1 Problem 1

1.1 Question a

1.1.1 rref

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 2 \\ 1 & 0 & 3 \end{bmatrix} A_{12}(1)$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix} M_2(\frac{1}{2})$$

$$\sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} A_{13}(-1)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} A_{31}(-2)$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A_{32}(-1)$$

A can be reduced to $I \implies A$ is invertible.

$$\Rightarrow A_{32}(-1)A_{31}(-2)A_{13}(-1)M_2(\frac{1}{2})A_{12}(1)A = I$$

$$\Rightarrow A^{-1} = A_{32}(-1)A_{31}(-2)A_{13}(-1)M_2(\frac{1}{2})A_{12}(1)$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3 & 0 & -2 \\ \frac{3}{2} & \frac{1}{2} & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

1.1.2 Caley-Hamilton Theorem

$$
\begin{aligned}
&p_A(\lambda) = \det(\lambda I - A) \\
\Rightarrow &p_A(\lambda) = \lambda^3 - 6\lambda^2 + 9\lambda - 2 \\
\Rightarrow &p_A(A) = A^3 - 6A^2 + 9A - 2I = 0 \\
&A^{-1} = \frac{1}{2}(A^2 - 6A + 9I) \\
\Rightarrow &A^{-1} = \begin{bmatrix} 3 & 0 & -2 \\ \frac{3}{2} & \frac{1}{2} & -1 \\ -1 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

1.2 Question b

$$
\begin{aligned}
&\beta = \{1, x, x^2\} \\
&T(1) = 2 \\
&T(x) = x + 1 \\
&T(x^2) = (x-1)^2 + 4 = x^2 - 2x + 5 \\
\Rightarrow &A = [T]_\beta^\beta = \begin{bmatrix} 2 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

1.2.1 rref

$$
\begin{aligned}
&\begin{bmatrix} 2 & 1 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \\
\sim &\begin{bmatrix} 1 & \frac{1}{2} & \frac{5}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} M_1(\frac{1}{2}) \\
\sim &\begin{bmatrix} 1 & 0 & \frac{7}{2} \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} A_{21}(-\frac{1}{2}) \\
\sim &\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} A_{31}(-\frac{7}{2}) \\
\sim &\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A_{32}(2) \\
&A \text{ can be reduced to } I \implies A \text{ is invertible.} \\
\Rightarrow &A_{32}(2)A_{31}(-\frac{7}{2})A_{21}(-\frac{1}{2})M_1(\frac{1}{2})A = I \\
\Rightarrow &A^{-1} = A_{32}(2)A_{31}(-\frac{7}{2})A_{21}(-\frac{1}{2})M_1(\frac{1}{2}) \\
\Rightarrow &A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

1.2.2 Caley-Hamilton Theorem

$$
\begin{aligned}
&p_A(\lambda) = \det(\lambda I - A) \\
\Rightarrow &p_A(\lambda) = \lambda^3 - 4\lambda^2 + 5\lambda - 2 \\
\Rightarrow &p_A(A) = A^3 - 4A^2 + 5A - 2I = 0 \\
\Rightarrow &A^{-1} = \frac{1}{2}(A^2 - 4A + 5I) \\
&A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

2 Problem 2

2.1 Question a

$$
\begin{aligned}
&A^2:\\
&(\lambda+2)(\lambda-3)=0\\
\Rightarrow&\lambda^2-\lambda-6=0\\
\Rightarrow&A^2-A-6I=0\\
\Rightarrow&A^2=A+6I\\
\Rightarrow&a_2=1,b_2=6\\
A^n&=A^{n-1}A\\
&=(a_{n-1}A+b_{n-1}I)A\\
&=a_{n-1}A^2+b_{n-1}A\\
&=a_{n-1}+6a_{n-1}I+b_{n-1}A\\
&=(a_{n-1}+b_{n-1})A+6a_{n-1}I\\
\Rightarrow&\begin{cases}a_n=a_{n-1}+b_{n-1}\\ b_n=6a_{n-1}\end{cases}\\
\Rightarrow&\begin{cases}a_n=a_{n-1}+6a_{n-2}\\ b_n=6a_{n-1}\end{cases}\\
&a_n=a_{n-1}+6a_{n-2}\\
\Rightarrow&r^2-r-6=0\\
\Rightarrow&\begin{cases}r_1=-2\\ r_2=3\end{cases}\\
\Rightarrow&a_n=(-2)^nA+3^nB\\
&a_1=\frac{1}{6}b_2=1=-2A+3B\\
&a_0=a_2-a_1=0=A+B\\
\Rightarrow&\begin{cases}A=-\frac{1}{5}\\ B=\frac{1}{5}\end{cases}\\
\Rightarrow&\begin{cases}a_n=-\frac{1}{5}(-2)^n+\frac{1}{5}3^n\\ b_n=6(-\frac{1}{5}(-2)^{n-1}+\frac{1}{5}3^{n-1})\end{cases}
\end{aligned}
$$

2.2 Question b

$$
\begin{aligned}
&A^{2023} = a_{2023}A + b_{2023}I \\
&a_{2023} = -\frac{1}{5}(-2)^{2023} + \frac{1}{5}3^{2023} \\
&b_{2023} = 6(-\frac{1}{5}(-2)^{2022} + \frac{1}{5}3^{2022}) \\
\Rightarrow &A^{2023} = (-\frac{1}{5}(-2)^{2023} + \frac{1}{5}3^{2023})A + 6(-\frac{1}{5}(-2)^{2022} + \frac{1}{5}3^{2022})I
\end{aligned}
$$

3 Problem 3

3.1 Question a

$$
\begin{aligned}
&p_A(\lambda) = \det(\lambda I - A) \\
\Rightarrow &p_A(\lambda) = a_0 + a_1\lambda + ... + a_n\lambda^n \\
\Rightarrow &a_0 I + a_1 A + ... + a_n A^n = 0 \\
\Rightarrow &A^n = -\frac{1}{a_n}(a_0 I + a_1 A + ... + a_{n-1}A^{n-1}) \\
\Rightarrow &A^n \in \text{span}(I, A, A^2 ..., A^{n-1}) \\
\Rightarrow &\forall k > n : A^k = A^n \cdot A^{k-n} \in \text{span}(I, A, A^2 ..., A^{n-1}) \\
&\dim(\text{span}(I, A, A^2 ..., A^{n-1})) = n \Leftrightarrow I, A, A^2 ..., A^{n-1} \text{ are linear independent} \\
\Rightarrow &\dim(W) \leq n
\end{aligned}
$$

3.2 Question b

The λ in $p_A(\lambda)$ should be strictly be scalar, or the deterninant cannot be calculated.

$$
\begin{aligned}
&A \coloneqq \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\
&p_A(\lambda) = \det(A - \lambda I) \\
\Rightarrow &p_A(\lambda) = \det \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix} \\
&\text{Subtitute } \lambda \text{ as } A : \\
&p_A(A) = \det \begin{bmatrix} 1 - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} & 2 \\ 3 & 4 - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \end{bmatrix} \\
&\text{This is not a valid matrix, which makes the proof a wrong one.}
\end{aligned}
$$

4 Problem 4

4.1 Question a

4.1.1 Conjugate symmetry

$$
\begin{aligned}
&\langle p(x), q(x)\rangle = p(-1)q(-1) + p(1)q(1) + p(2)q(2) \\
&\overline{\langle q(x), p(x)\rangle} = \overline{q(-1)p(-1) + q(1)p(1) + q(2)p(2)} \\
&p(x) \wedge q(x) \in \mathbb{R} \implies \overline{q(-1)p(-1) + q(1)p(1) + q(2)p(2)} = q(-1)p(-1) + q(1)p(1) + q(2)p(2) \\
\Rightarrow &\langle p(x), q(x)\rangle = \overline{\langle q(x), p(x)\rangle}
\end{aligned}
$$

4.1.2 Linearity

$$
\begin{aligned}
&\langle af(x) + bg(x), q(x)\rangle \\
=&(af(-1) + bg(-1))q(-1) + (af(1) + bg(1))q(1) + (af(2) + bg(2))q(2) \\
=&af(-1)q(-1) + af(1)q(1) + af(2)q(2) + bg(-1)q(-1) + bg(1)q(1) + bg(2)q(2) \\
=&a(f(-1)q(-1) + f(1)q(1) + f(2)q(2)) + b(g(-1)q(-1) + g(1)q(1) + g(2)q(2)) \\
=&a\langle f(x), q(x)\rangle + b\langle g(x), q(x)\rangle
\end{aligned}
$$

4.1.3 Positive

$$
\begin{aligned}
&\langle p(x), p(x)\rangle \\
=&(p(-1))^2 + (p(1))^2 + (p(2))^2 \geq 0
\end{aligned}
$$

4.1.4 Conclusion

Since it satisfies all three axioms, it is a valid inner product.

4.2 Question b

4.2.1 Positive

$$
\begin{aligned}
&A \coloneqq \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \\
&\langle A, A \rangle \\
=&\mathrm{Tr}(A^2) \\
=&\mathrm{Tr}(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}) \\
=&\mathrm{Tr}(\begin{bmatrix} a^2 + bd + cg & ad + be + ch & ac + bf + ci \\ da + ed + fg & bd + e^2 + fh & dc + ef + fi \\ ga + hd + ig & gb + he + ih & cg + fh + i^2 \end{bmatrix}) \\
=&a^2 + bd + cg + bd + e^2 + fh + cg + fh + i^2 \\
=&a^2 + e^2 + i^2 + 2(bd + cg + fh) \\
\Rightarrow&\langle A, A \rangle \geq 0 \Leftrightarrow a^2 + e^2 + i^2 + 2(bd + cg + fh) \geq 0 \\
&\exists A \in M_{3\times 3}(\mathbb{R}) : a^2 + e^2 + i^2 + 2(bd + cg + fh) \leq 0 \\
\Rightarrow&\langle A, A \rangle \text{ is not always } \geq 0
\end{aligned}
$$

5 Problem 5

5.1 Question a

$$
\begin{aligned}
&\|u+v\|^2 + \|u-v\|^2 \\
=&\langle u+v, u+v\rangle + \langle u-v, u-v\rangle \\
=&\langle u, u+v\rangle + \langle v, u+v\rangle + \langle u, u-v\rangle - \langle v, u-v\rangle \\
=&\overline{1}\langle u,u\rangle + \overline{1}\langle u,v\rangle + \overline{1}\langle v,u\rangle + \overline{1}\langle v,v\rangle + \overline{1}\langle u,u\rangle + \overline{-1}\langle u,v\rangle - \overline{1}\langle v,u\rangle - \overline{-1}\langle v,v\rangle \\
=&2\langle u,u\rangle + 2\langle v,v\rangle \\
=&2\|u\|^2 + 2\|v\|^2 \\
=&2(\|u\|^2 + \|v\|^2)
\end{aligned}
$$

5.2 Question b

5.2.1 proof 1

$$
\begin{aligned}
&\text{Cauchy-Schwatz Inequality :} \\
&|\langle u,v\rangle|^2 \leq \|u\|^2\|v\|^2 \\
\Rightarrow &\frac{|\langle u,v\rangle|^2}{\|u\|^2\|v\|^2} \leq 1 \\
\Rightarrow &-1 \leq \frac{|\langle u,v\rangle|}{\|u\|\|v\|} \leq 1
\end{aligned}
$$

5.2.2 proof 2

$$
\begin{aligned}
&\|u-v\|^2 \\
=&\langle u-v, u-v\rangle \\
=&\langle u, u-v\rangle - \langle v, u-v\rangle \\
=&\overline{1}\langle u,u\rangle + \overline{-1}\langle u,v\rangle - \overline{1}\langle v,u\rangle - \overline{-1}\langle v,v\rangle \\
=&\langle u,u\rangle + \langle v,v\rangle - 2\langle u,v\rangle \\
=&\|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\frac{|u,v|}{\|u\|\|v\|} \\
=&\|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta
\end{aligned}
$$

5.3 Question c

$$
\begin{aligned}
\|x^2+1\| =& \sqrt{\langle x^2+1, x^2+1\rangle} \\
=& \sqrt{\int_0^1 (x^2+1)^2 dx} \\
=& \sqrt{\frac{28}{15}} \\
\|x-1\| =& \sqrt{\langle x-1, x-1\rangle} \\
=& \sqrt{\int_0^1 (x-1)^2 dx} \\
=& \sqrt{\frac{1}{3}} \\
\|x+1\| =& \sqrt{\langle x+1, x+1\rangle} \\
=& \sqrt{\int_0^1 (x+1)^2 dx} \\
=& \sqrt{\frac{7}{3}}
\end{aligned}
$$

$$
\begin{aligned}
\langle x^2+1, x-1\rangle &= \int_0^1 (x^2+1)(x-1)dx \\
&= -\frac{7}{12} \\
\langle x^2+1, x+1\rangle &= \int_0^1 (x^2+1)(x+1)dx \\
&= \frac{25}{12} \\
\langle x-1, x+1\rangle &= \int_0^1 (x-1)(x+1)dx \\
&= -\frac{2}{3}
\end{aligned}
$$

$$
\begin{aligned}
\theta_{x^2+1,x-1} &= \arccos(\frac{\langle x^2+1, x-1\rangle}{\|x^2+1\|\|x-1\|}) \\
&= \arccos(\frac{-\frac{7}{12}}{\sqrt{\frac{28}{15}}\sqrt{\frac{1}{3}}}) \\
&= \arccos(-\frac{\sqrt{35}}{8}) \\
\theta_{x^2+1,x+1} &= \arccos(\frac{\langle x^2+1, x+1\rangle}{\|x^2+1\|\|x+1\|}) \\
&= \arccos(\frac{\frac{25}{12}}{\sqrt{\frac{28}{15}}\sqrt{\frac{7}{3}}}) \\
&= \arccos(\frac{25\sqrt{5}}{56}) \\
\theta_{x-1,x+1} &= \arccos(\frac{\langle x-1, x+1\rangle}{\|x-1\|\|x+1\|}) \\
&= \arccos(\frac{-\frac{2}{3}}{\sqrt{\frac{1}{3}}\sqrt{\frac{7}{3}}}) \\
&= \arccos(-\frac{2}{\sqrt{7}})
\end{aligned}
$$

6 Reference

Jeffery Shu

Frank Zhu