Problem 1

Question a

$$
\begin{aligned}
&a \coloneqq (0,1)\\
&b \coloneqq (1,5)\\
\Rightarrow &f(x) \coloneqq \frac{y_b - y_a}{x_b - x_a}x + (y_a - \frac{y_b - y_a}{x_b - x_a}x_a)\\
\Rightarrow &f = 4x + 1\\
&f(x_1) = f(x_2)\\
\Rightarrow &4x_1 + 1 = 4x_2 + 1\\
\Rightarrow &x_1 = x_2\\
\Rightarrow &f \text{ is one-to-one}\\
&\forall y \in [1,5], \exists x \in [0,1] : x = \frac{y-1}{4}\\
\Rightarrow &f \text{ is onto}\\
\Rightarrow &f \text{ is bijective}\\
\Rightarrow &|[0,1]| = |[1,5]|
\end{aligned}
$$

Question b

$$
\begin{aligned}
&a \coloneqq (2,6)\\
&b \coloneqq (4,26)\\
\Rightarrow &f(x) \coloneqq \frac{y_b - y_a}{x_b - x_a}x + (y_a - \frac{y_b - y_a}{x_b - x_a}x_a)\\
\Rightarrow &f = 10x - 14\\
&f(x_1) = f(x_2)\\
\Rightarrow &10x_1 - 14 = 10x_2 - 14\\
\Rightarrow &x_1 = x_2\\
\Rightarrow &f \text{ is one-to-one}\\
&\forall y \in (6,26), \exists x \in (2,4) : x = \frac{y+14}{10}\\
\Rightarrow &f \text{ is onto}\\
\Rightarrow &f \text{ is bijective}\\
\Rightarrow &|(2,4)| = |(6,26)|
\end{aligned}
$$

Question c

$$
\begin{aligned}
&a \coloneqq (a, c)\\
&b \coloneqq (b, d)\\
\Rightarrow &f(x) \coloneqq \frac{y_b - y_a}{x_b - x_a}x + (y_a - \frac{y_b - y_a}{x_b - x_a}x_a)\\
\Rightarrow &f = \frac{d-c}{b-a}x + (c - \frac{d-c}{b-a}a)\\
&f(x_1) = f(x_2)\\
\Rightarrow &\frac{d-c}{b-a}x_1 + (c - \frac{d-c}{b-a}a) = \frac{d-c}{b-a}x_1 + (c - \frac{d-c}{b-a}a)\\
\Rightarrow &x_1 = x_2\\
\Rightarrow &f \text{ is one-to-one}\\
&\forall y \in (c, d], \exists x \in (a, b] : x = \frac{y - c + \frac{d-c}{b-a}a}{\frac{d-c}{b-a}}\\
\Rightarrow &f \text{ is onto}\\
\Rightarrow &f \text{ is bijective}\\
\Rightarrow &|(a, b]| = |(c, d]|
\end{aligned}
$$

Problem 2

$$
\begin{aligned}
&f:\mathbb{R}\to(-\frac{\pi}{2},\frac{\pi}{2})\\
&f(x)\coloneqq\arctan(x)\\
&f(x_1)=f(x_2)\\
\Rightarrow&\arctan(x_1)=\arctan(x_2)\\
\Rightarrow&\tan\arctan(x_1)=\tan\arctan(x_2)\\
\Rightarrow&x_1=x_2\\
\Rightarrow&f\text{ is one-to-one}\\
&\forall y\in(-\frac{\pi}{2},\frac{\pi}{2}),\exists x\in\mathbb{R}:x=\tan(y)\\
\Rightarrow&f\text{ is onto}\\
\Rightarrow&f\text{ is bijective}\\
\Rightarrow&|\mathbb{R}|=|(-\frac{\pi}{2},\frac{\pi}{2})|
\end{aligned}
$$

$$
\begin{aligned}
&g:(-\frac{\pi}{2},\frac{\pi}{2})\to(0,1)\\
&g(x)\coloneqq\frac{1}{\pi}x+\frac{1}{2}\\
&g(x_1)=g(x_2)\\
\Rightarrow&\frac{1}{\pi}x_1+\frac{1}{2}=\frac{1}{\pi}x_2+\frac{1}{2}\\
\Rightarrow&x_1=x_2\\
\Rightarrow&g\text{ is one-to-one}\\
&\forall y\in(0,1),\exists x\in(-\frac{\pi}{2},\frac{\pi}{2}):x=(y-\frac{1}{2})\pi\\
\Rightarrow&g\text{ is onto}\\
\Rightarrow&g\text{ is bijective}\\
\Rightarrow&|(-\frac{\pi}{2},\frac{\pi}{2})|=|(0,1)|\\
\Rightarrow&|\mathbb{R}|=|(0,1)|
\end{aligned}
$$

Problem 3

Question a

Numbers of subsets of A is the size of the power set of A

$|\wp(A)| = 2^4 = 16$

$\Rightarrow$Numbers of subsets of $A = 16$

Question b

1 partition :

$\{a, b, c, d\}$

2 partitions :

$\{a\}, \{b, c, d\}$

$\{b\}, \{a, c, d\}$

$\{c\}, \{a, b, d\}$

$\{d\}, \{a, b, c\}$

$\{a, b\}, \{c, d\}$

$\{a, c\}, \{b, d\}$

$\{a, d\}, \{b, c\}$

3 patitions :

$\{a\}, \{b\}, \{c, d\}$

$\{a\}, \{c\}, \{b, d\}$

$\{a\}, \{d\}, \{b, c\}$

$\{b\}, \{c\}, \{a, d\}$

$\{b\}, \{d\}, \{a, c\}$

$\{c\}, \{d\}, \{a, b\}$

4 partitions :

$\{a\}, \{b\}, \{c\}, \{d\}$

$\Rightarrow$Total numbers are 15

Question c

For each of the values in the domain, there are four possible values in the codomain to match, each choice for the element can produce a unique function, so there are $4^4 = 256$ distinct functions mapping from A to A.

Question d

The first element has 4 values to choose from, for the second elment, there are only 3 images to choosse from. By induction and multiplication principle, the number of all possible injective functions mapping from A to A is $4 \times 3 \times 2 \times 1 = 24$.

Question e

The relations of A must be a subset of $A \times A$. $|A \times A| = 4^2 = 16$ And the number of the subsets in $A \times A$ is the size of the power set of $A \times A$. That is: $|\wp(A \times A)| = 2^{16} = 65536$

Question f

For every element that can form an equivalence relation in A, they can form a cell in A. This means that every kind of equivalence relation in A is a type of partition in A, and this correspondence is bijective. Since there are 15 partitions in A, there are 15 equivalence relations in A.

Problem 4

Question a

$$
\begin{aligned}
& m \geqslant n \\
\Rightarrow & m \neg \mathcal{R} n \\
& n \mathcal{R} m \\
\Rightarrow & \mathcal{R} \text{ is not symmetric} \\
& m = m \\
\Rightarrow & m \neg \mathcal{R} m \\
\Rightarrow & \mathcal{R} \text{ is not reflexive}
\end{aligned}
$$

Question b

$$
\begin{aligned}
& z = z \coloneqq 0 \\
\Rightarrow & zz = 0 \\
\Rightarrow & z \neg \mathcal{R} z \\
\Rightarrow & \mathcal{R} \text{ is not reflexive}
\end{aligned}
$$

Question c

$$\begin{aligned}
&m = m \\
\Rightarrow &|m| = |m| \\
\Rightarrow &m\mathcal{R}m \\
\Rightarrow &\mathcal{R} \text{ is reflexive} \\
&\text{Suppose} : |m| = |n| \\
\Rightarrow &m\mathcal{R}n \\
&\text{Also} : |n| = |m| \\
\Rightarrow &n\mathcal{R}m \\
\Rightarrow &\mathcal{R} \text{ is symmetric} \\
&\text{Suppose} : |m| = |n| \wedge |n| = |p| \\
\Rightarrow &m\mathcal{R}n \wedge n\mathcal{R}p \\
&|m| = |n| \wedge |n| = |p| \\
\Rightarrow &|m| = |p| \\
\Rightarrow &m\mathcal{R}p \\
\Rightarrow &\mathcal{R} \text{ is transitive}
\end{aligned}$$

Question d

$$\begin{aligned}
&|z - z| < 1 \\
\Rightarrow &z \neg\mathcal{R} z \\
\Rightarrow &\mathcal{R} \text{ is not reflexive} \\
&\text{suppose} |z - w| \geqslant 1 \wedge |w - y| \geqslant 1 \\
\Rightarrow &z\mathcal{R}w \wedge w\mathcal{R}y \\
&z \coloneqq 5 \wedge w \coloneqq -1 \wedge y \coloneqq 4.5 \\
\Rightarrow &|z - w| \geqslant 1 \wedge |w - y| \geqslant 1 \wedge |z - y| = 0.5 < 1 \\
\Rightarrow &z \neg\mathcal{R} y \\
\Rightarrow &\mathcal{R} \text{ is not transitive}
\end{aligned}$$

Question e

$$
\begin{aligned}
&\exists A, B, C \in \mathrm{FS}(\mathbb{N}) \\
&|A| = |A| \\
\Rightarrow &A\mathcal{R}A \\
\Rightarrow &\mathcal{R} \text{ is reflexive} \\
&\text{Suppose} : |A| = |B| \\
\Rightarrow &A\mathcal{R}B \\
&|B| = |A| \leftarrow |A| = |B| \\
\Rightarrow &B\mathcal{R}A \\
\Rightarrow &\mathcal{R} \text{ is symmetric} \\
&\text{Suppose} : |A| = |B| \land |B| = |C| \\
\Rightarrow &A\mathcal{R}B \land B\mathcal{R}C \\
&|A| = |C| \leftarrow |A| = |B| \land |B| = |C| \\
\Rightarrow &A\mathcal{R}C \\
\Rightarrow &\mathcal{R} \text{ is transitive}
\end{aligned}
$$

Question f

$$
\begin{aligned}
&f(1) = f(1) \\
\Rightarrow &f\mathcal{R}f \\
\Rightarrow &\mathcal{R} \text{ is reflexive} \\
&\text{Suppose} : f(1) = g(1) \\
\Rightarrow &f\mathcal{R}g \\
&g(1) = f(1) \leftarrow f(1) = g(1) \\
\Rightarrow &g\mathcal{R}f \\
\Rightarrow &\mathcal{R} \text{ is symmetric} \\
&\text{Suppose} : f(1) = g(1) \land g(1) = h(1) \\
\Rightarrow &f\mathcal{R}g \land g\mathcal{R}h \\
&f(1) = h(1) \leftarrow f(1) = g(1) \land g(1) = h(1) \\
\Rightarrow &f\mathcal{R}h \\
\Rightarrow &\mathcal{R} \text{ is transitive}
\end{aligned}
$$

Problem 5

Question a

$$
\begin{aligned}
&a - a = 0 = 0 \times n \\
\Rightarrow &a \sim a \\
\Rightarrow &\sim \text{ is reflexive} \\
&\text{Suppose} : a - b = q \times n \\
\Rightarrow &a \sim b \\
&b - a = -q \times n \leftarrow a - b = q \times n \\
&-q \in \mathbb{Z} \\
\Rightarrow &b \sim a \\
\Rightarrow &\sim \text{ is symmetric} \\
&\text{Suppose} : a - b = q \times n \wedge b - c = p \times n \\
\Rightarrow &a \sim b \wedge b \sim c \\
&a - c = (p + q) \times n \\
&p + q \in \mathbb{Z} \\
\Rightarrow &a \sim c \\
\Rightarrow &\sim \text{ is transitive} \\
\Rightarrow &\sim \text{ is an equivalence relation}
\end{aligned}
$$

Question b

$$
\begin{aligned}
&\text{There are 4 cells for } n = 4 \\
&\text{class of remainder } 0 : \{\cdots, -4, 0, 4, \cdots\} = \{n | n = 4k, k \in \mathbb{Z}\} \\
&\text{class of remainder } 1 : \{\cdots, -3, 1, 5, \cdots\} = \{n | n = 4k + 1, k \in \mathbb{Z}\} \\
&\text{class of remainder } 2 : \{\cdots, -2, 2, 6, \cdots\} = \{n | n = 4k + 2, k \in \mathbb{Z}\} \\
&\text{class of remainder } 3 : \{\cdots, -1, 3, 7, \cdots\} = \{n | n = 4k + 3, k \in \mathbb{Z}\}
\end{aligned}
$$

Problem 6

Question a

$$
\begin{aligned}
&S = \{a, b, c\} \\
&R = \{(a,a),(b,b),(c,c),(a,b),(b,a)\}, \text{relation denoted as } = \\
&\{(a,a),(b,b),(c,c)\} \subseteq R \\
\Rightarrow &R \text{ is reflexive} \\
&\{(a,b),(b,a)\} \implies a = b \rightarrow b = a \\
\Rightarrow &R \text{ is symmetric} \\
&\text{In order for } a = c, \text{ we need } a = b \wedge b = c, butb \neq c \\
\Rightarrow &R \text{ is transitive}
\end{aligned}
$$

Question b

$$
\begin{aligned}
&S = \{a, b, c\} \\
&R = \{(a,a),(b,b),(c,c),(a,b),(b,c),(a,c)\}, \text{relation denoted as } = \\
&\{(a,a),(b,b),(c,c)\} \subseteq R \\
\Rightarrow &R \text{ is reflexive} \\
&\{(a,b),(b,c),(a,c)\} \subseteq R \\
\Rightarrow &a = b \wedge b = c \rightarrow a = c \\
\Rightarrow &R \text{ is transitive} \\
&(b,a) \text{ is not an element in } R \text{ whereas } (a,b) \in R \\
\Rightarrow &a = b \nrightarrow b = a \\
\Rightarrow &R \text{ is not symmetric}
\end{aligned}
$$

Question c

$S = \{a, b, c\}$

$R = \{(a, b), (b, a), (b, c), (c, b), (a, c), (c, a)\}$, relation denoted as $=$

$\{(a, b), (b, a)\} \implies a = b \to b = a$

$\{(b, c), (c, b)\} \implies b = c \to c = b$

$\{(a, c), (c, a)\} \implies a = c \to c = a$

$\Rightarrow R$ is symmetric

$\{(a, b), (b, c), (a, c)\} \in R \implies a = b \wedge b = c \implies c = a$

vice versa $\leftarrow R$ is symmetric

$\Rightarrow R$ is transitive

$\{(a, a), (b, b), (c, c)\} \nsubseteq R$

$\Rightarrow a \neq a \wedge b \neq b \wedge c \neq c$

$\Rightarrow R$ is not reflexive

Reference

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