1 Problem 1

$$
\begin{aligned}
T(p_1) =&(p_1(1), \int_0^1 p_1(x)dx) \\
=&(3, \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 + x\right]_0^1) \\
=&(3, \frac{11}{6}) \\
=&-\frac{11}{6}(1,-1) + \frac{29}{12}(2,0) \\
T(p_2) =&(p_2(1), \int_0^1 p_2(x)dx) \\
=&(2, \left[\frac{1}{2}x^2 + x\right]_0^1) \\
=&(2, \frac{3}{2}) \\
=&-\frac{3}{2}(1,-1) + \frac{7}{4}(2,0) \\
T(p_3) =&(p_3(1), \int_0^1 p_3(x)dx) \\
=&(1, [x]_0^1) \\
=&(1,1) \\
=&-(1,-1) + (2,0) \\
\Rightarrow [T]_{\beta_1}^{\gamma_1} =& \begin{bmatrix} -\frac{11}{6} & -\frac{3}{2} & -1 \\ \frac{29}{12} & \frac{7}{4} & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
T(f_1) =&(f_1(1), \int_0^1 f_1(x)dx) \\
=&(0, \left[\frac{1}{3}x^3 - \frac{1}{2}x^2\right]_0^1) \\
=&(0, -\frac{1}{6}) \\
=&\frac{1}{6}(-1,2) + \frac{1}{6}(1,-3) \\
T(f_2) =&(f_2(1), \int_0^1 f_2(x)dx) \\
=&(2, \left[\frac{1}{2}x^2 + x\right]_0^1) \\
=&(2, \frac{3}{2}) \\
=&-\frac{15}{2}(-1,2) - \frac{11}{2}(1,-3) \\
T(f_3) =&(f_3(1), \int_0^1 f_3(x)dx) \\
=&(2, \left[x^2\right]_0^1) \\
=&(2,1) \\
=&-7(-1,2) - 5(1,-3) \\
\Rightarrow [T]_{\beta_2}^{\gamma_2} =& \begin{bmatrix} \frac{1}{6} & -\frac{15}{2} & -7 \\ \frac{1}{6} & -\frac{11}{2} & -5 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
p_1(x) =&x^2 + x + 1 \\
=&1 \cdot (x^2 - x) + 1 \cdot (x+1) + \frac{1}{2}(2x) \\
p_2(x) =&x + 1 \\
=&0 \cdot (x^2 - x) + 1 \cdot (x+1) + 0 \cdot (2x) \\
p_3(x) =&1 \\
=&0 \cdot (x^2 - x) + 1 \cdot (x+1) - \frac{1}{2} \cdot (2x) \\
\Rightarrow Q_{\beta_1}^{\beta_2} =& \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
(-1,2) =& -2\cdot(1,-1)+\frac{1}{2}\cdot(2,0)\\
(1,-3) =& 3\cdot(1,-1)-1\cdot(2,0)\\
\Rightarrow Q^{\gamma_1}_{\gamma_2} =& \begin{bmatrix} -2 & 3 \\ \frac{1}{2} & -1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
& Q^{\gamma_1}_{\gamma_2}\,[T]^{\gamma_2}_{\beta_2}\,Q^{\beta_2}_{\beta_1}\\
=& \begin{bmatrix} -2 & 3 \\ \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & -\frac{15}{2} & -7 \\ \frac{1}{6} & -\frac{11}{2} & -5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}\\
=& \begin{bmatrix} \frac{1}{6} & -\frac{3}{2} & -1 \\ \frac{1}{12} & \frac{27}{4} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}\\
=& \begin{bmatrix} -\frac{11}{6} & -\frac{3}{2} & -1 \\ \frac{29}{12} & \frac{7}{4} & 1 \end{bmatrix}\\
=& [T]^{\gamma_1}_{\beta_1}\\
\Rightarrow & [T]^{\gamma_1}_{\beta_1} = Q^{\gamma_1}_{\gamma_2}\,[T]^{\gamma_2}_{\beta_2}\,Q^{\beta_2}_{\beta_1}
\end{aligned}
$$

2 Problem 2

2.1 Question a

$$
\begin{aligned}
L_A(p_1) &= \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \\
&= \frac{12}{5} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{2}{5} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{4}{5} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \\
L_A(p_2) &= \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \\
&= \frac{7}{5} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{8}{5} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{4}{5} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \\
L_A(p_3) &= \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \\
&= \frac{7}{5} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{7}{5} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \\
\Rightarrow [L_A]_\beta^\beta &= \begin{bmatrix} \frac{12}{5} & \frac{7}{5} & \frac{7}{5} \\ \frac{2}{5} & -\frac{8}{5} & \frac{7}{5} \\ -\frac{4}{5} & -\frac{4}{5} & \frac{1}{5} \end{bmatrix}
\end{aligned}
$$

2.2 Question b

$$
\begin{aligned}
[L_A]_\beta &= [I]_\alpha^\beta \, [L_A]_\alpha \, [I]_\beta^\alpha \rightarrow \alpha \text{ is the standard basis on } \mathbb{F}^3 \\
&= Q^{-1}AQ \\
\Rightarrow Q &= [I]_\beta^\alpha \\
&= \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -2 \end{bmatrix}
\end{aligned}
$$

3 Problem 3

3.1 Question a

The reflection is on the xy plane, so the x and y coordinates are the same, with z flips the sign.

So the formula is $R(x,y,z) = (x,y,-z)$

3.2 Question b

$$\begin{aligned} x + 2y + 3z =& 0 \\ x =& -2y - 3z \\ \Rightarrow (x,y,z) =& (-2y-3z, y, z) \\ =& y(-2,1,0) + z(-3,0,1) \end{aligned}$$

So the basis for P is $\{(-2,1,0),(-3,0,1)\}$, the normal vector is $(1,2,3)$.

3.3 Question c

$$\begin{aligned} D_p =& \frac{(x,y,z) \cdot (1,2,3)}{|< 1,2,3 >|^2} \\ R_p(x,y,z) =& (x,y,z) - 2D_p \cdot (1,2,3) \\ =& (x,y,z) - 2 \cdot \frac{(x,y,z) \cdot (1,2,3)}{|< 1,2,3 >|^2} \cdot (1,2,3) \end{aligned}$$

4 Problem 4

4.1 Question a

$$\begin{aligned} f(A+B) =& \text{Trace}(A+B) \\ =& \text{Trace}(A) + \text{Trace}(B) \\ =& f(A) + f(B) \\ f(cA) =& \text{Trace}(cA) \\ =& c\text{Trace}(A) \\ =& cf(A) \end{aligned}$$

Therefore, Trace(A) fits the properties of linear functional and in V^*.

4.2 Question b

Proof. Assume there exists matrices: $AB - BA = I$

$$\begin{aligned} \text{Trace}(AB - BA) =& \text{Trace}(I) \\ \text{Trace}(AB) - \text{Trace}(BA) =& \text{Trace}(I) \end{aligned}$$

Trace(AB) − Trace(BA) is the trace of commutator, which is 0.

However, in this case, we assume that Trace(AB) − Trace(BA) = Trace(I), which is not 0.

There is the contradiction.

Therefore there cannot be matrices that $AB - BA = I$. □

4.3 Question c

Define $p(x) = g(x) - g(I)f(x)$

We need to first prove the commutative property for $p(x)$

$$\begin{aligned} p(AB) =& g(AB) - g(I)f(AB) \\ =& g(BA) - g(I)f(BA) \\ =& h(BA) \end{aligned}$$

Then substitute x as kI, where k is an arbitrary real number and I is identity matrix.

$$
\begin{aligned}
p(kI) =& g(kI) - g(I)f(kI) \\
=& kg(I) - kg(I)f(I) \\
=& 0
\end{aligned}
$$

Therefore, $p(x) = 0 \forall x \in V$

$$
\begin{aligned}
g(x) - g(I)f(x) =& 0 \\
g(x) =& g(I)f(x)
\end{aligned}
$$

Since $g(I)$ is a constant, $g(x) = cf(x)$

5 Problem 5

5.1 Question a

$$f_1(x) = \begin{cases} 1, x = -1 \\ 0, x \neq -1 \end{cases}$$
$$f_2(x) = \begin{cases} 1, x = 2 \\ 0, x \neq -2 \end{cases}$$

In this case, both functions satisfy the requirements, however, they are linear independent because they have different non-zero points, for example, there is never a k for $kf_1(2) = f_2(2)$.

5.2 Question b

Assume $q_1(x) = 1$ and $q_2(x) = \frac{1}{2}x - \frac{1}{4}$

$$f(q_1(x)) = \int_0^1 1dx = 1$$
$$g(q_2(x)) = q_2(1) - q_2(-1) = 1$$
$$f(q_2(x)) = \int_0^1 \frac{1}{2}x - \frac{1}{4}dx = 0$$
$$g(q_1(x)) = q_1(1) - q_1(-1) = 0$$

The functions satisfy the requirements. and they are linear independent because they are in different dimensions so that they are not scalar multiples.

6 Reference

6.1 Collaborators

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