Question 1

a

Proof.

$x = m^2 - n^2$
$y = 2mn$
$z = m^2 + n^2$
$x^2 + y^2 = (m^2 - n^2)^2 + (2mn)^2$
$= m^4 - 2m^2n^2 + n^4 + 4m^2n^2$
$= m^4 + 2m^2n^2 + n^4$
$= (m^2 + n^2)^2$
$= z^2$
$\Rightarrow x^2 + y^2 = z^2$

$\Rightarrow$
Suppose : (x, y, z) are primitive triple
$\Rightarrow x, y, z$ are pairwise co-prime
Suppose : m, n are not co-prime or both odd
case 1 : not co-prime
$\exists p, q, k \neq 1 \in \mathbb{Z} : m = pk, n = qk$
$x = m^2 - n^2 = (pk)^2 - (qk)^2 = (p^2 - q^2)k^2$
$y = 2mn = 2pk \cdot qk = 2pqk^2$
$\gcd(x, y) \geqslant k^2 > 1$
$\Rightarrow x, y$ are not co-prime $\not\Leftrightarrow$
case 2 : both odd
$\exists a, b \in \mathbb{Z} : m = 2a + 1, n = 2b + 1$
$x = m^2 - n^2 = (2a + 1)^2 - (2b + 1)^2 = 4a^2 + 4a - 4b^2 - 4b = 4(a^2 + a - b^2 - b)$
$y = 2mn = 2(2a + 1)(2b + 1)$
$\gcd(x, y) \geqslant 2 > 1$
$\Rightarrow x, y$ are not co-prime $\not\Leftrightarrow$
$\Rightarrow (x, y, z)$ are primitive triple $\implies$ m, n are co-prime and not both odd

$\Leftarrow$

Suppose : m, n are co-prime and not both odd

Without loss of generality, suppose : $\exists p, q \in \mathbb{Z} : m = 2p, n = 2q+1, \gcd(2p, 2q+1) = 1$

Suppose : x, y, z are not pairwise co-prime

case 1 : $\gcd(x, y) > 1$

$a \coloneqq \gcd(x, y)$

$\Rightarrow a|2mn$

$a|2 \vee a|m \vee a|n$

$a \not| 2$ since m, n are not both even

$a|m$:

$a|m^2 - n^2$

$\Rightarrow a|n^2$

$\Rightarrow a|n$

$\Rightarrow \gcd(m, n) \geqslant a > 1 \not\Leftrightarrow \gcd(m, n) = 1$

$a|n$ same condition as $a|m$

case 2 : $\gcd(x, z) > 1$

$a \coloneqq \gcd(x, z)$

$\Rightarrow a|m^2 - n^2 \wedge a|m^2 + n^2$

$a|2m^2 \wedge a|2n^2$

$\Rightarrow a|2 \vee (a|m^2 \wedge a|n^2)$

$a \not| 2$ since m, n are not both even

$\Rightarrow a|m^2 \wedge a|n^2$

$a|m \wedge a|n$

$\Rightarrow \gcd(m, n) \geqslant a > 1 \not\Leftrightarrow \gcd(m, n) = 1$

case 3 : $\gcd(y, z) > 1$

$a \coloneqq \gcd(y, z)$

$\Rightarrow a|2mn$

$a|2 \vee a|m \vee a|n$

$a \not| 2$ since m, n are not both even

$a|m$:

$a|m^2 + n^2$

$\Rightarrow a|n^2$

$\Rightarrow a|n$

$\Rightarrow \gcd(m, n) \geqslant a > 1 \not\Leftrightarrow \gcd(m, n) = 1$

$a|n$ same condition as $a|m$

$\Rightarrow m, n$ are co-prime and not both odd $\implies (x, y, z)$ are primitive triple
$\Rightarrow m, n$ are co-prime and not both odd $\Leftrightarrow (x, y, z)$ are primitive triple

$\square$

b

$$\begin{aligned}
&2mn \text{ must be odd} \\
\Rightarrow &2mn = 24 \\
&x < z \\
\Rightarrow &x = 7, z = 25 \\
&x = m^2 - n^2, z = m^2 + n^2 \\
&x + z = 2m^2 = 32 \\
&m = 4 \vee m = -4 \\
&m = 4: \\
&2mn = 24 \\
&n = 3 \\
&m = -4: \\
&2mn = 24 \\
&n = -3 \\
\Rightarrow &(m, n) = (4, 3) \vee (-4, -3)
\end{aligned}$$

c

$$\begin{aligned}
&(x, y, z) \text{ is pairwise co-prime} \\
\Rightarrow &\text{without loss of generality} : \exists p, q \in \mathbb{Z} : m = 2p, n = 2q + 1, \gcd(2p, 2q + 1) = 1 \\
&z = m^2 + n^2 \\
&\ = (2p)^2 + (2q + 1)^2 \\
&\ = 4p^2 + 4q^2 + 4q + 1
\end{aligned}$$

$$= 2(2p^2 + 2q^2 + 2q) + 1$$

$$2p^2 + 2q^2 + 2q \in \mathbb{Z}$$

$$\Rightarrow 2(2p^2 + 2q^2 + 2q) + 1 \text{ is odd}$$

z is odd

z cannot be even

Question 2

Draw perpendicular line of a line at a given point A:

Use compass to draw two arcs by center A intersecting the line, the two intersections are marked as B and C. At B and C, use compass to draw one arc by each B and C as the center, make sure the two circles containing the two arcs have the same radius and the two arcs intersects at one point, marking D, then draw a line crossing A and D, line AD is the perpendicular line of the orignal line at A.

Draw parallel line of a line at a given point A:

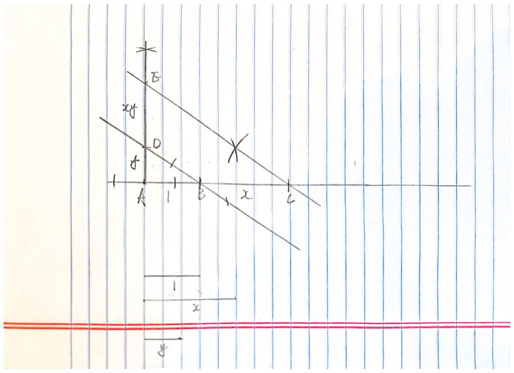
Use compass to draw an arc by center A intersecting the line, marking B. Then use the same distance on the compass to draw an arc by center B intersecting the line at C, note that $\angle ABC > 90°$. Then use the same distance on the compass to draw an arc by center C intersecting the arc centered by A, marking the intersection D. Then draw a line crossing A and D, then line AD is parallel to BC, the original line.

a

Draw a line on the paper, and select a point A on the line. Use the compass to record the length x, then use A as the start point and mark the endpoint of the compass as B, then the segment AB has length x. Use the compass to record the length y, then use B as the start point and mark the endpoint of the compass in the same direction of $\overrightarrow{AB}$ as C, then the segment BC has length y and segment AC has length $x + y$.

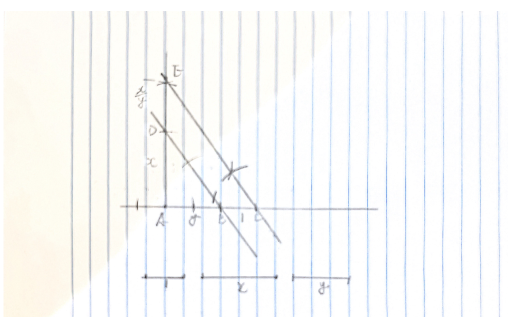
Draw a line on the paper, and select a point A on the line. Use the compass to record the length x, then use A as the start point and mark the endpoint of the compass as B, then the segment AB has length x. Use the compass to record the length y, then use B as the start point and mark the endpoint of the compass in the opposite direction of $\overrightarrow{AB}$ as C, then the segment BC has length y and segment AC has length $x - y$.

b



Draw a line on the paper, and select a point A on the line. Use the compass to record the length 1 and then use A as the start point and mark the endpoint of the compass as B, then the segment AB has length 1. Use the compass to record the length x, then use B as the start point and mark the endpoint of the compass in the same direction of $\overrightarrow{AB}$ as C, then the segment BC has length x. Draw a perpendicular line of the line AC at point A. Use the compass to record the length y, then use A as the start point and mark the endpoint of the compass on the perpendicular line as D, then the segment AD has length y and connect BD. At point C, draw a line parallel to BD, and intersects line AD at E, then line segment DE has length xy.

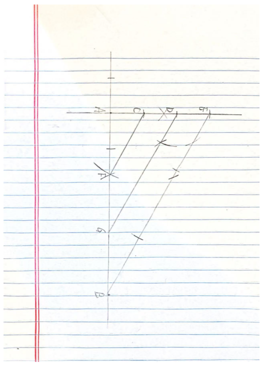
c



Draw a line on the paper, and select a point A on the line. Use the compass to record the length y and then use A as the start point and mark the endpoint of the compass as B, then the segment AB has length y. Use the compass to record the length 1, then use B as the start point and mark the endpoint of the compass in the same direction of $\overrightarrow{AB}$ as C, then the segment BC has length 1. Draw a perpendicular line of the line AC at point A. Use the compass to record the length x, then use A as the start point and mark the endpoint of the compass on the perpendicular line as D, then the segment AD has length x and connect BD. At point C, draw a line parallel to BD, and intersects line AD at

E, then line segment DE has length $\frac{x}{y}$.

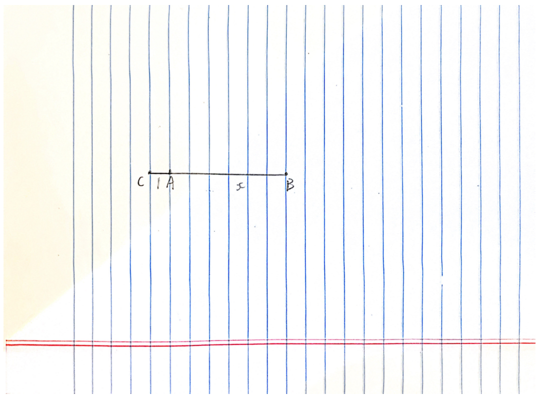
d



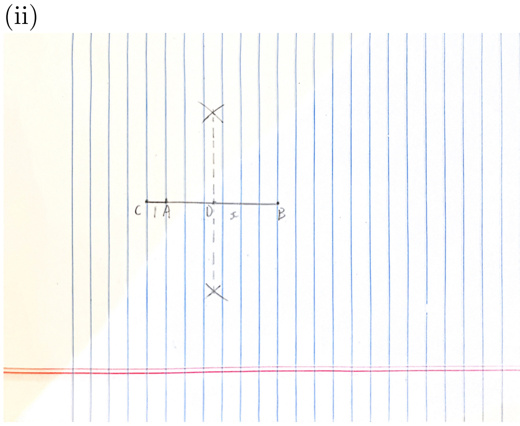
To triply divide a line segment of length x, draw a line on the paper, and select a point A on the line. Use the compass to record the length x and then use A as the start point and mark the endpoint of the compass as B, then the segment AB has length x. Draw a perpendicular line of the line AB at point A. Use any length to draw a line segment starting at A on the perpendicular line, marking the endpoint C, and draw two more line segments with the same length as line segment AC on line AC, marking the two endpoints D and E. Make sure that $AC = CD = DE$ and $\overrightarrow{AC}$, $\overrightarrow{CD}$, $\overrightarrow{DE}$ have the same direction. Connect BE, and draw two lines parallel to BE at C and D, intersecting AB at F, G separately. Then $AF = FG = GB = \frac{1}{3}AB$.

Question 3

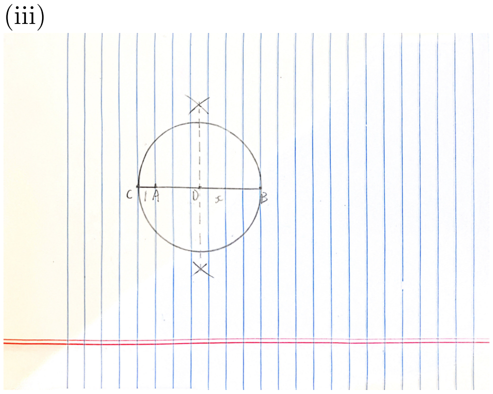
(i)



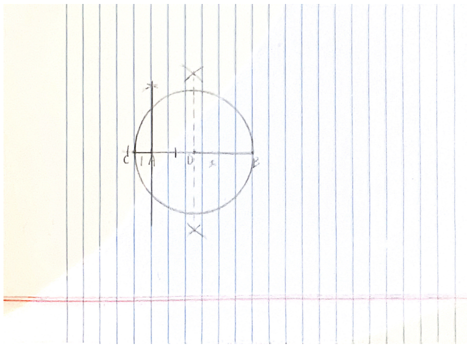
(ii)



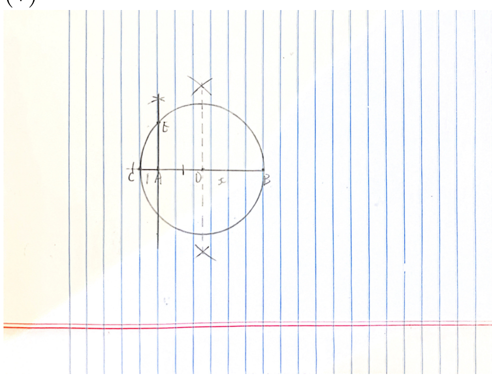
(iii)



(iv)



(v)



Proof.

$$
\begin{aligned}
&\text{Connects } OE \\
\Rightarrow &OE = \frac{x+1}{2} \\
&OA = OC - AC = \frac{x+1}{2} - 1 = \frac{x-1}{2} \\
&AE \perp BC \\
\Rightarrow &\Delta OAE \text{ is a right triangle} \\
&OA^2 + AE^2 = OE^2 \\
&\left(\frac{x-1}{2}\right)^2 + AE^2 = \left(\frac{x+1}{2}\right)^2 \\
&\frac{x^2-2x+1}{4} + AE^2 = \frac{x^2+2x+1}{4} \\
&AE^2 = x \\
&AE = \sqrt{x} \quad \text{(negative root is meaningless)}
\end{aligned}
$$

□

Question 4

a

i

Proof.

$$
\begin{aligned}
&\Delta ABC \text{ is isosceles by } AB = AC \\
\Rightarrow &\angle ACB = \angle ABC \\
&\angle BAC = 36^\circ \wedge \angle ACB + \angle ABC + \angle BAC = 180^\circ \\
\Rightarrow &\angle ACB = \angle ABC = 72^\circ \\
&BE \text{ bisects } \angle ABC \\
\Rightarrow &\angle CBE = 36^\circ \\
&\angle BEC = 72^\circ \\
\Rightarrow &\Delta BCE \text{ is isosceles by } BC = BE \\
&\Delta ABC \text{ is similar to } \Delta BCE \\
\Rightarrow &\frac{CE}{BC} = \frac{BC}{AC} \\
&BE \text{ bisects } \angle ABC \\
&\angle ABE = 36^\circ \\
\Rightarrow &\angle ABE = \angle BAE \\
\Rightarrow &\Delta ABE \text{ is isosceles by } AE = BE \\
&BC = AE \\
\Rightarrow &\frac{CE}{AE} = \frac{AE}{AC} \\
&CE : AE = AE : AC
\end{aligned}
$$

□

ii

$$
\begin{aligned}
&BC = AE \coloneqq x\\
&CE = 1 - x\\
\Rightarrow &\frac{1-x}{x} = \frac{x}{1}\\
&x^2 + x - 1 = 0\\
&x = \frac{\sqrt{5}-1}{2}\text{(the negative value is meaningless)}\\
\Rightarrow &BC = \frac{\sqrt{5}-1}{2}
\end{aligned}
$$

iii

By connecting every vertex of the decagon to the center, there are 10 congruent isosceles triangles, making the acutest angle in the triangles 36°. So every triangle is similar to ΔABC. And with the radius 1, the ten triangles are all congruent to ΔABC, so the length of the side of the decagon should be the same as BC, $\frac{\sqrt{5}-1}{2}$.

b

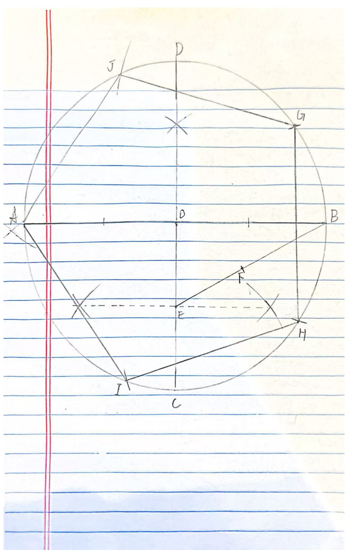
i

$$
\begin{aligned}
&\text{By construction :}\\
&\angle EOB = 90^\circ\\
&OE = EF = \frac{1}{2}\\
&OB = 1\\
&\text{Since } \Delta BOE \text{ is a right triangle}\\
&BE = \sqrt{OE^2 + OB^2} = \frac{\sqrt{5}}{2}\\
&BF = BE - FE = \frac{\sqrt{5}-1}{2}
\end{aligned}
$$

ii

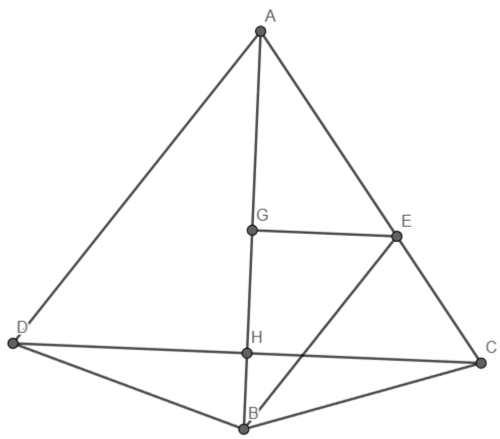
Connect BG, $BG = BF = \frac{\sqrt{5}-1}{2}$, and $OB = OG = 1$, so ΔBOG is congruent to the triangle ABC in part a. So $\angle BOG = \angle BAC = 36°$

iii



Use the steps up to ii to get G, then mark the other intersection of the circle O and circle B as H. Then draw an arc with center H and radius GH that intersects the circle with I different than G. Draw an arc with center G and radius HG that intersects the circle with J different than H. Then connect AI, IH, HG, GJ, JA. The pentagon $AIHGJ$ is a regular pentagon.

c



ΔABC is the same triangle described in 4(a). ΔABD is ΔABC reflected along AB. So $\angle CAD = 72^\circ$ and $AD = AC$, making CD the length of a regular pentagon inscribed in circle of radius 1. Since the coefficient given by the radius of the circle can be offset, only considering the circle with radius of 1 is efficient for the proof

Proof.

$$AB = AC = 1$$

$$AE = BE = BC = \frac{\sqrt{5}-1}{2}$$

EG is perpendicular to AB

$$\Rightarrow BG = \frac{1}{2}AB = 0.5$$

$$CH = \sin 36^\circ \cdot AC = \sin 36^\circ$$

$$\sin 36^\circ = \frac{GE}{BE} = \frac{\sqrt{BE^2 - BG^2}}{BE}$$

$$= \frac{\sqrt{\left(\frac{\sqrt{5}-1}{2}\right)^2 - \frac{1}{4}}}{\frac{\sqrt{5}-1}{2}}$$

$$= \frac{\sqrt{\frac{5-2\sqrt{5}}{4}}}{\frac{\sqrt{5}-1}{2}}$$

$$= \frac{\sqrt{5-2\sqrt{5}}}{\sqrt{5}-1}$$

$$= \frac{\sqrt{5-2\sqrt{5}}(\sqrt{5}+1)}{4}$$

$$= \frac{\sqrt{10-2\sqrt{5}}}{4}$$

$$\Rightarrow CH = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

$$AD = AC \wedge \angle CAH = \frac{1}{2}\angle CAD$$

$$\Rightarrow CD = 2CH = \frac{\sqrt{10-2\sqrt{5}}}{2}$$

Side of a regular pentagon inscribed in circle of radius 1 is $\frac{\sqrt{10-2\sqrt{5}}}{2}$

hexagon :

the central angle formed by two near vertices is 60°

$\Rightarrow$The triangle formed by two near vertices and the center is equilateral

$\Rightarrow$Side length of regular hexagon inscribed in circle of radius 1 is 1

Side length of regular decagon inscribed in circle of radius 1 is $\frac{\sqrt{5}-1}{2}$

$$S_{pentagon} = \frac{\sqrt{10-2\sqrt{5}}}{2}$$

$$S_{hexagon} = 1$$

$$S_{decagon} = \frac{\sqrt{5}-1}{2}$$

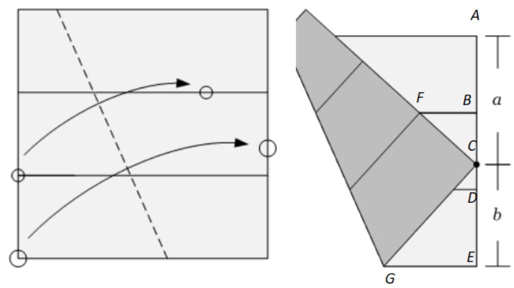
$$S_{decagon}{}^2 + S_{hexagon}{}^2 = \left(\frac{\sqrt{5}-1}{2}\right)^2 + 1^2 = \frac{5-\sqrt{5}}{2}$$

$$S_{pentagon}{}^2 = \left(\frac{\sqrt{10-2\sqrt{5}}}{2}\right)^2 = \frac{5-\sqrt{5}}{2}$$

$$\Rightarrow S_{decagon}{}^2 + S_{hexagon}{}^2 = S_{pentagon}{}^2$$

$\square$

Question 5



Proof.

$$
\begin{aligned}
& EG \coloneqq x \\
& CG + EG = a + 1 \\
\Rightarrow & CG = a + 1 - x \\
& \Delta CEG \text{ is right triangle} \\
\Rightarrow & x^2 + 1^2 = (a + 1 - x)^2 \\
& x^2 + 1 = 1 + 2a + a^2 - 2x - 2ax + x^2 \\
& (2a + 2)x = a^2 + 2a \\
& x = \frac{a^2 + 2a}{2a + 2} \\
\Rightarrow & EG = \frac{a^2 + 2a}{2a + 2} \\
& BC = a - \frac{1}{3}(a + 1) = \frac{2}{3}a - \frac{1}{3} \\
& CF = \frac{1}{3}(a + 1) \\
& \Delta BCF \text{ is right triangle} \\
\Rightarrow & BF = \sqrt{\left(\frac{1}{3}(a + 1)\right)^2 - \left(\frac{2}{3}a - \frac{1}{3}\right)^2} \\
& \ = \frac{\sqrt{6a - 3a^2}}{3} \\
& \angle BCF + \angle GCE + 90^\circ = 180^\circ \\
& \angle CGE + \angle GCE + 90^\circ = 180^\circ \\
\Rightarrow & \angle BCF = \angle CGE \\
& \angle CBF = \angle GEC = 90^\circ \\
& \text{So } \Delta BCF \text{ and } \Delta EGC \text{ are similar}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow &\frac{EG}{CE} = \frac{BC}{BF} \\
&\frac{a^2+2a}{2a+2} = \frac{\frac{2}{3}a - \frac{1}{3}}{\frac{\sqrt{6a-3a^2}}{3}} \\
&(a^2+2a)\sqrt{6a-3a^2} = (2a+2)(2a-1) \\
&(a^2+2a)^2(6a-3a^2) = ((2a+2)(2a-1))^2 \\
&-3a^6 - 6a^5 + 12a^4 + 24a^3 = 16a^4 + 16a^3 - 12a^2 - 8a - 4 \\
&3a^6 + 6a^5 + 4a^4 - 8a^3 - 12a^2 - 8a + 4 = 0 \\
&(a^3-2)(3a^3+6a^2+4a-2) = 0 \\
&a = \sqrt[3]{2} \\
\Rightarrow &\frac{a}{b} = \sqrt[3]{2}
\end{aligned}
$$

$\square$