Problem 1

Question a

$$
\begin{aligned}
&X \setminus \emptyset = X \leftarrow \text{a member of the set} \\
\Rightarrow &\emptyset \in \mathcal{T}_f \\
&X \setminus X = \emptyset \leftarrow \text{ is finite} \\
\Rightarrow &X \in \mathcal{T}_f \\
&\exists U_i \in \mathcal{T}_f \\
\Rightarrow &X \setminus U_i \text{ is finite} \\
\Rightarrow &\cap (X \setminus U_i) \text{ is finite} \\
\Rightarrow &X \setminus \cup U_i \text{ is finite} \leftarrow \text{ De Morgan's Law} \\
\Rightarrow &\cup U_i \in \mathcal{T}_f \\
&X \setminus U_i \text{ is finite} \\
\Rightarrow &\cup (X \setminus U_i) \text{ is finite} \leftarrow \text{finite unions of finite set is finite} \\
\Rightarrow &X \setminus \cap U_i \text{ is finite} \leftarrow \text{ De Morgan's Law} \\
\Rightarrow &\cap U_i \in \mathcal{T}_f \\
\Rightarrow &\mathcal{T}_f \text{ is a topology on } X
\end{aligned}
$$

Question b

$$
\begin{aligned}
&X \setminus \emptyset = X \leftarrow \text{a member of the set} \\
\Rightarrow &\emptyset \in \mathcal{T}_c \\
&X \setminus X = \emptyset \leftarrow \text{ is countable} \\
\Rightarrow &X \in \mathcal{T}_c \\
&\exists U_i \in \mathcal{T}_c \\
\Rightarrow &X \setminus U_i \text{ is countable} \\
\Rightarrow &\cap (X \setminus U_i) \text{ is countable} \\
\Rightarrow &X \setminus \cup U_i \text{ is countable} \leftarrow \text{ De Morgan's Law} \\
\Rightarrow &\cup U_i \in \mathcal{T}_c \\
&X \setminus U_i \text{ is countable} \\
\Rightarrow &\cup (X \setminus U_i) \text{ is countable} \leftarrow \text{countable unions of countable set is countable} \\
\Rightarrow &X \setminus \cap U_i \text{ is countable} \leftarrow \text{ De Morgan's Law} \\
\Rightarrow &\cap U_i \in \mathcal{T}_c \\
\Rightarrow &\mathcal{T}_c \text{ is a topology on } X
\end{aligned}
$$

Question c

$$
\begin{aligned}
&\forall U \in \mathcal{T}_f : \\
&X \setminus U \text{ is finite} \\
\Rightarrow &X \setminus U \text{ is countable} \\
\Rightarrow &U \in \mathcal{T}_c \\
\Rightarrow &\mathcal{T}_f \subseteq \mathcal{T}_c \\
\Rightarrow &\mathcal{T}_c \text{ is finer than } \mathcal{T}_f
\end{aligned}
$$

Problem 2

Question a

$$
\begin{aligned}
& a, b \in \mathbb{Q} \\
& \mathcal{B} = \{(a,b) | a < b, a, b \in \mathbb{Q}\} \\
\Rightarrow & |\mathcal{B}| = |\mathbb{Q}^2| \\
& \mathbb{Q} \text{ is countable} \\
\Rightarrow & \mathbb{Q}^2 \text{ is countable} \\
\Rightarrow & \mathcal{B} \text{ is countable}
\end{aligned}
$$

Question b

$$
\begin{aligned}
& \forall n \in \mathbb{R} : n \in (n-1, n+1) \\
& \exists B \in \mathcal{B} : (n-1, n+1) \subseteq B \in \mathcal{B} \\
\Rightarrow & \forall n \in \mathbb{R}, \exists B \in \mathcal{B} : n \in B \\
& \text{Suppose} : B_1 = (a,b), B_2 = (c,d) \in \mathcal{B} \\
& x \in B_1 \cap B_2 \\
\Rightarrow & x \in (\max(a,c), \min(b,d)) \\
& B_3 \coloneqq (\max(a,c), \min(b,d)) \\
\Rightarrow & x \in B_3 \\
& B_1, B_2 \in \mathcal{B} \\
\Rightarrow & a, b, c, d \in \mathbb{Q} \\
\Rightarrow & \max(a,c), \min(b,d) \in \mathbb{Q} \\
\Rightarrow & B_3 \in \mathcal{B} \\
\Rightarrow & \forall x \in B_1 \cap B_2, \exists B_3 \in \mathcal{B} : x \in B_3 \\
\Rightarrow & \mathcal{B} \text{ is a basis for the standard topology on } \mathbb{R}
\end{aligned}
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