1 Problem 1

Define: $u = (a, b) \in V \wedge v = (c, d) \in V \wedge w = (e, f) \in V \wedge n \in \mathbb{R} \wedge m \in \mathbb{R}$

1.1 Closure

1.1.1 Addition

$u \boxplus v = (a + c + 1, b + d + 1)$.
$u = (a, b) \in V \wedge v = (c, d) \in V \wedge V = \mathbb{R}^2 \Rightarrow a + c + 1 \in \mathbb{R}$.
For the same reason, $b + d + 1 \in \mathbb{R}$.
As a result, $(a + c + 1, b + d + 1) \in \mathbb{R}^2 \Rightarrow u \boxplus v \in V$.

1.1.2 Multiplication

$n \boxdot u = (n \cdot a - n + 1, n \cdot b - n + 1)$.
$u = (a, b) \in V \wedge V = \mathbb{R}^2 \wedge n \in \mathbb{R} \Rightarrow n \cdot a - n + 1 \in \mathbb{R}$.
For the same reason, $n \cdot b - n + 1 \in \mathbb{R}$.
As a result, $(n \cdot a - n + 1, n \cdot b - n + 1) \in \mathbb{R}^2 \Rightarrow n \boxdot u \in V$.

1.2 Commutative Addition

$u \boxplus v = (a + c + 1, b + d + 1)$.
$v \boxplus u = (c + a + 1, d + b + 1)$.
$c + a + 1 = a + c + 1 \wedge d + b + 1 = b + d + 1 \Rightarrow u \boxplus v = v \boxplus u$

1.3 Associative Addition

$$\begin{aligned}
&(u \boxplus v) \boxplus w \\
=&(a + c + 1, b + d + 1) \boxplus w \\
=&(a + c + 1 + e + 1, b + d + 1 + f + 1) \\
=&(a + c + e + 2, b + d + f + 2)
\end{aligned} \tag{1.3-1}$$

$$\begin{aligned}
&u \boxplus (v \boxplus w) \\
=&u \boxplus (c + e + 1, d + f + 1) \\
=&(a + c + e + 1 + 1, b + d + f + 1 + 1) \\
=&(a + c + e + 2, b + d + f + 2)
\end{aligned} \tag{1.3-2}$$

The results from 1.3-1 and 1.3-2 are the same. $\Rightarrow (u \boxplus v) \boxplus w = u \boxplus (v \boxplus w)$

1.4 Identity for Addition

$$\begin{aligned}
&\overrightarrow{0} \boxplus u = (0,0) \\
&\text{Let } \overrightarrow{0} = (x,y) \\
&a + x + 1 = a \wedge b + y + 1 = b \\
&x = -1 \wedge y = -1 \\
&\overrightarrow{0} = (-1,-1)
\end{aligned} \tag{1.4}$$

1.5 Inverse

$$\begin{aligned}
&\overline{u} + u = \overrightarrow{0} \\
&\overline{u} + (a,b) = (-1,-1) \\
&(\overline{a} + a + 1, \overline{b} + b + 1) = (-1,-1) \\
&\overline{a} = -a - 2 \wedge \overline{b} = -b - 2 \\
\Rightarrow &\overline{u} = (-a-2, -b-2) \in V
\end{aligned} \tag{1.5}$$

1.6 Unit Property

$$\begin{aligned}
1 \boxdot u &= (1 \times a - 1 + 1, 1 \times b - 1 + 1) \\
&= (a,b) \\
&= u
\end{aligned} \tag{1.5}$$

1.7 Associative Multiplication

$$\begin{aligned}
(m \cdot n) \boxdot u &= (mnx - mn + 1, mny - mn + 1) \\
m \boxdot (n \boxdot u) &= m \boxdot (nx - n + 1, ny - n + 1) \\
&= (mnx - mn + m - m + 1, mny - mn + m - m + 1) \\
&= (mnx - mn + 1, mny - mn + 1) \\
&= (m \cdot n) \boxdot u
\end{aligned} \tag{1.6}$$

1.8 First Distributive

$$\begin{aligned}
m \boxdot (u \boxplus v) &= m \boxdot (a + c + 1, b + d + 1) \\
&= (m(a + c + 1) - m + 1, m(b + d + 1) - m + 1) \\
&= (ma + mc + 1, mb + md + 1) \\
m \boxdot u \boxplus m \boxdot v &= (ma - m + 1, mb - m + 1) \boxplus (mc - m + 1, md - m + 1) \\
&= ((ma - m + 1) + (mc - m + 1) + 1, (mb - m + 1) + (md - m + 1) + 1) \\
&= (ma + mc - 2m + 3, mb + md - 2m + 3) \\
&\neq m \boxdot (u \boxplus v)
\end{aligned} \tag{1.7}$$

The distributive property is not verified.

1.9 Conclusion

Since V does not have every identities for vector space, V is not a vector space over $\mathbb{R}$.

2 Problem 2

Define $u = (u_1, u_2) \in V \wedge v = (v_1, v_2) \in V \wedge w = (w_1, w_2) \in V \wedge c \in \mathbb{R} \wedge d \in \mathbb{R}$

2.1 Closure

2.1.1 Addition

$u \boxplus v = (u_1 + v_1, u_2 + v_2)$.
$u = (u_1, u_2) \in V \wedge v = (v_1, v_2) \in V \wedge V = \mathbb{R}^2 \Rightarrow u_1 + u_2 \in \mathbb{R}$.
For the same reason, $u_2 + v_2 \in \mathbb{R}$.
As a result, $(u_1 + v_1, u_2 + v_2) \in \mathbb{R}^2 \Rightarrow u \boxplus v \in V$.

2.1.2 Multiplication

$c \boxdot u = (|c|u_1, |c|u_2)$.
$u = (u_1, u_2) \in V \wedge V = \mathbb{R}^2 \wedge c \in \mathbb{R} \Rightarrow |c|u_1 \in \mathbb{R}$.
For the same reason, $|c|u_2 \in \mathbb{R}$.
As a result, $(|c|u_1, |c|u_2) \in \mathbb{R}^2 \Rightarrow c \boxdot u \in V$.

2.2 Commutative Addition

$$\begin{aligned} u \boxplus v &= (u_1, u_2) \boxplus (v_1, v_2) \\ &= (u_1 + v_1, u_2 + v_2) \end{aligned} \tag{2.2-1}$$

$$\begin{aligned} v \boxplus u &= (v_1, v_2) \boxplus (u_1, u_2) \\ &= (v_1 + u_1, v_2 + u_2) \end{aligned} \tag{2.2-2}$$

$(u_1 + v_1, u_2 + v_2) = (v_1 + u_1, v_2 + u_2) \Rightarrow u \boxplus v = v \boxplus u$

2.3 Associative Addition

$$\begin{aligned} (u\boxplus)v \boxplus w &= (u_1 + v_1, u_2, v_2) \boxplus w \\ &= (u_1 + v_1 + w_1, u_2 + v_2 + w_2) \end{aligned} \tag{2.3-1}$$

$$\begin{aligned} u \boxplus (v \boxplus w) &= u \boxplus (v_1 + w_1, v_2 + w_2) \\ &= (u_1 + v_1 + w_1, u_2 + u_2 + w_2) \end{aligned} \tag{2.3-2}$$

$(u_1 + v_1 + w_1, u_2 + u_2 + w_2) = (u_1 + v_1 + w_1, u_2 + u_2 + w_2) \Rightarrow (u\boxplus)v \boxplus w = u \boxplus (v \boxplus w)$

2.4 Identity of Addition

$$\begin{aligned} \overrightarrow{0} + u &= (0, 0) + (u_1, u_2) \\ &= (u_1, u_2) \end{aligned} \tag{2.4-1}$$

$(u_1, u_2) = (u_1, u_2) \Rightarrow \overrightarrow{0} + u = u$

2.5 Inverse

Define $\overline{u}$ such that $\overline{u} + u = \overrightarrow{0}$

$$\begin{aligned} \overline{u} + u &= \overrightarrow{0} \\ \overline{u} &= \overrightarrow{0} - u \\ &= (0, 0) - (u_1, u_2) \\ &= (-u_1, -u_2) \in V \end{aligned} \tag{2.5-1}$$

$\Rightarrow$ inverse of $u \in V$

2.6 Unit Property

$$\begin{aligned}1 \boxdot u &= (|1|u_1, |1|u_2) \\ &= (u_1, u_2) \\ &= u\end{aligned} \tag{2.6-1}$$

2.7 Associative Multiplication

$$\begin{aligned}(c \cdot d) \boxdot u &= (|cd|u_1, |cd|u_2) \\ c \boxdot (d \boxdot u) &= c \boxdot (|d|u_1, |d|u_2) \\ &= (|c||d|u_1, |c||d|u_2) \\ &= (|cd|u_1, |cd|u_2) \\ &= (c \cdot d) \boxdot u\end{aligned} \tag{2.7-1}$$

2.8 First Distributive

$$\begin{aligned}c \boxdot (u \boxplus v) &= c \boxdot (u_1 + v_1, u_2 + v_2) \\ &= (|c|(u_1 + v_1), |c|(u_2 + v_2)) \\ &= (|c|u_1 + |c|v_1, |c|u_2 + |c|v_2) \\ c \boxdot u \boxplus c \boxdot v &= (|c|u_1, |c|u_2) \boxplus (|c|v_1, v_2) \\ &= (|c|u_1 + |c|v_1, |c|u_2 + |c|v_2) \\ \Rightarrow c \boxdot (u \boxplus v) &= c \boxdot u \boxplus c \boxdot v\end{aligned} \tag{2.8-1}$$

2.9 Second Distributive

$$\begin{aligned}(c + d) \boxdot u &= (|c + d|u_1, |c + d|u_2) \\ c \boxdot u \boxplus d \boxdot u &= (|c|u_1, |c|u_2) \boxplus (|d|u_1, |d|u_2) \\ &= ((|c| + |d|)u_1, (|c| + |d|)u_2) \\ |c + d| &\neq |c| + |d| \\ \Rightarrow (c + d) \boxdot u &\neq c \boxdot u \boxplus d \boxdot u\end{aligned} \tag{2.9-1}$$

2.10 Conclusion

Since V does not fit in the Second Distributive, V is not a vector space over $\mathbb{R}$.

3 Problem 3

3.1 Question a

3.1.1 Proof

In $\boxplus$ test for W:
Define $a = (a_1, a_2) \in W \wedge b = (b_1, b_2) \in W$.
$W = \{(x, y)|xy \geqslant 0\} \Rightarrow a_1a_2 \geqslant 0 \wedge b_1b_2 \geqslant 0$.
$a \boxplus b = (a_1 + b_1, a_2 + b_2)$
According to the definition:

$$\begin{aligned} &(a_1 + b_1)(a_2 + b_2) \\ =&a_1a_2 + a_1b_2 + b_1a_2 + b_1b_2 \\ =&(a_1a_2 + b_1b_2) + (a_1b_2 + b_1a_2) \end{aligned} \tag{3.1.1-1}$$

Only the first bracket in 3.1.1-1 must $\geqslant 0$, where the second bracket may $\leqslant 0$, making $(a_1 + b_1)(a_2 + b_2) \leqslant 0$.

3.1.2 Conclusion

So W is not closed, and is not a subspace for $V = \mathbb{R}^2$

3.2 Question b

Define $n = (n_1, n_2, n_3) \in W \wedge m = (m_1, m_2, m_3) \in W$

3.2.1 Zero Vector

For $n = (n_1, n_2, n_3) \in W$, we can let $n_1 = n_2 = n_3 = 0$ so that $n = (0, 0, 0) = \overrightarrow{0} \wedge 2n_2 = n_1 - 3n_3 \Rightarrow \overrightarrow{0} \in W$

3.2.2 Addition

$$n \boxplus m = (n_1 + m_1, n_2 + m_2, n_3 + m_3) \tag{3.2.2-1}$$

$$\begin{aligned} W &= \{(x,y,z)|2y = x - 3z\} \\ \Rightarrow 2n_2 &= n_1 - 3n_3 \\ 2m_2 &= m_1 - 3m_3 \end{aligned} \tag{3.2.2-2}$$

We can see that the vector in 3.2.2-1 has the requirement that $2(n_2+m_2) = (n_1+m_1) - 3(n_3+m_3)$.

Adding the two equations in 3.2.2-2: $2(n_2+m_2) = (n_1+m_1) - 3(n_3+m_3)$, which fills in the requirement.

So $n \boxplus m \in W$

3.2.3 Multiplication

Define $c \in \mathbb{R}$: $c \boxdot n = (3n_1, 3n_2, 3n_3)$.

We can see that $3n_2 = 3n_1 - 3n_3$ is required for $\boxdot$ test.

Dividing both sides by 3: $2n_2 = n_1 - 3n_3$, which equality is established.

3.2.4 Conclusion

Since W with these operations are valid under addition and multiplication with $\overrightarrow{0} \in W$, W is a subspace of $V = \mathbb{R}^3$.

3.3 Question c

Define $p_1(x) = a_1x^4 + b_1x^3 + c_1x^2 + d_1x + e_1 \wedge p_2(x) = a_2x^4 + b_2x^3 + c_2x^2 + d_2x + e_2 \wedge n \in \mathbb{R}$

3.3.1 Zero Vector

$\overrightarrow{0} \in W$ since $0x^4 + 0x^3 + 0x^2 + 0x + 0 = 0 \in W$.

3.3.2 Addition

$$
\begin{aligned}
p_1(x) \boxplus p_2(x) &= a_1x^4 + b_1x^3 + c_1x^2 + d_1x + e_1 + a_2x^4 + b_2x^3 + c_2x^2 + d_2x + e_2 \\
&= (a_1 + a_2)x^4 + (b_1 + b_2)x^3 + (c_1 + c_2)x^2 + (d_1 + d_2)x + (e_1 + e_2) \\
p_1(1) \boxplus p_2(1) &= a_1 + a_2 + b_1 + b_2 + c_1 + c_2 + d_1 + d_2 + e_1 + e_2 \\
&= (a_1 + b_1 + c_1 + d_1 + e_1) + (a_2 + b_2 + c_2 + d_2 + e_2) \\
p_1(1) &= a_1 + b_1 + c_1 + d_1 + e_1 = 0 \\
p_2(1) &= a_2 + b_2 + c_2 + d_2 + e_2 = 0 \\
&\Rightarrow (a_1 + b_1 + c_1 + d_1 + e_1) + (a_2 + b_2 + c_2 + d_2 + e_2) = 0 \\
&\Rightarrow p_1(1) \boxplus p_2(1) = 0 \\
&\Rightarrow p_1(x) \boxplus p_2(x) \in W
\end{aligned}
\tag{3.3.2-1}
$$

3.3.3 Multiplication

$$
\begin{aligned}
n \boxdot p_1(x) &= n \cdot (a_1x^4 + b_1x^3 + c_1x^2 + d_1x + e_1) \\
p_1(1) &= a_1 + b_1 + c_1 + d_1 + e_1 = 0 \\
\Rightarrow n \boxdot p_1(1) &= n \cdot (a_1 + b_1 + c_1 + d_1 + e_1) \\
&= n \cdot 0 \\
&= 0 \\
\Rightarrow n \boxdot p_1(x) \in W
\end{aligned}
\tag{3.3.3-1}
$$

3.3.4 Conclusion

Since W with these operations are valid under addition and multiplication with $\overrightarrow{0} \in W$, W is a subspace of $V = P_4(\mathbb{R})$.

3.4 Question d

3.4.1 Proof

In Zero Vector test, $\det \overrightarrow{0} = 0 \neq 1$.
So W dos not meet the zero vector requirement.

3.4.2 Conclusion

W is not a subspace of $V = M_{n\times n}(\mathbb{R})$.

4 Problem 4

4.1 Question a

Define $V = \mathbb{R}^2 \wedge a \in V_1 = \{(x, y)|y = 0\} \wedge b \in V_2 = \{(x, y)|x = 0\}$.

Then we can define $a = (a, 0) \wedge b = (0, b)$.But $a+b = (a, b) \notin V_1 \cup V_2 \Rightarrow V_1 \cup V_2$ is not a subspace.

4.2 Question b

Define $V = \mathbb{C} \wedge a \in V_1 = \{x|x \in \mathbb{R}\} \wedge b \in V_2 = \{x|x \in \mathbb{Q}\}$.

According to the definition: $V_2 \subset V_1 \Rightarrow V_2 \cup V_1 = V_2$, which is a subspace of V.

5 Problem 5

5.1 Question a

$W_1 \cap W_2 = \{x + y + z = 0 \wedge x + 2y - z = 0\}$

$\begin{bmatrix} x & y & z \\ x & 2y & -z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{bmatrix}$

The vector that spans the matrix above is $[-3, 2, 1]$ which $\in V = \mathbb{R}^3$.

5.2 Question b

5.2.1 Zero Vector

W_1 is a subspace of $V \Rightarrow w_1$ can be $\overrightarrow{0}$.

For the same reason, $w_2 = \overrightarrow{0}$.

$\Rightarrow w_1 + w_2 = \overrightarrow{0}$

5.2.2 Addition

Define $u_1 \in W \wedge u_2 \in W$, then there must be $a + b = u_1$ such that $a \in W_1 \wedge b \in W_2$.

For the same reason, $u_2 = c + d$, where $c \in W_1 \wedge d \in W_2$.

$u_1 \boxplus u_2 = a + b + c + d = (a + c) + (b + d)$, where $a + c \in W_1 \land b + d \in W_2$.
$\Rightarrow u_1 \boxplus u_2 \in W$.

5.2.3 Multiplication

Define $u_1 = a + b \in W \land a \in W_1 \land b \in W_2 \land c \in \mathbb{R}$.

$$c \boxdot u_1 = c(a + b) = c \cdot a + c \cdot b$$

$a \in W_1 \Rightarrow c \cdot a \in W_1$
For the same reason: $c \cdot b \in W_2$
$\Rightarrow c \boxdot u_1 \in W$

5.2.4 Conclusion

Since W with these operations are valid under addition and multiplication with $\overrightarrow{0} \in W$, W is a subspace of V.

6 Reference

6.1 Collaborator

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