Question 1

a

i

Proof.

1. $\exists x, y \in \mathbb{Z} : ax + by = c$
2. $a = \gcd(a, b) \cdot \dfrac{a}{\gcd(a, b)}$ and $b = \gcd(a, b) \cdot \dfrac{b}{\gcd(a, b)}$
3. Substituting a and b into the equation:
   (a) $\gcd(a, b) \cdot \dfrac{a}{\gcd(a, b)} \cdot x + \gcd(a, b) \cdot \dfrac{b}{\gcd(a, b)} \cdot y = c$
   (b) $c = \gcd(a, b) \left(\dfrac{a}{\gcd(a, b)} \cdot x + \dfrac{b}{\gcd(a, b)} \cdot y \right)$
4. Since $x, y \in \mathbb{Z}$, $\dfrac{a}{\gcd(a, b)} \cdot x + \dfrac{b}{\gcd(a, b)} \cdot y \in \mathbb{Z}$
5. So $\gcd(a, b) | c$

$\square$

ii

Proof.

1. $ax_1 + by_1 = c$
2. Substituting x_1 to $x_1 + mb$ and y_1 to $y_1 - ma$:
   (a) $LHS = a(x_1 + mb) + b(y_1 - ma) = ax_1 + mab + by_1 - mab = ax_1 + by_1$
   (b) $ax_1 + by_1 = c$
3. So $a(x_1 + mb) + b(y_1 - ma) = c$, and $(x_1 + mb, y_1 - ma)$ is a solution to the equation.

4. if (x_1, y_1) is a solution, (x_1+mb, y_1-ma) is also a solution to the equation.

$\square$

b

1. Change the words into mathematical language: find x: $63x + 7 \equiv 0 \pmod{23}$
2. $63 = 23 \times 2 + 17$
3. So the equation can be reduced to $17x + 7 \equiv 0 \pmod{23}$
4. $17x \equiv -7 \pmod{23}$
5. $-7 \mod 23 = 16 \pmod{23}$
6. So $17x \equiv 16 \pmod{23}$
7. $17^{-1} = \frac{1}{17} \equiv \frac{1}{-6} \equiv \frac{24}{-6} = -4 \equiv 19 \pmod{23}$
8. So $17^{-1} \equiv 19 \pmod{23}$
9. Multiplying 19 on both sides: $x \equiv 16 \times 19 = 304 \equiv 5 \pmod{23}$
10. So the smallest solution is 5

Question 2

a

1. (a) $N = 24a + 6$
   (b) $N = 98b + 18$
2. So $24a - 98b = 12$, $12a - 49b = 6$
3. $49 - 12 \times 4 = 1$
4. So $12 \times -24 - 49 \times -6 = 6$
5. $(-24, -6)$ is a solution
6. by 1 a ii,$(-24 + 49, -6 + 12) = (25, 6)$ is a solution.
7. $N = 24 \times 25 + 6 = 606 = 98 \times 6 + 18$
8. $N = 606$ is the smallest positive solution.

b

1. (a) $N = 11a + 2$
   (b) $N = 13b + 3$
   (c) $N = 18c + 5$
2. Use the first two equations:
   (a) $11a - 13b = 1$
   (b) $11 \times 6 - 13 \times 5 = 1$
   (c) $(6, 5)$ is a solution
   (d) $(6 + 13d, 5 + 11d)$ is also a solution
   (e) $N = 11(6 + 13d) + 2 = 143d + 68$
3. Use the third equation:
   (a) $143d - 18c = -63$
   (b) $143d \equiv -63 \pmod{18}$
   (c) $17n \equiv 9 \pmod{18}$

(d) $17^{-1} \equiv 17 \pmod{18}$

(e) $d \equiv 9 \times 17 = 153 \equiv 9 \pmod{18}$

(f) $d = 9 \implies c = 75$

(g) $(9, 75)$ is a solution, which is also the smallest positive solution for N since all the other smallers go to negative.

4. $N = 143 \times 9 + 68 = 1355 = 18 \times 75 + 5$

5. $N = 1355$ is the smallest positive solution.

Question 3

a

Proof.

1. case 1 (v is even):

   (a) $u_1 = \dfrac{1}{2}uv(v^2+1)(v^2+3)$:

      i. $u \in \mathbb{Z}$;

      ii. $\frac{1}{2}v \in \mathbb{Z}$ since v is even;

      iii. $v^2+1 \in \mathbb{Z}$;

      iv. $v^2+3 \in \mathbb{Z}$;

      v. So $u_1 \in \mathbb{Z}$.

   (b) $v_1 = (v^2+2)\left[\dfrac{1}{2}(v^2+1)(v^2+3)-1\right]$:

      i. $v_1 = \dfrac{1}{2}(v^2+1)(v^2+3)(v^2+2)-(v^2+2)$

      ii. v^2+2 is even, so $\dfrac{1}{2}(v^2+1)(v^2+3)(v^2+2) \in \mathbb{Z}$;

      iii. $v^2+2 \in \mathbb{Z}$;

      iv. So $v_1 \in \mathbb{Z}$.

   (c) So $u_1, v_1 \in \mathbb{Z}$

2. case 2 (v is odd):

   (a) $u_1 = \dfrac{1}{2}uv(v^2+1)(v^2+3)$:

      i. $u \in \mathbb{Z}$;

      ii. $v \in \mathbb{Z}$;

      iii. v is odd, so v^2+1 is even, $\dfrac{1}{2}(v^2+1) \in \mathbb{Z}$;

      iv. $v^2+3 \in \mathbb{Z}$;

      v. So $u_1 \in \mathbb{Z}$.

   (b) $v_1 = (v^2+2)\left[\dfrac{1}{2}(v^2+1)(v^2+3)-1\right]$:

      i. v is odd, so v^2+1 is even, $\dfrac{1}{2}(v^2+1) \in \mathbb{Z}$;

      ii. $v^2+3 \in \mathbb{Z}$;

      iii. So $\dfrac{1}{2}(v^2+1)(v^2+3)-1 \in \mathbb{Z}$;

iv. $v^2 + 2 \in \mathbb{Z}$;

v. So $v_1 \in \mathbb{Z}$.

(c) So $u_1, v_1 \in \mathbb{Z}$

3. So (u_1, v_1) is a pair of integers.

$\square$

b

1. test for small x:
2. $x = 1$:
   (a) $13 - 4 = y^2$
   (b) $y = 3$
   (c) $x = 1$ stands.
3. $(u, v) = (1, 3)$

c

1. $(u, v) = (1, 3)$
2. $u_1 = \frac{1}{2}uv(v^2 + 1)(v^2 + 3) = \frac{1}{2} \cdot 1 \cdot 3(3^2 + 1)(3^2 + 3) = 180$
3. $v_1 = (v^2+2)\left[\frac{1}{2}(v^2 + 1)(v^2 + 3) - 1\right] = (3^2+2)\left[\frac{1}{2}(3^2 + 1)(3^2 + 3) - 1\right] =$ 649
4. So $(u_1, v_1) = (180, 649)$ is a solution to the equation.

Question 4

Proof.

1. Suppose $\square ABCD$ which diagonals intersect at O
2. $AC \perp BD \implies AB^2 + CD^2 = AD^2 + BC^2$:
   (a) $AC \perp BD$;
   (b) So $\angle AOB = \angle BOC = \angle COD = \angle DOA = 90°$;
   (c) By Pythagorean Theorem:
      i. $AB^2 = AO^2 + BO^2$
      ii. $BC^2 = BO^2 + CO^2$
      iii. $CD^2 = CO^2 + DO^2$
      iv. $DA^2 = DO^2 + AO^2$
   (d) $AB^2 + CD^2 = AO^2 + BO^2 + CO^2 + DO^2$
   (e) $AD^2 + BC^2 = BO^2 + CO^2 + DO^2 + AO^2$
   (f) So $AB^2 + CD^2 = AD^2 + BC^2$
3. $AB^2 + CD^2 = AD^2 + BC^2 \implies AC \perp BD$:
   (a) $\alpha \coloneqq \angle AOB = \angle COD, \beta \coloneqq \angle BOC = \angle DOA$
   (b) By Law of Cosines:
      i. $AB^2 = AO^2 + BO^2 - 2AO \cdot BO \cos\alpha$
      ii. $BC^2 = BO^2 + CO^2 - 2BO \cdot CO \cos\beta$
      iii. $CD^2 = CO^2 + DO^2 - 2CO \cdot DO \cos\alpha$
      iv. $DA^2 = DO^2 + AO^2 - 2DO \cdot AO \cos\beta$
   (c) By $AB^2 + CD^2 = AD^2 + BC^2$:
   $AO^2 + BO^2 - 2AO \cdot BO \cos\alpha + CO^2 + DO^2 - 2CO \cdot DO \cos\alpha = BO^2 + CO^2 - 2BO \cdot CO \cos\beta + DO^2 + AO^2 - 2DO \cdot AO \cos\beta$
   (d) $(AO \cdot BO + CO \cdot DO) \cos\alpha = (BO \cdot CO + DO \cdot AO) \cos\beta$
   (e) $\alpha + \beta = 180° \implies \cos\alpha = -\cos\beta$
   (f) $(AO \cdot BO + CO \cdot DO + BO \cdot CO + DO \cdot AO) \cos\alpha = 0$
   (g) $AO, BO, CO, DO > 0 \implies AO \cdot BO + CO \cdot DO + BO \cdot CO + DO \cdot AO \neq 0$
   (h) So $\cos\alpha = 0 \implies \alpha = 90°$
   (i) $AC \perp BD$
4. $AC \perp BD \iff AB^2 + CD^2 = AD^2 + BC^2$

$\square$

Question 5

a

Proof.

1. $LHS > 0$
2. $\sqrt{\frac{a+\sqrt{a^2-b}}{2}} > \sqrt{\frac{a-\sqrt{a^2-b}}{2}} \implies RHS > 0$
3. So squaring does not affect the positivity of both sides
   i.e. $LHS^2 = RHS^2 \implies LHS = RHS$
4. $$\begin{aligned}
&RHS^2\\
=&\left(\sqrt{\frac{a+\sqrt{a^2-b}}{2}} \pm \sqrt{\frac{a-\sqrt{a^2-b}}{2}}\right)^2\\
=&\frac{a+\sqrt{a^2-b}}{2} + \frac{a-\sqrt{a^2-b}}{2} \pm 2\sqrt{\frac{a+\sqrt{a^2-b}}{2}}\sqrt{\frac{a-\sqrt{a^2-b}}{2}}\\
=&a \pm \sqrt{a^2-(a^2-b)}\\
=&a \pm \sqrt{b}
\end{aligned}$$
5. $LHS^2 = a \pm \sqrt{b}$
6. So $RHS^2 = LHS^2$
7. $RHS = LHS$
8. $\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a+\sqrt{a^2-b}}{2}} \pm \sqrt{\frac{a-\sqrt{a^2-b}}{2}}$

□

b

1. $a = 17, b = 240$
2. $\sqrt{17+\sqrt{240}} = \sqrt{\frac{17+\sqrt{17^2-240}}{2}} + \sqrt{\frac{17-\sqrt{17^2-240}}{2}}$
3. $\sqrt{\frac{17+\sqrt{17^2-240}}{2}} + \sqrt{\frac{17-\sqrt{17^2-240}}{2}} = \sqrt{\frac{17+7}{2}} + \sqrt{\frac{17-7}{2}}$
4. $\sqrt{\frac{17+7}{2}} + \sqrt{\frac{17-7}{2}} = \sqrt{12} + \sqrt{5}$
5. So $\sqrt{17+\sqrt{240}} = \sqrt{12} + \sqrt{5}$

Question 6

a

1. $L(x) = kx + b$
2. $k = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$
3. $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \cdot x_1 + b = f(x_1)$
4. $b = f(x_1) - \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \cdot x_1$
5. $L(x) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \cdot x + f(x_1) - \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} \cdot x_1$
6. $L(x_3) = 0$
7. $x_3 = \dfrac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} \cdot x_1 - f(x_1)}{\frac{f(x_2) - f(x_1)}{x_2 - x_1}} = \dfrac{f(x_2)x_1 - f(x_1)x_2}{f(x_2) - f(x_1)}$

b

1. $x_1 = 2, f(x_1) = 8$
2. $x_2 = 3, f(x_2) = -9$
3. $x_3 = \dfrac{-9 \cdot 2 - 8 \cdot 3}{-9 - 8} = \dfrac{42}{17}$
4. $f(x_3) = -\dfrac{9144}{4913}$
5. Use x_1 and x_3 to get x_4:
6. $x_4 = \dfrac{-\frac{9144}{4913} \cdot 2 - 8 \cdot \frac{42}{17}}{-\frac{9144}{4913} - 8} = \dfrac{1803}{757}$
7. $f(x_4) = -\dfrac{100793169}{433798093}$

8. Use x_1 and x_4 to get x_5:

9. $x_5 = \dfrac{-\dfrac{100793169}{433798093} \cdot 2 - 8 \cdot \dfrac{1803}{757}}{-\dfrac{100793169}{433798093} - 8} = \dfrac{29298426}{12357017} \approx 2.370995 \approx 2.37$

10. So the root is approximately 2.37.