Question 1

Problem a

$$\begin{aligned}
&M_X(t) \\
=&E[e^{tx}] \\
=&\sum_{x=0}^{n} e^{tx}\binom{n}{x}p^x(1-p)^{n-x} \\
=&\sum_{x=0}^{n} (pe^t)^x\binom{n}{x}(1-p)^{n-x} \\
=&(1-p+pe^t)^n
\end{aligned}$$

Problem b

$$\begin{aligned}
{M_X}'(t) &= \frac{d}{dt}(1-p+pe^t)^n = n(1-p+pe^t)^{n-1}pe^t \\
{M_X}''(t) &= \frac{d}{dt}n(1-p+pe^t)^{n-1}pe^t = (pe^t)^2n(n-1)(1-p+pe^t)^{n-2} + pe^tn(1-p+pe^t)^{n-1} \\
{M_X}'''(t) &= \frac{d}{dt}((pe^t)^2n(n-1)(1-p+pe^t)^{n-2} + pe^tn(1-p+pe^t)^{n-1}) \\
&= (pe^t)^3n(n-1)(n-2)(1-p+pe^t)^{n-3} + 3(pe^t)^2n(n-1)(1-p+pe^t)^{n-2} + pe^tn(1-p+pe^t)^{n-1} \\
E[X^3] &= {M_X}'''(0) = np + 3(n-1)np^2 + n(n-1)(n-2)p^3
\end{aligned}$$

Question 2

Problem a

$$
\begin{aligned}
&{Z_i}^2 \sim {\chi_{(1)}}^2 \\
&Z_i \text{ are independent} \\
\Rightarrow &Y = \sum_{i=1}^{n} {Z_i}^2 \sim {\chi_{(n)}}^2 \\
\Rightarrow &\overline{Y} = n \\
&\sigma^2 = 2n
\end{aligned}
$$

Problem b

$$
\begin{aligned}
M_Y(t) &= E[e^{tY}] \\
&= E[e^{t({Z_1}^2+{Z_2}^2+\ldots+{Z_n}^2)}] \\
&= E[\Pi_{i=1}^{n} e^{t{Z_i}^2}] \\
&= \Pi_{i=1}^{n} E[e^{t{Z_i}^2}] \\
&= \Pi_{i=1}^{n} M_{{Z_i}^2}(t) \\
&= (1-2t)^{-\frac{n}{2}}, t < \frac{1}{2}
\end{aligned}
$$

problem c

$$
\begin{aligned}
E[Y^3] &= {M_Y}'''(0) \\
&= \frac{d^3}{dt^3}(1-2t)^{-\frac{n}{2}}|_{t=0} \\
&= 4n(-\frac{1}{2}n-1)(-\frac{1}{2}n-2)(1-2t)^{-\frac{1}{2}n-3}|_{t=0} \\
&= n^3+6n^2+8n
\end{aligned}
$$

Question 3

Problem a

$$
\begin{aligned}
P(x > 15) &\leqslant \frac{E[X]}{15} \\
P(x > 15) &\leqslant \frac{12}{15} = \frac{2}{3} \\
\Rightarrow P(x > 15) &\leqslant \frac{2}{3}
\end{aligned}
$$

Problem b

$$
\begin{aligned}
P(x > 15) &= P(x - 10 > 5) \\
P(x - 10 > \frac{5}{\sqrt{3}} \times \sqrt{3}) &\leqslant \frac{1}{(\frac{5}{\sqrt{3}}^2)} = \frac{3}{25} \\
\Rightarrow P(x > 15) &\leqslant \frac{3}{25}
\end{aligned}
$$

Problem c

$$\sum_{i=1}^{300} Y_i \sim N(10 \times 300, 3 \times 300)$$

$$\sigma = \sqrt{900} = 30$$

$$P(\sum_{i=1}^{300} Y_i > 3030) = P(z > \frac{3030 - 3000}{30}) = P(z > 1)$$

$$P(z > 1) = 1 - P(z \leqslant 1) = 1 - 0.84134 = 0.15866$$

$$\Rightarrow P(\sum_{i=1}^{300} Y_i > 3030) = 0.15866$$