Book

Euclid's Elements, Book I

Problem 1

(a)

Theorem 1. *Let AB and CD be two line segments of equal length. If AB is closer to the eye at point O than CD, then the angle subtended by AB at O is larger than the angle subtended by CD at O by Assumption (iv).*

Proof. 1. Let O be the position of the eye. Let AB and CD be two line segments of equal length, with AB closer to O than CD.

2. Connect OA, OB, OC, and OD.

3. Since AB is closer to O, the distances OA and OB are shorter than OC and OD, respectively.

4. The angle subtended by AB at O is $\angle AOB$, and the angle subtended by CD at O is $\angle COD$.

5. By Sine Theorem, $\sin \angle AOB > \sin \angle COD$, therefore $\angle AOB > \angle COD$.

6. Therefore, AB appears larger than CD when viewed from O by Assumption (iv). □

(b)

Theorem 2. *Let l and m be two parallel lines. When viewed from a point O not on either line, the distance between l and m appears to decrease as the lines extend further from O.*

Proof. 1. Let l and m be two parallel lines. Let O be the position of the eye, not on either line.

2. Draw visual rays from O to points A and B on l and m, respectively, noting that A O B should not be collinear.

3. As A and B move further away from O along l and m, the angle $\angle AOB$ decreases.

4. The apparent distance between l and m is related to the angle $\angle AOB$. As $\angle AOB$ decreases, the lines l and m appear to converge, making the distance between them appear smaller, as proved in (a).

5. Therefore, parallel lines appear to not be equally distant from each other when viewed from a distance. □

Problem 2

(a)

Theorem 3. *In a quadrilateral $ABCD$, if $\angle DAB$ and $\angle ABC$ are both right angles, and $BC = AD$, then $\angle BCD = \angle CDA$.*

Proof.

1. Consider quadrilateral $ABCD$, where $\angle DAB = \angle ABC = 90°$ and $BC = AD$.
2. find a point E on CD such that $AE = BE$.
3. So $\angle BAE = \angle ABE$.
4. $\angle DAE = \angle BAD - \angle BAE = \angle ABC - \angle ABE = \angle CBE$.
5. Triangles ADE and BCE are congruent by SAS:
    - $BC = AD$
    - $\angle DAE = \angle CBE$
    - $AE = BE$.
6. Since $\triangle ADE \cong \triangle BCE$, their corresponding angles are equal. Therefore, $\angle BCD = \angle CDA$.
7. Thus, $\angle BCD = \angle CDA$.

□

(b)

Theorem 4. *Let E be the midpoint of AB and F be the midpoint of CD. Then EF is perpendicular to both AB and CD.*

Proof.

1. Connect CE, DE, AF, BF,
2. Triangles ADE and BCE are congruent by SAS:
    - $BC = AD$
    - $\angle DAE = \angle CBE$

- $AE = BE$.

3. Since $\triangle ADE \cong \triangle BCE$, $DE = CE$ and $\angle AED = \angle BEC$.
4. Triangles DEF and CEF are congruent by SSS:
   - $DE = CE$
   - $DF = CF$ since F is the midpoint of CD
   - EF is the common side.
5. Since $\triangle DEF \cong \triangle CEF$, $\angle DFE = \angle CFE$ and $\angle FED = \angle FEC$.
6. $\angle DFE = \angle CFE$ and $\angle DFE + \angle CFE = 180°$
7. So $\angle DFE = \angle CFE = 90°$, and $EF \perp CD$
8. Since $\angle FED = \angle FEC$ and $\angle AED = \angle BEC$, $\angle FED + \angle AED = \angle AEF = \angle FEC + \angle BEC = \angle BEF$.
9. Since $\angle FEA = \angle FEB$ and $\angle FEA + \angle FEB = 180°$
10. So $\angle FEA = \angle FEB = 90°$, and $EF \perp AB$
11. Conclusion: $EF \perp AB$ and $EF \perp CD$

□

(c)

1. *Proof.* Triangles ADE and BCE are congruent by SAS:
   - $UX = YX$
   - $\angle UXV = \angle YXW$ since they are opposite angles
   - $VX = WX$ since X is the midpoint of VW

   □

2. (a) connect UZ.
   (b) find a point M on $\overline{UZ}$ such that $UZ = ZM$.
   (c) connect MW
   (d) like before as proved, $\triangle UZX \cong \triangle MZW$.
   (e) As $\angle XUW$ is divided into 2 angles, and one is at most $\frac{1}{2}\angle XUW$
   (f) Without loss of generality, suppose $\angle MUW \leq \frac{1}{2}\angle XUW \leq \frac{1}{4}\angle VUW$
   (g) By this repetition, there must be an angle that is $\frac{1}{2^n}\angle VUW < \alpha$

(d)

Theorem 5. *In quadrilateral $ABCD$, the two undetermined angles $\angle BCD$ and $\angle CDA$ cannot be obtuse.*

Proof.

1. From part (c), the angle sum of a triangle cannot exceed $180°$.
2. In quadrilateral $ABCD$, the sum of the angles cannot exceed $360°$. Since $\angle DAB = \angle ABC = 90°$, the sum of $\angle BCD$ and $\angle CDA$ cannot exceed $180°$.
3. Therefore, since $\angle BCD = \angle CDA$, neither $\angle BCD$ nor $\angle CDA$ can be obtuse, as their sum would exceed $180°$.

□

(e)

Proof.

1. Assume the undetermined angles $\angle BCD$ and $\angle CDA$ are acute.
2. In quadrilateral $FEBC$, $\angle BEF = \angle EFC = 90°$ and $\angle EBC > \angle BCF$, by the Lemma, $BE < CF$.
3. E, F are the midpoints, so $\frac{1}{2}AB < \frac{1}{2}CD$, leading to $AB < CD$.
4. Again, consider quadrilateral $FEBC$, $\angle BEF = \angle EBC = 90°$ and $\angle EFC > \angle BCF$, by the Lemma, $EF < BC$.
5. Since $AD = BC$, $EF < BC$ and $EF < AD$.

□

(f)

In hyperbolic geometry, parallel lines “diverge” as they extend further from the observer. This means that the distance between them increases, and they do not remain equidistant.

Problem 3

(a)

Proof.

1. Apply the Euclidean algorithm:

$$\begin{aligned} 32830 &= 3 \times 9147 + 5389 \\ 9147 &= 1 \times 5389 + 3758 \\ 5389 &= 1 \times 3758 + 1631 \\ 3758 &= 2 \times 1631 + 496 \\ 1631 &= 3 \times 496 + 143 \\ 496 &= 3 \times 143 + 67 \\ 143 &= 2 \times 67 + 9 \\ 67 &= 7 \times 9 + 4 \\ 9 &= 2 \times 4 + 1 \\ 4 &= 4 \times 1 + 0 \end{aligned}$$

2. $\gcd(32830, 9147) = 1$.
3. The coefficient in the continued fraction expansion are the quotients in the Euclidean algorithm in the same order.

□

(b)

Proof.

1. Apply the Euclidean algorithm:

$$\begin{aligned} 223 &= 3 \times 71 + 10 \\ 71 &= 7 \times 10 + 1 \\ 10 &= 10 \times 1 + 0 \end{aligned}$$

The continued fraction expansion is $[3; 7, 10]$.

□

(c)

Proof.

1. Base Case: The fraction can be calculated to an integer, then the continued fraction expansion and the Euclidean algorithm both have one step, which is the integer as the quotient.

2. Induction Hypothesis: For all continued fraction expansion of length $\leq k$, there are k steps in the Euclidean algorithm, with each quotient the same to the coeffient in the expansion with the same order.

3. Induction Step: suppose the final expansion is in the form $a_{k-1}+\frac{b_k}{c_k}$, which a_{k-1} is the quotient, c_k is the divisor, and b_k is the remainder in the kth step in the Euclidean algorithm. $\frac{b}{c_k} = \frac{1}{\frac{c_k}{b_k}} = \frac{1}{a_{k+1}+\frac{b_{k+1}}{b_k}}$. The $k+1$th step of the Euclidean algorithm is $c_k = a_{k+1}b_k + b_{k+1}$, which the quotient of $k+1$th step is a_{k+1}, the same as the coefficient of the $k+1$th coefficient of the fraction expansion.

4. Induction Hypothesis is proven, and the relationship is proven.

□

Problem 4

Theorem 6. *Proclus's assumption is equivalent to Euclid's Parallel Postulate.*

Proof.

1. **Proclus's Assumption Implies the Parallel Postulate:**
   (a) If a straight line intersects one of two parallel lines, it must intersect the other.
   (b) The interior angles of parallel lines are $180°$, now with one line intersect with one of the parallel lines, the interior angle of the line and the other parallel line is less than $180°$
   (c) Since the assumption states that they must intersect, two lines that the interior angle of the two is less than $180°$ must intersect, which is the Parallel Postulate.
   (d) Proclus's Assumption can imply the Parallel Postulate.
2. **Parallel Postulate Implies Proclus's Assumption:**
   (a) If the interior angles on one side of a transversal sum to less than $180°$, the lines must intersect.
   (b) For these two lines, pick a point on one line to create a line parallel to the other, so the line intersects one of the parallel lines.
   (c) As Parallel Postulate states, the lines must intersect, so the line intersects one parallel line must intersects the other parallel line, which is Proclus's Assumption suggests.
   (d) Parallel Postulate can imply Proclus's Assumption.
3. **Conclusion:**
   Proclus's assumption and the Parallel Postulate are equivalent.

□

Question 5

OK.