1 Problem 1

1.1 Question a

Define $f(x) = ax^2 + bx + c \in W_1 \wedge a, b, c \in \mathbb{R}$

$$\begin{aligned}
&\int_{-1}^{1} f(x)dx = 0 \\
\Rightarrow &\int_{-1}^{1} ax^2 + bx + c = 0 \\
\Rightarrow &\left[\frac{1}{3}ax^3 + \frac{1}{2}bx^2 + cx\right]_{-1}^{1} = 0 \\
\Rightarrow &(\frac{1}{3}a + \frac{1}{2}b + c) - (-\frac{1}{3}a + \frac{1}{2}b - c) = 0 \\
\Rightarrow &\frac{2}{3}a + 2c = 0 \\
\Rightarrow &a = -3c \\
\Rightarrow &\text{span}(W_1) = \{\langle 3, 0, 1\rangle, \langle 0, 1, 0\rangle\} \\
\Rightarrow &\dim(W_1) = 2
\end{aligned} \tag{1.1}$$

1.2 Question b

Define $f(x) = ax^2 + bx + c \in W_2 \wedge a, b, c \in \mathbb{R}$

$$\begin{aligned}
&f'(x) = 2ax + b \\
&f'(1) = 0 \\
\Rightarrow &2a + b = 0 \\
\Rightarrow &b = -2a \\
\Rightarrow &\text{span}(W_2) = \{\langle 1, -2, 0\rangle, \langle 0, 0, 1\rangle\} \\
\Rightarrow &\dim(W_2) = 2
\end{aligned} \tag{1.2}$$

1.3 Question c

Define $f(x) = ax^2 + bx + c \in W_1 \cap W_2 \wedge a, b, c \in \mathbb{R}$

From (1.1) and (1.2):

$$
\begin{aligned}
&a = -3c \wedge b = -2a \\
\Rightarrow &a = -3c, b = 6c, c = c \qquad (1.3) \\
\Rightarrow &\text{span}(W_1 \cap W_2) = \{\langle -3, 6, 1 \rangle\} \\
\Rightarrow &\dim(W_1 \cap W_2) = 1
\end{aligned}
$$

2 Problem 2

Proof. Define span$(W_1 \cap W_2) = \{u_1, u_2, ..., u_p\}$

Since $W_1 \cap W_2 \subset W_1$, we can expand the basis of $W_1 \cap W_2$ to get the basis of span$(W_1) = \{u_1, u_2, ..., u_p, v_1, v_2, ..., v_m\}$.

For the same reason, span$(W_1) = \{u_1, u_2, ..., u_p, w_1, w_2, ..., w_n\}$

$\Rightarrow$ span$(W_1 + W_2) = \{u_1, u_2, ..., u_p, v_1, v_2, ..., v_m, w_1, w_2, ..., w_n\}$

2.1 Linear Independence

We have to show that $\{u_1, u_2, ..., u_p, v_1, v_2, ..., v_m, w_1, w_2, ..., w_n\}$ is linear independent.

$$
\begin{aligned}
&a_1u_1 + a_2u_2 + ... + a_pu_p + b_1v_1 + b_2v_2 + ... + b_mv_m + c_1w_1 + c_2w_2 + ... + c_nw_n = 0 \\
&a_1, ..., a_p, b_1, ..., b_m, c_1, ..., c_n \in \mathbb{R} \\
&\Rightarrow \sum_{i=1}^{p} a_iu_i + \sum_{j=1}^{m} b_jv_j + \sum_{k=1}^{n} c_kw_k = 0 \\
&\Rightarrow -\sum_{k=1}^{n} c_kw_k = \sum_{i=1}^{p} a_iu_i + \sum_{j=1}^{m} b_jv_j \\
&\Rightarrow -\sum_{k=1}^{n} c_kw_k \in W_1 \\
&\text{But } -\sum_{k=1}^{n} c_kw_k \in W_2 \\
&\Rightarrow -\sum_{k=1}^{n} c_kw_k \in W_1 \cap W_2 \\
&\Rightarrow -\sum_{k=1}^{n} c_kw_k = d_1u_1 + d_2u_2 + ... + d_qu_q \\
&d_1, d_2, ..., d_q \in \mathbb{R} \\
&\Rightarrow c_1w_1 + c_2w_2 + ... + c_nw_n + d_1u_1 + d_2u_2 + ... + d_qu_q = 0 \\
&u_1, u_2, ..., u_q, w_1, w_2, ..., w_n \text{ is linear independent since they span } W_2 \\
&\Rightarrow d_1 = d_2 = ... = d_q = c_1 = c_2 + ... = c_n = 0 \\
&\text{For the same reason, } a_1 = a_2 = ... = a_p = b_1 = b_2 = b_m = 0 \\
&\Rightarrow \{u_1, u_2, ..., u_p, v_1, v_2, ..., v_m, w_1, w_2, ..., w_n\} \text{ is linear independent}
\end{aligned} \tag{2.1}
$$

2.2 Dimension

$\{u_1, u_2, ..., u_p, v_1, v_2, ..., v_m, w_1, w_2, ..., w_n\}$ is linear independent
$\Rightarrow \dim(W_1 + W_2) = p + m + n$
For the same reason:

$$\begin{aligned}
&\dim(W_1) = p + m \\
&\dim(W_2) = p + n \\
&\dim(W_1 \cap W_2) = p
\end{aligned} \tag{2.2-1}$$

$$\begin{aligned}
\dim(W_1 + W_2) &= p + m + n \\
&= (p + m) + (p + n) - p \\
&= \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)
\end{aligned} \tag{2.2-2}$$

□

3 Problem 3

3.1 Question a

Define $u = (a, b, c) \wedge v = (x, y, z) \wedge m \in \mathbb{R}$

3.1.1 Addition

$$\begin{aligned}
T(u+v) &= T(a+x, b+y, c+z) \\
&= (2(a+x)+(b+y)-(c+z), (a+x)(b+y)-(c+z)) \\
&= (2a+b-c+2x+y-z, ab+xy+ax+by-c-z) \\
T(u)+T(v) &= T(a,b,c)+T(x,y,z) \\
&= (2a+b-c, ab-c)+(2x+y-z, xy-z) \\
&= (2a+b-c+2x+y-z, ab+xy-c-z) \\
&\neq T(u+v)
\end{aligned} \tag{3.1.1}$$

3.1.2 Conclusion

Since $T(u+v) \neq T(u)+T(v)$, T is not a linear transformation.

3.2 Question b

Define $f(x) = ax^2 + bx + c \wedge g(x) = dx^2 + ex + f \wedge m \in \mathbb{R}$

3.2.1 Addition

$$
\begin{aligned}
T(f(x)+g(x)) =& T((a+d)x^2+(b+e)x+(c+f)) \\
=& x((a+d)x^2+(b+e)x+(c+f)) \\
& + \int_0^x (a+d)t^2+(b+e)t+(c+f)dt \\
=& (a+d)x^3+(b+e)x^2+(c+f)x \\
& + \left[\frac{1}{3}(a+d)t^3+\frac{1}{2}(b+e)t^2+(c+f)t\right]_0^x \\
=& (a+d)x^3+(b+e)x^2+(c+f)x \\
& + \frac{1}{3}(a+d)x^3+\frac{1}{2}(b+e)x^2+(c+f)x \\
=& \frac{4}{3}(a+d)x^3+\frac{3}{2}(b+e)x^2+2(c+f)x \\
T(f(x))+T(g(x)) =& x(ax^2+bx+c)+\int_0^x at^2+bt+cdt \\
& + x(dx^2+ex+f)+\int_0^x dt^2+et+fdt \\
=& ax^3+bx^2+cx+\left[\frac{1}{3}at^3+\frac{1}{2}bt^2+ct\right]_0^x \\
& + dx^3+ex^2+fx+\left[\frac{1}{3}dt^3+\frac{1}{2}et^2+ct\right]_0^x \\
=& (a+d)x^3+(b+e)x^2+(c+f)x \\
& + \frac{1}{3}(a+d)x^3+\frac{1}{2}(b+e)x^2+(c+f)x \\
=& \frac{4}{3}(a+d)x^3+\frac{3}{2}(b+e)x^2+2(c+f)x \\
=& T(f(x)+g(x))
\end{aligned}
\tag{3.2.1}
$$

3.2.2 Multiplication

$$
\begin{aligned}
mT(f(x)) =& m(x(ax^2+bx+c)+\int_0^x at^2+bt+cdt) \\
=& m(ax^3+bx^2+cx+\left[\frac{1}{3}at^3+\frac{1}{2}bt^2+ct\right]_0^x) \\
=& m(ax^3+bx^2+cx+\frac{1}{3}ax^3+\frac{1}{2}bx^2+cx) \\
=& \frac{4}{3}amx^3+\frac{3}{2}bmx^2+2cmx \\
T(mf(x)) =& x(m(ax^2+bx+c))+\int_0^x m(at^2+bt+c)dt \\
=& m(ax^3+bx^2+cx)+\left[\frac{1}{3}amt^3+\frac{1}{2}bmt^2+cmt\right]_0^x \\
=& amx^3+bmx^2+cmx+\frac{1}{3}amx^3+\frac{1}{2}bmx^2+cmx \\
=& \frac{4}{3}amx^3+\frac{3}{2}bmx^2+2cmx \\
=& mT(f(x))
\end{aligned}
\tag{3.2.2}
$$

3.2.3 Conclusion

Since T meets the requirement of both addition and multiplication, T is a linear transformation.

3.3 Question c

Define $P, Q \in M_{n\times n}(\mathbb{R}) \wedge m \in \mathbb{R}$

3.3.1 Addition

$$
\begin{aligned}
T(P+Q) &= (P+Q)M - M(P+Q) \\
&= PM + QM - MP - MQ \\
&= PM - MP + QM - MQ \\
T(P)+T(Q) &= PM - MP + QM - MQ \\
&= T(P+Q)
\end{aligned}
\tag{3.3.1}
$$

3.3.2 Multiplication

$$
\begin{aligned}
mT(P) &= m(PM - MP) \\
&= mPM - mMP \\
T(mP) &= (mP)M - M(mP) \\
&= mPM - mMP \\
&= mT(P)
\end{aligned}
\tag{3.3.2}
$$

3.3.3 Conclusion

Since T meets the requirement of both addition and multiplication, T is a linear transformation.

4 Problem 4

4.1 Question a

$$
\begin{aligned}
&T(a,b,c) \\
=&(a-2b+3c)x+(2c-a) \\
=&\begin{bmatrix}a-2b+3c, 2c-a\end{bmatrix}\cdot\begin{bmatrix}x\\1\end{bmatrix}
\end{aligned}
\tag{4.1}
$$

4.1.1 Kernel

$$
\begin{aligned}
&\begin{cases}a-2b+3c=0\\2c-a=0\end{cases}\\
\Rightarrow&\begin{cases}a=2t\\b=\frac{5}{2}t\\c=t\end{cases}\\
\Rightarrow&\text{Kernel } =\{\begin{bmatrix}2\\\frac{5}{2}\\1\end{bmatrix}\}\\
&\dim(\text{Kernel})=1
\end{aligned}
\tag{4.1.2}
$$

4.1.2 Range

$$
\begin{aligned}
&\begin{bmatrix} a-2b+3c \\ 2c-a \end{bmatrix} \\
=&\begin{bmatrix} 1 & -2 & 3 \\ -1 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\
&\text{Column space} \\
&\begin{bmatrix} 1 & -1 \\ 0 & -2 \\ 2 & -3 \end{bmatrix} \\
\sim&\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\
\Rightarrow&\text{Range } = \{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\} \\
&\dim(\text{Range}) = 2
\end{aligned} \tag{4.1.3}
$$

4.2 Question b

Define $A = \left[\begin{smallmatrix} a & b & c \\ d & e & f \\ g & h & i \end{smallmatrix}\right]$

4.2.1 Kernel

$$
\begin{aligned}
&A - A^T = 0 \\
\Rightarrow&A = A^T \\
\Rightarrow&A = \left[\begin{smallmatrix} a & b & c \\ b & e & f \\ c & f & i \end{smallmatrix}\right] \\
&\text{null space for symmetric matrix :} \\
&\{\left[\begin{smallmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0&0&1 \\ 0&0&0 \\ 1&0&0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0&0&0 \\ 0&1&0 \\ 0&0&0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0&0&0 \\ 0&0&1 \\ 0&1&0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&1 \end{smallmatrix}\right]\} \\
\Rightarrow&\text{Kernel}(A) = \{\left[\begin{smallmatrix} 1&0&0 \\ 0&0&0 \\ 0&0&0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0&0&1 \\ 0&0&0 \\ 1&0&0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0&0&0 \\ 0&1&0 \\ 0&0&0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0&0&0 \\ 0&0&1 \\ 0&1&0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&1 \end{smallmatrix}\right]\} \\
&\dim(A) = 6
\end{aligned} \tag{4.2.1}
$$

4.2.2 Range

$$
\begin{aligned}
&A - A^T \\
=&\begin{bmatrix} 0 & b-d & c-g \\ d-b & 0 & f-h \\ g-c & h-f & 0 \end{bmatrix} \\
=&\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}(b-d) + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}(c-g) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}(f-h) \qquad (4.2.2) \\
\Rightarrow&\mathrm{Range}(A) = \{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\} \\
&\dim(A) = 3
\end{aligned}
$$

5 Problem 5

5.1 Question a

$$
\begin{aligned}
&3x^2 - x + 2 \\
=&3(x^2 + x - 1) - 4x + 5 \\
=&3(x^2 + x - 1) - 4(x + 2) + 13 \\
\Rightarrow &T(3x^2 - x + 2) \\
=&T(3(x^2 + x - 1) - 4(x + 2) + 13) \\
=&3T(x^2 + x - 1) - 4T(x + 2) + 13T(1) \\
=&3 \times (2, 1) - 4 \times (1, -1) + 13 \times T(1) \\
=&(6, 3) - (4, -4) + (65, 0) \\
=&(67, -1)
\end{aligned} \tag{5.1}
$$

5.2 Question b

$$
\begin{aligned}
&\sum_{i=0}^{n} c_i v_i = 0 \\
&T(\sum_{i=0}^{n} c_i v_i) \\
=&\sum_{i=0}^{n} c_i T(v_i) \\
=&0 \\
&T(v_i) \text{ LI} \\
\Rightarrow &c_i \text{ can only } = 0 \\
\Rightarrow &v_i \text{ LI}
\end{aligned} \tag{5.2}
$$

5.3 Question c

$$
\begin{aligned}
&T \text{ is one-to-one}\\
\Rightarrow &\operatorname{nullity}(T) = 0\\
&\operatorname{nullity}(T) + \operatorname{rank}(T) = \dim(V)\\
\Rightarrow &\operatorname{rank}(T) = \dim(V) \qquad (5.3)\\
\Rightarrow &\dim(\operatorname{Range}(T)) = \dim(V)\\
\Rightarrow &\operatorname{Range}(T) = V\\
\Rightarrow &T \text{ is onto}
\end{aligned}
$$

6 Reference

6.1 Collaborator

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