Problem 1

$$
\begin{aligned}
P(X = k) &= (1-p)^{k-1} \cdot p \\
P(X = n+k | X > n) &= \frac{P(X = n+k \cap X > n)}{P(X > n)} \\
P(X = n+k | X > n) &= \frac{(1-p)^{n+k-1} \cdot p}{(1-p)^n} \\
P(X = n+k | X > n) &= (1-p)^{k-1} \cdot p = P(X = k) \\
\Rightarrow P(X = n+k | X > n) &= P(X = k), \forall n, k \geqslant 1
\end{aligned}
$$

Problem 2

Question a

$$p_k = P(Heads)p_{k+1} + P(Tails)p_{k-1}$$
$$p_k = \frac{1}{2}p_{k+1} + \frac{1}{2}p_{k-1}$$

Question b

$$\begin{aligned}
b_1 =& p_1 - p_0 \\
=& \frac{1}{2}p_2 + \frac{1}{2}p_0 - p_0 \\
=& \frac{1}{2}p_2 - \frac{1}{2}p_0 \\
=& \frac{1}{2}p_2 - \frac{1}{2} \\
b_2 =& p_2 - p_1 \\
=& p_2 - (\frac{1}{2}p_2 + \frac{1}{2}p_0) \\
=& \frac{1}{2}p_2 - \frac{1}{2} \\
\Rightarrow& b_1 = b_2
\end{aligned}$$

Question c

$$
\begin{aligned}
b_k =& p_k - p_{k-1} \\
=& p_k - (\frac{1}{2}p_k + \frac{1}{2}p_{k-2}) \\
=& \frac{1}{2}p_k - \frac{1}{2}p_{k-2} \\
b_k =& p_{k-1} - p_{k-2} \\
=& \frac{1}{2}p_k + \frac{1}{2}p_{k-2} - p_{k-2} \\
=& \frac{1}{2}p_k - \frac{1}{2}p_{k-2} \\
\Rightarrow& b_k = b_{k-1} \\
\Rightarrow& b_k = b_{k-1} = ... = b_1 \leftarrow \text{By induction} \\
\Rightarrow& b_k = b_1, \forall k \in (0, N]
\end{aligned}
$$

Question d

$$
\begin{aligned}
& b_k = b_1 \\
\Rightarrow & \sum_{i=1}^{k} b_i = k \cdot b_1 \\
& b_k = p_k - p_{k-1} \\
\Rightarrow & \sum_{i=1}^{k} b_i = \sum_{i=1}^{k} p_k - p_{k-1} = p_k - p_0 \\
\Rightarrow & k \cdot b_1 = p_k - p_0 \\
\Rightarrow & p_k = k \cdot b_1 + p_0
\end{aligned}
$$

Question e

$$
\begin{aligned}
& p_k = k \cdot b_1 + p_0 \\
& p_N = N \cdot b_1 + p_0 = 0 \\
\Rightarrow & N = -\frac{p_0}{b_1} = -\frac{1}{b_1} \\
\Rightarrow & p_k = k \cdot b_1 + p_0 \\
\Rightarrow & p_k = -\frac{k}{N} + 1
\end{aligned}
$$