1 Problem 1

1.1 Question a

Define U : span$U$ = span$(S_1 \cup S_2) \wedge V$: span$V$ = span(S_1) + span$(S_2) \wedge$ span$(S_1) = \sum_i m_i \wedge \text{span}(S_2) = \sum_j n_j$

1.1.1 $\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2)$

Let $x \in \text{span}(U)$, then x can be written as :

$$x = \sum_i a_i m_i + \sum_j b_j n_j$$

$\sum_i a_i m_i$ spans S_1 and $\sum_j b_j n_j$ spans $S_2 \Rightarrow x \in \text{span}(S_1) + \text{span}(S_2)$

1.1.2 $\text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2)$

Let $y \in \text{span}(V)$, then y can be written as:

$$y = \sum_i a_i m_i + \sum_j b_j n_j$$

$\sum_i a_i m_i$ spans S_1 and $\sum_j b_j n_j$ spans $S_2 \Rightarrow$ y is a linear combination of $S_1 \cup S_2 \Rightarrow x \in \text{span}(S_1) + \text{span}(S_2)$

1.1.3 Conclusion

$$\begin{aligned} &\text{span}(S_1 \cup S_2) \subseteq \text{span}(S_1) + \text{span}(S_2) \\ &\text{span}(S_1) + \text{span}(S_2) \subseteq \text{span}(S_1 \cup S_2) \\ \Rightarrow &\text{span}(S_1 \cup S_2) = \text{span}(S_1) + \text{span}(S_2) \end{aligned} \tag{1.1.3}$$

1.2 Question b

According to the question: $W_2 = A|A^T = -A$.
Define $m \in W_2 \wedge n \in W_2 \wedge a \in \mathbb{R}$

1.2.1 Zero

$$
\begin{aligned}
m &= \overrightarrow{0} \\
-m &= \overrightarrow{0} \\
m^T &= \overrightarrow{0} \\
\Rightarrow m^T &= -m
\end{aligned}
\tag{1.2.1}
$$

1.2.2 Addition

$$
\begin{aligned}
m \boxplus n &= \begin{bmatrix} m_{11} + n11 & ... & m_{1n} + n_{1n} \\ ... & ... & ... \\ m_{n1} + nn1 & ... & m_{nn} + n_{nn} \end{bmatrix} \\
(m \boxplus n)^T &= \begin{bmatrix} m_{11} + n11 & ... & m_{n1} + n_{n1} \\ ... & ... & ... \\ m_{1n} + n1n & ... & m_{nn} + n_{nn} \end{bmatrix} \\
&= \begin{bmatrix} m_{11} & ... & m_{n1} \\ ... & ... & ... \\ m_{1n} & ... & m_{nn} \end{bmatrix} \boxplus \begin{bmatrix} n_{11} & ... & n_{n1} \\ ... & ... & ... \\ n_{1n} & ... & n_{nn} \end{bmatrix} \\
&= \begin{bmatrix} -m_{11} & ... & -m_{1n} \\ ... & ... & ... \\ -m_{n1} & ... & -m_{nn} \end{bmatrix} \boxplus \begin{bmatrix} -n_{11} & ... & -n_{1n} \\ ... & ... & ... \\ -n_{n1} & ... & -n_{nn} \end{bmatrix} \\
&= -m \boxplus -n \\
&= -(m \boxplus n) \\
\Rightarrow (m \boxplus n)^T &= -(m \boxplus n)
\end{aligned}
\tag{1.2.2}
$$

1.2.3 Multiplication

$$
\begin{aligned}
(a \boxdot m)^T &= \begin{bmatrix} am_{11} & ... & am_{n1} \\ ... & ... & ... \\ am_{1n} & ... & am_{nn} \end{bmatrix} \\
&= a \boxdot \begin{bmatrix} m_{11} & ... & m_{n1} \\ ... & ... & ... \\ m_{1n} & ... & m_{nn} \end{bmatrix} \\
&= a \boxdot \begin{bmatrix} -m_{11} & ... & -m_{1n} \\ ... & ... & ... \\ -m_{n1} & ... & -m_{nn} \end{bmatrix} \\
&= -a \boxdot \begin{bmatrix} m_{11} & ... & m_{1n} \\ ... & ... & ... \\ m_{n1} & ... & m_{nn} \end{bmatrix} \\
&= -a \boxdot m \\
\Rightarrow (a \boxdot m)^T &= -a \boxdot m
\end{aligned}
\tag{1.2.3}
$$

1.2.4 Conclusion

Since W_2 with these operations are valid under addition and multiplication with $\overrightarrow{0} \in W_2$, W_2 is a subspace of $M_{n\times n(\mathbb{R})}$.

1.3 Question c

1.3.1 $M_{n\times n}(\mathbb{R}) = W_1 + W_2$

Define $A \in M_{n\times n}(\mathbb{R})$

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

In this place, A is an arbitrary matrix.
Define $\frac{1}{2}(A + A^T) = M \wedge \frac{1}{2}(A - A^T) = N$

$$A = M + N$$

$$
\begin{aligned}
M^T &= (\frac{1}{2}(A + A^T))^T \\
&= \frac{1}{2}(A + A^T)^T \\
&= \frac{1}{2}(A^T + (A^T)^T) \\
&= \frac{1}{2}(A^T + A) \\
&= M \\
\Rightarrow & M \in W_1
\end{aligned} \tag{1.3.1-1}
$$

$$
\begin{aligned}
N^T &= (\frac{1}{2}(A - A^T))^T \\
&= \frac{1}{2}(A - A^T)^T \\
&= \frac{1}{2}(A^T - (A^T)^T) \\
&= \frac{1}{2}(A^T - A) \\
&= -\frac{1}{2}(A - A^T) \\
&= -N \\
\Rightarrow & N \in W_2
\end{aligned} \tag{1.3.1-2}
$$

$$
\begin{aligned}
A = M + N \\
A \in M_{n\times n}(\mathbb{R}) \\
M \in W_1 \\
N \in W_2 \\
\Rightarrow M_{n\times n}(\mathbb{R}) = W_1 + W_2
\end{aligned} \tag{1.3.1-3}
$$

1.3.2 $W_1 \cap W_2$

Define $P = W_1 \cap W_2$

$$
\begin{aligned}
P = \{A | A^T = A \cap A^T = -A\} \\
\Rightarrow P = \{A | A = -A\} \\
\Rightarrow P = \{A | A = \overrightarrow{0}\} \\
\Rightarrow P = \overrightarrow{0} \\
\Rightarrow W_1 \cap W_2 = \overrightarrow{0}
\end{aligned} \tag{1.3.2}
$$

2 Problem 2

2.1 Question a

Define $s = (1, 0, 1, 1) \wedge t = (2, 0, 2, 3)$
If $v \in \text{span}(S)$, then $v = m \cdot s + n \cdot t$, where $m \in \mathbb{R} \wedge b \in \mathbb{R}$.

$$m \cdot s + n \cdot t = (m + 2n, 0, m + 2n, m + 3n)$$

$$\begin{cases} m + 2n = 0 \\ 0 = 1 \\ m + 2n = 4 \\ m + 3n = 2 \end{cases} \tag{2.1}$$

It is easy to see that there is no solution for the set of equations 2.1 $\Rightarrow v \notin$ span(S)

2.2 Question b

If $v \in \text{span}(S)$, then $v = a(x^3) + b(2x + x^2) + c(x + x^3)$, where $a \in \mathbb{R} \wedge b \in \mathbb{R} \wedge c \in \mathbb{R}$

$$a(x^3) + b(2x + x^2) + c(x + x^3) = x(2b + c) + x^2(b) + x^3(a + c)$$

$$\begin{aligned} &\begin{cases} 2b + c = 1 \\ b = 0 \\ a + c = -1 \end{cases} \\ \Rightarrow &\begin{cases} a = -2 \\ b = 0 \\ c = 1 \end{cases} \\ \Rightarrow &v = -2(x^3) + (x + x^3) \\ \Rightarrow &v \in \text{span}(S) \end{aligned} \tag{2.2}$$

2.3 Question c

$$v = 1 + \cos(2x) = 1 + 2\cos^2(x) - 1 = 2\cos^2(x)$$

$$\text{Define } s = \sin^2(x) \wedge t = \cos^2(x)$$

$$v = 2t \Rightarrow v \in \mathrm{span}(s)$$

3 Problem 3

3.1 Question a

Define $S = a(1,2,-1) + b(2,-3,1) + c(2,3,-5)$,where $a \in \mathbb{R} \wedge b \in \mathbb{R} \wedge c \in \mathbb{R}$.

$$S = \begin{bmatrix} 1 & 2 & 2 \\ 2 & -3 & 3 \\ -1 & 1 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{3.1-1}$$

$$\begin{aligned} &\begin{cases} a + 2b + 2c = 0 \\ 2a - 3b + 3c = 0 \\ -a + b - 5c = 0 \end{cases} \\ \Rightarrow &\begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases} \end{aligned} \tag{3.1-2}$$

Since $a = 0 \wedge b = 0 \wedge c = 0$, S is linear independent.

3.2 Question b

Define $S = a(1 + x) + b(1 + x^2) + c(x + x^2)$,where $a \in \mathbb{R} \wedge b \in \mathbb{R} \wedge c \in \mathbb{R}$.

$$S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \tag{3.2-1}$$

$$\begin{aligned} &\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\ \sim &\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \Rightarrow &\begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases} \end{aligned} \tag{3.2-2}$$

Since $a = 0 \wedge b = 0 \wedge c = 0$, S is linear independent.

3.3 Question c

Define $S = a\left[\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right] + b\left[\begin{smallmatrix} -1 & 0 \\ 1 & 1 \end{smallmatrix}\right] + c\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right] + d\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, where $a \in \mathbb{R} \wedge b \in \mathbb{R} \wedge c \in \mathbb{R} \wedge d \in \mathbb{R}$

$$S = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \tag{3.3-1}$$

$$\begin{aligned} &\begin{cases} b + c + d = 0 \\ a + c + d = 0 \\ a + b + d = 0 \\ a + b + c = 0 \end{cases} \\ \Rightarrow &\begin{cases} a = 0 \\ b = 0 \\ c = 0 \\ d = 0 \end{cases} \end{aligned} \tag{3.3-2}$$

Since $a = 0 \wedge b = 0 \wedge c = 0 \wedge d = 0$, S is linear independent.

4 Problem 4

4.1 Question a

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{4.1-1}$$

$$\begin{aligned} &\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \\ \sim &\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \Rightarrow &\begin{cases} u = 0 \\ v = 0 \\ w = 0 \end{cases} \end{aligned} \tag{4.1-2}$$

Since $u = 0 \land v = 0 \land w = 0$, S_1 is linear independent.

4.2 Question b

$$S_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{4.2-1}$$

$$\begin{aligned} &\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \\ \sim &\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{4.2-2}$$

We can see from that the set of equations have infinite solutions. $\Rightarrow$ The vectors are linear dependent.

5 Problem 5

5.1 Question a

$$\begin{aligned}&\begin{bmatrix}1 & 0 & -1 & 0 & -3\\ 0 & 2 & 2 & -1 & 3\\ -1 & 0 & 0 & 1 & 0\end{bmatrix}\\ \sim&\begin{bmatrix}1 & 0 & 0 & -1 & 0\\ 0 & 1 & 0 & \frac{1}{2} & \frac{9}{2}\\ 0 & 0 & 1 & -1 & 3\end{bmatrix}\to \text{RREF}\end{aligned} \tag{5.1}$$

Since there are three pivot columns in the RREF, these vectors span $\mathbb{R}^3$.

5.2 Question b

Define $a \in \mathbb{R} \wedge b \in \mathbb{R} \wedge c \in \mathbb{R} \wedge d \in \mathbb{R} \wedge e \in \mathbb{R}$.

$$\begin{aligned}&\begin{bmatrix}1 & 0 & -1 & 0 & -3\\ 0 & 2 & 2 & -1 & 3\\ -1 & 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}a\\ b\\ c\\ d\\ e\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}\\ &\begin{cases}a-c+3e=0\\ 2b+2c-d+3e=0\\ -a+d=0\end{cases}\\ \Rightarrow & a=d=c-5e \wedge b=-\frac{c+6e}{2}\end{aligned} \tag{5.2}$$

The equations have infinite solutions. $\Rightarrow$ The vectors are linear dependent.

5.3 Question c

$$\begin{aligned}B&=\{u_1,u_3,u_5\}\\ &=\begin{bmatrix}1 & -1 & 3\\ 0 & 2 & 3\\ -1 & 0 & 0\end{bmatrix}\end{aligned} \tag{5.3-1}$$

$$
\begin{aligned}
&\begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 3 \\ -1 & 0 & 0 \end{bmatrix} \\
\sim &\begin{bmatrix} 1 & 0 & \frac{9}{2} \\ 0 & 2 & 3 \\ 0 & 0 & 9 \end{bmatrix} \rightarrow \text{RREF}
\end{aligned}
\tag{5.3-2}
$$

Since the RREF form has 3 pivot columns, the vectors form a basis of $\mathbb{R}^3$.

6 Reference

6.1 Collaborators

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