Question 1

Problem a

$$\begin{array}{c|ccc} * & e & a & b \\ \hline e & e & a & b \\ a & a & b & e \\ b & b & e & a \end{array}$$

This is the only way to fill the group table : $\mathbb{Z}_3$

$\Rightarrow$True

Problem b

$$\mathbb{V}_4 : \begin{array}{c|cccc} * & e & a & b & c \\ \hline e & e & a & b & c \\ a & a & e & c & b \\ b & b & c & e & a \\ c & c & b & a & e \end{array}$$

$$\mathbb{Z}_4 : \begin{array}{c|cccc} * & e & a & b & c \\ e & e & a & b & c \\ a & a & e & c & b \\ b & b & c & a & e \\ c & c & b & e & a \end{array}$$

$\mathbb{V}_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

$\Rightarrow$ $\mathbb{Z}_4$ and $\mathbb{V}_4$ have elements of different orders

not isomorphic

false

Problem c

Order 2

$\{e\} \Rightarrow$ abelian

Order 2

$$\begin{array}{c|cc} * & e & a \\ e & e & a \\ a & a & e \end{array} \Rightarrow \text{abelian}$$

Order 3

$$\begin{array}{c|ccc} * & e & a & b \\ \hline e & e & a & b \\ a & a & b & e \\ b & b & e & a \end{array} \Rightarrow \text{abelian}$$

Order 4

$$\mathbb{Z}_4 : \begin{array}{c|cccc} * & e & a & b & c \\ e & e & a & b & c \\ a & a & e & c & b \\ b & b & c & a & e \\ c & c & b & e & a \end{array} \Rightarrow \text{abelian}$$

Other forms of $\mathbb{Z}_4$ use the same method as this

$$\mathbb{V}_4 : \begin{array}{c|cccc} * & e & a & b & c \\ \hline e & e & a & b & c \\ a & a & e & c & b \\ b & b & c & e & a \\ c & c & b & a & e \end{array} \Rightarrow \text{abelian}$$

$\Rightarrow$True

Question 2

Problem 1

$|\mathbb{Q}| \neq |\mathbb{R}|$
There cannot be a bijection map between $\mathbb{Q}$ and $\mathbb{R}$
$\langle \mathbb{Q}, + \rangle$ and $\langle \mathbb{R}, + \rangle$ are not isomorphic

Problem 2

$$
\begin{aligned}
&e_{\langle \mathbb{R}^*, \cdot \rangle} = 1 \\
&-1 \cdot -1 = 1 \\
&e_{\langle \mathbb{R}, + \rangle} = 0 \\
&\nexists n \neq 0 \in \mathbb{R}, n + n = 0 \\
\Rightarrow &\nexists f : \mathbb{R} \to \mathbb{R}^*, f(x+y) \coloneqq f(x)f(y) \\
&\text{not isomorphic}
\end{aligned}
$$

Problem 3

$$
\begin{aligned}
&\text{Suppose} : \phi : \mathbb{C}^* \to \mathbb{R}^* \\
&\phi(1) = 1 = \phi((-1)(-1)) = (\phi(-1))^2 \\
&\phi(-1) = -1 \leftarrow \phi \text{ is injective} \\
&\phi(-1) = \phi(i^2) = (\phi(i))^2 = -1 \\
&\phi(i) = \pm i \\
&\phi(i) = \pm i \not\Leftrightarrow \phi : \mathbb{C}^* \to \mathbb{R}^* \\
&\text{They are not isomorphic}
\end{aligned}
$$

Problem 4

$$f : \mathbb{R} \to \mathbb{R}^+, f \coloneqq e^x$$
$$f(x+y) = e^{x+y} = e^x e^y = f(x)f(y)$$
$\Rightarrow$is isomorphic

Problem 5

$f : G \to H$

$f(a_1) = f(a_2)$

$\Rightarrow$all the terms in $f(a_1)$ and $f(a_2)$ are equal

$\Rightarrow a_1 = a_2$

$\Rightarrow$injective

$\forall h \in H, \exists g \in G : f(g) = h$

$\Rightarrow$surjective

$\Rightarrow$bijective

$f(a) \coloneqq u, f(b) \coloneqq v$

$f(a) + f(b) = u + v$

$f(a+b) = f((a_1 + b_1, a_2 + b_2, ...))$

$f(a+b)(q) = f((a_1 + b_1, a_2 + b_2, ...))q = f(a_1, a_2, ...) + f(b_1, b_2, ...)q = u(q) + v(q)$

$\Rightarrow f(a+b) = f(a) + f(b)$

$\Rightarrow$isomorphism

Question 3

Problem 1

$$
\begin{aligned}
&\text{Closure :}\\
&\exists a, b \in H_1 \cap H_2\\
&H_1, H_2 \text{ are subgroups}\\
&a * b \in H_1 \land a * b \in H_2\\
\Rightarrow &a * b \in H_1 \cap H_2\\
&\text{Identity :}\\
&H_1, H_2 \text{ are subgroups}\\
\\
&e \in H_1 \land e \in H_2\\
&e \in H_1 \cap H_2\\
&\text{Inverse :}\\
&H_1, H_2 \text{ are subgroups}\\
&\exists a \in H_1 \land a \in H_2\\
\Rightarrow &a^{-1} \in H_1 \land a^{-1} \in H_2\\
\Rightarrow &a^{-1} \in H_1 \cap H_2\\
\Rightarrow &H_1 \cap H_2 \text{ is a subgroup}
\end{aligned}
$$

Problem 2

$$
\begin{aligned}
&G \coloneqq \langle \mathbb{Z}_6, +_6 \rangle\\
&G_1 \coloneqq \{0, 3\} \text{ is a subgroup of } G\\
&G_2 \coloneqq \{0, 2, 4\} \text{ is a subgroup of } G\\
&G_1 \cup G_2 = \{0, 2, 3, 4\}\\
&2 +_6 3 = 5 \notin G_1 \cup G_2\\
\Rightarrow &G_1 \cup G_2 \text{ is not a subgroup}
\end{aligned}
$$

Question 4

Problem 1

$$
\begin{aligned}
&\text{Closure :}\\
&\exists a, b \in G\\
&a^2, b^2 \in H\\
&a^2b^2 = (ab)^2 \leftarrow G \text{ is abelian}\\
&(ab)^2 \in H\\
&\text{Identity :}\\
&e \in G\\
&e^2 = e\\
&e^2 \in H\\
\Rightarrow &e \in H\\
\text{Inverse :} &\forall a^2 \in H, \exists a^{-2} : a^2a^{-2} = e\\
&a^{-2} = (a^{-1})^{-2}\\
&(a^{-1})^{-2} \in H\\
\Rightarrow &H \text{ is a subgroup}
\end{aligned}
$$

Problem 2

$$
\begin{aligned}
&\text{Closure :}\\
&\exists a, b \in H\\
&(ab)^2 = a^2b^2\\
&a^2b^2 = e^2 = e\\
\Rightarrow &ab \in H\\
&\text{Identity :}\\
&e^2 = e\\
\Rightarrow &e \in H\\
&\text{Inverse :}\\
&\forall a \in H, \exists a^{-1} : aa^{-1} = e\\
&(aa^{-1})^2 = e^2 = e\\
&a^2(a^{-1})^2 = e \leftarrow G \text{ is abelian}\\
&(a^{-1})^2 = e \leftarrow a^2 = e\\
\Rightarrow &a^{-1} \in H\\
\Rightarrow &H \text{ is a subgroup}
\end{aligned}
$$

Question 5

Reflexivity :

$a^{-1}a = e$

H is a subgroup

$\Rightarrow e \in H$

$\Rightarrow a^{-1}a \in H$

$\Rightarrow a \sim a$

Symmetry :

$\exists a \sim b$

$\Rightarrow a^{-1}b \in H$

H is a subgroup

$(a^{-1}b)^{-1} \in H$

$\Rightarrow b^{-1}a \in H$

$\Rightarrow b \sim a$

Transitivity :

$\exists a \sim b, b \sim c$

$\Rightarrow a^{-1}b, b^{-1}c \in H$

H is a subgroup

$\Rightarrow a^{-1}bb^{-1}c \in H$

$\Rightarrow a^{-1}c \in H$

$\Rightarrow a \sim c$

$\Rightarrow \sim$ is an equivalence relation

$\bar{e} = \{x | x^{-1}e \in H\}$

$\Rightarrow x^{-1}e \in H$

$x^{-1} \in H$

$x \in H$

$\Rightarrow \bar{e} = H$

Question 6

$$
\begin{aligned}
& a \in G \\
& a * a = a^2 \in G \\
\Rightarrow & \forall i \in \mathbb{Z}^+, a^i \in G \\
& G \text{ is finite} \\
\Rightarrow & \exists u > v \in \mathbb{Z}^+, a^u = a^v \\
\Rightarrow & a^{u-v} = e
\end{aligned}
$$

Question 7

Problem a

$$
\begin{aligned}
&\exists x \in H \\
&x \times x = x^2 \in H \\
\Rightarrow &\forall i \in \mathbb{Z}^+, x^i \in H \\
&H \text{ is finite} \\
\Rightarrow &\exists u > v \in \mathbb{Z}^+, x^u = x^v \\
\Rightarrow &x^{u-v} = e \\
&u - v \in \mathbb{Z}^+ \\
&e = x^{u-v} \in \mathbb{Z}^+ \\
&x \times x^{-1} = e \\
&x \times x^{-1} = x^{u-v} \\
\Rightarrow &x^{-1} = x^{u-v-1} \in \mathbb{Z}^+ \\
&\exists x^a, x^b \in H, a, b \in \mathbb{Z}^+ \\
&x^a \times x^b = x^{a +_{u-v} b} \in H \\
\Rightarrow &H \text{ is closed} \\
\Rightarrow &H \text{ is a subgroup}
\end{aligned}
$$

Problem b

$$
\begin{aligned}
&H \coloneqq \{x^n | n \in \mathbb{Z}^+\} \\
&H \text{ is infinite} \\
\Rightarrow &\nexists u, v \in \mathbb{Z}^+ : x^u = x^v \\
\Rightarrow &e \notin H \\
\Rightarrow &H \text{ is not a group}
\end{aligned}
$$

Question 8

$\exists a \in G$

$g \neq e$

$ag \neq ae$

$ae = ea$

$\Rightarrow ag \neq ea$

$aga^{-1} \neq eaa^{-1} = e$

$(aga^{-1})^{-1} = ag^{-1}a^{-1} = aga^{-1}$

g is the unique element that equals to inverse other than e

$\Rightarrow aga^{-1} = g$

$ag = ga$

Question 9

Problem a

$$
\begin{aligned}
&\text{Closure :}\\
&\exists a_1, b_1, a_2, b_2 \in \mathbb{Z}\\
&x \coloneqq a_1 m + b_1 n, y \coloneqq a_2 m + b_2 n\\
&x, y \in H\\
&x + y = a_1 m + b_1 n + a_2 m + b_2 n = (a_1 + a_2)m + (b_1 + b_2)n\\
&a_1 + a_2 \in \mathbb{Z}, b_1 + b_2 \in \mathbb{Z}\\
\Rightarrow &x + y \in H\\
&\text{Identity :}\\
&x + e = x\\
&a_1 m + b_1 n + e = a_1 m + b_1 n\\
\Rightarrow &e = 0 = 0m + 0n \in H \Leftarrow 0 \in \mathbb{Z}\\
&\text{Inverse :}\\
&x + x^{-1} = e\\
&a_1 m + b_1 n + x^{-1} = 0\\
&x^{-1} = -a_1 m - b_1 n = -x \in H \Leftarrow -a_1, -b_1 \in \mathbb{Z}\\
\Rightarrow &H_{m,n} \text{ is a subgroup}
\end{aligned}
$$

Problem b

$$
\begin{aligned}
&\text{Suppose } K \text{ is a subgroup of } \langle \mathbb{Z}, + \rangle : m, n \in K\\
&0 = 0m + 0n \in K \leftarrow K \text{ is a subgroup}\\
&m \in K \implies m + m = 2m \in K \wedge -m \in K \leftarrow K \text{ is a subgroup}\\
&\text{By induction : } am \in K, a \in \mathbb{Z}\\
&\text{By the same method : } bn \in K, b \in \mathbb{Z}\\
\Rightarrow &am + bn \in K, a, b \in \mathbb{Z} \leftarrow K \text{ is a subgroup, closed under operation}\\
&\{am + bn, a, b \in \mathbb{Z}\} \text{ is generated only by } m, n \text{ and property of identity, inverse and closure under operation}\\
\Rightarrow &K \geqslant \{am + bn, a, b \in \mathbb{Z}\}\\
\Rightarrow &H_{m,n} \leqslant K
\end{aligned}
$$

Reference

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