Integral de Fourier

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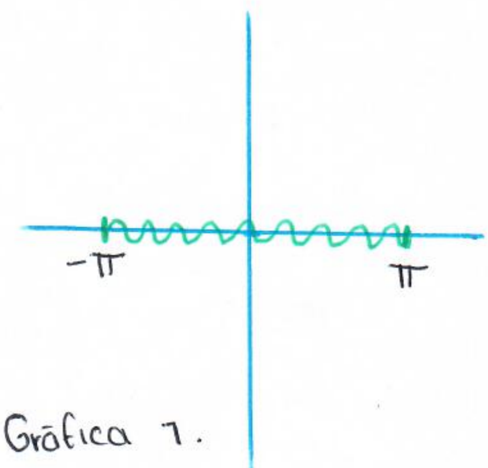
En cada problema del 1 al 10, desarrolle la función en una integral de Fourier y determine a que converge esta integral.

$$f(x) \begin{cases} x & \text{para } -\pi \leq x \leq \pi \\ 0 & \text{para } |x| > \pi \end{cases}$$

Donde: $f(t)$ $\begin{matrix} t \text{ para } -\pi \leq t \leq \pi \\ 0 \text{ para } |t| > \pi \end{matrix}$



Gráfica 1.

Tenemos

$$A\omega = \frac{1}{\pi} \int_{-\infty}^{\infty} f(z) \cos \omega t \, dt$$

$$A\omega = \frac{1}{\pi} \left[\int_{-\infty}^{\infty} 0 \cos \omega t \, dt + \ldots \right.$$

$$\left. \int_{-\pi}^{\pi} t \cos \omega t \, dt + \int_{\pi}^{\infty} 0 \cos \omega t \, dt \right] = 0$$

Para

$$B\omega = \frac{1}{\pi} \left[\int_{-\infty}^{\infty} 0 \operatorname{sen} \omega t \, dt + \int_{-\pi}^{\pi} t \operatorname{sen} \omega t \, dt + \int_{\pi}^{\infty} 0 \operatorname{sen} \omega t \, dt \right]$$

Entonces

$$B\omega = \frac{2}{\pi} \int_{0}^{\pi} t \operatorname{sen} \omega t \, dt = 2 \left[-\frac{t}{\omega} \cos(\omega t) + \frac{\operatorname{sen}(\omega t)}{\omega^2} \right]_0$$

$$= 2 \left[\frac{\operatorname{sen}(\omega \pi)}{\omega^2} - \frac{\pi}{\omega} \cos(\omega \pi) \right]$$

Entonces tenemos que la integral de fourier representada en f es.

$$\int_0^\infty \left[\frac{2\,\text{sen}(\omega\pi)}{\pi\omega^2} - \frac{2\cos(\omega\pi)}{\omega}\right]\text{sen}(\omega x)\,d\omega$$

2. $f(x) \begin{cases} K & \text{para } -10 \leq x \leq 10 \\ 0 & \text{para } |x| > 10 \end{cases}$

Tenemos:

$$f(t) \begin{cases} K & \text{para } -10 \leq x \leq 10 \\ 0 & \text{para } |t| > 10 \end{cases}$$

Entonces

Para

$$A\omega = \int_{-\infty}^{-10} 0\cos\omega t\,dt + \int_{-10}^{10} K\cos\omega t\,dt + \int_{10}^{\infty} 0\cos\omega t\,dt$$

$$A\omega = 2\int_0^{10} K\cos(\omega t)\,dt = \frac{2K}{\omega}\,\text{sen}(10\omega) \quad \text{si } |\omega| > 0$$

Para

$$B\omega = \int_{-\infty}^{\infty} 0\,\text{sen}(\omega t)\,dt + \int_{-10}^{10} K\,\text{sen}(\omega t)\,dt + \int_{10}^{\infty} 0\,\text{sen}(\omega t)\,dt$$

Entonces $f(t)\,\text{sen}(\omega t)$ es impar, por lo tanto $B\omega = 0$

Así la integral de Fourier es representada por

$$\int_0^\infty \frac{2K}{\pi\omega}\,\text{sen}(10\omega)\cos(\omega x)\,d\omega$$

3 $f(x) \begin{cases} -1 & \text{para} \quad -\pi \le x \le 0 \\ 1 & \text{para} \quad 0 < x \le \pi \\ 0 & \text{para} \quad |x| > \pi \end{cases}$

Entonces

$$f(t) = \begin{cases} -1 & \text{para} \quad -\pi \le t \le 0 \\ 1 & \text{para} \quad 0 < t \le \pi \\ 0 & \text{para} \quad |t| > \pi \end{cases}$$

Para

$$A\omega = \int_{-\infty}^{\infty} 0 \cos(\omega t)dt - \int_{-\pi}^{0} 1 \cos(\omega t)dt + \int_{0}^{\pi} \cos(\omega t)dt + \int_{\pi}^{\infty} 0 \cos(\omega t)dt$$

Tenemos

como f es impar, entonces $A\omega = 0$

Para

$$B\omega = \int_{-\pi}^{\pi} f(t)\,\text{sen}(\omega t)dt = 2\int_{0}^{\pi} \text{sen}(\omega t)\,dt = \frac{-2}{\omega} \cos(\omega t)\Big|_{0}^{\pi}$$

Entonces

$$B\omega = \frac{2}{\omega}\left[-\cos(\omega\pi) + 1\right] = \frac{2}{\omega}\left[1 - \cos(\omega\pi)\right]$$

Por lo tanto la integral

de Fourier es

$$\int_{0}^{\infty} \frac{2}{\pi\omega}\left[1 - \cos(\pi\omega)\right] \text{sen}(\omega x)d\omega$$

4.

$$f(x)=\begin{cases}\operatorname{sen}(x) & \text{para } -4\le x\le 0\\ \cos(x) & \text{para } 0<x\le 4\\ 0 & \text{para } |x|>4\end{cases}$$

$$f(t)=\begin{cases}\operatorname{sen}(t) & \text{para } -4\le x\le 0\\ \cos(t) & \text{para } 0<x\le 4\\ 0 & \text{para } |x|>4\end{cases}$$

Para

$$A\omega=\int_{-\infty}^{-4} 0\cos(\omega t)\,dt+\int_{-4}^{0}\operatorname{sen}(t)\cos(\omega t)\,dt+\int_{0}^{4}\cos(t)\cos(\omega t)\,dt+\int_{4}^{\infty}0\cos(\omega t)\,dt$$

Entonces

$$A\omega=\int_{-4}^{0}\operatorname{sen}(t)\cos(\omega t)\,dt+\int_{0}^{4}\cos(t)\cos(\omega t)\,dt$$

$$=\frac{1}{2}\left[\frac{1-\cos[4(\omega-1)]}{\omega-1}-\frac{1-\cos[4(\omega+1)]}{\omega+1}+\frac{\operatorname{sen}[4(\omega-1)]}{\omega-1}+\frac{\operatorname{sen}[4(\omega+1)]}{\omega+1}\right]\text{ donde } |\omega|\neq 1$$

Para

$$B\omega=\frac{1}{2}\left[\frac{1-\cos[4(\omega-1)]}{\omega-1}+\frac{1-\cos[4(\omega+1)]}{\omega+1}+\frac{\operatorname{sen}[4(\omega-1)]}{\omega-1}-\frac{\operatorname{sen}[4(\omega+1)]}{\omega+1}\right]\text{ donde } |\omega|\neq 1$$

Entonces si $|\omega|=1$, entonces $A\omega$ y $B\omega$ deberan reemplazarse por los limites Así

$$A(1)=\lim_{\omega\to 1}A\omega$$

Por lo tanto la integral de fourier es

$$\frac{1}{\pi}\int_{0}^{\infty}\left[A\omega\cos(\omega x)+B\omega\operatorname{sen}(\omega x)\right]d\omega$$

5.4

$f(x) = x^2$ para $-100 \le x \le 100$
0 para $|x| > 100$

$$f(t) = \begin{cases} t^2 & \text{para } -100 \le x \le 100 \\ 0 & \text{para } |x| > 100 \end{cases}$$

Entonces

Para

$$A\omega = \frac{1}{\pi}\left[\int_{-\infty}^{-\infty} 0 \cos(\omega t)\,dt + \int_{-100}^{100} t^2 \cos(\omega t)\,dt + \int_{100}^{\infty} 0 \cos(\omega t)\,dt\right]$$

Entonces

$$A\omega = \frac{1}{\pi}\int_{-100}^{100} t^2 \cos(\omega t)\,dt = \frac{2}{\pi}\int_{0}^{100} t^2 \cos(\omega t)\,dt$$

donde

$$A\omega = \frac{2}{\pi}\left[\frac{t^2 \operatorname{sen}(\omega t)}{\omega} + \frac{2t\cos(\omega t)}{\omega^2} - \frac{2\operatorname{sen}(\omega t)}{\omega^3}\Big|_0^{100}\right]$$

Nos queda

$$A\omega = \frac{20\,000 \operatorname{sen}(100\omega)}{\pi\omega} - \frac{4\operatorname{sen}(100\omega)}{\pi\omega^3} + \frac{400\cos(100\omega)}{\pi\omega^2}$$

Por lo tanto

$$\int_0^{\infty}\left[\frac{400\cos(100\omega)}{\pi\omega^2} + \frac{20000\,\omega^2 - 4}{\pi\omega^3}\operatorname{sen}(100\omega)\right]\cos(\omega x)\,d\omega$$

6.- $f(x) = \begin{cases} |x| & \text{para } -\pi \leq t \leq 2\pi \\ 0 & \text{para } x < -\pi \text{ y para } x > 2\pi \end{cases}$

$$f(t) = \begin{cases} |t| & \text{para } -\pi \leq t \leq 2\pi \\ 0 & \text{para } t < -\pi \text{ y para } t > 2\pi \end{cases}$$

Para

$$A\omega = \int_{-\infty}^{\pi} 0 \cos(\omega t) dt + \int_{-\pi}^{2\pi} |t| \cos(\omega t) dt + \int_{2\pi}^{\infty} 0 \cos(\omega t)$$

Entonces

$$A\omega = \int_{-\pi}^{2\pi} |t| \cos(\omega t) dt = 2\int_{0}^{\pi} t \cos(\omega t) dt + \int_{\pi}^{2\pi} t \cos(\omega t) dt$$

Nos queda

$$A\omega = \frac{\pi}{\omega} \operatorname{sen}(\pi\omega) + \frac{2\pi}{\omega} \operatorname{sen}(2\pi\omega) + \frac{1}{\omega^2} \cos(\omega\pi) + \frac{\cos(2\pi\omega)}{\omega^2} - \frac{1}{\omega^2}$$

Para

$$B\omega = \int_{-\infty}^{-\pi} 0 \operatorname{sen}(\omega t) dt + \int_{-\pi}^{2\pi} |t| \operatorname{sen}(\omega t) dt + \int_{2\pi}^{\infty} 0 \operatorname{sen}(\omega t) dt$$

Entonces

$$B\omega = \int_{-\pi}^{2\pi} |t| \operatorname{sen}(\omega t) dt = \int_{\pi}^{2\pi} t \operatorname{sen}(\omega t) dt$$

Nos queda

$$B\omega = \frac{\operatorname{sen}(2\pi\omega)}{\omega^2} - \frac{\operatorname{sen}(\omega\pi)}{\omega^2} - \frac{2\pi\cos(2\pi\omega)}{\omega} + \frac{\pi}{\omega}\cos(\omega\pi)$$

Por lo tanto

$$\frac{1}{\pi} \int_{0}^{\infty} \left[A\omega \cos(\omega x) + B\omega \operatorname{sen}(\omega x) \right] dx$$

7.

$$f(x) = \begin{cases} \operatorname{sen}(x) & \text{para } -3\pi \\ 0 & \text{para } x < 3\pi \text{ y para } x > \pi \end{cases}$$

$$f(t) = \begin{cases} \operatorname{sen}(t) & \text{para } -3\pi \\ 0 & \text{para } t < -3\pi \text{ y para } t > \pi \end{cases}$$

// Nos queda de la siguiente forma

Para

$$A\omega = \int_{-\infty}^{-3\pi} 0 \cancel{\cos(\omega t)}^{0}\, dt + \int_{-3\pi}^{\pi} \operatorname{sen}(t)\cos(\omega t)\, dt + \int_{\pi}^{\infty} 0 \cancel{\cos(\omega t)}^{0}\, dt$$

Entonces

$$A\omega = \int_{-3\pi}^{\pi} \operatorname{sen}(t)\cos(\omega t)\, dt = \frac{2\operatorname{sen}(\omega\pi)\operatorname{sen}(2\omega\pi)}{\omega^2 - 1}$$

Para

$$B\omega = \int_{-3\pi}^{\pi} \operatorname{sen}(t)\operatorname{sen}(\omega t)\, dt = -\frac{2\cos(\pi\omega)\operatorname{sen}(2\pi\omega)}{\omega^2 - 1}$$

Por lo tanto

$$\int_0^{\infty} \frac{2}{\pi(\omega^2-1)}\left[\operatorname{sen}(\omega\pi)\operatorname{sen}(2\omega\pi)\cos(\omega x) - \cos(\omega\pi)\operatorname{sen}(2\omega\pi) \cdots \operatorname{sen}(\omega x)\right] d\omega$$

8.

$$f(x) = \begin{cases} \frac{1}{2} & \text{para } -5 \leq x < 1 \\ 1 & \text{para } 1 \leq x \leq 5 \\ 0 & \text{para } |x| > 5 \end{cases}$$

//Nos queda de la siguiente manera

$$f(x) = \begin{cases} \frac{1}{2} & \text{para } -5 \leq x < 1 \\ 1 & \text{para } 1 \leq x \leq 5 \\ 0 & \text{para } |x| > 5 \end{cases}$$

Para

$$A\omega = \frac{1}{\pi}\left[\int_{-\infty}^{-5} 0 \cos(\omega t)dt + \int_{-5}^{1} \frac{1}{2}\cos(\omega t)dt + \int_{1}^{5}\cos(\omega t)dt + \int_{5}^{\infty} 0\cos(\omega t)dt\right]$$

Nos queda

$$\int_{-5}^{1} \frac{\cos(\omega t)}{2}dt + \int_{1}^{5}\cos(\omega t)dt$$

Entonces

$$A\omega = \frac{1}{\pi}\left(\frac{\operatorname{sen}(\omega t)}{2\omega}\Bigg|_{-5}^{1} + \frac{\operatorname{sen}(\omega t)}{\omega}\Bigg|_{1}^{5}\right)$$

Tenemos

$$A\omega = \frac{1}{\pi}\left(\frac{\operatorname{sen}\omega - \operatorname{sen}(-5\omega)}{2\omega} + \frac{\operatorname{sen}(5\omega) - \operatorname{sen}\omega}{\omega}\right)$$

Nos queda

$$A\omega = \frac{1}{\pi}\left(\frac{\operatorname{sen}\omega + \operatorname{sen}(5\omega)}{2\omega} + \frac{\operatorname{sen}(5\omega) - \operatorname{sen}\omega}{\omega}\right)$$

Finalmente

$$= \frac{3\ \operatorname{sen}(5\omega) - \operatorname{sen}\omega}{2\pi\omega}$$

Para

$$B\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,\mathrm{sen}(\omega t)\,dt$$

Entonces

$$\frac{1}{\pi}\left[\int_{-5}^{1}\frac{\mathrm{sen}(\omega t)}{2}\,dt + \int_{1}^{5}\frac{\mathrm{sen}\,(\omega t)}{2}\,dt + \int_{1}^{5}\mathrm{sen}\,(\omega t)\,dt\right]$$

Nos queda

$$B\omega = -\frac{1}{\pi}\left(\frac{(\cos(\omega t)}{2\omega}\Big|_{-5}^{1} + \frac{\cos\,(\omega t)}{\omega}\Big|_{1}^{5}\right)$$

$$B\omega = -\frac{1}{\pi}\left(\frac{\cos\omega - \cos(-5\omega)}{2\omega} + \frac{\cos(5\omega) - \cos\omega}{\omega}\right)$$

Entonces

$$B\omega = -\frac{1}{\pi}\left(\frac{\cos\omega - \cos\,(5\omega)}{2\omega} + \frac{\cos\,(5\omega) - \cos\omega}{\omega}\right)$$

queda

$$= \frac{\cos(5\omega) - \cos\omega}{2\pi\omega}$$

Al final nos queda de la siguiente manera

$$\frac{1}{2\pi}\int_{0}^{\infty}\left[\frac{3\,\mathrm{sen}\,(5\omega - \mathrm{sen}\,\omega)}{\omega}\cos(\omega x) + \frac{\cos(5\omega) - \cos\omega}{\omega}\mathrm{sen}(\omega x)\right]d\omega$$

9. $f(x) = e^{-|x|}$

Nos queda

$f(t) = e^{|t|}$

Tenemos

Para

$$A\omega = \frac{2}{\pi}\int_0^{\infty} e^{-|t|}\cos(\omega t)$$

$$= \frac{2}{\pi}\int_0^{\infty} e^{-t}\cos(\omega t)dt$$

//Debemos resolver la integral por partes

Nos queda

$u = \cos(\omega t)$ || $dv = e^{-t}dt$

$du = -\omega\,\text{sen}(\omega t)dt$ || $v = -e^{-t}$

tenemos

$$I = \int e^{-t}\cos(\omega t)dt$$

Entonces

$$= \frac{2}{\pi}\left[-\cos(\omega t)e^{-t} - \omega\left(-\text{sen}(\omega t)e^{-t}\right) + \omega\int \cos(\omega t)e^{-t}dt\right]_0^{\infty}$$

simplificando

$$= \frac{2}{\pi}\left[\omega\,\text{sen}(\omega t)e^{-t} - \cos(\omega t)e^{-t} - \omega^2 I\right]$$

// Ahora resolvemos para I

Tenemos

$$(1+\omega^2)I = e^{-t}(\omega \operatorname{sen}(\omega t) - \cos(\omega t))$$

Nos queda

$$I = e^{-t}\,\frac{(\omega \operatorname{sen}(\omega t) - \cos(\omega t))}{1+\omega^2}$$

Entonces

Para

$$A\omega = \frac{2}{\pi}\, e^{-t}\, \frac{(\omega \operatorname{sen}(\omega t) - \cos(\omega t)}{1+\omega^2}\Bigg|_0^{\infty}$$

Nos queda

$$= \frac{2}{\pi(1+\omega^2)}\left[\lim_{t\to\infty} e^{-t}(\omega \operatorname{sen}(\omega t) - \cos(\omega t)) - 1(0-1)\right]$$

El Resultado

$$A\omega = \frac{2}{\pi(1+\omega^2)}$$

Para

$$B\omega = \frac{2}{\pi}\int_0^{\infty} e^{-|t|} \operatorname{sen}(\omega t)\, dt$$, entonces es producto impar

Nos queda

$$B\omega = 0$$

Por lo tanto

$$\frac{2}{\pi}\int_0^{\infty} \frac{\cos(\omega x)}{1+\omega^2}\, d\omega$$

finalmente tenemos

$$e^{-|t|} = \frac{2}{\pi(1+\omega^2)} \cos(\omega x)\, d\omega$$