Question 1

Problem a

$$
\begin{aligned}
&G \text{ is finite} \\
\Rightarrow &|G| \text{ is finite} \\
&N \text{ is normal subgroup of } G \\
\Rightarrow &|G/N| = [G : N] = \frac{|G|}{|N|} \text{ is finite} \\
\Rightarrow &G/N \text{ is finite} \\
&\text{True}
\end{aligned}
$$

Problem b

$$
\begin{aligned}
&G \coloneqq \mathbb{Z} \\
&N \coloneqq 2\mathbb{Z} \\
&G/N \cong \mathbb{Z}_2 \\
\Rightarrow &|G/N| = 2 \\
&\text{False}
\end{aligned}
$$

Problem c

$$
\begin{aligned}
&G \text{ is abelian} \\
&\exists aN, bN \in G/N \\
&(aN)(bN) = aNbN = abN \\
&abN = baN = bNaN = (bN)(aN) \\
\Rightarrow &G/N \text{ is abelian} \\
&\text{True}
\end{aligned}
$$

Problem d

A_3 is normal in S_3
$S_3/A_3 \cong A_2$
A_2 is abelian
S_3 is not abelian
False

Question 2

$$
\begin{aligned}
&\sigma \coloneqq (1, 2, 3, ..., n) \\
\Rightarrow &\sigma \in S_{n+1} \\
&\tau \coloneqq (n, n+1) \in S_{n+1} \\
&\tau^{-1} = (n+1, n) \\
&\tau\sigma\tau^{-1} \\
= &(n, n+1)(1, 2, 3, ..., n)(n+1, n) \\
= &(n, n+1)(1, 2, 3, ..., n, n+1) \\
= &(1, 2, 3, ..., n-1, n+1) \notin S_n \\
\Rightarrow &S_n \text{ is not a normal subgroup of } S_{n+1}
\end{aligned}
$$

Question 3

Problem a

$$
\begin{aligned}
&H \cong K \text{ if } H \text{ is conjugate to } K\\
\\
&\text{Reflexive :}\\
&e \in G\\
&eHe^{-1} = H\\
\Rightarrow &H \cong K\\
\\
&\text{Transitive :}\\
&H \cong K \wedge K \cong P\\
&gHg^{-1} = K \wedge pKp^{-1} = P\\
\Rightarrow &pgHg^{-1}p^{-1} = P\\
&pgH(pg)^{-1} = P\\
\rightarrow &H \cong P\\
\\
&\text{Symmetry :}\\
&H \cong K\\
\Rightarrow &gHg^{-1} = K\\
&g^{-1}Kg = H\\
&g^{-1}K(g^{-1})^{-1} = H\\
&g^{-1} \in G \text{ since } G \text{ is a group}\\
\Rightarrow &K \cong H\\
\Rightarrow &\cong \text{ is an equivalence relation}
\end{aligned}
$$

Problem b

$$
\begin{aligned}
& H \text{ is normal} \\
\Rightarrow & \forall g \in G, gHg^{-1} = H \\
\Rightarrow & H \text{ can only be conjugate to } H \\
\Rightarrow & \text{equivalence class of } H \text{ is } \{H\}
\end{aligned}
$$

Problem c

$$
\begin{aligned}
& \{e\} \\
& \{(1,2),(1,3),(2,3),\} \\
& \{(1,2,3),(1,3,2)\}
\end{aligned}
$$

Question 4

$$
\begin{aligned}
&\phi[G] \text{ is ableian} \implies xyx^{-1}y^{-1} \in \ker\phi : \\
&\phi[G] \text{ is abelian} \\
\Rightarrow &\phi(x)\phi(y) = \phi(y)\phi(x) \\
&\phi(xyx^{-1}y^{-1}) \\
= &\phi(x)\phi(y)\phi(x)^{-1}\phi(y)^{-1} \\
= &(\phi(x)\phi(x)^{-1})(\phi(y)\phi(y)^{-1}) \\
= &e \times e \\
= &e \\
\Rightarrow &xyx^{-1}y^{-1} \in \ker\phi
\end{aligned}
$$

$$
\begin{aligned}
&xyx^{-1}y^{-1} \in \ker\phi \implies \phi[G] \text{ is ableian} : \\
&xyx^{-1}y^{-1} \in \ker\phi \\
\Rightarrow &\phi(xyx^{-1}y^{-1}) = e \\
\Rightarrow &\phi(x)\phi(y)\phi(x)^{-1}\phi(y)^{-1} = e \\
\Rightarrow &\phi(x)\phi(y) = \phi(y)\phi(x) \\
\Rightarrow &\phi[G] \text{ is abelian}
\end{aligned}
$$

Question 5

$$
\begin{aligned}
&|G/N| = r \\
\Rightarrow &G/N \cong \mathbb{Z}_r \\
&\text{There are } r \text{ cosets} \\
&\text{Suppose} : x \in aN \\
\Rightarrow &x^r \in (aN)^r \\
\Rightarrow &x^r \in a^r N \\
&a^r \equiv 0 \in \mathbb{Z}_r \\
\Rightarrow &\forall x \in G, x^r \in N
\end{aligned}
$$

Question 6

Problem a

$$
\begin{aligned}
&\phi GL_2(\mathbb{R}) \to \mathbb{R}^* \\
&\phi(A) \coloneqq \det(A) \\
&\phi(A \times B) = \det(A \times B) = \det(A) \times \det(B) = \phi(A) \times \phi(B) \\
\Rightarrow &\phi \text{ is a homomorphism} \\
&\ker \phi = \{A \in G | \det(A) = 1\} \\
\Rightarrow &\ker \phi = H \\
\Rightarrow &H \text{ is normal in } G
\end{aligned}
$$

Problem b

$$
\begin{aligned}
&\ker \phi = H \\
&G/H \cong \phi(G) \\
&\forall a \in \mathbb{R}^*, \exists A = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \in G : \phi(A) = a \\
\Rightarrow &\phi \text{ is surjective} \\
\Rightarrow &\phi(G) = \mathbb{R}^* \\
\Rightarrow &G/H \cong \mathbb{R}^*
\end{aligned}
$$

Question 7

Problem a

$$
\begin{aligned}
&\phi : F \to \mathbb{R} \\
&\phi(f) \coloneqq f(0) \\
&\phi(f+g) = (f+g)(0) = f(0) + g(0) \to F \text{ is under addition} \\
\Rightarrow &\phi \text{ is homomorphism} \\
&\text{By contruction} : \phi \text{ is onto} \\
&\text{By definition of } F : \{f(0) | f \in F\} = \mathbb{R} \\
\Rightarrow &\text{im}\phi = \mathbb{R} \\
&\ker \phi = \{f(0) = 0 | f \in F\} = H \\
\Rightarrow &F/H \cong \mathbb{R}
\end{aligned}
$$

Problem b

$$
\begin{aligned}
&\phi : F \to H \\
&\phi(f) \coloneqq f(x) - f(0) \\
&\phi(f+g) = (f+g)(x) - (f+g)(0) = f(x) + g(x) - f(0) - g(0) \\
\Rightarrow &\phi(f+g) = \phi(f) + \phi(g) \\
\Rightarrow &\phi \text{ is a homomorphism} \\
&\forall f(x) \in F, h(x) = f(x) - f(0), h(0) = f(0) - f(0) = 0 \in H \\
\Rightarrow &\text{im}\phi \subseteq H \\
&\phi \text{ is homomorphism} \\
\Rightarrow &Id \in \text{im}\phi \wedge \forall f, g \in F, \phi(f+g) \in \text{im}\phi \\
\Rightarrow &\text{im}\phi \text{ is a subgroup of } H \\
&\ker \phi = \{f(x) = f(0) | f \in F\} = C \\
\Rightarrow &F/C \cong \text{im}\phi \leqslant H
\end{aligned}
$$

Problem c

Suppose $\exists f \in F : f$ is not continuous with order 2 in F/K

$\rightarrow 2f \in K$

$h \coloneqq 2f$

h is continuous

$\Rightarrow f = \frac{1}{2}h$

$\Rightarrow f$ is continuous

f is continuous $\not\Leftrightarrow f$ is not continuous

$\Rightarrow \nexists f \in F/K$ with order of 2

Question 8

Problem a

$$
\begin{aligned}
&h \coloneqq ax \in H \\
&g \coloneqq x + b \\
\Rightarrow &g^{-1} = x - b \\
&g \circ h \circ g^{-1} = g \circ h(x-b) = g(a(x-b)) = a(x-b) + b = ax - ab + b \notin H \\
\Rightarrow &H \text{ is not normal}
\end{aligned}
$$

Question b

$$
\begin{aligned}
&\forall g \in G, g \coloneqq ax + b, a \in \mathbb{R}^*, b \in \mathbb{R} \\
&\forall k \in k, k \coloneqq x + c, c \in \mathbb{R} \\
&g^{-1} = \frac{x}{a} - \frac{b}{a} \\
&g \circ k \circ g^{-1} \\
= &g \circ k(\frac{x}{a} - \frac{b}{a}) \\
= &g(\frac{x}{a} - \frac{b}{a} + c) \\
= &a(\frac{x}{a} - \frac{b}{a} + c) + b \\
= &x + ac \in K \\
\Rightarrow &\forall k \in K, \forall g \in G, g \circ k \circ g^{-1} \in K \\
\Rightarrow &K \text{ is normal}
\end{aligned}
$$

Question c

$\phi : G \to \mathbb{R}^*$
$f(x) \coloneqq ax + b$
$g(x) \coloneqq cx + d$
$\phi(f(x)) = f'(x) = a$
By construction, ϕ is onto
$\Rightarrow \mathrm{im}\phi = \mathbb{R}^*$
$\phi(f \circ g(x)) = \phi(a(cx + d) + b) = ac$
$\phi(f(x))\phi(g(x)) = a \times c = ac$
$\Rightarrow \phi(f \circ g(x)) = \phi(f(x))\phi(g(x))$
$\Rightarrow \phi$ is homomorphic
$\ker\phi = \{f'(x) = 1 | f(x) \in G\} = K$
$\Rightarrow G/K \cong \mathbb{R}^*$

Question 9

Problem a

$$
\begin{aligned}
&\langle(1,1)\rangle \text{ generates the whole group}\\
\Rightarrow &|G/\langle(1,1)\rangle| = 1\\
&(\langle 12\rangle, \langle 10\rangle)\\
=&\{(12,10),(6,20),(0,6),(12,16),(6,2),(0,12),(12,22),(6,8),(0,18),(12,4),(6,14),(0,0)\}\\
\Rightarrow &|(\langle 12\rangle, \langle 10\rangle)| = 12\\
\Rightarrow &|G/(\langle 12\rangle, \langle 10\rangle)| = 18 \times 24/12 = 36
\end{aligned}
$$

Problem b

$$
\begin{aligned}
&(1,7) + \langle(1,1)\rangle = (0,6) + \langle(1,1)\rangle\\
&(2,5) + \langle(1,1)\rangle = (0,3) + \langle(1,1)\rangle\\
&(3,3) + \langle(1,1)\rangle = \langle(1,1)\rangle\\
\Rightarrow &\text{Order of } (1,7) + \langle(1,1)\rangle \text{ in } \mathbb{Z}_6 \times \mathbb{Z}_9/\langle(1,1)\rangle \text{ is } 3\\
&(2,1) + \langle(2,3)\rangle = (0,1) + \langle(2,3)\rangle\\
&(4,2) + \langle(2,3)\rangle = (0,2) + \langle(2,3)\rangle\\
&(6,3) + \langle(2,3)\rangle = \langle(2,3)\rangle\\
\Rightarrow &\text{Order of } (2,1) + \langle(2,3)\rangle \text{ in } \mathbb{Z}_6 \times \mathbb{Z}_9/\langle(2,3)\rangle \text{ is } 3
\end{aligned}
$$

Problem c

$H_1:$

$(1,0)+\langle(2,1)\rangle$

$(2,0)+\langle(2,1)\rangle$

$(3,0)+\langle(2,1)\rangle$

$(4,0)+\langle(2,1)\rangle=\langle(2,1)\rangle$

$\Rightarrow G/H_1 \cong \mathbb{Z}_4$

$H_2:$

$(1,0)+\langle(2,0)\rangle$

$(2,0)+\langle(2,0)\rangle=\langle(2,0)\rangle$

$(0,1)+\langle(2,0)\rangle$

$(0,2)+\langle(2,0)\rangle=\langle(2,0)\rangle$

$\Rightarrow G/H_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

Reference

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