Question 1

Problem a

$$\begin{aligned}
&\text{There are } n \text{ inputs to } n \text{ outputs}\\
\Rightarrow&\text{Number of combinations : } n^2\\
\Rightarrow&\text{Number of binary operation : } n^{n^2}
\end{aligned}$$

Problem b

$$\begin{aligned}
&\text{There are } n \text{ inputs to } n-1 \text{ outputs}\\
\Rightarrow&\text{Number of combinations : } \frac{n(n-1)}{2}\\
\Rightarrow&\text{Also add the ordered pairs} x=y : \frac{n(n-1)}{2}+n=\frac{n(n+1)}{2}\\
\Rightarrow&\text{Number of binary operation : } n^{\frac{n(n+1)}{2}}
\end{aligned}$$

Question 2

$$
\begin{aligned}
&(b*d)*a = b*(d*a)\\
&c*a = b*d\\
&c*a = c\\
&(b*d)*b = b*(d*b)\\
&c*b = b*b\\
&c*b = b\\
&(b*d)*c = b*(d*c)\\
&c*c = b*c\\
&c*c = c\\
&(b*d)*d = b*(d*d)\\
&c*d = b*a\\
&c*d = b
\end{aligned}
$$

$$
\Rightarrow \begin{array}{c|cccc}
* & a & b & c & d\\
\hline
a & a & b & c & d\\
b & b & b & c & c\\
c & c & b & c & b\\
d & d & b & c & a
\end{array}
$$

Question 3

Problem a

$$
\begin{aligned}
&a * b \\
=&5ab \\
=&5ba \\
=&b * a \leftarrow \text{commutative} \\
&a * (b * c) \\
=&a * (5bc) \\
=&5a(5bc) \\
=&25abc \\
&(a * b) * c \\
=&(5ab) * c \\
=&5 \times 5abc \\
=&25abc \\
=&a * (b * c) \leftarrow \text{associative}
\end{aligned}
$$

Problem b

$$
\begin{aligned}
&a * b \\
=&5ab - 1 \\
=&5ba - 1 \\
=&b * a \leftarrow \text{commutative} \\
&a * (b * c) \\
=&a * (5bc - 1) \\
=&5a(5bc - 1) - 1 \\
=&25abc - 5a - 1 \\
&(a * b) * c \\
=&(5ab - 1) * c \\
=&5 \times (5ab - 1)c - 1 \\
=&25abc - 5c - 1 \\
\neq&a * (b * c) \leftarrow \text{not associative}
\end{aligned}
$$

Problem c

$$
\begin{aligned}
&a * b = a - b + 1\\
&b * a = b - a + 1\\
\Rightarrow &a * b \neq b * a\\
\Rightarrow &\text{not commutative}\\
&a * (b * c) = a * (b - c + 1) = a - (b - c + 1) + 1 = a - b + c\\
&(a * b) * c = (a - b + 1) * c = a - b + 1 - c + 1 = a - b - c + 2\\
\Rightarrow &a * (b * c) \neq (a * b) * c\\
\Rightarrow &\text{not associative}
\end{aligned}
$$

Problem d

$$
\begin{aligned}
&a * b = 3^{a+b}\\
&b * a = 3^{b+a}\\
\Rightarrow &a * b = b * a\\
\Rightarrow &\text{commutative}\\
&a * (b * c) = a * (3^{b+c}) = 3^{a+3^{b+c}}\\
&(a * b) * c = (3^{a+b}) * c = 3^{3^{a+b}+c}\\
\Rightarrow &a * (b * c) \neq (a * b) * c\\
\Rightarrow &\text{not associative}
\end{aligned} \tag{1}
$$

Question 4

$$\langle A, * \rangle \coloneqq \begin{array}{c|cc} * & a & b \\ \hline a & b & a \\ b & a & a \end{array}$$

The operation is commutative since it is symmetric along the diagonal

$a * (a * b) = a * a = b$

$(a * a) * b = b * b = a$

$\Rightarrow a * (a * b) \neq (a * a) * b$

$\Rightarrow \langle A, * \rangle$ is not associative

Question 5

Problem 1

$$\langle G, * \rangle := \begin{array}{c|cc} * & a & b \\ \hline a & a & b \\ b & b & a \end{array}$$

$$\Rightarrow a^2 \neq b \land b^2 \neq b$$

$$\Rightarrow \nexists x \in G : x^2 = b$$

Problem 2

$$
\begin{aligned}
&\text{Suppose} : x \in G \text{ appears twice in one row} \\
&u * v := x \land u * w := x, v \neq w \in G \\
\Rightarrow &u * v = u * w \\
&u^{-1} * (u * v) = u^{-1} * (u * w) \\
&(u^{-1} * u) * v = (u^{-1} * u) * w \\
&e * v = e * w \\
&v = w \not\Leftrightarrow v \neq w \\
\Rightarrow &\forall x \in G \text{ appears at most once in one row} \\
&\text{Since each row has } |G| \text{ different outputs for } |G| \text{ different inputs} \\
\Rightarrow &\text{There must exactly one element in each row} \\
&\text{The logic is the same for the proposition in column.}
\end{aligned}
$$

Question 6

Problem 1

$$
\begin{aligned}
&\text{identity value :}\\
&a * e = a = e * a\\
&a * e = a + e\\
\Rightarrow &a + e = a\\
\Rightarrow &e = 0\\
&\text{inverse :}\\
&a * a^{-1} = e\\
\Rightarrow &a + a^{-1} = 0\\
\Rightarrow &a^{-1} = -a\\
&\text{associative :}\\
&a * (b * c)\\
= &a * (b + c)\\
= &a + (b + c)\\
= &(a + b) + c\\
= &(a * b) * c\\
&\text{closure :}\\
&\exists a, b \in G\\
\Rightarrow &a \coloneqq 3m, b \coloneqq 3n\\
&a * b\\
= &a + b\\
= &3m + 3n\\
= &3(m + n) \in G\\
\Rightarrow &\langle G, * \rangle \text{ is a group}
\end{aligned}
$$

Problem 2

$$
\begin{aligned}
&\text{identity value :}\\
&a * e\\
=&a + 2e\\
=&a\\
\Rightarrow&e = 0\\
&e * a\\
=&e + 2a\\
=&a\\
\Rightarrow&e = -a\\
&e = 0 \not\Leftrightarrow e = -a\\
\Rightarrow&\text{There is no identity value for } *\\
\Rightarrow&\langle G, * \rangle \text{ is not a group}
\end{aligned}
$$

Problem 3

$$
\begin{aligned}
&\text{identity value :}\\
&a * e\\
=&|ae|\\
&|ae| = a\\
&LHS \geqslant 0\\
\Rightarrow&a < 0 \implies |ae| = a \text{ has no solution for } e\\
\Rightarrow&\langle G, * \rangle \text{ is not a group}
\end{aligned}
$$

Problem 4

$$
\begin{aligned}
&\text{identity value}:\\
&a * e\\
=&ae\\
=&a\\
\Rightarrow &e = 1 = 1 + 0 \times \sqrt{2} \in G\\
&e * a = ea = a\\
&\text{inverse}:\\
&a * a^{-1} = e\\
\Rightarrow &a \cdot a^{-1} = 1\\
\Rightarrow &a^{-1} = \frac{1}{a}\\
&\text{associative}:\\
&a * (b * c)\\
=&a * (bc)\\
=&abc\\
=&(ab)c\\
=&(a * b) * c\\
&\text{closure}:\\
&a \coloneqq a_1 + a_2\sqrt{2}, b \coloneqq b_1 + b_2\sqrt{2}\\
&a * b\\
=&ab\\
=&(a_1 + a_2\sqrt{2})(b_1 + b_2\sqrt{2})\\
=&(a_1b_1 + 2a_2b_2) + (a_1b_2 + b_1a_2)\sqrt{2} \in G\\
\Rightarrow &\langle G, * \rangle \text{ is a group}
\end{aligned}
$$

Problem 5

$$
\begin{aligned}
&\text{identity value}:\\
&a * e\\
=&\sqrt{ae}\\
&\sqrt{ae} = a\\
\Rightarrow &e = a\\
\Rightarrow &\text{there cannot be an identity}\\
\Rightarrow &\langle G, * \rangle \text{ is not a group}
\end{aligned}
$$

Problem 6

$$
\begin{aligned}
&\text{identity value}: \\
&A * E \\
=&AE \\
=&A \\
\Rightarrow&E = I_2 \\
&\text{inverse}: \\
&A * A^{-1} = E \\
\Rightarrow&A \cdot A^{-1} = I_2 \\
&A^{-1} \cdot A \cdot A^{-1} = A^{-1} \cdot I_2 \\
&A^{-1} = A^{-1} \\
&\text{associative}: \\
&A * (B * C) \\
=&A * (BC) \\
=&ABC \\
=&(AB)C \\
=&(A * B) * C \\
&\text{closure}: \\
&C \coloneqq A * B \\
\Rightarrow&C = AB \\
&M_{a\times b} \times M_{b\times c} = M_{a\times c} \\
\Rightarrow&C \in M_{2\times 2}(\mathbb{R}) \\
\Rightarrow&\langle G, *\rangle \text{ is a group}
\end{aligned}
$$

Problem 7

$$
\begin{aligned}
&\text{identity value}:\\
&A * E\\
=&AE\\
=&A\\
\Rightarrow&E = I_2 \in G\\
&\text{inverse}:\\
&A * A^{-1} = E\\
\Rightarrow&A \cdot A^{-1} = I_2\\
&A^{-1} \cdot A \cdot A^{-1} = A^{-1} \cdot I_2\\
&A^{-1} = A^{-1}\\
&\text{associative}:\\
&A * (B * C)\\
=&A * (BC)\\
=&ABC\\
=&(AB)C\\
=&(A * B) * C\\
&\text{closure}:\\
&C \coloneqq A * B\\
\Rightarrow&C = AB\\
&\det(C) = \det(A)\det(B)\\
\Rightarrow&\det(C) = 1 \times 1 = 1\\
\Rightarrow&C \in G\\
\Rightarrow&\langle G, * \rangle \text{ is a group}
\end{aligned}
$$

Problem 8

$$
\begin{aligned}
&\text{identity value :}\\
&A * E\\
=&AE\\
=&A\\
\Rightarrow &E = I_2 \in G\\
&\text{inverse :}\\
&A * A^{-1} = E\\
\Rightarrow &A \cdot A^{-1} = I_2\\
&A^{-1} \cdot A \cdot A^{-1} = A^{-1} \cdot I_2\\
&A^{-1} = A^{-1}\\
&\text{associative :}\\
&A * (B * C)\\
=&A * (BC)\\
=&ABC\\
=&(AB)C\\
=&(A * B) * C\\
&\text{closure :}\\
&A \coloneqq \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, B \coloneqq \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}\\
&A^2 = B^2 = I_2\\
&C \coloneqq A * B\\
&C = AB = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \neq I_2\\
\Rightarrow &\langle G, * \rangle \text{is not closed}\\
\Rightarrow &\langle G, * \rangle \text{is not a group}
\end{aligned}
$$

Question 7

Problem 1

$$
\begin{aligned}
&\exists a, b \in G\\
&a = a^{-1}\\
\Rightarrow &a^2 = e\\
&c \coloneqq a * b\\
&\text{Consider} : a * b * b * a\\
&a * b * b * a\\
= &a * (b * b) * a\\
= &a * e * a\\
= &a * a\\
= &e\\
\Rightarrow &c * (b * a) = e\\
&c \text{ has only one inverse } a * b\\
\Rightarrow &a * b = b * a\\
\Rightarrow &\langle G, * \rangle \text{ is commutative}
\end{aligned}
$$

Problem 2

$$
\begin{aligned}
&\text{Suppose} : \forall a \neq e \in G, |G| = 2n, n \in \mathbb{Z}^+ : a \neq a^{-1}\\
&\forall a \neq b \in G : a, a^{-1}, b, b^{-1} \text{ are all distinct} \leftarrow \text{properties of group}\\
\Rightarrow &\text{ Every } a \text{ and } a^{-1} \text{ can form a pair without repetition}\\
\Rightarrow &|G| = 2n + 1, n \in \mathbb{Z}^+ \leftarrow e^{-1} = e\\
&|G| = 2n + 1 \nLeftrightarrow |G| = 2n\\
\Rightarrow &\exists a \neq e \in G : a^{-1} = a
\end{aligned}
$$

Reference

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