Question 1

a

$$E[Y|X=x] = \int_0^x y f_{Y|X}(y|x) dy$$
$$= \int_0^x y \frac{1}{x} dy$$
$$= \frac{y^2}{2}\frac{1}{x}|_0^x$$
$$= \frac{x}{2}$$
$\Rightarrow E[Y|X=x]$ is a linear function of x
$$E[Y^2|X=x] = \int_0^x y^2 f_{Y|X}(y|x) dy$$
$$= \left[\frac{y^3}{3}\frac{1}{x}\right]_0^x$$
$$= \frac{x^2}{3}$$
$$V[Y|X=x] = E[Y^2|X=x] - (E[Y|X=x])^2 = \frac{x^2}{3} - frac x^2 4 = \frac{x^2}{12}$$

b

$$f(x,y) = f_{Y|X}(y|x) f_X(x) = \frac{1}{x} \times 1 = \frac{1}{x}$$

c

$$
\begin{aligned}
&0 \leqslant y \leqslant x \leqslant 1 \\
&\Rightarrow f_Y(y) = \int_y^1 f(x,y)dx \\
&= \int_y^1 \frac{1}{x}dx \\
&= [\ln(x)]_y^1 \\
&= -\ln(y), y \in [0,1]
\end{aligned}
$$

d

$$
\begin{aligned}
&E[Y] = \int_0^1 y f_Y(y)dy \\
&= \int_0^1 y(-\ln(y))dy \\
&= \left[-\ln(y)\int ydy + \int(\frac{d}{dy}\ln(y)\int ydy)dy\right]_0^1 \\
&= \left[-\frac{x^2}{2}\ln(y) + \frac{y^2}{4}\right]_0^1 \\
&= \frac{1}{4}
\end{aligned}
$$

Question 2

a

$$
\begin{aligned}
&f(x,y) = 2e^{-(x+y)} \\
&f_X(x) = \int_x^{+\infty} f(x,y)dy \\
&= \int_x^{+\infty} 2e^{-(x+y)}dy \\
&= \left[-2e^{-(x+y)}\right]_x^{+\infty} \\
&= 2e^{-2x}, x > 0
\end{aligned}
$$

b

$$
\begin{aligned}
&f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} \\
&= \frac{2e^{-(x+y)}}{2e^{-2x}} = e^{x-y}, x < y
\end{aligned}
$$

c

$$
\begin{aligned}
&f_{Y|X}(y|x=1) = e^{1-y}, y > 1\\
&P(y > 2|x = 1) = \int_2^{+\infty} f_{Y|X}(y|x=1)dy\\
&= \int_2^{+\infty} e^{1-y}dy\\
&= \left[-e^{1-y}\right]_2^{+\infty}\\
&= e^{-1}
\end{aligned}
$$

d

$$
\begin{aligned}
&f_Y(y) = \int_0^y f(x,y)dx\\
&= \int_0^y 2e^{-(x+y)}dx\\
&= \left[-2e^{-(x+y)}\right]_0^y\\
&= 2e^{-y} - 2e^{-2y}\\
&f_X(x)f_Y(y) = 2e^{-2x}(2e^{-y} - 2e^{-2y}) \neq 2e^{-(x+y)}\\
\Rightarrow &f_X(x)f_Y(y) \neq f(x,y) \implies \text{dependent}
\end{aligned}
$$

e

$$E[Y|X=x] = \int_x^{+\infty} y f_{Y|X}(y|x)dy$$
$$= \int_x^{+\infty} y e^{x-y} dy$$
$$= \left[y \int e^{x-y} dy - \int (\frac{d}{dy} y \int e^{x-y} dy) dy \right]_x^{+\infty}$$
$$= \left[-y e^{x-y} - e^{x-y} \right]_x^{+\infty}$$
$$= x+1$$

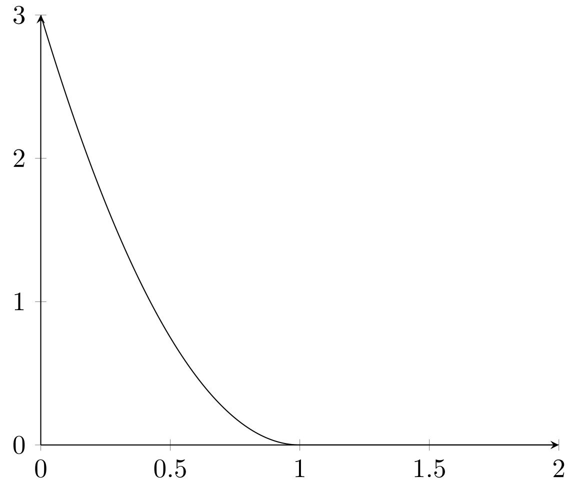
$\Rightarrow$it is linear

f

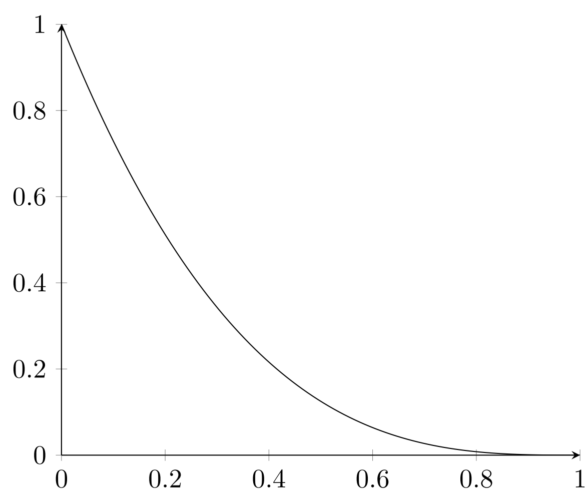
$$f_{Y|X}(y|x) = e^{x-y}$$
$$E[Y^2|X=x] = \int_x^{\infty} y^2 f_{Y|X}(y|x)dy$$
$$= \int_x^{\infty} y^2 e^{x-y} dy$$
$$= \left[y^2 \int e^{x-y} dy - \int (\frac{d}{dy} y^2 \int e^{x-y} dy) dy \right]_x^{+\infty}$$
$$= \left[-y^2 e^{x-y} + 2 \int y e^{x-y} dy \right]_x^{+\infty}$$
$$= \left[-y^2 e^{x-y} + 2(-y e^{x-y} - e^{x-y}) \right]_x^{+\infty}$$
$$= x^2 + 2x + 2$$
$$V[Y|X=x] = E[Y^2|X=x] - (E[Y|X=x])^2$$
$$= x^2 + 2x + 2 - (x+1)^2$$
$$= 1$$

Question 3

a



b

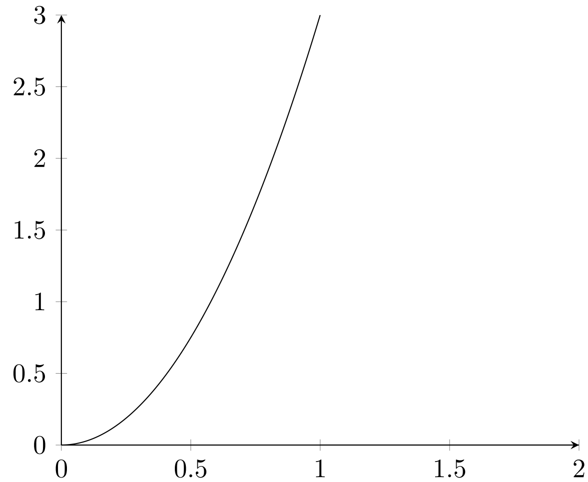


c

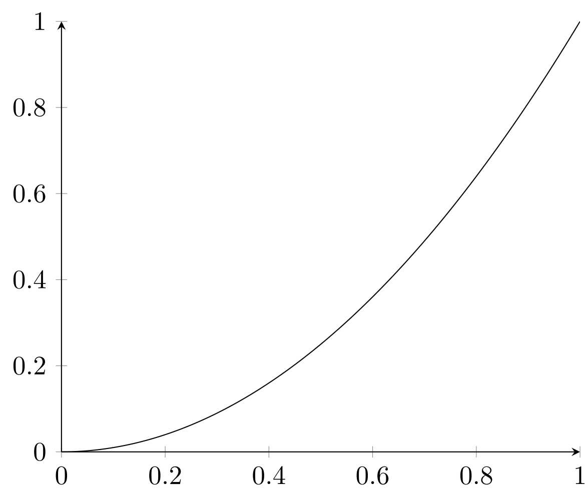
$$
\begin{aligned}
&Y = (1-X)^3 \\
\Rightarrow &X = 1 - Y^{\frac{1}{3}} \\
&\frac{dx}{dy} = -\frac{1}{3}y^{-\frac{2}{3}} \\
&f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| \\
&= 3(1-(1-y^{\frac{1}{3}}))^2\frac{1}{3}y^{-frac23} \\
&= 1
\end{aligned}
$$

Question 4

d



e



f

$$
\begin{aligned}
&y = x^2, x, y \in [0,1] \\
&x = \sqrt{y}, x, y \in [0,1] \\
\Rightarrow &g_Y(y) = \sqrt{y}
\end{aligned}
$$

g

$$
{g_Y}'(y) = \frac{1}{2\sqrt{y}}, y \in [0,1]
$$

h

$$
\begin{aligned}
&f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| \\
&= 3(\sqrt{y})^2 \frac{1}{2\sqrt{y}} \\
&= \frac{3}{2}\sqrt{y}, y \in [0,1]
\end{aligned}
$$

Question 5

a

$$P(X = 1, Y = 1) = 0.030$$

b

$$\begin{aligned}
&P(X \leqslant 1, Y \leqslant 1) \\
&= P(X = 0, Y = 0) + P(X = 0, Y = 1) \\
&+ P(X = 1, Y = 0) + P(X = 1, Y = 1) \\
&= 0.025 + 0.015 + 0.050 + 0.030 = 0.120
\end{aligned}$$

c

$$\begin{aligned}
&P(X = 1) = P(X = 1, Y = 0) + P(X = 1, Y = 1) + P(X = 1, Y = 2) \\
&= 0.050 + 0.030 + 0.020 = 0.100 \\
&P(Y = 1) = P(X = 0, Y = 1) + P(X = 1, Y = 1) + P(X = 2, Y = 1) \\
&+ P(X = 3, Y = 1) + P(X = 4, Y = 1) + P(X = 5, Y = 1) \\
&= 0.015 + 0.030 + 0.075 + 0.090 + 0.060 + 0.030 \\
&= 0.300
\end{aligned}$$

d

$X + 3Y > 5$
$Y = 0$: no overflow
$Y = 1 : X = 3, 4, 5$
$P(X \leqslant 3, Y = 1) = 0.090 + 0.060 + 0.030 = 0.18$
$Y = 2 : X = 1, 2, 3, 4, 5$
$P(Y = 2) = 0.010 + 0.020 + 0.050 + 0.060 + 0.040 + 0.020$
$= 0.200$
$\Rightarrow P(X + 3Y > 5) = P(X \leqslant 3, Y = 1) + P(Y = 2) = 0.180 + 0.200 = 0.380$

e

$$\forall x \in [0, 5], y \in [0, 2], P(X = x, Y = y) = P(X = x)P(Y = y)$$
$\Rightarrow$Independent

Question 6

$P(X = 0) = 0.025 + 0.015 + 0.010 = 0.050$
$P(X = 1) = 0.050 + 0.030 + 0.020 = 0.100$
$P(X = 2) = 0.125 + 0.075 + 0.050 = 0.250$
$P(X = 3) = 0.150 + 0.090 + 0.060 = 0.300$
$P(X = 4) = 0.100 + 0.060 + 0.040 = 0.200$
$P(X = 5) = 0.050 + 0.030 + 0.020 = 0.100$
$E[X] = 0P(X = 0) + 1P(X = 1) + 2P(X = 2) + 3P(X = 3) + 4P(X = 4) + 5P(X = 5)$
$= 0 \times 0.05 + 1 \times 0.1 + 2 \times 0.25 + 3 \times 0.3 + 4 \times 0.2 + 5 \times 0.1 = 2.8$
$P(Y = 0) = 0.025 + 0.050 + 0.125 + 0.150 + 0.100 + 0.050 = 0.500$
$P(Y = 1) = 0.015 + 0.030 + 0.075 + 0.090 + 0.060 + 0.030 = 0.300$
$P(Y = 2) = 0.010 + 0.020 + 0.050 + 0.060 + 0.040 + 0.020 = 0.200$
$E[Y] = 0P(Y = 0) + 1P(Y = 1) + 2P(Y = 2)$
$= 0 \times 0.5 + 1 \times 0.3 + 2 \times 0.2 = 0.7$
$E[3X + 10Y] = 3E[X] + 10E[Y] = 3 \times 2.8 + 10 \times 0.7 = 15.4$

Question 7

$F_X(x) = 3x^2, x \in [0,1]$

$F_Y(y) = 2y, y \in [0,1]$

Independent $\implies F(x,y) = f_X(x)F_Y(y) = 3x^2 2y = 6x^2 y, x, y \in [0,1]$

$E[|X-Y|] = \int_0^1 \int_0^x (x-y)6x^2 ydydx + \int_0^1 \int_x^1 (y-x)6x^2 ydydx$

$$\int_0^1 \int_0^x (x-y)6x^2 ydydx$$
$$= \int_0^1 \int_0^x 6x^3 y - 6x^2 y^2 dydx$$
$$= \int_0^1 \left[3x^3y^2 - 2x^2y^3\right]_0^x$$
$$= \int_0^1 x^5 dx$$
$$= \left[\frac{1}{6}x^6\right]_0^1$$
$$= \frac{1}{6}$$

$$\int_0^1 \int_x^1 (y-x)6x^2 ydydx$$
$$= \int_0^1 \int_x^1 6x^2y^2 - 6x^3 ydydx$$
$$= \int_0^1 \left[2x^2y^3 - 3x^3y^2\right]_x^1$$
$$= \int_0^1 2x^2 - 3x^3 + x^5 dx$$
$$= \left[\frac{2}{3}x^3 - \frac{3}{4}x^4 + \frac{1}{6}x^6\right]_0^1$$
$$= \frac{1}{12}$$

$$\Rightarrow E[|X-Y|] = \int_0^1 \int_0^x (x-y)6x^2 ydydx + \int_0^1 \int_x^1 (y-x)6x^2 ydydx$$
$$= \frac{1}{6} + \frac{1}{12}$$
$$= \frac{1}{4}$$

Collaborators

Frank Zhu

Jeffery Shu

Sam Sun