Question 1

Problem a

$$\begin{aligned}&\text{False}:\\&\exists G/H \text{ as a factor group} \Leftrightarrow H \trianglelefteq G\end{aligned}$$

Problem b

$$\begin{aligned}&\text{True}:\\&G \text{ is abelian}\\\Rightarrow&\forall H \text{ as a subgroup}, H \trianglelefteq G\\\Rightarrow&G/H \text{ is a group}\end{aligned}$$

Problem c

$$\begin{aligned}&\text{True}:\\&|G/N| = \infty\\\Rightarrow&\text{There are infinitely many left cosets of } N \text{ in } G\\\Rightarrow&|G| = \infty\end{aligned}$$

Problem d

$$\begin{aligned}&\text{False}:\\&\forall G \text{ nonabelian}:\\&G/\{e\} \text{ is abelian}\end{aligned}$$

Problem e

$$\begin{aligned}
&\text{True :}\\
&\forall g \in G : gN \in G/N\\
&(gN)^a = g^a N\\
\Rightarrow &\exists n : g^n = g \leftarrow G \text{ is cyclic}\\
\Rightarrow &g^n N = gN \in G/N\\
\Rightarrow &G/N \text{ is cyclic}
\end{aligned}$$

Problem f

$$\begin{aligned}
&\text{False :}\\
&\{e\} \text{ is abelian}\\
&G/\{e\} \text{ must be abelian}\\
&G \text{ does not need to be abelian}
\end{aligned}$$

Problem g

$$\begin{aligned}
&\text{True :}\\
&N \text{ is normal}\\
\Rightarrow &\forall g \in G, gN = Ng\\
\Rightarrow &gNg^{-1} = Ngg^{-1} = N
\end{aligned}$$

Problem h

$$
\begin{aligned}
&\text{True :}\\
&|H| = d\\
\Rightarrow &\forall h \in H, g \in G, (ghg^{-1})^d = gh^dg^{-1} = geg^{-1} = e\\
\Rightarrow &gHg^{-1} \text{ has the same order as } H\\
\Rightarrow &gHg^{-1} = H \text{ by assumption}\\
\Rightarrow &gH = Hg\\
&H \trianglelefteq G
\end{aligned}
$$

Problem i

$$
\begin{aligned}
&\text{True :}\\
&\exists n \in H \cap K\\
\Rightarrow &\forall g \in G, gng^{-1} \in H \wedge gng^{-1} \in K \leftarrow H \trianglelefteq G \wedge K \trianglelefteq G\\
\Rightarrow &\forall g \in G, n \in H \cap K : gng^{-1} \in H \cap K\\
\Rightarrow &H \cap K \trianglelefteq G
\end{aligned}
$$

Question 2

Problem a

$$
\begin{aligned}
&|\mathbb{Z}_9 \times \mathbb{Z}_{35}| = 9 \times 35 \\
&|\langle (3) \times (25) \rangle| = \frac{9}{\gcd(9,3)} \times \frac{35}{\gcd(35,25)} = 3 \times 7 \\
\Rightarrow &|\mathbb{Z}_9 \times \mathbb{Z}_{35} / \langle (3), (25) \rangle| \\
= &\frac{|\mathbb{Z}_9 \times \mathbb{Z}_{35}|}{|\langle (3), (25) \rangle|} \\
= &\frac{9 \times 35}{3 \times 7} \\
= &15
\end{aligned}
$$

Problem b

$$
\begin{aligned}
&|\mathbb{Z}_9 \times \mathbb{Z}_{35}| = 9 \times 35 \\
&\langle (0) \times (11) \rangle \text{ only affects} \mathbb{Z}_{35} \\
\Rightarrow &|\langle (0) \times (11) \rangle| = \frac{35}{\gcd(35,11)} = 35 \\
\Rightarrow &|\mathbb{Z}_9 \times \mathbb{Z}_{35} / \langle (0) \times (11) \rangle| \\
= &\frac{9 \times 35}{35} \\
= &9
\end{aligned}
$$

Problem c

$$
\begin{aligned}
&\langle (1,1) \rangle \text{ generates the whole group} \\
\Rightarrow &|\mathbb{Z}_{19} \times \mathbb{Z}_{24} / \langle (1,1) \rangle| = 1
\end{aligned}
$$

Question 3

Problem a

$$
\begin{aligned}
&|\mathbb{Z}_2 \times \mathbb{Z}_4| = 8 \\
&|\langle(0,2)\rangle| = 2 \\
\Rightarrow &|\mathbb{Z}_2 \times \mathbb{Z}_4/\langle(0,2)\rangle| = 4 \\
&\mathbb{Z}_2 \text{ is untouched} \\
\Rightarrow &\mathbb{Z}_2 \text{ remains the same} \\
\Rightarrow &\mathbb{Z}_2 \times \mathbb{Z}_4/\langle(0,2)\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2
\end{aligned}
$$

Problem b

$$
\begin{aligned}
&\langle(3,0,0)\rangle \text{ is infinite in } \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_4 \\
&\mathbb{Z}_4 \text{ remains untouched} \\
\Rightarrow &\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_4/\langle(3,0,0)\rangle \cong \mathbb{Z}
\end{aligned}
$$

Problem c

$$
\begin{aligned}
&(1,0) \in \mathbb{Z} \times \mathbb{Z} \\
&(1,0) \notin \langle(2,2)\rangle \\
\Rightarrow &\forall m, (m,0) \notin \langle(2,2)\rangle \\
\Rightarrow &\mathbb{Z} \times \mathbb{Z}/\langle(2,2)\rangle \cong \mathbb{Z}
\end{aligned}
$$

Problem d

$$
\begin{aligned}
&(1,0) \in \mathbb{Z} \times \mathbb{Z} \\
&(1,0) \notin \langle(1,2)\rangle \\
\Rightarrow &\forall m, (m,0) \notin \langle(1,2)\rangle \\
\Rightarrow &\mathbb{Z} \times \mathbb{Z}/\langle(1,2)\rangle \cong \mathbb{Z}
\end{aligned}
$$

Question 4

Problem a

$$
\begin{aligned}
&10 + \langle 8 \rangle \\
=&2 + \langle 8 \rangle \\
&4(2 + \langle 8 \rangle) \\
=&8 + \langle 8 \rangle \\
=&\langle 8 \rangle \\
\Rightarrow &\operatorname{ord}(10 + \langle 8 \rangle) = 4
\end{aligned}
$$

Problem b

$$
\begin{aligned}
&(1,7) + \langle (1,1) \rangle \\
&(2,14) + \langle (1,1) \rangle = (2,5) + (2,14) + \langle (1,1) \rangle \\
&(3,21) + \langle (1,1) \rangle = (3,3) + \langle (1,1) \rangle = \langle (1,1) \rangle \\
\Rightarrow &\operatorname{ord}((1,7) + \langle (1,1) \rangle) = 3
\end{aligned}
$$

Problem c

$$
\begin{aligned}
&(2,3) \in \mathbb{Z}_4 \times \mathbb{Z}_8 \\
&|(2,3)| = \operatorname{lcm}(|\langle 2 \rangle|, |\langle 3 \rangle|) = 8 \\
\Rightarrow &\operatorname{ord}((2,3) + \langle (1,2) \rangle) | 8 \\
&\operatorname{ord}((2,3) + \langle (1,2) \rangle) = 1, 2, 4, 8 \\
&(2,3) + \langle (1,2) \rangle \neq \langle (1,2) \rangle \\
&2(2,3) + \langle (1,2) \rangle = (0,6) + \langle (1,2) \rangle \neq \langle (1,2) \rangle \\
&4(2,3) + \langle (1,2) \rangle = (0,4) + \langle (1,2) \rangle \neq \langle (1,2) \rangle \\
&8(2,3) + \langle (1,2) \rangle = (0,0) + \langle (1,2) \rangle = \langle (1,2) \rangle \\
\Rightarrow &\operatorname{ord}((2,3) + \langle (1,2) \rangle) = 8
\end{aligned}
$$

Problem d

$$\begin{aligned}
&(3,2) \in \mathbb{Z}_4 \times \mathbb{Z}_8 \\
&|(3,2)| = \mathrm{lcm}(|\langle 3 \rangle|, |\langle 2 \rangle|) = 4 \\
\Rightarrow &\mathrm{ord}((3,2) + \langle (1,2) \rangle) | 4 \\
&\mathrm{ord}((3,2) + \langle (1,2) \rangle) = 1, 2, 4 \\
&(3,2) + \langle (1,2) \rangle \neq \langle (1,2) \rangle \\
&2(3,2) + \langle (1,2) \rangle = (2,4) + \langle (1,2) \rangle = \langle (1,2) \rangle \\
\Rightarrow &\mathrm{ord}((3,2) + \langle (1,2) \rangle) = 2
\end{aligned}$$

Question 5

Problem a

$$
\begin{aligned}
&S_3 = \{Id, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\} \\
&G_1 = \{g \in G, g \cdot 1 = 1\} \\
\Rightarrow &G_1 = \{Id, (2,3)\} \\
&G_2 = \{g \in G, g \cdot 2 = 2\} \\
\Rightarrow &G_2 = \{Id, (1,3)\} \\
&G_3 = \{g \in G, g \cdot 3 = 3\} \\
\Rightarrow &G_3 = \{Id, (1,2)\}
\end{aligned}
$$

Problem b

$$
\begin{aligned}
&X_{Id} = \{1,2,3\} \\
&X_{(1,2)} = \{3\} \\
&X_{(1,3)} = \{2\} \\
&X_{(2,3)} = \{1\} \\
&X_{(1,2,3)} = X_{(1,3,2)} = \emptyset
\end{aligned}
$$

Problem c

$Id \cdot 1 = 1$
$(1,2) \cdot 1 = 2$
$(1,3) \cdot 1 = 3$
$G \cdot 1 = \{1,2,3\} = X$
$Id \cdot 2 = 2$
$(1,2) \cdot 2 = 1$
$(2,3) \cdot 2 = 3$
$G \cdot 2 = \{1,2,3\} = X$
$Id \cdot 3 = 3$
$(1,3) \cdot 3 = 1$
$(2,3) \cdot 3 = 2$
$G \cdot 3 = \{1,2,3\} = X$
$\Rightarrow \forall x \in X, G \cdot x = X$
$\Rightarrow$There is only one orbit
$\Rightarrow$The action is transitive

Question 6

Problem a

$$
\begin{aligned}
&X = \{G, \{e\}, \langle\rho\rangle, \langle\rho^2\rangle, \langle\tau\rangle, \langle\tau\rho\rangle, \langle\tau\rho^2\rangle, \langle\tau\rho^3\rangle, \langle\rho^2,\tau\rangle, \langle\rho^2,\tau\rho\rangle\}\\
&\text{center of } D_4 : e, \rho^2\\
&\forall g \in G, \rho g \rho^{-1} \in G\\
\Rightarrow &G \in X_\rho\\
&\rho e \rho^{-1} = e\\
\Rightarrow &\{e\} \in X_\rho\\
&\forall h \in \langle\rho\rangle, h = \rho^n, n \in \{1,2,3,4\}, \rho^4 = e\\
&\rho h \rho^{-1} = \rho\rho^n\rho^{-1} = \rho^n = h \in \langle\rho\rangle\\
\Rightarrow &\langle\rho\rangle \in X_\rho\\
&\rho\rho^2\rho^{-1} = \rho^2\\
\Rightarrow &\langle\rho^2\rangle \in X_\rho\\
&\rho\tau\rho^{-1} = \tau\rho^3\rho^{-1} = \tau\rho^2 \notin \langle\tau\rangle\\
\Rightarrow &\langle\tau\rangle \notin X_\rho\\
&\rho\tau\rho\rho^{-1} = \rho\tau = \tau\rho^3 \notin \langle\tau\rho\rangle\\
\Rightarrow &\langle\tau\rho\rangle \notin X_\rho\\
&\rho\tau\rho^2\rho^{-1} = \rho\tau\rho = \tau\rho^3\rho = \tau \notin \langle\tau\rho^2\rangle\\
\Rightarrow &\langle\tau\rho^2\rangle \notin X_\rho\\
&\rho\tau\rho^3\rho^{-1} = \rho\tau\rho^2 = \tau\rho^3\rho^2 = \tau\rho \notin \langle\tau\rho^3\rangle\\
\Rightarrow &\langle\tau\rho^3\rangle \notin X_\rho\\
&\rho\tau\rho^{-1} = \tau\rho^2 \in \langle\rho^2,\tau\rangle\\
&\rho\tau\rho^2\rho^{-1} = \tau \in \langle\rho^2,\tau\rangle\\
&e, \rho^2 \text{are centers}\\
\Rightarrow &\langle\rho^2,\tau\rangle \in X_\rho\\
&\rho\tau\rho\rho^{-1} = \tau\rho^3 \in \langle\rho^2,\tau\rho\rangle\\
&\rho\tau\rho^3\rho^{-1} = \tau\rho \in \langle\rho^2,\tau\rho\rangle\\
&e, \rho^2 \text{are centers}\\
\Rightarrow &\langle\rho^2,\tau\rho\rangle \in X_\rho\\
\Rightarrow &X_\rho = \{G, \{e\}, \langle\rho\rangle, \langle\rho^2\rangle, \langle\rho^2,\tau\rangle, \langle\rho^2,\tau\rho\rangle\}
\end{aligned}
$$

Problem b

$$
\begin{aligned}
&X=\{G,\{e\},\langle\rho\rangle,\langle\rho^2\rangle,\langle\tau\rangle,\langle\tau\rho\rangle,\langle\tau\rho^2\rangle,\langle\tau\rho^3\rangle,\langle\rho^2,\tau\rangle,\langle\rho^2,\tau\rho\rangle\}\\
&\forall g\in G, g=\tau^m\rho^n, m\in\{0,1\}, n\in\{0,1,2,3\}\\
&\text{orb}(G)=\{G\}\\
&\text{center of } D_4: e,\rho^2\\
\Rightarrow&\text{orb}(\{e\})=\{\{e\}\}\\
&\text{orb}(\langle\rho^2\rangle)=\{\langle\rho^2\rangle\}\\
&\forall\rho^k\in\langle\rho\rangle:\\
&\tau^m\rho^n\rho^k(\tau^m\rho^n)^{-1}\\
=&\tau^m\rho^n\rho^k\rho^{-n}\tau^{-m}\\
=&\tau^m\rho^k\tau^{-m}\\
=&\rho^k\tau^{-2m}\\
=&\rho^k\\
\Rightarrow&\text{orb}(\langle\rho\rangle)=\{\langle\rho\rangle\}\\
&\forall\tau^k\in\langle\tau\rangle:\\
&\tau^m\rho^n\tau^k(\tau^m\rho^n)^{-1}\\
=&\tau^m\rho^n\tau^k\rho^{-n}\tau^{-m}\\
=&\rho^{-n}\tau^{-m}\tau^k\tau^m\rho^n\\
=&\rho^{-n}\tau^k\rho^n\\
=&\tau^k\rho^{2n}\\
&n\text{ is odd}: \tau^m\rho^n\tau^k(\tau^m\rho^n)^{-1}\in\langle\tau\rho^2\rangle\\
&n\text{ is even}: \tau^m\rho^n\tau^k(\tau^m\rho^n)^{-1}\in\langle\tau\rangle\\
\Rightarrow&\text{orb}(\langle\tau\rangle)=\{\langle\tau\rangle,\langle\tau\rho^2\rangle\}\\
&\forall\tau^k\rho^k\in\langle\tau\rho\rangle:\\
&\tau^m\rho^n\tau^k\rho^k(\tau^m\rho^n)^{-1}\\
=&\tau^m\rho^n\tau^k\rho^k\rho^{-n}\tau^{-m}\\
=&\tau^{m-k}\rho^{k-2n}\tau^{-m}\\
=&\tau^{2m-k}\rho^{2n-k}\\
=&\tau^{-k}\rho^{2n-k}\\
&n\text{ is odd}: \tau^m\rho^n\tau^k\rho^k(\tau^m\rho^n)^{-1}\in\langle\tau\rho\rangle\\
&n\text{ is even}: \tau^m\rho^n\tau^k\rho^k(\tau^m\rho^n)^{-1}\in\langle\tau\rho^3\rangle\\
\Rightarrow&\text{orb}(\langle\tau\rho\rangle)=\{\langle\tau\rho\rangle,\langle\tau\rho^3\rangle\}
\end{aligned}
$$

$$
\begin{aligned}
&\forall \tau^p\rho^{2q} \in \langle \rho^2, \tau\rangle :\\
&\tau^m\rho^n\tau^p\rho^{2q}(\tau^m\rho^n)^{-1}\\
=&\tau^m\rho^n\tau^p\rho^{2q}\rho^{-n}\tau^{-m}\\
=&\tau^{m-p}\rho^{2q-2n}\tau^{-m}\\
=&\tau^{-p}\rho^{2n-2q} \in \langle \rho^2, \tau\rangle\\
\Rightarrow&\mathrm{orb}(\langle \rho^2, \tau\rangle) = \{\langle \rho^2, \tau\rangle\}\\
&\forall \tau^p\rho^{2q+p} \in \langle \rho^2, \tau\rho\rangle :\\
&\tau^m\rho^n\tau^p\rho^{2q+p}(\tau^m\rho^n)^{-1}\\
=&\tau^m\rho^n\tau^p\rho^{2q+p}\rho^{-n}\tau^{-m}\\
=&\tau^{m-p}\rho^{2q-2n+p}\tau^{-m}\\
=&\tau^{-p}\rho^{-p+2n-2q} \in \langle \rho^2, \tau\rho\rangle\\
\Rightarrow&\mathrm{orb}(\langle \rho^2, \tau\rho\rangle) = \{\langle \rho^2, \tau\rho\rangle\}\\
&\mathrm{orb}(X) = \{G\}, \{\{e\}\}, \{\langle \rho^2\rangle\}, \{\langle \rho\rangle\}, \{\langle \tau\rangle, \langle \tau\rho^2\rangle\}, \{\langle \tau\rho\rangle, \langle \tau\rho^3\rangle\}, \{\langle \rho^2, \tau\rangle\}, \{\langle \rho^2, \tau\rho\rangle\}
\end{aligned}
$$

Problem 3

$$
\begin{aligned}
&G \trianglelefteq G\\
&\{e\} \trianglelefteq G\\
&\langle \rho^2\rangle \trianglelefteq G\\
&\langle \rho\rangle \trianglelefteq G\\
&\langle \rho^2, \tau\rangle \trianglelefteq G\\
&\langle \rho^2, \tau\rho\rangle \trianglelefteq G
\end{aligned}
$$

Question 7

$Id : n^4$

$90^\circ : 0 \leftarrow$ The sides cannot be the same since they are not fixed

$180^\circ : n^2$

Flip :

Opposite edges : $2n^2$

Diagonal : $2n^2$

$$\frac{1}{|D_4|}(n^4 + 0 + n^2 + 2n^2 + 2n^2)$$
$$= \frac{1}{8}(n^4 + 5n^2)$$
$$= \frac{n^4}{8} + \frac{5n^2}{8}$$
$$\Rightarrow \frac{n^4}{8} + \frac{5n^2}{8} \text{ non-indistinguishable squares}$$

Reference

Jeffery Shu