Question 1

Proof.

$p(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$

by substituting x :

$p(y) = (y - \frac{a_{n-1}}{n})^n + a_{n-1}(y - \frac{a_{n-1}}{n})^{n-1} + ... + a_1(y - \frac{a_{n-1}}{n}) + a_0$

To look at the $(n-1)$th term, it is sufficient to look at the nth term and $(n-1)$th term

Since all terms after $n-1$ cannot produce a term with degree $n-1$

$\Rightarrow p(y) = \sum_{i=0}^{n} \binom{n}{i} y^i (-\frac{a_{n-1}}{n})^{n-i} + a_{n-1} \sum_{i=0}^{n} \binom{n-1}{i} y^i (-\frac{a_{n-1}}{n})^{n-1-i} + ... + a_1(y - \frac{a_{n-1}}{n}) + a_0$

Looking at $(n-1)$th term :

$(y - \frac{a_{n-1}}{n})^n : \binom{n}{n-1} y^{n-1}(-\frac{a_{n-1}}{n}) = -a_{n-1}y^{n-1}$

$a_{n-1}(y - \frac{a_{n-1}}{n})^{n-1} : a_{n-1}\binom{n-1}{n-1} y^{n-1}(-\frac{a_{n-1}}{n})^0 = a_{n-1}y^{n-1}$

$\Rightarrow p(y) = b_ny^n + (-a_{n-1} + a_{n-1})y^{n-1} + b_{n-2}y^{n-2} + ... + b_1y + b_0$

$p(y) = b_ny^n + b_{n-2}y^{n-2} + ... + b_1y + b_0$

So the $(n-1)$th term is depressed.

$\square$

Question 2

a

$$
\begin{aligned}
&x = -2 \text{ is a root}\\
\Rightarrow &(x^3 + 6x^2 + x - 14) \div (x + 2) = x^2 + 4x - 7\\
&\text{The other two roots can be found by solving } x^2 + 4x - 7 = 0\\
&x = \frac{-4 \pm \sqrt{16 + 28}}{2} = 2 \pm \sqrt{11}\\
&x_1 = -2, x_2 = 2 + \sqrt{11}, x_3 = 2 - \sqrt{11}
\end{aligned}
$$

b

$$
\begin{aligned}
&x^3 = 3x^2 + 27x + 41\\
&x^3 - 3x^2 = 27x + 41\\
&x^3 - 3x^2 + 3x - 1 = 30x + 40\\
&(x - 1)^3 = 30x + 40\\
&y \coloneqq x - 1\\
&x = y + 1\\
&y^3 = 30y + 70\\
&y^3 - 30y - 70 = 0\\
&\text{Applying Cardano's formula :}\\
&y = \sqrt[3]{-\frac{-70}{2} + \sqrt{\frac{(-30)^3}{27} + \frac{(-70)^2}{4}}} + \sqrt[3]{-\frac{-70}{2} - \sqrt{\frac{(-30)^3}{27} + \frac{(-70)^2}{4}}}\\
&y = \sqrt[3]{35 + \sqrt{-1000 + 1225}} + \sqrt[3]{35 - \sqrt{-1000 + 1225}} = \sqrt[3]{50} + \sqrt[3]{20}\\
\Rightarrow &x = \sqrt[3]{50} + \sqrt[3]{20} + 1
\end{aligned}
$$

Question 3

Proof.

$$x^2 + 3x - 36 = 0$$
$$p = 3, q = -36$$
Applying Cardano's formula :
$$x = \sqrt[3]{-\frac{-36}{2} + \sqrt{\frac{3^3}{27} + \frac{36^2}{4}}} + \sqrt[3]{-\frac{-36}{2} - \sqrt{\frac{3^3}{27} + \frac{36^2}{4}}}$$
$$x = \sqrt[3]{18 + \sqrt{325}} - \sqrt[3]{-18 + \sqrt{325}}$$
$$x = \sqrt[3]{18 + 5\sqrt{13}} - \sqrt[3]{-18 + 5\sqrt{13}}$$
$$a + b\sqrt{13} = \sqrt[3]{18 + 5\sqrt{13}}$$
$$(a + b\sqrt{13})^3 = 18 + 5\sqrt{13}$$
$$a^3 + 3a^2b\sqrt{13} + 39ab^2 + 13b^3\sqrt{13} = 18 + 5\sqrt{13}$$
$$\begin{cases} a^3 + 39ab^2 = 18 \\ 3a^2b + 13b^3 = 5 \end{cases}$$
$$\Rightarrow \begin{cases} a = \frac{3}{2} \\ b = \frac{1}{2} \end{cases}$$
$$\Rightarrow \sqrt[3]{18 + 5\sqrt{13}} = \frac{3}{2} + \frac{1}{2}\sqrt{13}$$
By the same method, $\sqrt[3]{-18 + 5\sqrt{13}} = -\frac{3}{2} + \frac{1}{2}\sqrt{13}$
$$\Rightarrow x = \sqrt[3]{18 + 5\sqrt{13}} - \sqrt[3]{-18 + 5\sqrt{13}} = \frac{3}{2} + \frac{1}{2}\sqrt{13} - (-\frac{3}{2} + \frac{1}{2}\sqrt{13}) = 3$$

□

Question 4

a

$$
\begin{aligned}
&(x^2+6)^2 = x^4+12x^2+36\\
&x^4+6x^2+8x+21=0\\
&x^4+12x^2+36 = 6x^2-8x+15\\
&(x^2+6)^2 = 6x^2-8x+15
\end{aligned}
$$

b

$$
\begin{aligned}
&(x^2+6+z)^2 = x^4+12x^2+36+2x^2z+z^2+12z = (x^2+6)^2+2x^2z+z^2+12z\\
&(x^2+6+z)^2 = 6x^2-8x+15+2x^2z+z^2+12z = (6+2z)x^2-8x+(z^2+12z+15)
\end{aligned}
$$

c

$$
\begin{aligned}
&(6+2z)x^2-8x+(z^2+12z+15)\\
&\Delta = 0\\
\Rightarrow&(-8)^2-4(6+2z)(z^2+12z+15)=0\\
&-8z^3-120z^2-408z-296=0
\end{aligned}
$$

d

$$
\begin{aligned}
&-8z^3 - 120z^2 - 408z - 296 = 0 \\
&z^3 + 15z^2 + 51z + 37 = 0 \\
&z \coloneqq y - \frac{15}{3} = y - 5 \\
&(y-5)^3 + 15(y-5)^2 + 51(y-5) + 37 = 0 \\
&y^3 - 24y + 32 = 0
\end{aligned}
$$

e

$$
\begin{aligned}
&y = 4 \text{ is a root of } y^3 - 24y + 32 = 0 \\
\Rightarrow &z = y - 5 = -1 \\
&(x^2 + 6 + z)^2 = (6 + 2z)x^2 - 8x + (z^2 + 12z + 15) \\
\Rightarrow &(x^2 + 5)^2 = 4x^2 - 8x + 4 \\
&(x^2 + 5)^2 = (2x - 2)^2 \\
&(x^2 + 5)^2 - (2x - 2)^2 = 0 \\
&(x^2 + 2x + 3)(x^2 - 2x + 7) = 0 \\
&(x + 1 + \sqrt{2}i)(x + 1 - \sqrt{2}i)(x - 1 + \sqrt{6}i)(x - 1 - \sqrt{6}i) = 0 \\
\Rightarrow &\begin{cases} x_1 = -1 - \sqrt{2}i \\ x_2 = -1 + \sqrt{2}i \\ x_3 = 1 - \sqrt{6}i \\ x_4 = 1 + \sqrt{6}i \end{cases}
\end{aligned}
$$