Book

Dunham_Chapter6.pdf

Question 1

Proof.

$p(x) = x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$

by substituting x :

$p(y) = (y - \frac{a_{n-1}}{n})^n + a_{n-1}(y - \frac{a_{n-1}}{n})^{n-1} + ... + a_1(y - \frac{a_{n-1}}{n}) + a_0$

To look at the $(n-1)$th term, it is sufficient to look at the nth term and $(n-1)$th term

Since all terms after $n-1$ cannot produce a term with degree $n-1$

$\Rightarrow p(y) = \sum_{i=0}^{n} \binom{n}{i} y^i(-\frac{a_{n-1}}{n})^{n-i} + a_{n-1}\sum_{i=0}^{n-1} \binom{n-1}{i} y^i(-\frac{a_{n-1}}{n})^{n-1-i} + ... + a_1(y - \frac{a_{n-1}}{n}) + a_0$

Looking at $(n-1)$th term :

$(y - \frac{a_{n-1}}{n})^n : \binom{n}{n-1} y^{n-1}(-\frac{a_{n-1}}{n}) = -a_{n-1}y^{n-1}$

$a_{n-1}(y - \frac{a_{n-1}}{n})^{n-1} : a_{n-1}\binom{n-1}{n-1} y^{n-1}(-\frac{a_{n-1}}{n})^0 = a_{n-1}y^{n-1}$

$\Rightarrow p(y) = b_ny^n + (-a_{n-1} + a_{n-1})y^{n-1} + b_{n-2}y^{n-2} + ... + b_1y + b_0$

$p(y) = b_ny^n + b_{n-2}y^{n-2} + ... + b_1y + b_0$

So the $(n-1)$th term is depressed.

$\square$

Question 2

a

$$
\begin{aligned}
&x = -2 \text{ is a root}\\
\Rightarrow &(x^3 + 6x^2 + x - 14) \div (x + 2) = x^2 + 4x - 7\\
&\text{The other two roots can be found by solving } x^2 + 4x - 7 = 0\\
&x = \frac{-4 \pm \sqrt{16 + 28}}{2} = 2 \pm \sqrt{11}\\
&x_1 = -2, x_2 = 2 + \sqrt{11}, x_3 = 2 - \sqrt{11}
\end{aligned}
$$

b

$$
\begin{aligned}
&x^3 = 3x^2 + 27x + 41\\
&x^3 - 3x^2 = 27x + 41\\
&x^3 - 3x^2 + 3x - 1 = 30x + 40\\
&(x - 1)^3 = 30x + 40\\
&y \coloneqq x - 1\\
&x = y + 1\\
&y^3 = 30y + 70\\
&y^3 - 30y - 70 = 0\\
&\text{Applying Cardano's formula :}\\
&y = \sqrt[3]{-\frac{-70}{2} + \sqrt{\frac{(-30)^3}{27} + \frac{(-70)^2}{4}}} + \sqrt[3]{-\frac{-70}{2} - \sqrt{\frac{(-30)^3}{27} + \frac{(-70)^2}{4}}}\\
&y = \sqrt[3]{35 + \sqrt{-1000 + 1225}} + \sqrt[3]{35 - \sqrt{-1000 + 1225}} = \sqrt[3]{50} + \sqrt[3]{20}\\
\Rightarrow &x = \sqrt[3]{50} + \sqrt[3]{20} + 1
\end{aligned}
$$

Question 3

Proof.

$$
\begin{aligned}
&x^2+3x-36=0\\
&p=3, q=-36\\
&\text{Applying Cardano's formula :}\\
&x=\sqrt[3]{-\frac{-36}{2}+\sqrt{\frac{3^3}{27}+\frac{36^2}{4}}}+\sqrt[3]{-\frac{-36}{2}-\sqrt{\frac{3^3}{27}+\frac{36^2}{4}}}\\
&x=\sqrt[3]{18+\sqrt{325}}-\sqrt[3]{-18+\sqrt{325}}\\
&x=\sqrt[3]{18+5\sqrt{13}}-\sqrt[3]{-18+5\sqrt{13}}\\
&a+b\sqrt{13}=\sqrt[3]{18+5\sqrt{13}}\\
&(a+b\sqrt{13})^3=18+5\sqrt{13}\\
&a^3+3a^2b\sqrt{13}+39ab^2+13b^3\sqrt{13}=18+5\sqrt{13}\\
&\begin{cases} a^3+39ab^2=18\\ 3a^2b+13b^3=5\end{cases}\\
\Rightarrow&\begin{cases} a=\frac{3}{2}\\ b=\frac{1}{2}\end{cases}\\
\Rightarrow&\sqrt[3]{18+5\sqrt{13}}=\frac{3}{2}+\frac{1}{2}\sqrt{13}\\
&\text{By the same method, }\sqrt[3]{-18+5\sqrt{13}}=-\frac{3}{2}+\frac{1}{2}\sqrt{13}\\
\Rightarrow&x=\sqrt[3]{18+5\sqrt{13}}-\sqrt[3]{-18+5\sqrt{13}}=\frac{3}{2}+\frac{1}{2}\sqrt{13}-(-\frac{3}{2}+\frac{1}{2}\sqrt{13})=3
\end{aligned}
$$

□

Question 4

a

$$
\begin{aligned}
&(x^2+6)^2 = x^4 + 12x^2 + 36\\
&x^4 + 6x^2 + 8x + 21 = 0\\
&x^4 + 12x^2 + 36 = 6x^2 - 8x + 15\\
&(x^2+6)^2 = 6x^2 - 8x + 15
\end{aligned}
$$

b

$$
\begin{aligned}
&(x^2+6+z)^2 = x^4 + 12x^2 + 36 + 2x^2z + z^2 + 12z = (x^2+6)^2 + 2x^2z + z^2 + 12z\\
&(x^2+6+z)^2 = 6x^2 - 8x + 15 + 2x^2z + z^2 + 12z = (6+2z)x^2 - 8x + (z^2 + 12z + 15)
\end{aligned}
$$

c

$$
\begin{aligned}
&(6+2z)x^2 - 8x + (z^2 + 12z + 15)\\
&\Delta = 0\\
\Rightarrow &(-8)^2 - 4(6+2z)(z^2+12z+15) = 0\\
&-8z^3 - 120z^2 - 408z - 296 = 0
\end{aligned}
$$

d

$$
\begin{aligned}
&-8z^3 - 120z^2 - 408z - 296 = 0\\
&z^3 + 15z^2 + 51z + 37 = 0\\
&z \coloneqq y - \frac{15}{3} = y - 5\\
&(y-5)^3 + 15(y-5)^2 + 51(y-5) + 37 = 0\\
&y^3 - 24y + 32 = 0
\end{aligned}
$$

e

$$
\begin{aligned}
&y = 4 \text{ is a root of } y^3 - 24y + 32 = 0\\
\Rightarrow &z = y - 5 = -1\\
&(x^2 + 6 + z)^2 = (6 + 2z)x^2 - 8x + (z^2 + 12z + 15)\\
\Rightarrow &(x^2 + 5)^2 = 4x^2 - 8x + 4\\
&(x^2 + 5)^2 = (2x - 2)^2\\
&(x^2 + 5)^2 - (2x - 2)^2 = 0\\
&(x^2 + 2x + 3)(x^2 - 2x + 7) = 0\\
&(x + 1 + \sqrt{2}i)(x + 1 - \sqrt{2}i)(x - 1 + \sqrt{6}i)(x - 1 - \sqrt{6}i) = 0\\
\Rightarrow &\begin{cases} x_1 = -1 - \sqrt{2}i\\ x_2 = -1 + \sqrt{2}i\\ x_3 = 1 - \sqrt{6}i\\ x_4 = 1 + \sqrt{6}i \end{cases}
\end{aligned}
$$