

Question 1

Problem a

True :

$$\phi : A \times B \rightarrow B \times A$$

$$\phi((a, b)) = (b, a)$$

Homomorphism :

$$\exists a, a' \in A, b, b' \in B :$$

$$\phi((a, b)(a', b')) = \phi(aa', bb') = (bb', aa')$$

$$(bb', aa') = (b, a)(b', a') = \phi(a, b)\phi(a', b')$$

bijjective :

$$\exists a, a' \in A, b, b' \in B : \phi(a, b) = \phi(a', b')$$

$$\Rightarrow (b, a) = (b', a')$$

$$\Rightarrow \begin{cases} a = a' \\ b = b' \end{cases}$$

$$\Rightarrow (a, b) = (a', b')$$

$$\Rightarrow \phi(a, b) = \phi(a', b') \implies (a, b) = (a', b')$$

\Rightarrow injective

$$|A \times B| = |B \times A|$$

\Rightarrow surjective since injective

\Rightarrow bijective

\Rightarrow isomorphism

Problem b

True :

$$|\mathbb{Z}_m \times \mathbb{Z}_n| = \text{lcm}(m, n)$$

$$\Rightarrow \text{lcm}(m, n) = mn \Leftrightarrow \text{gcd}(m, n) = 1$$

$$|\mathbb{Z}_m \times \mathbb{Z}_n| = mn \Leftrightarrow m, n \text{ are relatively prime}$$

There is an element of order mn in a group of order mn

Problem c

False :

$$\mathbb{Z}_2 \times \mathbb{Z}_4$$

$$|\mathbb{Z}_2 \times \mathbb{Z}_4| = 8$$

$$2 \times 4 = 8$$

Problem d

True :

	*	e	a	b	c
	e	e	a	b	c
Klein-4 :	a	a	e	c	b
	b	b	c	e	a
	c	c	b	a	e

	*	e	a	b	c
	e	e	a	b	c
$\mathbb{Z}_2 \times \mathbb{Z}_2 :$	a	a	e	c	b
	b	b	c	e	a
	c	c	b	a	e

$$\Rightarrow \text{Klein-4} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

Problem e

False :

$$\gcd(3, 3) = 3 \neq 1$$

$$\Rightarrow \mathbb{Z}_3 \times \mathbb{Z}_3 \text{ is not cyclic}$$

$$\gcd(3, 5) = 1$$

$$\Rightarrow \mathbb{Z}_3 \times \mathbb{Z}_5 \text{ is cyclic}$$

Problem f

False :

$\mathbb{Z}_2 \times \mathbb{Z}_6$ is not cyclic

$\nexists n \in \mathbb{Z} : n^2 = 2 \times 6 = 12$

Problem g

False :

$\mathbb{Z}_2 \times \mathbb{Z}_2$

$|\mathbb{Z}_2 \times \mathbb{Z}_2| = 4 = 2^2$

$\mathbb{Z}_2 \times \mathbb{Z}_2$ is not cyclic

Problem h