Question 1

Proof.

$$\begin{aligned}
&\text{Suppose} : \exists f : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \text{ is a ring isomorphism} \\
\Rightarrow &f(e_{\mathbb{R}\times\mathbb{R}}) = e_{\mathbb{C}} \\
&\exists f((a, b))^4 = f(e_{\mathbb{R}\times\mathbb{R}}) = e_{\mathbb{C}} = (c + di)^4 \\
&(a, b)^2 = 1 \\
&(c + di)^2 = \pm 1 \\
\Rightarrow &\nexists f : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \text{ is a bijection} \\
\Rightarrow &\mathbb{R} \times \mathbb{R} \text{ and } \mathbb{C} \text{ are not ring isomorphic}
\end{aligned}$$

□

Proof.

$$\begin{aligned}
&f : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \\
&f : (a, b) \mapsto a + bi \\
&\text{By contruction, } f \text{ is bijective} \\
&f((a, b) + (c, d)) \\
=&f((a + c, b + d)) \\
=&a + c + (b + d)i \\
=&a + bi + c + di \\
=&f((a, b)) + f((c + d)) \\
\Rightarrow &f \text{ is homomorphic} \\
\Rightarrow &f \text{ is isomorphic}
\end{aligned}$$

□

Question 2

Problem a

Proof.

$$
\begin{aligned}
&\exists \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \in S \\
&\text{Identity :} \\
&a = 1, b = 0 \\
\Rightarrow &Id \in S \\
&\text{addition :} \\
&\exists \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \\
&\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a+c & -b-d \\ b+d & a+c \end{bmatrix} \in S \\
&\exists \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \\
&\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \\
= &\begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix} \in S \\
\Rightarrow &S \text{ is a subring}
\end{aligned}
$$

□

Problem 2

Proof.

$$f : S \to \mathbb{C}$$
$$f : \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \mapsto a + bi$$
Identity :
$$a = 1, b = 0$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = Id_S$$
$$1 + 0i = 1 = Id_{\mathbb{C}}$$
$$\Rightarrow f(Id_S) = Id_{\mathbb{C}}$$
Addition :
$$\exists A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, B = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$$
$$f(A + B) = f(\begin{bmatrix} a + c & -b - d \\ b + d & a + c \end{bmatrix}) = a + c + (b + d)i$$
$$f(A) + f(B) = a + bi + c + di = a + c + (b + d)i$$
$$\Rightarrow f(A + B) = f(A) + f(B)$$
Multiplication
$$f(AB) = f(\begin{bmatrix} ac - bd & -(ad + bc) \\ ad + bc & ac - bd \end{bmatrix}) = ac - bd + (ad + bc)i$$
$$f(A)f(B) = (a + bi)(c + di) = ac - bd + (ad + bc)i$$
$$\Rightarrow f(AB) = f(A)f(B)$$
By construction, f us bijective
$\Rightarrow f$ is a ring isomorphism

□

Question 3

$f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is a ring homomorphism

$f(1) =\coloneqq (a, b)$

$f(mn) = f(m)f(n) = m(a, b)n(a, b) = mn(a, b)^2 = mn(a^2, b^2)$

$f(mn) = mn(a, b)$

$\Rightarrow (a, b) = (a^2, b^2)$

$a = 0, 1 \wedge b = 0, 1$

$\Rightarrow$4 homomorphisms :

$f_1(1) = (1, 0) \to f_1(m) = (m, 0)$

$f_2(1) = (1, 1) \to f_2(m) = (m, m)$

$f_3(1) = (0, 1) \to f_3(m) = (0, m)$

$f_4(1) = (0, 0) \to f_4(m) = (0, 0)$

By construction, f_1, f_2, f_3 are injective

$f_4 : \forall n \in \mathbb{Z}, f(n) = (0, 0)$

$\Rightarrow f_4$ is not injective

$\forall n \in \mathbb{Z} \forall a \neq b \neq 0, f_1(n) \neq f_2(n) \neq f_3(n) \neq f_4(n) \neq (a, b), (a, b) \in \mathbb{Z} \times \mathbb{Z}$

$\Rightarrow f_1, f_2, f_3, f_4$ are all not surjective

Question 4

Problem a

Proof.

$$
\begin{aligned}
&\forall n \in R:\\
&(n+n) = (n+n)^2\\
&n+n = n^2+2n+n^2\\
&n+n = n+2n+n\\
&2n = 0\\
&n \text{ is arbitrary}\\
\Rightarrow &\operatorname{char}(R) = 2
\end{aligned}
$$

□

Problem b

Proof.

$$
\begin{aligned}
&\forall x, y \in R\\
&x+y = (x+y)^2\\
&x+y = (x+y)(x+y)\\
&x+y = x(x+y)+y(x+y)\\
&x+y = x^2+xy+yx+y^2\\
&x+y = x+xy+yx+y\\
&xy+yx = 0\\
&\operatorname{char}(R) = 2\\
\Rightarrow &xy+xy = 0\\
\Rightarrow &xy = yx\\
\Rightarrow &R \text{ is commutative}
\end{aligned}
$$

□

Problem c

Proof.

$$
\begin{aligned}
&D \text{ is a division ring}\\
&x \text{ is idempotent } \in D\\
\Rightarrow &x = x^2\\
&x^2 - x = 0\\
&x = 0, 1\\
&\text{Division ring cannot have zero divisors}\\
\Rightarrow &x = 1 (\text{multiplicative identity})\\
&x = 0 (\text{additive identity})
\end{aligned}
$$

□

Reference

Jeffery Shu