1 Problem 1

1.1 Question a

1.1.1 $(3, -4, 7)$

$$
\begin{aligned}
\text{Suppose: } & (3,-4,7) \in \operatorname{im}(T) \\
& \Rightarrow T(a_1, a_2, a_3) = (3, -4, 7) \\
& \Rightarrow \begin{cases} a_1 + a_2 = 3 \\ a_1 - a_2 - a_3 = -4 \\ 2a_2 + a_3 = 7 \end{cases} \\
& \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 7 \end{bmatrix} \\
& \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 1 & -1 & -1 & -4 \\ 0 & 2 & 1 & 7 \end{array}\right] \\
& \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -2 & -1 & -7 \\ 0 & 2 & 1 & 7 \end{array}\right] A_{21}(-1) \\
& \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & 0 & 0 \end{array}\right] A_{32}(1) \\
& \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & \frac{1}{2} & \frac{7}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] M_2(-\frac{1}{2}) \\
& \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{7}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] A_{12}(-1) \\
& \Rightarrow \begin{cases} a_1 = \frac{1}{2}a_3 - \frac{1}{2} \\ a_2 = -\frac{1}{2}a_3 + \frac{7}{2} \\ a_3 = a_3 \end{cases} \\
& \Rightarrow \text{There are solutions to the system} \\
& \Rightarrow (3,-4,7) \in \operatorname{im}(T)
\end{aligned}
$$

1.1.2 $(0, 2, 2)$

Suspose: $(0,2,2) \in \mathrm{im}(T)$

$$\Rightarrow T(b_1, b_2, b_3) = (0,2,2)$$

$$\Rightarrow \begin{cases} b_1 + b_2 = 0 \\ b_1 - b_2 - b_3 = 2 \\ 2b_2 + b_3 = 2 \end{cases}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & -1 & -1 & 2 \\ 0 & 2 & 1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 2 \\ 0 & 2 & 1 & 2 \end{array}\right] A_{21}(-1)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & 0 & 4 \end{array}\right] A_{32}(1)$$

$\Rightarrow$The final row implies that $0 = 4 \Rightarrow\!\Leftarrow$

$\Rightarrow (0,2,2) \notin \mathrm{im}(T)$

1.2 Question b

$x_1 + 3x_2 + x_3 - 2x_4 = 0$

$\Rightarrow x_1 = -3x_2 - x_3 + 2x_4$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_2 - x_3 + 2x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$\Rightarrow x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

It is easy to see that the three matrices are linear independent since is a unique value in every matrix

$$\Rightarrow \text{span}(B) = \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\Rightarrow a_1 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -3a_1 - a_2 + 2a_3 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{cases} a_1 = 2 \\ a_2 = -1 \\ a_3 = 3 \end{cases}$$

$\Rightarrow \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}$ can be obtained by linear combinations of the other three

and is not a scalar multiplication of one of the matrices

$\Rightarrow$the four matrices can form the basis by either three in the four

$$\Rightarrow \text{span}(B) = \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix} \right\}$$

2 Problem 2

2.1 Zero Matrix

$$
\begin{aligned}
&\overrightarrow{0}\,\overrightarrow{v} = \overrightarrow{0} \\
\Rightarrow &\overrightarrow{0} \in W
\end{aligned}
$$

2.2 Addition

$$
\begin{aligned}
&\exists X_1 \wedge X_2 \in W \\
&(X_1 + X_2)\overrightarrow{v} \\
=&X_1\overrightarrow{v} + X_2\overrightarrow{v} \\
=&\overrightarrow{0} + \overrightarrow{0} \\
=&\overrightarrow{0} \\
\Rightarrow &X_1 + X_2 \in W
\end{aligned}
$$

2.3 Scalar multiplication

$$
\begin{aligned}
&\exists X_1 \in W \wedge a \in \mathbb{R} \\
&(aX_1)\overrightarrow{v} \\
=&a(X_1\overrightarrow{v}) \\
=&a \cdot 0 \\
=&0 \\
\Rightarrow\ &aX_1 \in W
\end{aligned}
$$

2.4 Conclusion and Dimension

Since W meets all the requirements of a subspace, W is a subspace of $\mathbb{R}$

$$X \coloneqq \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$$X\overrightarrow{v} = \overrightarrow{0}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \overrightarrow{0}$$

$$\Rightarrow \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} - \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + 2 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} = \overrightarrow{0}$$

$$s \coloneqq \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \wedge t \coloneqq \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = s - 2t$$

$\Rightarrow X = [s - 2t, s, t]$

$X = [1, 1, 0]s + [-2, 0, 1]t$

It is easy to see that the two matrices are linear independent

$\Rightarrow \text{basis}(W) = \{[1, 1, 0], [-2, 0, 1]\}$

$\Rightarrow \dim(W) = 2$

3 Problem 3

3.1 Question a

3.1.1 $\textbf{rank}(A) = 2 \implies b \neq 0$

$$
\begin{aligned}
A &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \\
&\text{rank}(A) = 2 \\
\Rightarrow &\text{There must be 2 linear independent columns} \\
\Rightarrow &b_1, b_2, b_3 \text{ must have one } \neq 0 \\
\Rightarrow &b \neq 0
\end{aligned}
$$

3.1.2 $b \neq 0 \implies \textbf{rank}(A) = 2$

$$
\begin{aligned}
&b \neq 0 \\
\Rightarrow &\text{At least one of } b_1, b_2, b_3 \neq 0 \\
&Suppose b_1 \neq 0 \\
\Rightarrow &\begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}, \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \text{ Linear Independent} \\
\Rightarrow &\left\{ \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \right\} \text{ is a basis of } A \\
\Rightarrow &\text{rank}(A) = 2
\end{aligned}
$$

3.1.3 Conclusion

$$
\text{rank}(A) = 2 \Leftrightarrow b \neq 0
$$

3.2 Question b

$$\begin{cases} b_1 = a_{12}a_{23} - a_{22}a_{13} \\ b_2 = a_{13}a_{21} - a_{11}a_{23} \\ b_3 = a_{11}a_{22} - a_{12}a_{21} \end{cases}$$

$Ax = 0$

$$\Rightarrow \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \end{array}\right] M_1(\frac{1}{a_{11}})$$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & 0 \\ 0 & \frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}} & \frac{a_{23}a_{11}-a_{13}a_{21}}{a_{11}} & 0 \end{array}\right] A_{12}(-a_{21})$$

$$\sim \left[\begin{array}{ccc|c} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & 0 \\ 0 & 1 & \frac{a_{23}a_{11}-a_{13}a_{21}}{a_{11}a_{22}-a_{12}a_{21}} & 0 \end{array}\right] M_2(\frac{a_{11}}{a_{11}a_{22}-a_{12}a_{21}})$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{a_{13}(a_{11}a_{22}-a_{12}a_{21})-a_{12}(a_{23}a_{11}-a_{13}a_{21})}{a_{11}(a_{11}a_{22}-a_{12}a_{21})} & 0 \\ 0 & 1 & \frac{a_{23}a_{11}-a_{13}a_{21}}{a_{11}a_{22}-a_{12}a_{21}} & 0 \end{array}\right] A_{21}(-\frac{a_{12}}{a_{11}})$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & \frac{-a_{12}a_{23}+a_{22}a_{13}}{a_{11}a_{22}-a_{12}a_{21}} & 0 \\ 0 & 1 & \frac{a_{23}a_{11}-a_{13}a_{21}}{a_{11}a_{22}-a_{12}a_{21}} & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & \frac{-b_1}{b_3} & 0 \\ 0 & 1 & \frac{-b_2}{b_3} & 0 \end{array}\right]$$

Set $\mathrm{col}(3) = b_3t$

$$\Rightarrow \begin{cases} \mathrm{col}(1) = b_1t \\ \mathrm{col}(2) = b_2t \\ \mathrm{col}(3) = b_3t \end{cases}$$

$\Rightarrow \mathrm{nullspace}(A) = (b_1, b_2, b_3) = b$

4 Problem 4

4.1 Question a

$$
\begin{aligned}
A^T &= -A \\
\det(A^T) &= \det(-A) \\
\det(A) &= (-1)^n \det(A) \\
\det(A) + (-1)^{n+1} \det(A) &= 0 \\
\det(A) &= 0 \Leftrightarrow n = 2k+1, k \in \mathbb{Z}^+
\end{aligned}
$$

4.2 Question b

$$
\begin{aligned}
A^{-1} &= A^T \\
\det(A^{-1}) &= \det(A^T) \\
\frac{1}{\det(A)} &= \det(A) \\
(\det(A))^2 &= 1 \\
\det(A) &= \pm 1
\end{aligned}
$$

4.3 Question c

$$
\begin{aligned}
&\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = -2 \\
&\begin{vmatrix} 3d & 3c - 6d \\ b + 2d & a - 2b + 2c - 4d \end{vmatrix} \\
=&3d(a - 2b + 2c - 4d) - (b + 2d)(3c - 6d) \\
=&3ad - 6bd + 6bc - 12d^2 - (3bc - 6bd + 6cd - 12d^2) \\
=&3ad - 3bc \\
=&3(ad - bc) \\
=&-6
\end{aligned}
$$

5 Problem 5

5.1 Question a

$$
\begin{aligned}
\beta &= \left\{x^2, x, 1\right\} \\
\gamma &= \{(1,0,0),(0,1,0),(0,0,1)\} \\
T(1) &= (1,1,1) \\
&= 1(1,0,0) + 1(0,1,0) + 1(0,0,1) \\
T(x) &= (a,b,c) \\
&= a(1,0,0) + b(0,1,0) + c(0,0,1) \\
T(x^2) &= (a^2, b^2, c^2) \\
&= a^2(1,0,0) + b^2(0,1,0) + c^2(0,0,1) \\
\Rightarrow A = [T]_\beta^\gamma &= \begin{bmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{bmatrix}
\end{aligned}
$$

5.2 Question b

5.2.1 $\det(A) \neq 0 \implies a \neq b \neq c$

$$
\begin{aligned}
&\text{Suppose } b = c \\
\Rightarrow A &= \begin{bmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ b^2 & b & 1 \end{bmatrix} \\
\det(A) &= \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ b^2 & b & 1 \end{vmatrix} \\
&= \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0 \\
&\text{The cases of } a = b \vee a = c \text{ is the same for } \det(A) \\
\Rightarrow & a = b \vee b = c \vee a = c \implies \det(A) = 0 \\
&\text{Since } a = b \vee b = c \vee a = c \implies \det(A) = 0 \\
&\text{ and } \det(A) \neq 0 \implies a \neq b \neq c \text{ are contrapositive} \\
\Rightarrow & \det(A) \neq 0 \implies a \neq b \neq c
\end{aligned}
$$

5.2.2 $a \neq b \neq c \implies \det(A) \neq 0$

$$
\begin{aligned}
&\text{Suppose } \det(A) = 0 \\
&\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = 0 \\
&\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 - b^2 & c - b & 0 \end{vmatrix} = 0 \\
&b^2(c-b) + a(c^2 - b^2) - (b(c^2 - b^2) - a^2(c-b)) = 0 \\
&(b^2 - a^2)(c-b) + (a-b)(c^2 - b^2) = 0 \\
&(b+a)(b-a)(c-b) + (a-b)(c+b)(c-b) = 0 \\
&(c-b)(b-a)(a-c) = 0 \\
&b = c \vee a = b \vee a = c \\
&\det(A) = 0 \implies b = c \vee a = b \vee a = c \\
&\text{Since } \det(A) = 0 \implies b = c \vee a = b \vee a = c \\
&\text{ and } a \neq b \neq c \implies \det(A) \neq 0 \text{ are contrapositive} \\
\Rightarrow &a \neq b \neq c \implies \det(A) \neq 0
\end{aligned}
$$

5.2.3 Conclusion

$$\det(A) \neq 0 \Leftrightarrow a \neq b \neq c$$

6 Reference

6.1 Collaborators

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