Problem 1

Proof.

1. By Eudoxus's definition of equality of ratios, $a : b = c : d \iff (\forall m, n \in \mathbb{Z}, ma >=< nb \iff mc >=< nd)$
2. $\forall m, n \in \mathbb{Z}, ma >=< nc \iff mab >=< ncb \iff (mb)a >=< (nc)b$
3. So by transitivity of $\iff$: $a : b = c : d \iff (\forall m, n \in \mathbb{Z}, (mb)a >=< (nc)b \iff (mb)c >=< (nc)d \iff (mc)b >=< (nc)d) \iff mb >=< md)$
4. So by transitivity of $\iff$: $a : b = c : d \iff (\forall m, n \in \mathbb{Z}, ma >=< nc \iff mb >=< md)$
5. By Eudoxus's definition of equality of ratios, $a : c = b : d \iff (\forall m, n \in \mathbb{Z}, ma >=< nc \iff mb >=< md)$
6. By transitivity of $\iff$: $a : b = c : d \iff a : c = b : d$

$\square$

Problem 2

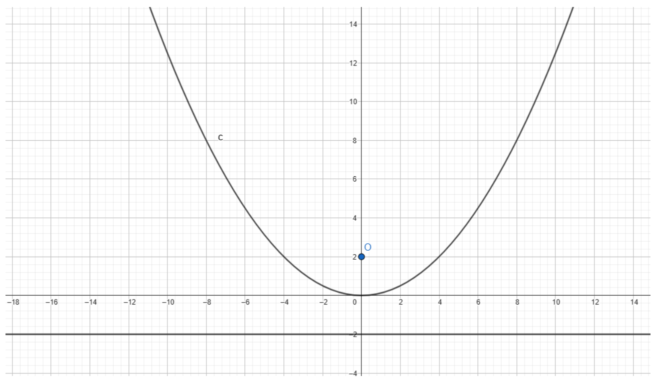
Proof.

1. Connect every vertex with the center of the circle, so there are n congruent triangles (SSS).
2. The central angle is $\frac{2\pi}{n}$.
3. Draw a perpendicular line from the center to the other side, since it is an isosceles triangle, the line also bisects the central angle and the side.
4. Since the perpendicular line is also the radius, half of the side is $r \tan \frac{\pi}{n}$, the side is $2r \tan \frac{\pi}{n}$.
5. The area of the triangle is $r^2 \tan \frac{\pi}{n}$.
6. The polygon has n congruent triangles, so the area of the polygon is $nr^2 \tan \frac{\pi}{n}$.
7. $\lim_{n\to\infty}(nr^2 \tan \frac{\pi}{n}) = nr^2 \lim_{n\to\infty}(\tan \frac{\pi}{n}) = nr^2 \frac{\pi}{n} = \pi r^2$
8. As n tends to infinity, a regular polygon becomes a circle, so the area of a circle is πr^2.

$\square$

Problem 3

a



$$O \text{ is the focus} : (0, \frac{p}{2})$$

$$l \text{ is the directrix} : y = \frac{-}{p}2$$

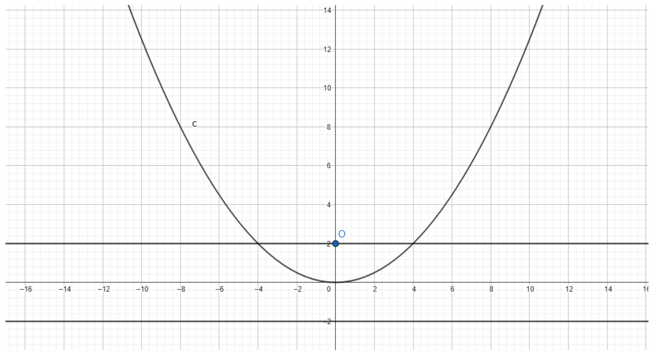
$$D_{OP} = D_{lP}$$

$$\sqrt{x^2 + (y - \frac{p}{2})^2} = y + \frac{p}{2}$$

$$x^2 + (y - \frac{p}{2})^2 = (y + \frac{p}{2})^2$$

$$x^2 = 2yp$$

b



$$\ell : y = \frac{p}{2}$$

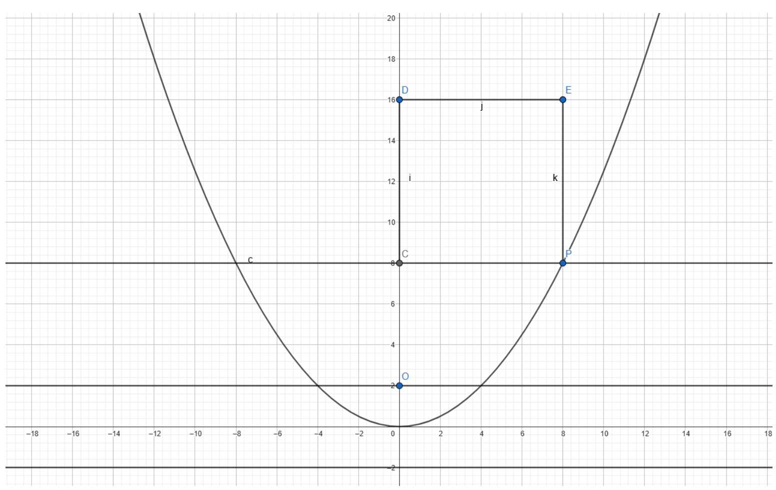
$$x^2 = 2\frac{p}{2}p$$

$$x^2 = p^2$$

$$x = \pm p$$

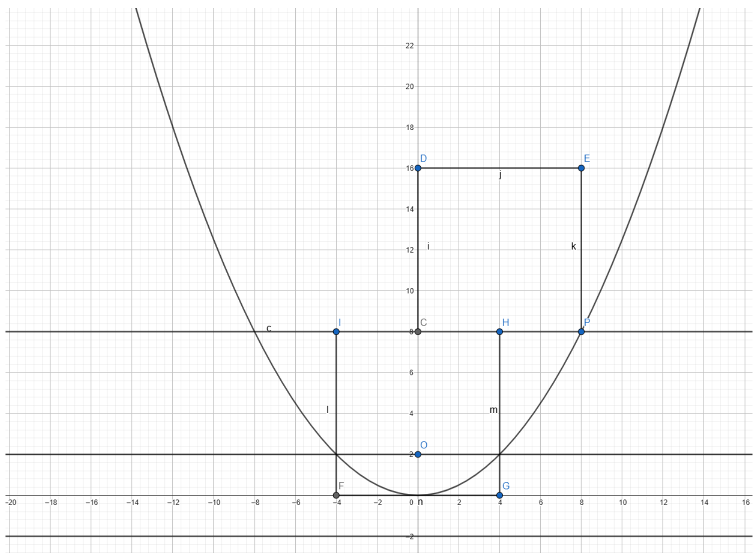
$$\Rightarrow D = 2p$$

c



$$
\begin{aligned}
&\text{perpendicular is } y\text{-axis} \\
\Rightarrow &D_P = |x| \\
&A_{PCDE} = |x|^2 = x^2
\end{aligned}
$$

d



$$
\begin{aligned}
&D_\ell = 2p \\
&H = y \\
\Rightarrow &A_{FGHI} = 2py \\
&x^2 = 2py \\
\Rightarrow &A_{FGHI} = A_{PCDE}
\end{aligned}
$$

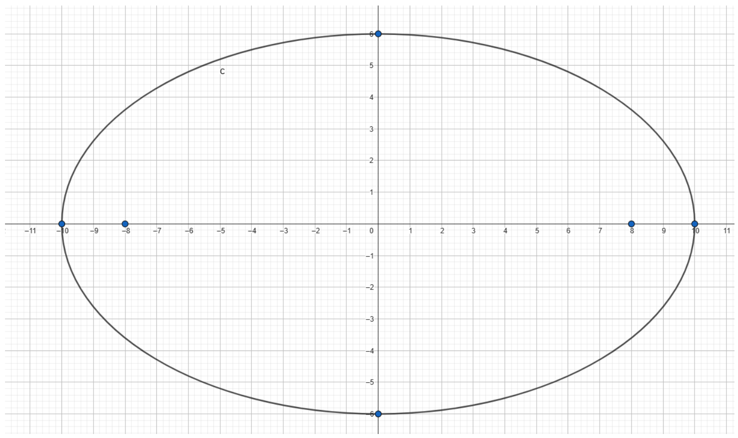
Problem 4

Proof.

1. $y = x^2$, so $2p = 1$, $p = \frac{1}{2}$, so focus is $(0, \frac{1}{2}p) = (0, \frac{1}{4})$
2. Directrix is $y = -\frac{1}{4}$
3. Suppose $A : (a, a^2)$, then tangent of $y = x^2$ at A is $y - a^2 = \frac{dy}{dx}(x - a) \implies y = 2ax - a^2$
4. Since perpendicular to directrix is vertical, the incoming angle is $\theta = \arctan(\frac{1}{2a})$.
5. By reflection law, reflecting angle is also $\theta = \arctan(\frac{1}{2a})$.
6. So the angle between the ray through focus and y-axis is $2\arctan(\frac{1}{2a})$
7. $\tan(2\arctan(\dfrac{1}{2a})) = \dfrac{2\tan(\arctan(\dfrac{1}{2a}))}{1 - \tan^2(\arctan(\dfrac{1}{2a}))} = \dfrac{a}{a^2 - \frac{1}{4}}$
8. So the equation for reflecting ray is $y - a^2 = \dfrac{a^2 - \frac{1}{4}}{a}(x - a)$
9. Substituting $y = \frac{1}{4}, x = 0$:
   (a) $LHS = \dfrac{1}{4} - a^2$
   (b) $RHS = \dfrac{a^2 - \frac{1}{4}}{a} \cdot (-a) = \frac{1}{4} - a^2$
   (c) $LHS = RHS$
10. The focus lands on the ray.
11. All rays perpendicular to the directrix reflect through the focus.

$\square$

Problem 5



a

1. The sum of the distances from any point on the ellipse to the foci is $2a$.
2. At $(0, b)$, the sum of the distances are $\sqrt{(b-0)^2 + (0-c)^2} + \sqrt{(b-0)^2 + (0+c)^2} = 2\sqrt{b^2 + c^2}$.
3. $2\sqrt{b^2 + c^2} = 2a \implies a^2 = b^2 + c^2$

b

1. Since the ellipse is symmetry, the two latus recta have the same length.
2. Substitute $x = c$ into ellipse equation:
   (a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$
   (b) $\dfrac{c^2}{a^2} + \dfrac{y^2}{b^2} = 1$
3. Substitute $c^2 = a^2 - b^2$:
   (a) $\dfrac{a^2 - b^2}{a^2} + \dfrac{y^2}{b^2} = 1$
   (b) $\dfrac{y^2}{b^2} = \dfrac{b^2}{a^2}$

(c) $y = \pm\frac{b^2}{a}$

4. $\ell = \frac{b^2}{a} - (-\frac{b^2}{a}) = \frac{2b^2}{a}$

c

Proof.

1. WLOG: Suppose $x \in [0, a)$ so that the distance to the nearest vertex is $a - x$, in $x < 0$ it is $a + x$
2. From ellipse equation:
   (a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
   (b) $y^2 = b^2(1 - \frac{x^2}{a^2})$
3. $\ell(a - x) = \frac{2b^2(a - x)}{a}$
4. $\frac{\ell(a - x)}{y^2}$:

$$\begin{aligned}
&\frac{\ell(a-x)}{y^2} \\
=&\frac{\frac{2b^2(a-x)}{a}}{b^2(1-\frac{x^2}{a^2})} \\
=&\frac{2a(a-x)}{a^2-x^2} \\
=&\frac{2a}{a+x}
\end{aligned}$$

5. $x \in [0, a)$, so $\frac{2a}{a+x} > 1$
6. So $y^2 < \ell(a - x)$

□