1 Problem 1

1.1 Question a

1.1.1 rank$(A) = 0 \implies A$ is a zero matrix

$$\begin{aligned}
\text{Premise: } & \text{rank}(A) = 0 \\
& \text{Suppose } A \text{ is not a zero matrix} \\
\Rightarrow & \text{There is at least a non-zero column for the reduced row-echelon form of } A \\
\Rightarrow & \text{rank}(A) \neq 0 \\
& \text{rank}(A) \neq 0 \Rightarrow\Leftarrow \text{rank}(A) = 0 \\
\Rightarrow & \text{rank}(A) = 0 \implies A \text{ is a zero matrix}
\end{aligned}$$

1.1.2 A is a zero matrix $\implies$ rank$(A) = 0$

$$\begin{aligned}
\text{Premise: } & A \text{ is a zero matrix} \\
\Rightarrow & \text{There is no non-zero columns or rows in } A \\
\Rightarrow & \text{rank}(A) = 0 \\
\Rightarrow & A \text{ is a zero matrix} \implies \text{rank}(A) = 0
\end{aligned}$$

1.1.3 Conclusion

$$\text{rank}(A) = 0 \iff A \text{ is a zero matrix}$$

1.2 Question b

$$\begin{aligned}
& \text{rank}(cA) \\
= & \text{rank}(cI_m A) \\
= & \text{rank}((cI_m)A) \\
& cI_m \text{ is invertible since } I_m \text{ is invertible} \\
\Rightarrow & \text{rank}((cI_m)A) = \text{rank}(A) \\
\Rightarrow & \text{rank}(cA) = \text{rank}(A)
\end{aligned}$$

2 Problem 2

2.1 Question a

$$
\begin{aligned}
&\dim(\mathrm{im}(S)+\mathrm{im}(T)) \leq \dim(\mathrm{im}(S))+\dim(\mathrm{im}(T))\\
\rightarrow &\text{There can be common elements between } \mathrm{im}(S) \text{ and } \mathrm{im}(T)\\
&\text{It is easy to see that } \mathrm{im}(S+T) \subseteq \mathrm{im}(S)+\mathrm{im}(T)\\
\rightarrow &\text{ Because of the similar reason above}\\
&\mathrm{im}(S+T) \subseteq \mathrm{im}(S)+\mathrm{im}(T)\\
\Rightarrow &\dim(\mathrm{im}(S+T)) \leq \dim(\mathrm{im}(S)+\mathrm{im}(T))\\
\Rightarrow &\dim(\mathrm{im}(S+T)) \leq \dim(\mathrm{im}(S))+\dim(\mathrm{im}(T))
\end{aligned}
$$

2.2 Question b

$$
\begin{aligned}
&S \coloneqq Ax \wedge T \coloneqq Bx \wedge S+T=(A+B)x\\
\Rightarrow &\dim(\mathrm{im}(S))=\mathrm{rank}(A) \wedge \dim(\mathrm{im}(T))=\mathrm{rank}(B) \wedge \dim(\mathrm{im}(S+T))=\mathrm{rank}(A+B)\\
\Rightarrow &\mathrm{rank}(A+B) \leq \mathrm{rank}(A)+\mathrm{rank}(B)
\end{aligned}
$$

3 Problem 3

3.1 Question a

$$\begin{bmatrix}0&1&0\\2&1&0\\1&0&1\end{bmatrix}$$
$$\sim\begin{bmatrix}2&1&0\\0&1&0\\1&0&1\end{bmatrix} S_{12}$$
$$\sim\begin{bmatrix}2&0&0\\0&1&0\\1&0&1\end{bmatrix} A_{12}(-1)$$
$$\sim\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix} M_1(\frac{1}{2})$$
$$\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} A_{31}(-1)$$

The row-echelon form has full-rank, meaning it is invertible

3.2 Question b

$$\left[\begin{array}{ccc|ccc}0&1&0&1&0&0\\2&1&0&0&1&0\\1&0&1&0&0&1\end{array}\right]$$
$$\sim\left[\begin{array}{ccc|ccc}2&1&0&0&1&0\\0&1&0&1&0&0\\1&0&1&0&0&1\end{array}\right] S_{12}$$
$$\sim\left[\begin{array}{ccc|ccc}2&0&0&-1&1&0\\0&1&0&1&0&0\\1&0&1&0&0&1\end{array}\right] A_{12}(-1)$$
$$\sim\left[\begin{array}{ccc|ccc}1&0&0&-\frac{1}{2}&\frac{1}{2}&0\\0&1&0&1&0&0\\1&0&1&0&0&1\end{array}\right] M_1(\frac{1}{2})$$
$$\sim\left[\begin{array}{ccc|ccc}1&0&0&-\frac{1}{2}&\frac{1}{2}&0\\0&1&0&1&0&0\\0&0&1&\frac{1}{2}&-\frac{1}{2}&1\end{array}\right] A_{31}(-1)$$

3.3 Question c

$$
\begin{aligned}
&S_{12}A_{12}(-1)M_1(\frac{1}{2})A_{31}(-1)A = I_3\\
\Rightarrow &A^{-1} = S_{12}A_{12}(-1)M_1(\frac{1}{2})A_{31}(-1)\\
&A = (A^{-1})^{-1} = (S_{12}A_{12}(-1)M_1(\frac{1}{2})A_{31}(-1))^{-1}\\
= &A_{31}(-1)^{-1}M_1(\frac{1}{2})^{-1}A_{12}(-1)^{-1}{S_{12}}^{-1}\\
&S_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A_{12}(-1) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M_1(\frac{1}{2}) = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A_{31}(-1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

4 Problem 4

4.1 Question a

$$
\begin{bmatrix}
1 & 0 & -1 & 2 & 1 \\
-1 & 1 & 3 & -1 & 0 \\
-2 & 1 & 4 & -1 & 3 \\
3 & -1 & -5 & 1 & -6
\end{bmatrix}
$$

$$
\sim \begin{bmatrix}
1 & 0 & -1 & 2 & 1 \\
0 & 1 & 2 & 1 & 1 \\
-2 & 1 & 4 & -1 & 3 \\
3 & -1 & -5 & 1 & -6
\end{bmatrix} A_{21}(1)
$$

$$
\sim \begin{bmatrix}
1 & 0 & -1 & 2 & 1 \\
0 & 1 & 2 & 1 & 1 \\
0 & 1 & 2 & 3 & 5 \\
3 & -1 & -5 & 1 & -6
\end{bmatrix} A_{31}(2)
$$

$$
\sim \begin{bmatrix}
1 & 0 & -1 & 2 & 1 \\
0 & 1 & 2 & 1 & 1 \\
0 & 1 & 2 & 3 & 5 \\
0 & -1 & -2 & -5 & -9
\end{bmatrix} A_{41}(-3)
$$

$$
\sim \begin{bmatrix}
1 & 0 & -1 & 2 & 1 \\
0 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 2 & 4 \\
0 & -1 & -2 & -5 & -9
\end{bmatrix} A_{32}(-1)
$$

$$
\sim \begin{bmatrix}
1 & 0 & -1 & 2 & 1 \\
0 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 2 & 4 \\
0 & 0 & 0 & -4 & -8
\end{bmatrix} A_{42}(1)
$$

$$
\sim \begin{bmatrix}
1 & 0 & -1 & 2 & 1 \\
0 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & -4 & -8
\end{bmatrix} M_3(\frac{1}{2})
$$

$$\begin{aligned}
&\sim \begin{bmatrix} 1 & 0 & -1 & 0 & -3 \\ 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -4 & -8 \end{bmatrix} A_{13}(-2) \\
&\sim \begin{bmatrix} 1 & 0 & -1 & 0 & -3 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -4 & -8 \end{bmatrix} A_{23}(-1) \\
&\sim \begin{bmatrix} 1 & 0 & -1 & 0 & -3 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} A_{43}(4)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow &\text{nullspace}(A) = \begin{bmatrix} s+3t \\ -2s+t \\ s \\ -2t \\ t \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 3 \\ 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} t \\
\Rightarrow &M = \begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \\
&\dim(\text{col}(M)) = 2 \\
\Rightarrow &\text{rank}(M) = 2
\end{aligned}$$

4.2 Question b

$AB = 0_{4\times 5}$

$\Rightarrow \text{col}(B) \subset \text{nullspace}(A)$

$\Rightarrow \text{rank}(B) \leq \dim(\text{nullspace}(A))$

$\dim(\text{nullspace}(A)) + \text{rank}(A) = 5 \rightarrow$ rank-nullity theorem

$\text{rank}(A) = 3 \rightarrow$ There is 3 pivot columns in row-echelon form

$\Rightarrow \dim(\text{nullspace}(A)) = 2$

$\Rightarrow \text{rank}(B) \leq 2$

5 Problem 5

Denote the first to the sixth columns $c_1, ..., c_6$

Since a_1, a_3, a_5 are the pivot columns, we can use the three columns to represent the others.

$$\begin{cases} c_2 = -3c_1 \\ c_4 = 4c_1 + 3c_3 \\ c_6 = 5c_1 + 2c_3 - c_5 \end{cases}$$

$$\Rightarrow \begin{cases} c_2 = \begin{bmatrix} 3 \\ -6 \\ -3 \\ 9 \end{bmatrix} \\ c_4 = \begin{bmatrix} 1 \\ -5 \\ 2 \\ 0 \end{bmatrix} \\ c_6 = \begin{bmatrix} 0 \\ 1 \\ -3 \\ 2 \end{bmatrix} \end{cases}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 3 & -1 & 1 & 3 & 0 \\ -2 & -6 & 1 & -5 & -9 & 1 \\ -1 & -3 & 2 & 2 & 2 & -3 \\ 3 & 9 & -4 & 0 & 5 & 2 \end{bmatrix}$$

6 Reference

6.1 Collaborators

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