1 Problem 1

1.1 Question a

$$\begin{cases} v_1 = (0,1,1) \\ v_2 = (1,1,1) \\ v_3 = (1,-2,2) \end{cases}$$

$$\begin{aligned}
u_1 =& v_1 = (0,1,1) \\
u_2 =& v_2 - \mathrm{proj}_{u_1} v_2 \\
=& (1,1,1) - \frac{\langle (1,1,1),(0,1,1) \rangle}{\langle (0,1,1),(0,1,1) \rangle}(0,1,1) \\
=& (1,1,1) - (0,1,1) \\
=& (1,0,0) \\
u_3 =& v_3 - \mathrm{proj}_{u_1} v_3 - \mathrm{proj}_{u_2} v_3 \\
=& (1,-2,2) - \frac{\langle (1,-2,2),(0,1,1) \rangle}{\langle (0,1,1),(0,1,1) \rangle}(0,1,1) - \frac{\langle (1,-2,2),(1,0,0) \rangle}{\langle (1,0,0),(1,0,0) \rangle}(1,0,0) \\
=& (1,-2,2) - (0,0,0) - (1,0,0) \\
=& (0,-2,2) \\
e_1 =& \frac{u_1}{\|u_1\|} \\
=& (0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \\
e_2 =& \frac{u_2}{\|u_2\|} \\
=& (1,0,0) \\
e_3 =& \frac{u_3}{\|u_3\|} \\
=& (0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \\
\Rightarrow& \overline{\beta} = \{(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), (1,0,0), (0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})\}
\end{aligned}$$

1.2 Question b

$$\begin{cases} v_1 = 1 + x^2 \\ v_2 = -1 + x \\ v_3 = 1 + x \end{cases}$$

$$\begin{aligned}
u_1 =& v_1 = 1 + x^2 \\
u_2 =& v_2 - \mathrm{proj}_{u_1} v_2 \\
=& -1 + x - \frac{\langle -1 + x, 1 + x^2 \rangle}{\langle 1 + x^2, 1 + x^2 \rangle} 1 + x^2 \\
=& -1 + x + \frac{5}{16}(1 + x^2) \\
=& -\frac{11}{16} + x + \frac{5}{16}x^2 \\
u_3 =& v_3 - \mathrm{proj}_{u_1} v_3 - \mathrm{proj}_{u_2} v_3 \\
=& 1 + x - \frac{\langle 1 + x, 1 + x^2 \rangle}{\langle 1 + x^2, 1 + x^2 \rangle}(1 + x^2) - \frac{\langle 1 + x, -\frac{11}{16} + x + \frac{5}{16}x^2 \rangle}{\langle -\frac{11}{16} + x + \frac{5}{16}x^2, -\frac{11}{16} + x + \frac{5}{16}x^2 \rangle}(-\frac{11}{16} + x + \frac{5}{16}x^2) \\
=& -\frac{38}{203} + \frac{32}{29}x - \frac{220}{203}x^2 \\
e_1 =& \frac{u_1}{\|u_1\|} \\
=& \frac{\sqrt{15}}{2\sqrt{7}} + \frac{\sqrt{15}}{2\sqrt{7}}x^2 \\
e_2 =& \frac{u_2}{\|u_2\|} \\
=& -\frac{11\sqrt{3}}{2\sqrt{29}} + \frac{8\sqrt{3}}{\sqrt{29}}x + \frac{5\sqrt{3}}{2\sqrt{29}}x^2 \\
e_3 =& \frac{u_3}{\|u_3\|} \\
=& -\frac{19\sqrt{3}}{\sqrt{203}} + \frac{16\sqrt{21}}{\sqrt{29}}x - \frac{110\sqrt{3}}{\sqrt{203}}x^2 \\
\Rightarrow& \overline{\beta} = \{\frac{\sqrt{15}}{2\sqrt{7}} + \frac{\sqrt{15}}{2\sqrt{7}}x^2, -\frac{11\sqrt{3}}{2\sqrt{29}} + \frac{8\sqrt{3}}{\sqrt{29}}x + \frac{5\sqrt{3}}{2\sqrt{29}}x^2, -\frac{19\sqrt{3}}{\sqrt{203}} + \frac{16\sqrt{21}}{\sqrt{29}}x - \frac{110\sqrt{3}}{\sqrt{203}}x^2\}
\end{aligned}$$

2 Problem 2

2.1 Question a

$$
\begin{aligned}
&x \coloneqq (x_1, x_2, x_3, x_4) \\
&\begin{cases} \langle x, (1,0,1,2) \rangle = 0 \\ \langle x, (1,-1,0,1) \rangle = 0 \end{cases} \\
\Rightarrow &\begin{cases} x_1 + x_3 + 2x_4 = 0 \\ x_1 - x_2 + x_4 = 0 \end{cases} \\
\Rightarrow &\begin{cases} x_1 = x_1 \\ x_2 = x_1 + x_4 \\ x_3 = -x_1 - 2x_4 \\ x_4 = x_4 \end{cases} \\
\Rightarrow &W^{\perp} = \operatorname{Span}\{(1,1,-1,0), (0,1,-2,1)\}
\end{aligned}
$$

$$
\begin{aligned}
&W: \\
u_1 &= v_1 = (1,0,1,2) \\
u_2 &= v_2 - \operatorname{proj}_{u_1} v_2 \\
&= (1,-1,0,1) - \frac{\langle (1,-1,0,1), (1,0,1,2) \rangle}{\langle (1,0,1,2), (1,0,1,2) \rangle}(1,0,1,2) \\
&= (1,-1,0,1) - \frac{1}{2}(1,0,1,2) \\
&= (\frac{1}{2}, -1, -\frac{1}{2}, 0) \\
e_1 &= \frac{u_1}{\|u_1\|} \\
&= (\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}) \\
e_2 &= \frac{u_2}{\|u_2\|} \\
&= (\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0) \\
&\Rightarrow \text{Orthonormal basis for } W: \{(\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}), (\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0)\}
\end{aligned}
$$

$W^{\perp}$:

$$
\begin{aligned}
u_1 &= v_1 = (1,1,-1,0) \\
u_2 &= v_2 - \mathrm{proj}_{u_1} v_2 \\
&= (0,1,-2,1) - \frac{\langle (0,1,-2,1),(1,1,-1,0)\rangle}{\langle (1,1,-1,0),(1,1,-1,0)\rangle}(1,1,-1,0) \\
&= (0,1,-2,1) - (1,1,-1,0) \\
&= (-1,0,-1,1) \\
e_1 &= \frac{u_1}{\|u_1\|} \\
&= (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0) \\
e_2 &= \frac{u_2}{\|u_2\|} \\
&= (-\frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \\
&\Rightarrow \text{Orthonormal basis for } W^{\perp} : \{(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0), (-\frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})\}
\end{aligned}
$$

2.2 Question b

$$
\begin{aligned}
&w_1 \coloneqq a(\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}) + b(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0) \\
&w_2 \coloneqq c(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0) + d(-\frac{1}{\sqrt{3}}, 0, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \\
&w_1 + w_2 = (1,1,1,1) \\
&\Rightarrow \begin{cases} \frac{1}{\sqrt{6}}a + \frac{1}{\sqrt{6}}b + \frac{1}{\sqrt{3}}c - \frac{1}{\sqrt{3}}d = 1 \\ -\frac{2}{\sqrt{6}}b + \frac{1}{\sqrt{3}}c = 1 \\ \frac{1}{\sqrt{6}}a - \frac{1}{\sqrt{6}}b - \frac{1}{\sqrt{3}}c - \frac{1}{\sqrt{3}}d = 1 \\ \frac{2}{\sqrt{6}}a + \frac{1}{\sqrt{3}}d = 1 \end{cases} \\
&\Rightarrow \begin{cases} a = \frac{4}{\sqrt{6}} \\ b = -\frac{2}{\sqrt{6}} \\ c = \frac{1}{\sqrt{3}} \\ d = -\frac{1}{\sqrt{3}} \end{cases} \\
&\Rightarrow \begin{cases} w_1 = (\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}) \\ w_2 = (\frac{2}{3}, \frac{1}{3}, 0, -\frac{1}{3}) \end{cases}
\end{aligned}
$$

3 Problem 3

3.1 Question a

$$
\begin{aligned}
& u \in W \\
\Rightarrow & \langle u, v\rangle = 0 \forall v \in W^{\perp} \\
& \langle v, w\rangle = 0 \forall w \in {W^{\perp}}^{\perp} \\
\Rightarrow & W \subseteq {W^{\perp}}^{\perp} \\
& \text{Let } W \text{ be a subspace over } V \text{ with dimension } n \\
& \dim(W) + \dim(W^{\perp}) = n \\
& \dim(W^{\perp}) + \dim({W^{\perp}}^{\perp}) = n \\
\Rightarrow & \dim(W) + \dim(W^{\perp}) = \dim(W^{\perp}) + \dim({W^{\perp}}^{\perp}) \\
\Rightarrow & \dim(W) = \dim({W^{\perp}}^{\perp}) \\
\Rightarrow & W = {W^{\perp}}^{\perp}
\end{aligned}
$$

3.2 Question b

$(W_1 + W_2)^{\perp} \subseteq {W_1}^{\perp} \cap {W_2}^{\perp}$

$$
\begin{aligned}
& v \in (W_1 + W_2)^{\perp} \\
\Rightarrow & \langle v, w_1 + w_2\rangle = 0 \forall w_1 \in W_1 \wedge w_2 \in W_2 \\
& \text{Let } w_1 = 0 : \\
& \langle v, w_2\rangle = 0 \implies v \in {W_2}^{\perp} \\
& \text{Similarly } v \in {W_1}^{\perp} \\
\Rightarrow & v \in {W_1}^{\perp} \cap {W_2}^{\perp} \\
\Rightarrow & (W_1 + W_2)^{\perp} \subseteq {W_1}^{\perp} \cap {W_2}^{\perp}
\end{aligned}
$$

${W_1}^{\perp} \cap {W_2}^{\perp} \subseteq (W_1 + W_2)^{\perp}$

$$
\begin{aligned}
&v \in {W_1}^\perp \cap {W_2}^\perp \\
\Rightarrow &\langle v, w_1 \rangle = 0 \wedge \langle v, w_2 \rangle = 0 \forall w_1 \in W_1 \wedge w_2 \in W_2 \\
\Rightarrow &\langle v, w_1 \rangle + \langle v, w_2 \rangle = 0 \\
\Rightarrow &\langle v, w_1 + w_2 \rangle = 0 \\
\Rightarrow &v \in (W_1 + W_2)^\perp \\
\Rightarrow &{W_1}^\perp \cap {W_2}^\perp \subseteq (W_1 + W_2)^\perp \\
&(W_1 + W_2)^\perp \subseteq {W_1}^\perp \cap {W_2}^\perp \wedge {W_1}^\perp \cap {W_2}^\perp \subseteq (W_1 + W_2)^\perp \\
\Rightarrow &(W_1 + W_2)^\perp = {W_1}^\perp \cap {W_2}^\perp
\end{aligned}
$$

3.3 Question c

$$
\begin{aligned}
&(W_1 \cap W_2)^\perp \\
= &({W_1^\perp}^\perp \cap {W_2^\perp}^\perp)^\perp \\
= &((W_1^\perp + W_2^\perp)^\perp)^\perp \\
= &W_1^\perp + W_2^\perp \\
\Rightarrow &(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp
\end{aligned}
$$

4 Problem 4

4.1 Question a

$$
\begin{aligned}
& v = \sum_{i=1}^{n} c_i v_i \\
\Rightarrow & \langle v, v_i \rangle = \langle \sum_{i=1}^{n} c_i v_i, v_i \rangle = c_i \langle v_i, v_i \rangle \leftarrow v_i \perp v_j \forall i \neq j \in \mathbb{R} \\
\Rightarrow & c_i = \frac{\langle v, v_i \rangle}{\|v_i\|^2} \\
& \langle v, v \rangle = \langle \sum_{i=1}^{n} c_i v_i, \sum_{i=1}^{n} c_i v_i \rangle \\
= & \sum_{i=1}^{n} {c_i}^2 \|v_i\|^2 \\
= & \sum_{i=1}^{n} \frac{\langle v, v_i \rangle^2}{\|v_i\|^2} \\
= & \|v\|^2 \\
& \|v_i\|^2 = 1 \leftarrow \beta \text{ is orthonormal} \\
\Rightarrow & \|v\| = \sqrt{\sum_{i=1}^{n} \langle v, v_i \rangle^2}
\end{aligned}
$$

4.2 Question b

$$
\begin{aligned}
u &= \sum_{i=1}^{n} d_i v_i \\
\Rightarrow d_i &= \frac{\langle u, v_i \rangle}{\|v_i\|^2} \\
\langle u, v \rangle &= \langle \sum_{i=1}^{n} d_i v_i \sum_{i=1}^{n} c_i v_i \rangle \\
&= \sum_{i=1}^{n} c_i d_i \|v_i\|^2 \\
&= \sum_{i=1}^{n} \langle v, v_i \rangle \langle u, v_i \rangle \leftarrow \|v_i\|^2 = 1 \\
\Rightarrow \langle u, v \rangle &= \sum_{i=1}^{n} \langle v, v_i \rangle \langle u, v_i \rangle
\end{aligned}
$$

5 Problem 5

5.1 Question a

$$v = \sum_{i=1}^{n} c_i v_i$$

$$\Rightarrow \langle v, v_i \rangle = \langle \sum_{i=1}^{n} c_i v_i, v_i \rangle = c_i \langle v_i, v_i \rangle \leftarrow v_i \perp v_j \forall i \neq j \in \mathbb{R}$$

$$\Rightarrow c_i = \frac{\langle v, v_i \rangle}{\|v_i\|^2}$$

$$\|v_1\|^2 = 1$$

$$\Rightarrow c_i = \langle v, v_i \rangle$$

$$\Rightarrow v = \sum_{i=1}^{n} \langle v, v_i \rangle v_i$$

$$\Rightarrow f(v) = f(\sum_{i=1}^{n} \langle v, v_i \rangle v_i)$$

f is a linear transformation

$$\Rightarrow f(v) = \sum_{i=1}^{n} \langle v, v_i \rangle f(v_i)$$

$$\Rightarrow f(v) = \langle v, \sum_{i=1}^{n} f(v_i) v_i \rangle = \langle v, w \rangle$$

$$\Rightarrow w = \sum_{i=1}^{n} f(v_i) v_i$$

w is unique because v_i are orthonormal, there cannnot be another way to form w of the same value

5.2 Question b

$$\begin{aligned}
&\langle p(x), g(x)\rangle = p(0)g(0) + p(1)g(1) + p(2)g(2)\\
&f(p(x)) = \int_{-1}^{1} p(x)\\
&\beta = \{1, x, x^2\}\\
&u_1 = v_1 = 1\\
&u_2 = v_2 - \mathrm{proj}_{u_1} v_2\\
=&x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle} 1\\
=&-1 + x\\
&u_3 = v_3 - \mathrm{proj}_{u_1} v_3 - \mathrm{proj}_{u_2} v_3\\
=&x^2 - \frac{5}{3} - 2(x-1)\\
=&\frac{1}{3} - 2x + x^2\\
&e_1 = \frac{u_1}{\|u_1\|}\\
=&\frac{1}{\sqrt{3}}\\
&e_2 = \frac{u_2}{\|u_2\|}\\
=&-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}x\\
&e_3 = \frac{u_3}{\|u_3\|}\\
=&\frac{\sqrt{6}}{6} - \sqrt{6}x + \frac{\sqrt{6}}{2}x^2\\
\Rightarrow&\beta' = \{\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}x, \frac{\sqrt{6}}{6} - \sqrt{6}x + \frac{\sqrt{6}}{2}x^2\} = \{v_1, v_2, v_3\}\\
&g(x) = \sum_{i=1}^{3} f(v_i)v_i\\
=&\frac{1}{3}\int_{-1}^{1} \frac{1}{3}dx + \int_{-1}^{1} -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}xdx(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}x) + \int_{-1}^{1} \frac{\sqrt{6}}{6} - \sqrt{6}x + \frac{\sqrt{6}}{2}x^2dx(\frac{\sqrt{6}}{6} - \sqrt{6}x + \frac{\sqrt{6}}{2}x^2)\\
=&\frac{2}{9} - \sqrt{2}(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}x) + 2\sqrt{\frac{2}{3}}(\frac{\sqrt{6}}{6} - \sqrt{6}x + \frac{\sqrt{6}}{2}x^2)\\
=&\frac{17}{9} - 5x + 2x^2\\
\Rightarrow&g(x) = \frac{17}{9} - 5x + 2x^2
\end{aligned}$$

6 Reference

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