-\frac{\pi}{6}\right)}\right]^6$

$= \underline{2^6 \cdot e^{-i\pi}}$

2. Expresando los factores individuales de la izquierda en forma exponencial, efectuar las operaciones requeridas y cambiar finalmente a coordenadas rectangulares, para probar que:

a) $i(1-\sqrt{3}i)(\sqrt{3}+i)=2(1+\sqrt{3}i)$

$// \ 2(1+\sqrt{3}i)=4\left[\frac{1}{2}+\frac{\sqrt{3}i}{2}\right]$

$=2\left[1+\sqrt{3}i\right]$

Para $z_1=i$

calculamos su magnitud

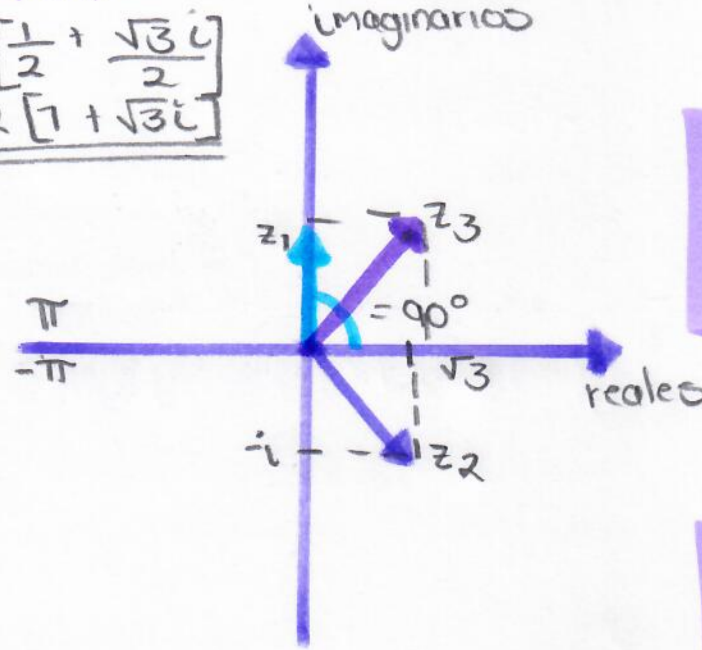
$|z_1|=\sqrt{i^2}=1$

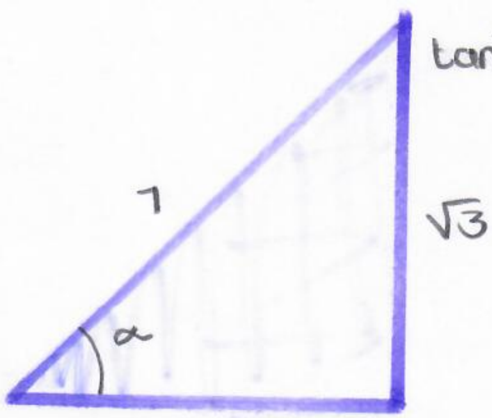
$\Theta=\frac{\pi}{2}$

$z_2=i-\sqrt{3}i$

$|z_2|=\sqrt{(1)^2+(\sqrt{3})^2}=\sqrt{4}=2$

$\Theta=$





$\tan^{-1}(\sqrt{3})=-\frac{\pi}{3}$

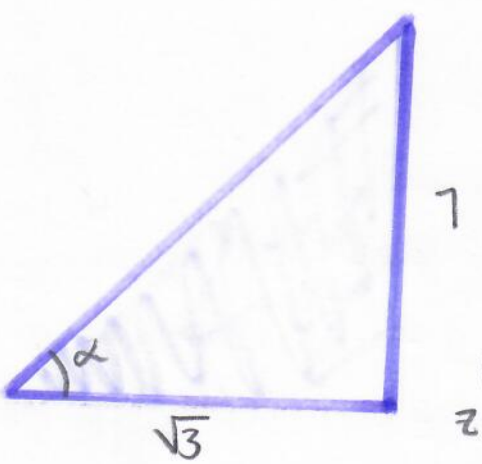
Para $z_3=\sqrt{3}+i$

calcular magnitud

$|z_3|=\sqrt{(\sqrt{3})^2+(1)^2}$

$|z_3|=\sqrt{4}=2$

$\Theta=-\frac{\pi}{6}$



$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=-\frac{\pi}{6}$

// multiplicamos los números complejos

$z_1 \cdot z_2 \cdot z_3 = z_0 \quad // \ z=|z|e^{i\Theta}$

$|e^{i\frac{\pi}{2}} \cdot 2e^{-i\frac{\pi}{3}} \cdot 2e^{i\frac{\pi}{6}}| = 4^{\frac{\pi}{3}i}$

$z=|z|[\cos\Theta + i\,\text{sen}\,\Theta]$

$=4\left[\cos\left(\frac{\pi}{3}\right)+i\,\text{sen}\left(\frac{\pi}{3}\right)\right]=4\left[\frac{1}{2}+i\frac{\sqrt{3}}{2}\right]$

b) $\frac{5i}{(2+i)} = 1+2i$

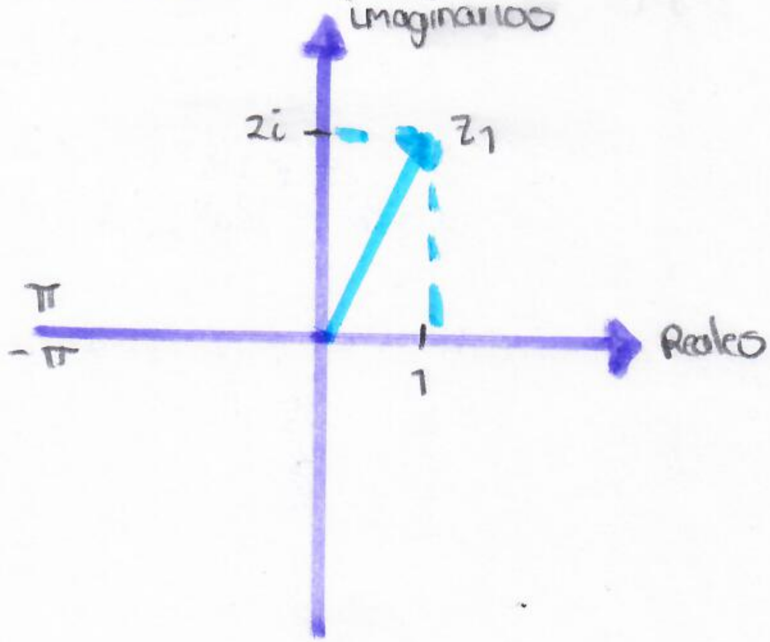
$$z_1 = \left(\frac{5i}{2+i}\right)\left(\frac{2-i}{2-i}\right) = \frac{10i+5}{4-2i+2i+1}$$

$$= 1+2i$$

//Se comprueba la igualdad

$$\underline{\underline{1+2i = 1+2i}}$$

coordenadas rectangulares



C) $(-1+i)^7 = -8(1+i)$

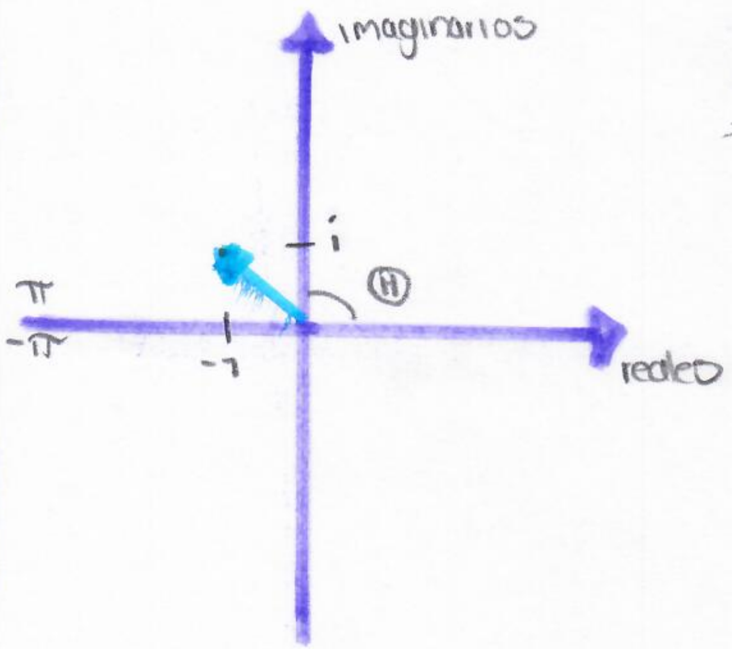
$$z = z_0^{c}$$

$$|z| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

$$\Theta = \frac{\pi}{4}$$

$$z_0 = \left(\sqrt{2}\, e^{\frac{i\pi}{4}}\right)^7$$

$$z_0 = 2^{\frac{7}{2}}\, e^{i\frac{7}{4}\pi}$$



$$\tan^{-1}\left(\frac{1}{1}\right) = 45 = \frac{1}{4}\pi$$

$$\Theta = \frac{7}{4}\pi$$

$$z = 2^{\frac{7}{2}} \cdot \left[\cos\left(\frac{7\pi}{4}\right) + i\,\text{sen}\left(\frac{7\pi}{4}\right)\right]$$

$$z = 2^{\frac{7}{2}}\left[\frac{\sqrt{2}}{2} - i\,\frac{\sqrt{2}}{2}\right]$$

//Reescribimos

$$= 2^{3} \cdot 2^{\frac{1}{2}}\left[\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}\right]$$

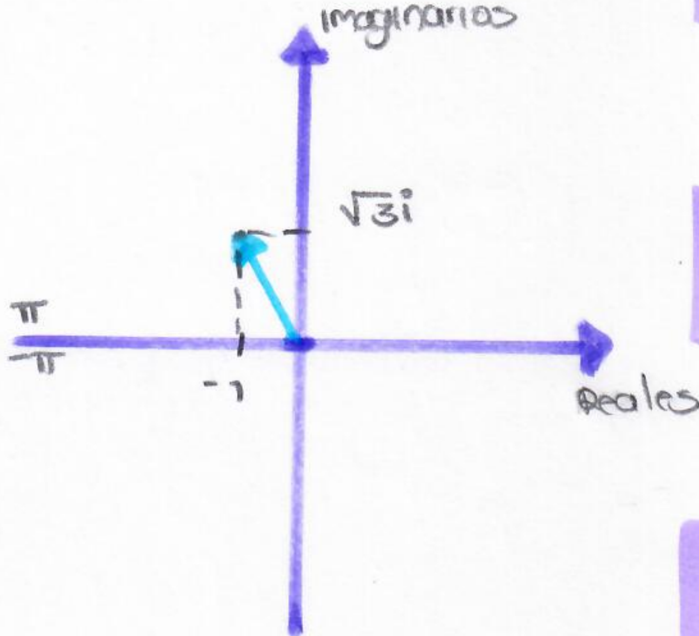
$$= 8[1 - i]$$

d) $(1+\sqrt{3i})^{-10} = 2^{-11}(-1+\sqrt{3i})$

$z = z_0{}^{c}$

$|z_0| = \sqrt{(1)^2 + (\sqrt{3})^2} = 2$

$\Theta = \frac{1}{3}\pi$



$z_0 = |z_0| e^{i\Theta}$

$z_0 = 2e^{i\frac{1}{3}\pi}$

$z_0^{-10} = (2e^{-i\frac{1}{3}\pi})^{-10}$

$z_0{}^{-10} = 2^{-10} \cdot e^{-\frac{10}{3}\pi}$

$\Theta_0 = -\frac{10}{3}\pi$

$z_0 = 2^{-10}(\cos(\frac{10}{3}\pi) + i\,\text{sen}(-\frac{10}{3}\pi))$

$= 2^{-10}(-\frac{1}{2} + \frac{\sqrt{3}i}{2})$

$\underline{\underline{= 2^{-11}(-1+\sqrt{3i})}}$

$\underline{\underline{2^{-11}(-1+\sqrt{3i}) = 2^{-11}(-1+\sqrt{3}i)}}$

3. a $(2i)^{\frac{1}{2}}$

$$z = z_0^{\frac{1}{k}}, \quad \frac{1}{k}, \quad k = \text{entero}$$

$$z = \exp\left[\frac{\ln|z_0|}{k} + i\left(\frac{\Theta_0}{k} + \frac{2n\pi}{k}\right)\right] \quad n = 0, 1, 2 \ldots k-1$$

$z_0 = 2i$, $k = 2$

Nota: La raíz principal es la que corresponde en el argumento principal, o cuando $n = 0$

Para $n = 0$

$$z_1 = \exp\left[\frac{\ln 2}{2} + i\left(\frac{\frac{\pi}{2}}{2} + 0\right)\right] = \exp\left[\ln 2^{\frac{1}{2}} + i\frac{\pi}{4}\right]$$

$$= e^{\ln 2^{\frac{1}{2}}} \cdot e^{\frac{\pi}{4}i}$$

$$z_1 = 2^{\frac{1}{2}}\left[\cos\frac{\pi}{4} + i\,\text{sen}\,\frac{\pi}{4}\right] = \sqrt{2}\left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right]$$

$$= [1 + i] \quad \text{Raíz Principal}$$

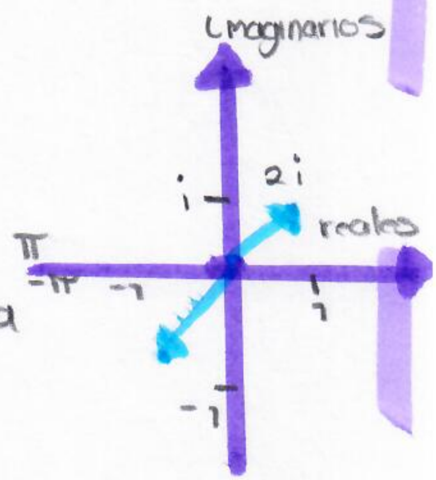
Para $n = 1$

$$z_2 = \exp\left[\frac{\ln 2}{2} + i\left(\frac{\frac{\pi}{2}}{2} + \frac{2\pi}{2}\right)\right]$$

$$z_2 = e^{\ln 2^{\frac{1}{2}}} \cdot e^{i\left(\frac{\pi}{4} + \pi\right)} = 2^{\frac{1}{2}} \cdot e^{\frac{5}{4}\pi i}$$

$$z_2 = \sqrt{2}\left[\cos\left(\frac{5}{4}\pi\right) + i\,\text{sen}\left(\frac{5}{4}\pi\right)\right]$$

$$z_2 = \sqrt{2}\left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right] = -[1 + i] \quad \text{segunda Raíz}$$



//Comprobando

$$z_2 \cdot z_2 = (-1 - i)(-1 - i) = 1 + i + i - 1 = 2i$$

b) $(1-\sqrt{3}i)^{\frac{1}{2}}$

$z = z_0^{\frac{1}{k}}$, $|z| = 2$, $\Theta = -\frac{\pi}{3}$ $k=2$

Para $n=0$

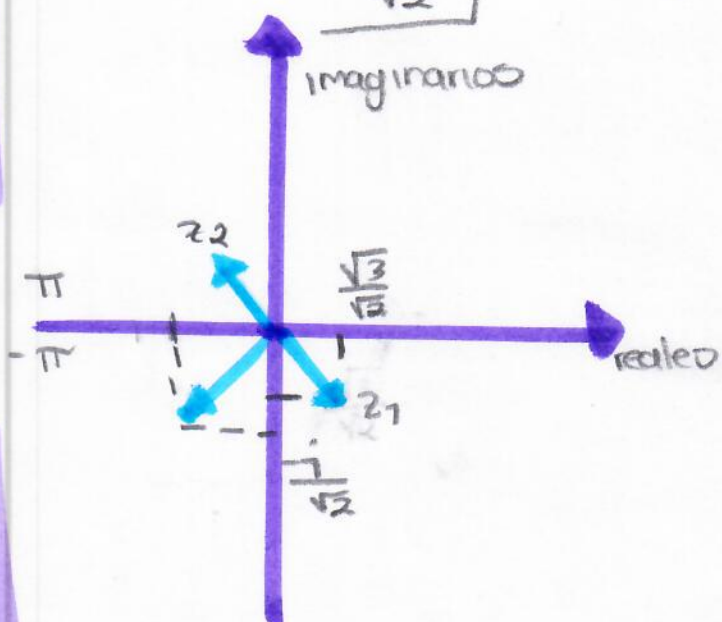
$$z_0 = \exp\left[\frac{\ln 2}{2} + i\left(\frac{\frac{\pi}{2}}{2} + 0\right)\right.$$

$$= e^{\ln 2 \frac{1}{2}} \cdot e^{-\frac{\pi}{6}i}$$

$$= \sqrt{2} \cdot e^{-\frac{\pi}{2}i}$$

$$= \sqrt{2}\left[\cos\frac{\pi}{6} + i\,\text{sen}\,-\frac{\pi}{6}\right] = \sqrt{2}\left[\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$= \boxed{\frac{\sqrt{3}-i}{\sqrt{2}}} \Rightarrow$$ Raíz principal.



Para $n=1$

$$z_2 = \exp\left[\frac{\ln 2}{2} + i\left(\frac{-\frac{\pi}{3}}{2} + \frac{2\pi}{2}\right)\right.$$

$$= e^{\ln 2 \frac{1}{2}} \cdot e^{\frac{5}{6}\pi i}$$

$$= \sqrt{2} \cdot e^{\frac{5}{6}\pi i}$$

$$= \sqrt{2}\left[\cos\left(\frac{5}{6}\pi\right) + i\,\text{sen}\left(\frac{5}{4}\pi\right)\right]$$

$$= \sqrt{2}\left[-\frac{\sqrt{3}}{2} + \frac{i}{2}\right]$$

$$z_2 = \boxed{\frac{-\sqrt{3}-1}{2}}$$

c) $(-1)^{\frac{1}{3}}$

$z = z_0^{\frac{1}{k}}$; $K=3$, $|z_0|=1$, $\Theta = 0$
$n=0$

La propiedad
$\sqrt{-x^n} = -x$
$n = \text{impar}$

$$z_0 = \exp\left[\frac{\ln 1}{3} + i\left(\frac{0}{2}+0\right)\right] = \exp\left[\frac{0}{3}\right]$$

$$z_0 = e^0 = 1$$

$$z_1 = \exp\left[\frac{\ln 1}{3} + i\left(\frac{0}{2}+\frac{2\pi}{3}\right)\right] = \exp\left[\frac{0}{3} + \frac{2}{3}\pi\right]$$

$$= e^{\frac{2}{3}\pi}$$

$$z_1 = \cos\left(\frac{2}{3}\pi\right) + i\,\text{sen}\left(\frac{2}{3}\pi\right)$$

$$z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

Para $n=2$

$$z_2 = \exp\left[\frac{\ln 1}{3} + i\left(\frac{0}{2} + \frac{4}{3}\pi\right)\right]$$

$$z_2 = \exp\left[\frac{0}{3} + \frac{4}{3}\pi i\right]$$

$$z_2 = \cos\left(\frac{4}{3}\pi\right) + \text{sen}\left(\frac{4}{3}\pi\right)i = -\frac{1}{2} - \frac{\sqrt{3}}{2}$$

$$z_3 = \exp\left[\frac{\ln 1}{3} + i\left(\frac{\pi}{3} + \frac{2\pi}{3}\right)\right] = \exp[\pi]$$

$$= e^{\pi}$$

$$z_1 = \cos\pi + i\,\text{sen}\,\pi = -1$$

d) $(-16)^{\frac{1}{4}}$

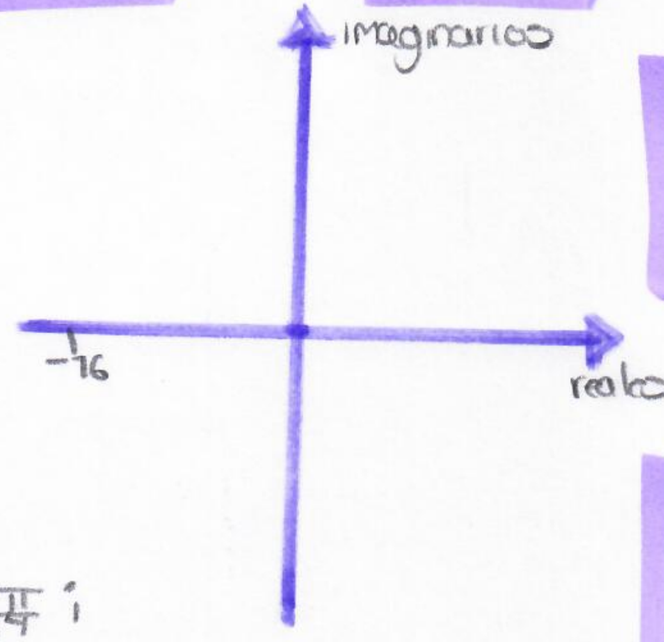
$|z| = 16, \quad \Theta = \pi, \quad k = 4$

Para $n=0$

$$z_0 = \exp\left[\frac{\ln 16}{4} + i\left(\frac{\pi}{4} + 0\right)\right]$$

$$= \exp\left[\ln 16^{1/4} + \frac{\pi}{4} i\right]$$

$$= e^{\ln 16^{\frac{1}{4}}} \cdot e^{\frac{\pi}{4} i} = 16^{\frac{1}{4}} \cdot e^{\frac{\pi}{4} i}$$



// Reescribimos

$$= 2 \cdot \left[\cos\frac{\pi}{4} + i \operatorname{sen}\frac{\pi}{4}\right] = 2$$

$$= \cancel{2}\left[\frac{\sqrt{2}}{\cancel{2}} + \frac{\sqrt{2}}{\cancel{2}} i\right]$$

// acomodando

$$= \sqrt{2}\,(1+i)$$ // Raíz

Para $n=1$

$$z_1 = \exp\left[\frac{\ln 6}{4} + i\left(\frac{\pi}{4} + \frac{2\pi}{4}\right)\right]$$

$$= \exp\left[\ln 16^{\frac{1}{4}} + \frac{3}{4}\pi i\right]$$

$$= e^{\ln 16^{\frac{1}{4}}} \cdot e^{\frac{3}{4}\pi i} = 2 \cdot e^{\frac{3}{4}\pi i}$$

$$z_1 = 2\left[\cos\left(\frac{3}{4}\pi\right) + i \operatorname{sen}\left(\frac{3}{4}\pi\right)\right] = 2\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right]$$

$$= -\sqrt{2}\,(1-i)$$ // Raíz

Para $n=2$

$$z_2 = 2\left(e^{i\frac{\pi}{4}}\right)\left(e^{i\pi}\right) = 2\left(e^{i\frac{5}{4}\pi}\right)$$

$$z_2 = 2\left[\cos\left(\frac{5}{4}\pi\right) + \operatorname{sen}\left(\frac{5}{4}\pi\right)i\right] = 2\left[-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right]$$

$$\underline{\underline{= -\sqrt{2}(1+i)}} \Big\} \text{ Raíz}$$

Para $n=3$

$$z_3 = 2\left(e^{i\frac{\pi}{4}}\right)\left(e^{i\frac{3\pi}{2}}\right) = 2\left(e^{i\frac{7}{4}\pi}\right)$$

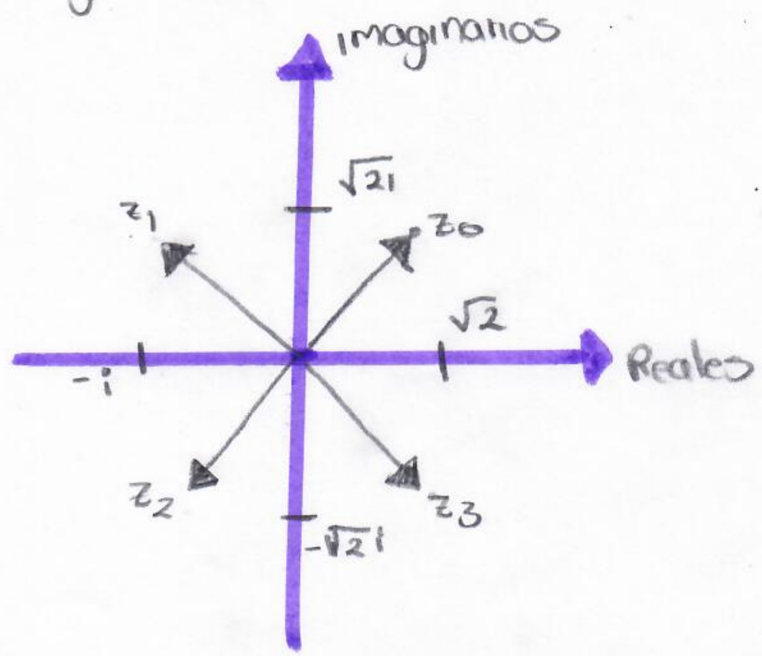
$$z_3 = 2\cos\left(\frac{3}{4}\pi\right) + i\operatorname{sen}\left(\frac{7}{4}\pi\right)]$$

$$= 2\left[\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right] = \sqrt{2}(1-i)$$

Las raíces

$$\pm\sqrt{2}(1+i),\ \pm\sqrt{2}(1-i)$$

Representación gráfica



c) Expresando los factores individuales de la izquierda en forma exponencial, efectuar las operaciones requeridas y cambiar finalmente a coordenadas rectangulares, para probar que:

$(8)^{\frac{1}{6}}$

$z = 8$

$|z| = 8$

$\theta = 0$

Calcular para $n=0$

$z_0 = \exp\left[\frac{\ln 8}{6} + i\,\frac{(\pi)}{6}\right] = \exp\left[\frac{\ln 8}{7}\right]$

$= e^{\ln 8^{\frac{1}{6}}} = 8^{\frac{1}{6}}$

$= 8^{\frac{1}{6}}\left[\cos(0) + \text{sen}(0)i\right] = 8^{\frac{1}{16}}\left[7 + 0\right]$

$= \left(8^{\frac{1}{3}}\right)^{\frac{1}{2}}\left[7 + 0\right] = \underline{\underline{\sqrt{2}}}$ Raiz principal

Calcular para $n=1$

$z_1 = \exp\left[\frac{\ln 8}{6} + i\left(0 + \frac{2\pi}{6}\right)\right] = \exp\left[\frac{\ln 8}{6} + \frac{\pi i}{6}\right]$

$= e^{\ln 8^{\frac{1}{6}}} \cdot e^{\frac{\pi}{3}i} = 8^{\frac{1}{6}} \cdot e^{\frac{\pi}{3}i}$

$= 8^{\frac{1}{6}}\left[\cos\left(\frac{\pi}{3}\right) + i\,\text{sen}\left(\frac{\pi}{3}\right)\right]$

$= 8^{\frac{1}{6}}\left[\frac{1}{2} + \frac{\sqrt{3}}{2}\right]$

$= \sqrt{2}\left[\frac{1}{2} + \frac{\sqrt{3}}{2}\right] \cdot \left(\frac{\sqrt{3}}{2}\right)$

$= \frac{1 + \sqrt{3}}{\sqrt{2}}$

Calcular para $n=2$

$$Z_2 = 8^{\frac{1}{16}} \left(e^{\frac{2}{3}\pi i}\right) \qquad 8^{\frac{1}{6}} = \sqrt{2}$$

Entonces

$$= \sqrt{2}\left[\left(\cos\left(\tfrac{2}{3}\pi\right) + i\,\text{sen}\left(\tfrac{2}{3}\pi\right)\right)\right]$$

$$= \sqrt{2}\left[-\frac{1}{2} + \frac{\sqrt{3}}{2}\right]$$

$$= \frac{-1+\sqrt{3}}{\sqrt{2}}$$

Calcular para $n=3$

$$Z_3 = 8^{\frac{1}{6}} \cdot e^{i\pi} = \sqrt{2} \cdot \left[\overset{-1}{\cancel{\cos\pi}} + \overset{0}{i\,\cancel{\text{sen}\pi}}\right] = \sqrt{2}\left[-1+0\right]$$

$$\boxed{Z_3 = -\sqrt{2}}$$

Calcular para $n=4$

$$Z_4 = 8^{\frac{1}{16}} \cdot e^{i\frac{4}{3}\pi} = \sqrt{2}\left[\cos\left(\tfrac{4}{3}\pi\right) + i\,\text{sen}\left(\tfrac{4}{3}\pi\right)\right]$$

$$Z_4 = \sqrt{2}\left[-\frac{1}{2} - \frac{\sqrt{3}}{2}\right]$$

$$\boxed{Z_4 = -\left(\frac{1+\sqrt{3}}{2}\right)}$$

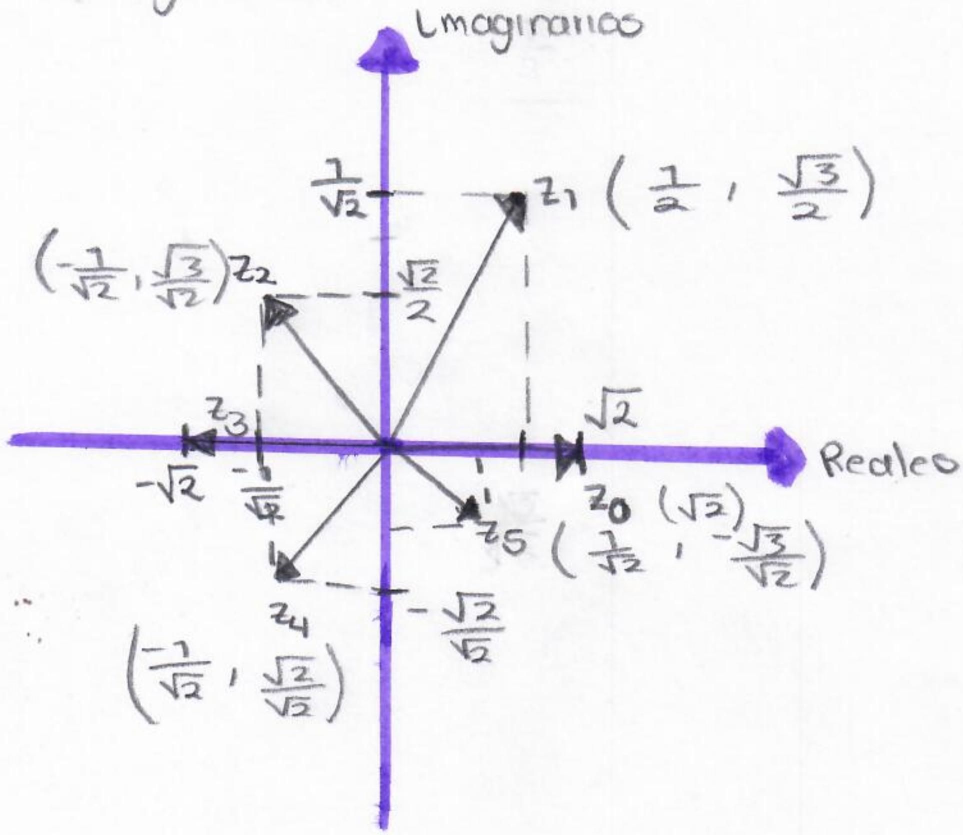
Calcular para $n=5$

$$z_5 = 8^{\frac{1}{6}} \cdot e^{i\frac{5}{3}\pi} = \sqrt{2}\left[\cos\left(\frac{5}{3}\pi\right) + i\,\mathrm{sen}\left(\frac{5}{3}\pi\right)\right]$$

$$z_5 = \sqrt{2}\left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right]$$

$$\boxed{z_5 = -\left(\frac{-1+\sqrt{3}}{\sqrt{2}}\right)}$$

Representación gráfica



f) Expresando los factores individuales de la izquierda en forma exponencial, efectuar las operaciones requeridas y cambiar finalmente a coordenadas rectangulares, para probar que:

$(-4\sqrt{2} + 4\sqrt{2}i)^{\frac{1}{3}}$ $K = 3$

$|z| = (4\sqrt{2})^2 = \sqrt{64} = 8$

$\theta = \tan^{-1}\left(\frac{4\sqrt{2}}{4\sqrt{2}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

Calcular para $n=0$

$z_0 = \exp\left[\frac{\ln 8}{3} + i\left(\frac{\frac{\pi}{4}}{3}\right) + 0\right] = \exp\left[\ln 8^{\frac{1}{3}} + \frac{\pi}{12}i\right]$

$z_0 = e^{\ln 8^{\frac{1}{3}}} + e^{\frac{\pi}{12}i} = 2\left[\cos\left(\frac{\pi}{4}\right) + i\,\text{sen}\left(\frac{\pi}{4}\right)\right]$

$= 2\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}i}{2}\right]$ $\boxed{z_0 = \sqrt{2}\,[1+i]}$

Calcular para $n=1$

$z_1 = \exp\left[\frac{\ln 8}{\frac{1}{3}} + i\left(\frac{\pi}{4} + \frac{2\pi}{3}\right)\right] =$

$= \exp\left[\ln 8^{\frac{1}{3}} + i\left(\frac{11}{12}\pi\right)\right]$

$= e^{\ln 8^{\frac{1}{3}}} \cdot e^{\frac{11}{12}\pi} = 2\left[\cos\frac{11}{12}\pi + i\,\text{sen}\,\frac{11}{12}\pi\right]$

$= 2\left[\frac{\sqrt{6} + \sqrt{2}}{4} + \frac{\sqrt{6} - \sqrt{2}i}{4}\right]\frac{\sqrt{6}}{\sqrt{2}}$

$\boxed{z_1 = \left[-(\sqrt{3}+1) + (\sqrt{3}-1)i\right]}$

4. Probar que:

a. $|e^{i\theta}| = \sqrt{|\cos\theta + \text{sen}\,\theta|}$

$= \sqrt{\cos^2\theta + \text{sen}2\theta}$

$= \sqrt{1}$

$$\boxed{|e^{i\theta}| = 1}$$

b) Probar que:

$\overline{e^{i\theta}} = e^{-i\theta}$

$z = x + yi$

$\bar{z} = x - yi$

$$\boxed{\overline{e^{i\theta}} = e^{-i\theta}}$$

c) Probar que:

$(e^{i\theta})^2 = e^{i2\theta}$

$(z)^n = z^n$

$$\boxed{(e^{i\theta})^2 = e^{i2\theta}}$$

d) Probar que:

$e^{i\theta_1} \cdot e^{i\theta_2} \cdots\cdots e^{i\theta_n} = e^{i\theta_1 + i\theta_2 + \cdots + i\theta_n} \quad (n = 2, 3 \ldots)$

$x^1 \cdot x^2 \cdots \cdots x^n = x^{1+2+\cdots+n}$

$e^{i\theta_1} \cdot e^{i\theta_2} \cdot e^{i\theta_3} \cdots e^{i\theta_n} = e^{i\theta_1 + i\theta_2 + i\theta_3 \cdots i\theta_n}$

8. Comprobar la afirmación (sec 5)

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 .$$

Se puede reescribir como

$$\arg\left(\frac{z_1}{z_2}\right) = \arg\left(z_1 \cdot z_2^{-1}\right)$$

// recordando que r y θ son coordenadas polares.

Para representar un número complejo en sus coordenadas polares

Nos queda:

$$z = r(\cos\theta + i\,\text{sen}\,\theta)$$

$$z^{-1} = \frac{1}{r}\left(\cos(-\theta) + i\,\text{sen}(-\theta)\right)$$

Podemos escribir

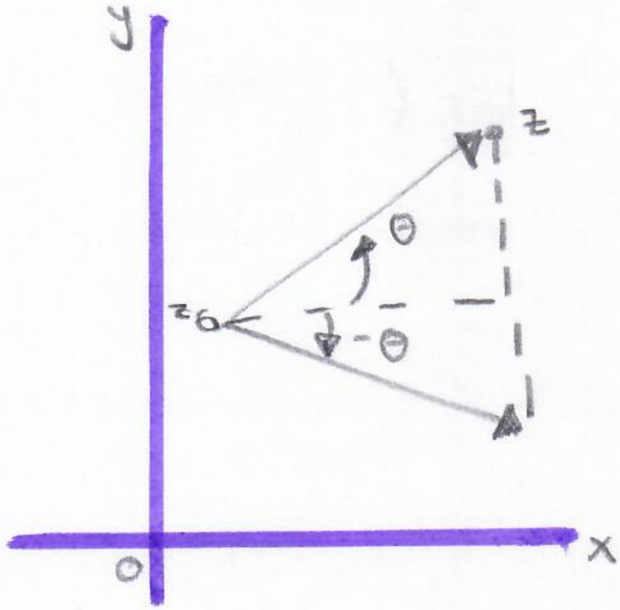
$$\arg(z_1 \cdot z_2^{-1}) = r_1\left[\cos(\theta_1) + i\,\text{sen}(\theta_1)\right] \cdot \frac{1}{r}\left[\cos(-\theta) + i\,\text{sen}(-\theta)\right]$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\,\text{sen}(\theta_1 - \theta_2)\right]$$

$$\boxed{\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2}$$

9. Por lo visto en la sección 3, el vector diferencia $z - z_0$ de dos números complejos puede interpretarse vectorialmente (Fig 11), donde Θ denota el ángulo de inclinación del vector $z - z_0$. Escribiendo el número $z - z_0$ con forma polar.

$$z - z_0 = R(\cos\Theta + i\,\mathrm{sen}\,\Theta)$$



Tenemos

$$z - z_0 = R(\cos\theta + i\,\mathrm{sen}\,\theta)$$

Entonces

$$\overline{z - z_0} = R(\cos\theta + i\,\mathrm{sen}(-\theta))$$

$$-(z - z_0) = R(\cos(-\theta) + i\,\mathrm{sen}(-\theta))$$

10.

a) dado un número real arbitrario a, probar que las dos raíces cuadradas de $a+i$ son $\pm\sqrt{A}\exp\left(\frac{1}{x}/2\right)$, siendo $A=\sqrt{a^2+1}$ y $x=\arg(a+i)$

$A=(a^2+1)^{\frac{1}{2}}$, $\operatorname{Arg}(a+1)$, $z=a+1$ $\|(a+1)^{\frac{1}{2}}$

$z_0=\left(\sqrt{a^2+1}\right)\cdot e^{i(\operatorname{Arg}(a+1)+0)}$

// se puede escribir.

$$z_0=\exp\left[\frac{\ln A}{2}+i\left(\frac{\operatorname{Arg}(a+1)}{2}+0\right)\right]$$

Tenemos:

$$=\exp\left[\ln A^{\frac{1}{2}}+\frac{\alpha}{2}i\right]$$

$$=e^{\ln A^{\frac{1}{2}}}\cdot e^{\frac{\alpha}{2}i}=\sqrt{A}\left[\cos\left(\frac{\alpha}{2}\right)+i\operatorname{sen}\left(\frac{\alpha}{2}\right)\right]$$

Entonces

$$=\pm\sqrt{A}\exp\frac{\alpha}{2}i$$

Nos queda demostrado

$$\boxed{=\pm\sqrt{A}\exp\frac{\alpha}{2}i}$$

12.

Hallar las cuatro raíces de la ecuación $z^4+4=0$, factorizar z^4+4 en factores cuadráticos con coeficientes reales.

Tenemos

$$z^4+4=0$$

Entonces:

$$z^4+4=(z-c_0)(z-c_1)(z-c_2)(z-c_3)$$

$$=[(z-c_1)(z-c_2)][(z-c_0)(z-c_3)]$$

$$=[(z+1)-i][(z+1)+i]\cdot[(z-1)-i][(z-1)+i]$$

$$=[(z+1)^2+1][(z-1)^2+1]$$

$$=\underbrace{(z^2+2z+2)(z^2-2z+2)}$$

18.

Deducir la identidad

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z} \quad (z \neq 1)$$

y usarla para probar la identidad trigonometrica de Lagrange:

$$1 + \cos\Theta + \cos 2\Theta + \cdots + \cos n\Theta = \frac{1}{2} + \frac{\operatorname{sen}[(n+\frac{1}{2})\Theta]}{2\operatorname{sen}(\Theta/2)} \quad (0 < \Theta < 2\pi)$$

Ayuda: Para la primera identidad, escribase $S = 1 + z + z^2 + \cdots + z^n$ y considerese la diferencia $S - zS$. Para la segunda, sustituir $z = e^{\Theta}$ en la primera.

Tenemos

$z = e^{i\Theta}$

la identidad nos queda sustituyendo $e^{i\Theta}$

$$1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{in\theta} = \frac{e^{(n+1)i\theta} - 1}{e^{i\theta} - 1}$$

// separamos reales de imaginarios

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{\left(e^{(n+1)i\frac{\theta}{2}}\right)\left(e^{(n+1)i\frac{\theta}{2}} - e^{-(n+1)i\frac{\theta}{2}}\right)}{e^{i\theta/2}\left(e^{i\theta/2} - e^{-i\theta/2}\right)}$$

// Por la propiedad trigonometrica.

$$\operatorname{Sen} x = \frac{e^{ix} - e^{-ix}}{2i}$$

// se puede escribir

$$2i\operatorname{sen} x = e^{ix} - e^{-ix}$$

nos queda

$$e^{ni\theta/2}\,\frac{2i\operatorname{sen}\frac{(n+1)\theta}{2}}{2i\operatorname{sen}\frac{\theta}{2}}$$

se puede escribir así..

Aplicamos sandiwsh

$$= \frac{\operatorname{sen}\left(\frac{(n+1)\theta}{2}\right)}{\operatorname{sen}\frac{\theta}{2}}\underbrace{\left(\cos\frac{n\theta}{2} + i\operatorname{sen}\frac{n\theta}{2}\right)}_{\text{no se considera}}$$

18.

$$\frac{\operatorname{sen}\frac{(2n+1)\theta}{2}+\operatorname{sen}\frac{\theta}{2}}{2\operatorname{sen}\frac{\theta}{2}}$$

$$=\frac{\operatorname{sen}\frac{(2n+1)}{2}\theta+\operatorname{sen}\frac{\theta}{2}}{2\operatorname{sen}\frac{\theta}{2}}$$

$$=\frac{1}{2}+\frac{\operatorname{sen}\frac{(2n+1)\theta}{2}}{2\operatorname{sen}\frac{\theta}{2}}$$

19.

Sea c una raíz n-ésima de la unidad, diferente de la unidad misma. Demostrar que

$$1 + c + c^2 + \dots + c^{n-1} = 0$$

Ayuda: usar la primera identidad del ejercicio 18.

Tenemos

$$1 + z + z^2 + \dots + z^{n-1} = \frac{1 - z^n}{1 - z}$$

Nos queda

$$1 + c + c^2 + \dots + c^{n-1} = \frac{1 - c^n}{1 - c} = \frac{\overbrace{1 - 1}^{\text{siempre va a ser 0 el numerador}}}{1 - c} = 0$$

20.

a) Probar que la fórmula cuadrática ordinaria resuelve la ecuación de segundo grado:

$$az^2 + bz + c = 0 \quad (a \neq 0)$$

con coeficientes a, b y c complejos. En concreto, completando el cuadrado en el miembro de la izquierda, demostrar que las raíces de la ecuación son:

$$z = \frac{-b + (b^2 - 4ac)^{1/2}}{2a}$$

// Tenemos

$$az^2 + bz + c = 0 \quad (a \neq 0)$$

// Entonces multiplicando a

$$a^2 z^2 + abz + ac = 0$$

// Dos veces el primero por el segundo

$$a^2z^2 + 2a\frac{zb}{2} + \frac{b^2}{4} + ac - \frac{b^2}{4} = 0$$

// Agrupamos terminos

$$\left(az + \frac{b}{2}\right)^2 + \frac{4ac - b^2}{4} = 0$$

// Despejando a z

$$\left(az + \frac{b}{2}\right)^2 = -\frac{4ac + b^2}{4}$$ // sacamos raíz

$$az + \frac{b}{2} = \left(-\frac{4ac + b^2}{4}\right)^{1/2}$$

// Pasar restando la b

$$az = \frac{-b + (b^2 - 4ac)^{1/2}}{2}$$ // Pasamos dividiendo a

$$z = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$ // queda demostrado.

20. b)

Usar el resultado de la parte a para hallar las raíces de la ecuación

$$z^2 + 2z + (1 - i) = 0$$

//tenemos

$a = 1, \quad b = 2, \quad c = 1 - i$

//utilizando el resultado en a tenemos

$$z = \frac{-2 \pm \sqrt{(2^2 - 4(1 - i)}}{2(1)}$$

$$z = \frac{-2 \pm \sqrt{4 - (4 - 4i)}}{2}$$

//obtenemos los resultados

$$\boxed{z_1 = -1 + \frac{\sqrt{-4i}}{2}}$$

$$\boxed{z_2 = -2 - \frac{\sqrt{4 - 4 - 4i}}{2}}$$

$$\boxed{z_2 = -1 - \frac{\sqrt{(4i)}}{2}}$$