1 Problem 1

1.1 Question a

According to the problem, we can have the equations:

$$
\begin{aligned}
T(x+1) &= \alpha_1(x^2+x+1) + \beta_1(x-1) + \gamma_1 \\
&= \alpha_1 x^2 + (\alpha_1+\beta_1)x + (\alpha_1 - \beta_1 + \gamma_1) \\
T(x-2) &= \alpha_2(x^2+x+1) + \beta_2(x-1) + \gamma_2 \\
&= \alpha_2 x^2 + (\alpha_2+\beta_2)x + (\alpha_2 - \beta_2 + \gamma_2)
\end{aligned}
$$

Also according to the linear transformation:

$$
\begin{aligned}
T(x+1) &= x((x+1)+1) + (1+1) + \int_0^x t + 1 dt \\
&= x^2 + 2x + 2 + \left[\frac{1}{2}t^2 + t\right]_0^x \\
&= x^2 + 2x + 2 + \frac{1}{2}x^2 + x \\
&= \frac{3}{2}x^2 + 3x + 2 \\
T(x-2) &= x((x+1)-2) + (1-2) + \int_0^x t - 2dt \\
&= x^2 - x - 1 + \left[\frac{1}{2}t^2 - 2t\right]_0^x \\
&= x^2 - x - 1 + (\frac{1}{2}x^2 - 2x) \\
&= \frac{3}{2}x^2 - 3x - 1
\end{aligned}
$$

From the two sets of equations above, we can see that:

$$\begin{cases}\alpha_1 = \frac{3}{2} \\ \alpha_1 + \beta_1 = 3 \\ \alpha_1 - \beta_1 + \gamma_1 = 2\end{cases}$$

$$\Rightarrow \begin{cases}\alpha_1 = \frac{3}{2} \\ \beta_1 = \frac{3}{2} \\ \gamma_1 = 2\end{cases}$$

$$\begin{cases}\alpha_2 = \frac{3}{2} \\ \alpha_2 + \beta_2 = -3 \\ \alpha_2 - \beta_2 + \gamma_2 = -1\end{cases}$$

$$\Rightarrow \begin{cases}\alpha_2 = \frac{3}{2} \\ \beta_2 = -\frac{9}{2} \\ \gamma_2 = -7\end{cases}$$

Hence:

$$[T]_\beta^\gamma = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{9}{2} \\ 2 & -7 \end{bmatrix}$$

1.2 Question b

$$
\begin{aligned}
T(3x-5) =& x(3(x+1)-5)+(3-5)+\int_0^x 3t-5dt \\
=& 3x^2-2x-2+\left[\frac{3}{2}t^2-5t\right]_0^x \\
=& 3x^2-2x-2+(\frac{3}{2}x^2-5x) \\
=& \frac{9}{2}x^2-7x-2 \\
\Rightarrow& \begin{cases} \alpha_p = \frac{9}{2} \\ \alpha_p+\beta_p = -7 \\ \alpha_p-\beta_p+\gamma_p = -2 \end{cases} \\
\Rightarrow& \begin{cases} \alpha_p = \frac{9}{2} \\ \beta_p = -\frac{23}{2} \\ \gamma_p = -18 \end{cases} \\
\Rightarrow& \left[T(3x-5)\right]_\gamma = \begin{bmatrix} \frac{9}{2} \\ -\frac{23}{2} \\ -18 \end{bmatrix} \\
3x-5 =& m(x+1)+n(x-2) \\
=& (m+n)x+(m-2n) \\
\Rightarrow& \begin{cases} m+n = 3 \\ m-2n = -5 \end{cases} \\
\Rightarrow& \begin{cases} m = \frac{1}{3} \\ n = \frac{8}{3} \end{cases} \\
\Rightarrow& \left[3x-5\right]_\beta = \begin{bmatrix} \frac{1}{3} \\ \frac{8}{3} \end{bmatrix} \\
\Rightarrow& \left[T\right]_\beta^\gamma \left[3x-5\right]_\beta \\
=& \begin{bmatrix} \frac{5}{2} & \frac{3}{2} \\ 3 & -\frac{9}{2} \\ 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{8}{3} \end{bmatrix} \\
& \left[T\right]_\beta^\gamma \left[p(x)\right]_\beta \\
=& \begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{9}{2} \\ 2 & -7 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{8}{3} \end{bmatrix} \\
=& \begin{bmatrix} \frac{9}{2} \\ -\frac{23}{2} \\ -18 \end{bmatrix} \\
=& \left[T(3x-5)\right]_\gamma \\
\Rightarrow& \left[T(3x-5)\right]_\gamma = \left[T\right]_\beta^\gamma \left[p(x)\right]_\beta
\end{aligned}
$$

2 Problem 2

Define $p(x) = ax^2 + bx + c$

2.1 Question a

$$\begin{aligned}
T(p(x)) &= \frac{1}{x}\int_0^x (a(t+1)^2 + b(t+1) + c)dt \\
&= \frac{1}{x}\int_0^x (at^2 + 2at + a + bt + b + c)dt \\
&= \frac{1}{x}\int_0^x (at^2 + (2a+b)t + (a+b+c))dt \\
&= \frac{1}{x}\left[\frac{1}{3}at^3 + \frac{1}{2}(2a+b)t^2 + (a+b+c)t\right]_0^x \\
&= \frac{1}{x}(\frac{1}{3}ax^3 + \frac{1}{2}(2a+b)x^2 + (a+b+c)x) \\
&= \frac{1}{3}ax^2 + \frac{1}{2}(2a+b)x + (a+b+c)
\end{aligned}$$

2.1.1 One-to-one

$$\begin{aligned}
&\frac{1}{3}ax^2 + \frac{1}{2}(2a+b)x + (a+b+c) = 0 \\
&\Rightarrow \begin{cases} \frac{1}{3}a = 0 \\ \frac{1}{2}(2a+b) = 0 \\ a+b+c = 0 \end{cases} \\
&\Rightarrow \begin{cases} a = 0 \\ b = 0 \qquad \rightarrow \text{trivial solution} \\ c = 0 \end{cases} \\
&\Rightarrow \text{One-to-one}
\end{aligned}$$

2.1.2 Onto

$$
\begin{bmatrix} \frac{1}{3}a \\ \frac{1}{2}(2a+b) \\ a+b+c \end{bmatrix}
$$

$$
= \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}
$$

$$
\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 1 & 1 & 1 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 1 & 1 & 1 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}
$$

$$
\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

Since there is a pivot in every row, T is onto.

2.2 Question b

$$
\beta = \{1, x, x^2\}
$$

Define $\beta_U = \{1, x, x^2\}$ and $\beta_V = \{1, x, x^2\}$ for $p(x)$ and $T(p(x))$ respectively.

$$
\begin{aligned}
T(1) =& \alpha_1 + \beta_1 x + \gamma_1 x^2 \\
T(x) =& \alpha_2 + \beta_2 x + \gamma_2 x^2 \\
T(x^2) =& \alpha_3 + \beta_3 x + \gamma_3 x^2
\end{aligned}
$$

According to the transformation:

$$
\begin{aligned}
T(1) =& \frac{1}{x}\int_0^x 1dt \\
=& \frac{1}{x}\left[t\right]_0^x \\
=& \frac{1}{x}x \\
=& 1 \\
T(x) =& \frac{1}{x}\int_0^x (t+1)dt \\
=& \frac{1}{x}\left[\frac{1}{2}t^2+t\right]_0^x \\
=& \frac{1}{x}(\frac{1}{2}x^2+x) \\
=& \frac{1}{2}x+1 \\
T(x^2) =& \frac{1}{x}\int_0^x (t+1)^2dt \\
=& \frac{1}{x}\int_0^x (t^2+2t+1)dt \\
=& \frac{1}{x}\left[\frac{1}{3}t^3+t^2+t\right]_0^x \\
=& \frac{1}{x}(\frac{1}{3}x^3+x^2+x) \\
=& \frac{1}{3}x^2+x+1
\end{aligned}
$$

From the two sets, we can see that:

$$
\begin{cases}
\alpha_1 = 1 \\
\beta_1 = 0 \\
\gamma_1 = 0
\end{cases}
$$

$$
\begin{cases}
\alpha_2 = 1 \\
\beta_2 = \frac{1}{2} \\
\gamma_2 = 0
\end{cases}
$$

$$
\begin{cases}
\alpha_3 = 1 \\
\beta_3 = 1 \\
\gamma_3 = \frac{1}{3}
\end{cases}
$$

$$
\Rightarrow M = [T]_\beta^\beta \begin{bmatrix} 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}
$$

2.2.1 Question c

$$
\begin{aligned}
&\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \end{array}\right] \\
\sim &\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right] \\
\sim &\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & -6 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right] \\
\sim &\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 & 2 & -6 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right] \\
\Rightarrow & M^{-1} = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -6 \\ 0 & 0 & 3 \end{bmatrix} \\
\Rightarrow & [T]_\beta^\beta = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -6 \\ 0 & 0 & 3 \end{bmatrix} \\
\Rightarrow & \begin{cases} T^{-1}(1) = 1 \\ T^{-1}(x) = 2x - 2 \\ T^{-1}(x^2) = 3x^2 - 6x + 3 \end{cases}
\end{aligned}
$$

So we can calculate $T^{-1}(ax^2 + bx + c)$:

$$
\begin{aligned}
T^{-1}(ax^2 + bx + c) = & T^{-1}(ax^2) + T^{-1}(bx) + T^{-1}(c) \\
= & aT^{-1}(x^2) + bT^{-1}(x) + cT^{-1}(1) \\
= & a(3x^2 - 6x + 3) + b(2x - 2) + c \\
= & 3ax^2 + (-6a + 2b)x + (3a - 2b + c)
\end{aligned}
$$

3 Problem 3

3.1 Question a

Let $x \in \ker(T)$, where x is arbitrary.

$$\begin{aligned}
&T(x) = 0_v \\
&S(0_v) = 0_w \\
\Rightarrow &S(T(x)) = 0_w \\
\Rightarrow &x \in \ker(S \circ T)
\end{aligned}$$

Since for arbitrary $x \in \ker(T)$ there is $x \in \ker(S \circ T) \Rightarrow \ker(T) \subseteq \ker(S \circ T)$

Let $w \in \operatorname{im}(S \circ T)$, where w is arbitrary.

$$\begin{aligned}
&\exists u : w = S \circ T(u) = S(T(u)) \\
\Rightarrow &w = S(v) \wedge v = T(u) \\
\Rightarrow &w \in \operatorname{im}(S)
\end{aligned}$$

Since for arbitrary $w \in \operatorname{im}(S \circ T)$ there is $w \in \operatorname{im}(S) \Rightarrow \operatorname{im}(S \circ T) \subseteq \operatorname{im}(S)$

3.2 Question b

According to rank-nullity theorem:

$$\dim(U) = \dim(\ker(T)) + \dim(\operatorname{im}(T))$$

Also:

$$\dim(U) = \dim(\ker(S \circ T)) + \dim(\operatorname{im}(S \circ T))$$

$$\begin{aligned}
&\Rightarrow \dim(\ker(T)) + \dim(\operatorname{im}(T)) = \dim(\ker(S \circ T)) + \dim(\operatorname{im}(S \circ T)) \\
&\Rightarrow \dim(\ker(S \circ T)) - \dim(\ker(T)) = \dim(\operatorname{im}(T)) - \dim(\operatorname{im}(S \circ T))
\end{aligned}$$

3.3 Question c

Let $u \in \ker(\hat{S}) \subseteq \hat{V} \subseteq V$

$\Rightarrow \hat{S}(u) = 0_w$

$\hat{S}(u) = S(u) = 0_w$

$\Rightarrow u \in \ker(S)$

Also, $u \in \hat{V} = \operatorname{im}(T)$

$\Rightarrow u \in \ker(S) \cap \operatorname{im}(T)$

$\Rightarrow \ker(\hat{S}) \subseteq \ker(S) \cap \operatorname{im}(T) \rightarrow u$ is arbitrary in $\ker(\hat{S})$

Let $x \in \ker(S) \cap \operatorname{im}(T)$

(conclusion 1)

$\Rightarrow x \in \ker(S) \wedge x \in \operatorname{im}(T)$

$\Rightarrow S(x) = 0_w \wedge x \in \hat{V}$

$S(x) = \hat{S}(x) = 0_w$

$\Rightarrow \hat{S}(x) = 0_w \wedge x \in \hat{V}$

$\Rightarrow x \in \ker(\hat{S})$

$\Rightarrow \ker(S) \cap \operatorname{im}(T) \subseteq \ker(\hat{S})$

$\ker(\hat{S}) \subseteq \ker(S) \cap \operatorname{im}(T)$

$\Rightarrow \ker(S) \cap \operatorname{im}(T) = \ker(\hat{S})$

$$
\begin{aligned}
&\text{Let } w \in \mathrm{im}(\hat{S}) \\
&\quad \Rightarrow \exists m \in \hat{V} = \mathrm{im}(T) : \hat{S}(m) = w \\
&\quad \quad \hat{S}(m) = S(m) = w \\
&\quad \quad m = T(k) \rightarrow m \in \mathrm{im}(T) \\
&\quad \Rightarrow S(T(k)) = w \\
&\quad \Rightarrow w \in \mathrm{im}(S \circ T) \\
&\quad \Rightarrow \mathrm{im}(\hat{S}) \subseteq \mathrm{im}(S \circ T) \\
&\text{Let } v \in \mathrm{im}(S \circ T) \\
&\quad \Rightarrow S \circ T(p) = v \\
&\quad \quad T(p) = q \in \hat{V} \\
&\quad \Rightarrow S(q) = w \\
&\quad \quad S(q) = \hat{S}(q) = w \\
&\quad \Rightarrow q \in \mathrm{im}(\hat{S}) \\
&\quad \Rightarrow \mathrm{im}(S \circ T) \subseteq \mathrm{im}(\hat{S}) \\
&\quad \quad \mathrm{im}(\hat{S}) \subseteq \mathrm{im}(S \circ T) \\
&\quad \Rightarrow \mathrm{im}(\hat{S}) = \mathrm{im}(S \circ T)
\end{aligned}
\tag{conclusion 2}
$$

$$\dim(\hat{V}) = \dim(\ker(\hat{S})) + \dim(\mathrm{im}(\hat{S}))$$

3.4 Question d

$$
\begin{aligned}
\ker(S) \cap \mathrm{im}(T) &= \ker(\hat{S}) \\
\dim(\ker(S) \cap \mathrm{im}(T)) &= \dim(\ker(\hat{S})) \\
\mathrm{im}(\hat{S}) &= \mathrm{im}(S \circ T) \\
\dim(\mathrm{im}(\hat{S})) &= \dim(\mathrm{im}(S \circ T)) \\
\dim(\ker(S \circ T)) - \dim(\ker(T)) &= \dim(\mathrm{im}(T)) - \dim(\mathrm{im}(S \circ T)) \\
&= \dim(\hat{V}) - \dim(\mathrm{im}(\hat{S})) \\
&= \dim(\ker(\hat{S})) \rightarrow \text{rank-nullity theorem} \\
&= \dim(\ker(S) \cap \mathrm{im}(T)) \\
\Rightarrow \dim(\ker(S \circ T)) - \dim(\ker(T)) &= \dim(\ker(S) \cap \mathrm{im}(T)) \\
\Rightarrow \dim(\ker(S \circ T)) &= \dim(\ker(T)) + \dim(\ker(S) \cap \mathrm{im}(T))
\end{aligned}
$$

4 Problem 4

4.1 Question a

$$
\begin{aligned}
T(a,b,c) &= (a,a,c+b-a) \\
T^2(a,b,c) &= T(T(a,b,c)) \\
&= T(a,a,c+b-a) \\
&= (a,a,c+b-a+a-a) \\
&= (a,a,c+b-a) \\
&= T(a,b,c) \\
&\Rightarrow T(a,b,c) \text{ is idempotent}
\end{aligned}
$$

4.2 Question b

Proof.
Proof by contradiction:

Suppose $u \neq 0_V \in \ker(T) \cap \operatorname{im}(T)$

$$
\begin{aligned}
u \in \ker(T) &\Rightarrow T(u) = 0_V \\
u \in \operatorname{im}(T) &\Rightarrow \exists v \in V : T(v) = u \\
T(u) &= T(T(v)) = T^2(v) = 0_V \\
u \neq 0 &\Rightarrow T(v) \neq 0 \\
\Rightarrow v \notin \ker(T) &\wedge v \in ker(T^2) \\
\Rightarrow \dim(\ker(T^2)) &\neq \dim(\ker(T)) \\
T^2 = T \Rightarrow \dim(\ker(T^2)) &= \dim(\ker(T)) \\
\dim(\ker(T^2)) = \dim(\ker(T)) &\wedge \dim(\ker(T^2)) \neq \dim(\ker(T)) \Rightarrow \perp \\
\Rightarrow \ker(T) &\cap \operatorname{im}(T) = 0_V
\end{aligned}
$$

□

4.3 Question c

$$
\begin{aligned}
&T \text{ is one-to-one}\\
\Rightarrow&\dim(\ker T) = 0\\
\Rightarrow&\dim(\operatorname{im} T) = \dim(V)\\
\Rightarrow&T \text{ is onto}\\
\Rightarrow&T \text{ is inversible}
\end{aligned}
$$

Hence define $T^{-1} : V \to V$ is the inverse of T.

$$
\begin{aligned}
T^2(v) &= T(v)\\
\Rightarrow T \circ T(v) &= T(v)\\
\Rightarrow T^{-1} \circ T \circ T(v) &= T^{-1} \circ T(v)\\
\Rightarrow T(v) &= v
\end{aligned}
$$

5 Problem 5

5.1 Question a

$$\begin{aligned}
\text{Define } T(x) = Ax \wedge S(x) &= Bx \\
(T+S)(x) =& T(x) + S(x) \\
=& Ax + Bx \\
=& (A+B)x \\
\Rightarrow \operatorname{im}(T+S) =& \operatorname{span}(A+B) \\
\operatorname{im}(T) =& \operatorname{span}(A) \\
\operatorname{im}(S) =& \operatorname{span}(B) \\
\text{Define } M =& \operatorname{span}(A \cap B) = \{v_1, v_2, ..., v_r\} \\
\text{Then we can also define:} & \\
\operatorname{span}(A) = A_1 + M, A_1 = \{u_1, u_2, ..., u_m\} \wedge \operatorname{span}(B) = B_1 + M, B_1 = \{w_1, w_2, ..., w_n\} & \\
\Rightarrow \operatorname{span}(A+B) =& \{v_1, v_2, ..., v_r, u_1, u_2, ..., u_m, w_1, w_2, ..., w_n\} \\
\{v_1, v_2, ..., v_r, u_1, u_2, ..., u_m, w_1, w_2, ..., w_n\} \subseteq& \{v_1, v_2, ..., v_r, u_1, u_2, ..., u_m\} + \{v_1, v_2, ..., v_r, w_1, w_2, ..., w_n\} \\
\Rightarrow \operatorname{span}(A+B) \subseteq& \operatorname{span}(A) + \operatorname{span}(B) \\
\Rightarrow \operatorname{im}(T+S) \subseteq& \operatorname{im}(T) + \operatorname{im}(S)
\end{aligned}$$

5.2 Question b

$$\begin{aligned}
A = [a_{mn}] \wedge B &= [b_{mn}] \wedge C = [c_{lm}] \\
C(A+B) &= [c_{lm}] ([a_{mn}] + [b_{mn}]) \\
&= [c_{lm}] [(a_{mn} + b_{mn})] \\
&= \left[\sum_{m=1}^{k} c_{lk}(a_{kn} + b_{kn}) \right] \\
&= \left[\sum_{m=1}^{k} c_{lk} a_{kn} + \sum_{m=1}^{k} c_{lk} b_{kn} \right] \\
&= \left[\sum_{m=1}^{k} c_{lk} a_{kn} \right] + \left[\sum_{m=1}^{k} c_{lk} b_{kn} \right] \\
&= [c_{lm} a_{mn}] + [c_{lm} b_{mn}] \\
&= CA + BA
\end{aligned}$$

6 Reference

6.1 Collaborators

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