1 Problem 1

1.1 Question a

$$
\begin{aligned}
T(a,b) =&(-2a+b,-10a+9b)\\
A =&\begin{bmatrix}-2 & 1\\ -10 & 9\end{bmatrix}\\
\det(A-I\lambda) =&(-2-\lambda)(9-\lambda)-(-10)\\
=&\lambda^2-7\lambda-8\\
=&0\\
\Rightarrow&\lambda=-1 \vee \lambda=8
\end{aligned}
$$

$$
\begin{aligned}
&\lambda=-1\\
&(A-I\lambda)v=\overrightarrow{0}\\
\Rightarrow&\begin{bmatrix}-1 & 1\\ -10 & 10\end{bmatrix}v=\overrightarrow{0}\\
\Rightarrow&v=\begin{bmatrix}1\\1\end{bmatrix}\\
&\lambda=-8\\
&(A-I\lambda)v=\overrightarrow{0}\\
\Rightarrow&\begin{bmatrix}-10 & 1\\ -10 & 1\end{bmatrix}v=\overrightarrow{0}\\
\Rightarrow&v=\begin{bmatrix}1\\10\end{bmatrix}\\
\Rightarrow&\beta=\{a+b,a+10b\}
\end{aligned}
$$

1.2 Question b

$$
\begin{aligned}
&B = 1, x, x^2, x^3 \\
&T(1) = -1 \\
&T(x) = x - 2 \\
&T(x^2) = 2x^2 - 2 \\
&T(x^3) = 3x^3 + 6x - 8 \\
&A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ -8 & 6 & 0 & 3 \end{bmatrix} \\
&\det(A - I\lambda) = 0 \\
\Rightarrow &(-1-\lambda)(1-\lambda)(2-\lambda)(3-\lambda) = 0 \\
\Rightarrow &\lambda = -1 \lor \lambda = 1 \lor \lambda = 2 \lor \lambda = 3 \\
&\lambda = -1 \\
&\begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ -2 & 0 & 3 & 0 \\ -8 & 6 & 0 & 4 \end{bmatrix} v = \overrightarrow{0} \\
\Rightarrow &v = \begin{bmatrix} 1 \\ 1 \\ \frac{2}{3} \\ \frac{1}{2} \end{bmatrix} \\
&\lambda = 1 \\
&\begin{bmatrix} -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -8 & 6 & 0 & 2 \end{bmatrix} v = \overrightarrow{0} \\
\Rightarrow &v = \begin{bmatrix} 0 \\ -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \\
&\lambda = 2 \\
&\begin{bmatrix} -3 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ -8 & 6 & 0 & 1 \end{bmatrix} v = \overrightarrow{0} \\
\Rightarrow &v = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}
\end{aligned}
$$

$\lambda = 3$

$$\begin{bmatrix} -4 & 0 & 0 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -1 & 0 \\ -8 & 6 & 0 & 0 \end{bmatrix} v = \overrightarrow{0}$$

$$\Rightarrow v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \beta = \{1 + x + \frac{2}{3}x^2 + \frac{1}{2}x^3, -\frac{1}{3}x + x^3, x^2, x^3\}$$

2 Problem 2

2.1 Question a

$$
\begin{aligned}
&B = \{1, x, x^2\} \\
&T(1) = 3 \\
&T(x) = 3x + 1 \\
&T(x^2) = 4x^2 \\
\Rightarrow &A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \\
&\det(A - I\lambda) = 0 \\
\Rightarrow &(3-\lambda)^2(4-\lambda) = 0 \\
\Rightarrow &\lambda = 3 \vee \lambda = 4 \\
&\lambda = 3 \\
&\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} v = \overrightarrow{0} \\
\Rightarrow &v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
&\lambda = 4 \\
&\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} v = \overrightarrow{0} \\
\Rightarrow &v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&\text{Eigenvectors cannot form a basis for } A \\
&T \text{ is not diagonizable}
\end{aligned}
$$

2.2 Question b

$$\det(A - I\lambda) = 0$$

$$(-1-\lambda)\det\begin{bmatrix} -2-\lambda & -3 & 1 \\ 1 & 2-\lambda & -1 \\ -3 & -3 & 2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (\lambda+1)^2(\lambda-1)(\lambda+2) = 0$$

$$\Rightarrow \lambda = -1 \vee \lambda = 1 \vee \lambda = -2$$

$$\lambda = -1$$

$$\begin{bmatrix} 0 & -4 & -4 & 3 \\ 0 & -1 & -3 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & -3 & -3 & 3 \end{bmatrix} v = \overrightarrow{0}$$

$$\Rightarrow v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = 1$$

$$\begin{bmatrix} -2 & -4 & -4 & 3 \\ 0 & -3 & -3 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & -3 & -3 & 1 \end{bmatrix} v = \overrightarrow{0}$$

$$\Rightarrow v = \begin{bmatrix} 5 \\ 3 \\ -1 \\ 6 \end{bmatrix}$$

$$\lambda = -2$$

$$\begin{bmatrix} 1 & -4 & -4 & 3 \\ 0 & 0 & -3 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & -3 & -3 & 4 \end{bmatrix} v = \overrightarrow{0}$$

$$\Rightarrow v = \begin{bmatrix} 7 \\ 3 \\ 1 \\ 3 \end{bmatrix}$$

Eigenvectors cannot form a basis for A

T is not diagonizable

3 Problem 3

3.1 Question a

$$
\begin{aligned}
&A = SDS^{-1} \\
\Leftrightarrow &AS = SD \\
\Leftrightarrow &S = A^{-1}SD \\
\Leftrightarrow &SD^{-1} = A^{-1}S \\
\Leftrightarrow &A^{-1} = SD^{-1}S^{-1} \\
\therefore &A = SDS^{-1} \Leftrightarrow A^{-1} = SD^{-1}S^{-1} \\
\Rightarrow &A \text{ is diagonizable } \Leftrightarrow \ A^{-1} \text{ is diagonizable}
\end{aligned}
$$

3.2 Question b

$$
\begin{aligned}
&A = SDS^{-1} \\
&A^2 = SD^2S^{-1} \\
&A^3 = SD^3S^{-1} \\
&A^3 + A^2 + A^1 + I \\
=&SD^3S^{-1} + SD^2S^{-1} + SDS^{-1} + SIS^{-1} \\
=&S(D^3S^{-1} + D^2S^{-1} + DS^{-1} + IS^{-1}) \\
=&S(D^3 + D^2 + D + I)S^{-1} \\
\therefore &A^3 + A^2 + A^1 + I \text{ is diagonizable}
\end{aligned}
$$

3.3 Question c

$$
\begin{aligned}
&A = SDS^{-1} \\
\Leftrightarrow &A^T = (SDS^{-1})^T \\
\Leftrightarrow &A^T = (S^{-1})^T D^T S^T \\
&P \coloneqq (S^{-1})^T \\
&PP^{-1} = I \\
&(S^{-1})^T P^{-1} = I \\
&((S^{-1})^T P^{-1})^T = I \\
&(P^{-1})^T S^{-1} = I \\
&(P^{-1})^T = S \\
&P^{-1} = S^T \\
\Rightarrow &A^T = PD^T P^{-1} \\
\therefore &A = SDS^{-1} \Leftrightarrow A^T = PD^T P^{-1} \\
&A \text{ is diagonizable } \Leftrightarrow \ A^T \text{ is diagonizable}
\end{aligned}
$$

4 Problem 4

4.1 Question a

$$\begin{aligned}
Tv =& \lambda v \\
Tv =& T^2 v \\
=& T \cdot Tv \\
=& T\lambda v \\
=& \lambda T v \\
=& \lambda\lambda v \\
=& \lambda^2 v \\
=& \lambda v \\
\Rightarrow& \lambda^2 - \lambda = 0 \rightarrow v \neq 0 \\
\Rightarrow& \lambda = 0 \vee \lambda = 1
\end{aligned}$$

Since all the eigenvalues are distinct, T is diagonizable

4.2 Question b

$$\begin{aligned}
& T(V) \coloneqq WV \\
\Rightarrow& WV = V^T \\
& T(WV) = (V^T)^T = V = WWV \\
\Rightarrow& WW = I \\
& Wv = \lambda v \\
\Rightarrow& WWv = W\lambda v = \lambda W v = \lambda^2 v \\
& WW = I \Rightarrow WWv = v
\end{aligned}$$

$$\Rightarrow \lambda^2 v = v$$

$$\Rightarrow \lambda = \pm 1$$

Since all the eigenvalues are distinct, T is diagonizable

5 Problem 5

5.1 Question a

$$\begin{aligned}
&D(f(x)) \coloneqq Af(x) \\
&D(e^{\lambda x}) = Ae^{\lambda x} \\
&(e^{\lambda x})' = \lambda e^{\lambda x} \\
\Rightarrow &Ae^{\lambda x} = \lambda e^{\lambda x} \\
&Af_\lambda(x) = \lambda f_\lambda(x) \\
\Rightarrow &f_\lambda(x) \text{ are eigenvectors}
\end{aligned}$$

5.2 Question b

$$e^{\lambda x} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Take the first $n-1$ terms of the expansion$T^{n-1}{}_\lambda$ as the approximation to $e^{\lambda x}$

$$\begin{bmatrix} f_{\lambda_1}x \\ f_{\lambda_2}x \\ f_{\lambda_3}x \\ \vdots \\ f_{\lambda_n}x \end{bmatrix} \approx \begin{bmatrix} T^{n-1}{}_{\lambda_1} \\ T^{n-1}{}_{\lambda_2} \\ T^{n-1}{}_{\lambda_3} \\ \vdots \\ T^{n-1}{}_{\lambda_n} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \lambda_1 x & (\lambda_1 x)^2 & \cdots & (\lambda_1 x)^{n-1} \\ 1 & \lambda_2 x & (\lambda_2 x)^2 & \cdots & (\lambda_2 x)^{n-1} \\ 1 & \lambda_3 x & (\lambda_3 x)^2 & \cdots & (\lambda_3 x)^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n x & (\lambda_n x)^2 & \cdots & (\lambda_n x)^{n-1} \end{bmatrix}}_{V} \begin{bmatrix} \frac{1}{0!} \\ \frac{1}{1!} \\ \frac{1}{2!} \\ \vdots \\ \frac{1}{(n-1)!} \end{bmatrix}$$

$$\det(V) = \prod_{0 \leqslant i < j \leqslant n} (\lambda_j x - \lambda_i x)$$

$\lambda_1, \ldots, \lambda_n$ are distinct values $\implies \lambda_j x - \lambda_i x \neq 0$

$\Rightarrow \det(V) \neq 0$

$\Rightarrow V$ is linear independent

By induction: $\nexists k \in \mathbb{Z}^+ : \exists m \in \mathbb{Z}^+ : T^k{}_{\lambda_m} = \sum_{i=0}^{m-1} a_i T^k{}_{\lambda_{m-1}}$

$$\Rightarrow \begin{bmatrix} f_{\lambda_1}x \\ f_{\lambda_2}x \\ \vdots \\ f_{\lambda_n}x \end{bmatrix} \text{ is linear independent}$$

5.3 Question c

$$
\begin{aligned}
&\text{Set } D(f(x)) = Af(x) = \lambda f(x) \\
&D(f(x)) = f'(x) \\
\Rightarrow &f'(x) = \lambda f(x) \\
&y \coloneqq f(x) \\
\Rightarrow &\frac{dy}{dx} - \lambda y = 0 \\
&I = e^{\int -\lambda \, dx} \\
\Rightarrow &I = e^{-\lambda x} \\
&e^{-\lambda x}\frac{dy}{dx} - e^{-\lambda x}\lambda y = 0 \\
&\frac{d}{dx}(e^{-\lambda x} y) = 0 \\
&e^{-\lambda x} y = C \\
&y = Ce^{\lambda x} \\
\Rightarrow &f(x) = Cf_\lambda(x)
\end{aligned}
$$

5.4 Question 4

$$
\begin{aligned}
&\forall \lambda \in \mathbb{Z}^+ \wedge C \in \mathbb{R} : Af(x) = \lambda f(x), \text{ where } f(x) = Ce^{\lambda x} \\
&\text{But } Ce^{\lambda x} \text{ are linear dependent, with only one independent vector} \\
\Rightarrow &\text{There is one eigenvector } f_\lambda(x) = e^{\lambda x} \text{ for every } \lambda \\
&\text{As proved before: } f_\lambda(x) \text{ with different } \lambda \text{ are linear independent} \\
\Rightarrow &\text{It can form an ordered basis for } D \\
\Rightarrow &D \text{ is diagonizable}
\end{aligned}
$$

6 Reference

Jeffery Shu

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