Question 1

$$f(x) = \frac{1}{10-0} = \frac{1}{10}$$
$$F(Y) = \frac{y}{10}$$

a

$$F(Y_L) = F(Y_1) \times F(Y_2) \times F(Y_3) \times F(Y_4) \times F(Y_5)$$
$$= F(Y)^5$$
$$= (\frac{y}{10})^5$$
$$= \frac{y^5}{10^5}$$
$$f(Y_L) = \frac{d}{dy}F(y_L) = \frac{5y^4}{10^5}$$
$$E(Y_L) = \int_0^{10} yf(Y_L)dy$$
$$= \int_0^{10} \frac{5y^5}{10^5}$$
$$= \left[\frac{5y^6}{6 \times 10^5}\right]_0^{10}$$
$$= \frac{5 \times 10^6}{6 \times 10^5}$$
$$= 8.33$$

b

$$G(Y_S) = P(Y_1 \leqslant Y)$$
$$= 1 - P(Y_1 > Y)$$
$$= 1 - P[X_1 > Y, X_2 > Y, X_3 > Y, X_4 > Y, X_5 > Y]$$
$$= 1 - P(X_1 > Y)P(X_2 > Y)P(X_3 > Y)P(X_4 > Y)P(X_5 > Y)$$
$$= 1 - (1 - P(X_1 \leqslant Y))(1 - P(X_2 \leqslant Y))(1 - P(X_3 \leqslant Y))(1 - P(X_4 \leqslant Y))(1 - P(X_5 \leqslant Y))$$
$$= 1 - (1 - F(Y))^5$$
$$= 1 - (1 - \frac{y}{10})^5$$
$$g(Y_S) = \frac{d}{dy} 1 - (1 - \frac{y}{10})^5$$
$$= -5(1 - \frac{y}{10})^4 \times (-\frac{1}{10})$$
$$= \frac{1}{2}(1 - \frac{y}{10})^4$$
$$E[Y_S] = \int_0^{10} y \frac{1}{2}(1 - \frac{y}{10})^4 dy$$
$$= \frac{1}{2 \times 10^4} \int_0^{10} y(y - 10)^4 dy$$
$$u \coloneqq y - 10$$
$$du = dy$$
$$\Rightarrow E[Y_S] = \frac{1}{2 \times 10^4} \int_{-10}^{0} (u^5 + 10u^4) du$$
$$= \frac{1}{2 \times 10^4} \frac{100000}{3}$$
$$= 1.67$$
$$E[Y_L] - E[Y_S] = \frac{50}{6} - \frac{10}{6} = \frac{20}{3}$$

c

$$h(Y_M) = \frac{n!}{(i-1)!(n-i)!}[F(Y)]^{i-1}[1-F(Y)]^{n-i}f(Y)$$
$$= \frac{5!}{2!2!}(\frac{y}{10})^2(1-\frac{y}{10})^2\frac{1}{10}$$
$$= 3(\frac{y}{10})^2(1-\frac{y}{10})^2$$
$$E[Y_M] = \int_0^{10} yh(Y_M)dy$$
$$= 0.03\int_0^{10} y^3(1-\frac{y}{10})^2dy$$
$$= 0.03 \times \frac{500}{3}$$
$$= 5$$

d

$$V[Y_L] = E[{Y_L}^2] - E[Y_L]^2$$
$$= \int_0^{10} y^2\frac{5y^4}{10^5}dy - (\frac{50}{6})^2$$
$$= \frac{5}{10^5}\int_0^{10} y^6dy - \frac{625}{9}$$
$$= \frac{500}{7} - \frac{625}{9}$$
$$= 1.984$$
$$\sigma = \sqrt{V[Y_L]} = \sqrt{1.984} = 1.409$$

Question 2

a

$$
\begin{aligned}
&P(X \geqslant 5) = 1 - P(X < 5) \\
&= 1 - P(X_1 < 5)P(X_2 < 5)P(X_3 < 5) \\
&= 1 - (P(X_i < 5))^3 \\
&= 1 - (\int_1^5 f(x)dx)^3 \\
&= 1 - (\int_1^5 \frac{3}{x^4}dx)^3 \\
&= 1 - (\left[-\frac{1}{x^3}\right]_1^5)^3 \\
&= 1 - (\frac{124}{125})^3 \\
&= 0.238
\end{aligned}
$$

b

$$F(x) = \int_1^x f(t)dt$$
$$= \int_1^x \frac{3}{t^4}dt$$
$$= \left[-t^{-3}\right]_1^x$$
$$= 1 - \frac{1}{x^3}$$
$$g(x_3) = \frac{3!}{2!0!}[F(x_3)]^2(\frac{3}{x^4})$$
$$= 3(1 - \frac{1}{x^3})^2(\frac{3}{x^4})$$
$$E[X_3] = \int_1^\infty xg(x_3)dx$$
$$= \int_1^\infty x3(1 - \frac{1}{x^3})^2(\frac{3}{x^4})dx$$
$$= 9\left[-\frac{1}{2x^2} - \frac{1}{8x^8} + \frac{2}{5x^5}\right]_1^\infty$$
$$= 9(\frac{1}{2} + \frac{1}{8} + \frac{2}{5})$$
$$= \frac{81}{40}$$
$$= 2.025$$

Question 3

a

$$\hat{\lambda} = \overline{x} = \frac{\sum f(x)}{\sum f}$$
$$\sum f(x) = 37 + 2 \times 42 + 3 \times 30 + 4 \times 13 + 5 \times 7 + 6 \times 2 + 7 \times 1$$
$$= 317$$
$$\sum f = 18 + 37 + 42 + 30 + 13 + 7 + 2 + 1 = 150$$
$$\Rightarrow \hat{\lambda} = \frac{317}{150}$$

b

$$V(X) = \lambda$$
$$V(\overline{X}) = \frac{\lambda}{n}$$
$$\Rightarrow \hat{\sigma} = \sqrt{\frac{\hat{\lambda}}{n}}$$
$$= \sqrt{\frac{\frac{317}{150}}{150}}$$
$$= 0.1187$$

Question 4

a

$$E[\frac{X_1}{n_1} - \frac{X_2}{n_2}] = \frac{1}{n_1}E[X_1] - \frac{1}{n_2}E[X_2]$$
$$= \frac{1}{n_1}(n_1p_1) - frac1n_2(n_2p_2)$$
$$= p_1 - p_2$$

b

$$\sigma = \sqrt{V[\frac{X_1}{n_1} - \frac{X_2}{n_2}]}$$
$$= \sqrt{\frac{1}{{n_1}^2}V(X_1) + \frac{1}{{n_2}^2}V(X_2)}$$
$$= \sqrt{\frac{1}{{n_1}^2}n_1p_1(1-p_1) + \frac{1}{{n_2}^2}n_2p_2(1-p_2)}$$
$$= \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

c

$$
\begin{aligned}
\hat{p_1} &= \frac{x_1}{n_1} \\
\hat{p_2} &= \frac{x_2}{n_2} \\
s\hat{igm}a &= \sqrt{\frac{\hat{p_1}(1-\hat{p_1})}{n_1} + \frac{\hat{p_2}(1-\hat{p_2})}{n_2}} \\
&= \sqrt{\frac{\frac{x_1}{n_1}(1-\frac{x_1}{n_1})}{n_1} + \frac{\frac{x_2}{n_2}(1-\frac{x_2}{n_2})}{n_2}} \\
&= \sqrt{\frac{\frac{x_1(n_1-x_1)}{{n_1}^2}}{n_1} + \frac{\frac{x_2(n_2-x_2)}{{n_2}^2}}{n_2}} \\
&= \sqrt{\frac{x_1(n_1-x_1)}{{n_1}^3} + \frac{x_2(n_2-x_2)}{{n_2}^3}}
\end{aligned}
$$

d

$$
\begin{aligned}
\hat{p_1} - \hat{p_2} &= \frac{x_1}{n_1} - \frac{x_2}{n_2} \\
&= \frac{127}{200} - \frac{176}{200} \\
&= -\frac{49}{200}
\end{aligned}
$$

e

$$
\begin{aligned}
\hat{\sigma} &= \sqrt{\frac{x_1(n_1-x_1)}{{n_1}^3} + \frac{x_2(n_2-x_2)}{{n_2}^3}} \\
&= \sqrt{\frac{127(200-127)}{200^3} + \frac{176(200-176)}{200^3}} \\
&= 0.0411
\end{aligned}
$$

Question 5

a

$$
\begin{aligned}
E[X] &= \int_0^1 xf(x)dx \\
&= \int_0^1 x(\theta+1)x^\theta dx \\
&= \int_0^1 (\theta+1)x^{\theta+1} dx \\
&= \left[\frac{\theta+1}{\theta+2}x^{\theta+2}\right]_0^1 \\
&= \frac{\theta+1}{\theta+2} \\
\Rightarrow \overline{X} &= \frac{\theta+1}{\theta+2} \\
\theta\overline{X} + 2\overline{X} &= \theta + 1 \\
\theta &= \frac{1-2\overline{X}}{\overline{X}-1} \\
\overline{X} &= \frac{\sum x}{n} \\
&= \frac{0.92+0.79+0.90+0.65+0.86+0.47+0.73+0.97+0.94+0.77}{10} \\
&= 0.8 \\
\theta &= \frac{1-2\times 0.8}{0.8-1} = 3
\end{aligned}
$$

b

$$L = \prod_{i=1}^{10} (\theta + 1) x_i^{\theta}$$

$$= (\theta + 1)^1 0 \prod_{i=1}^{10} x_i^{\theta}$$

$$\ln L = 10 \ln(\theta + 1) + \theta \sum_{i=1}^{10} \ln x_i$$

$$\frac{d \ln L}{theta} = \frac{10}{\theta + 1} + \sum_{i=1}^{10} \ln x_i = 0$$

$$\frac{10}{\theta + 1} = -\sum_{i=1}^{10} \ln x_i$$

$$\theta = -\frac{10}{\sum_{i=1}^{10} \ln x_i} - 1$$

$$\sum_{i=1}^{10} \ln x_i = \ln 0.92 + \ln 0.79 + \ln 0.90 + \ln 0.65$$

$$+ \ln 0.86 + \ln 0.47 + \ln 0.73 + \ln 0.97 + \ln 0.94 + \ln 0.77$$

$$= -2.4295$$

$$\theta = -\frac{10}{-2.4295} - 1$$

$$= 3.12$$

Collaborators

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