Problem 1

$\mathcal{B} \coloneqq \{\bigcap U_i | U_i \in \mathcal{S}\}$

$\Rightarrow \mathcal{B} = \{\emptyset, [0,1), \{\frac{1}{2}, \frac{\pi}{3}\}, [1,2], \{\frac{1}{2}\}, \{\frac{\pi}{3}\}\}$

$\mathcal{T} \coloneqq \{\bigcup U_i | U_i \in \mathcal{B}\}$

$\Rightarrow \mathcal{T} = \{\emptyset, [0,1), \{\frac{1}{2}, \frac{\pi}{3}\}, [1,2], \{\frac{1}{2}\}, \{\frac{\pi}{3}\}, [0,1) \cup \{\frac{\pi}{3}\}, \{\frac{1}{2}\} \cup [1,2], X\}$

$\emptyset, X \in \mathcal{T}$, Also according to the structure of $\mathcal{T}$: $\mathcal{T}$ is a topology on X

open sets :

$\emptyset$

$[0,1)$

$\{\frac{1}{2}, \frac{\pi}{3}\}$

$[1,2]$

$\{\frac{1}{2}\}$

$\{\frac{\pi}{3}\}$

$[0,1) \cup \{\frac{\pi}{3}\}$

$\{\frac{1}{2}\} \cup [1,2]$

X

Problem 2

$$\begin{aligned}
&\forall x \in A, \exists U_x \in X : x \in U_x \subseteq A \\
\Rightarrow &A = \bigcup_{x \in A} U_x \\
&U_x \in X \\
\Rightarrow &U_x \text{ is open} \\
\Rightarrow &A \text{ is open} \leftarrow \text{finite union of open sets is open}
\end{aligned}$$

Problem 3

Question a

$$
\begin{aligned}
&\forall n \in \mathbb{R} : n \in [n-1, n+1), n-1 < n+1 \\
&\exists B \in \mathcal{B} : [n-1, n+1) \subseteq B \in \mathcal{B} \\
\Rightarrow &\forall n \in \mathbb{R}, \exists B \in \mathcal{B} : n \in B \\
&\text{Suppose} : B_1 = [a, b), B_2 = [c, d) \in \mathcal{B} \\
&x \in B_1 \cap B_2 \\
\Rightarrow &x \in [\max(a, c), \min(b, d)) \\
&B_3 \coloneqq [\max(a, c), \min(b, d)) \\
\Rightarrow &x \in B_3 \\
&B_1, B_2 \in \mathcal{B} \\
\Rightarrow &a, b, c, d \in \mathbb{R} \\
\Rightarrow &\max(a, c), \min(b, d) \in \mathbb{R} \\
\Rightarrow &B_3 \in \mathcal{B} \\
\Rightarrow &\forall x \in B_1 \cap B_2, \exists B_3 \in \mathcal{B} : x \in B_3 \\
\Rightarrow &\mathcal{B} \text{ is a basis of a topology on } \mathbb{R}
\end{aligned}
$$

Question b

$\forall n \in \mathbb{R} : n \in [n-1, n+1), n-1 < n+1$

$\exists C \in \mathcal{C} : [n-1, n+1) \subseteq C \in \mathcal{C}$

$\Rightarrow \forall n \in \mathbb{R}, \exists C \in \mathcal{C} : n \in C$

Suppose : $C_1 = [a, b), C_2 = [c, d) \in \mathcal{C}$

$x \in C_1 \cap C_2$

$\Rightarrow x \in [\max(a, c), \min(b, d))$

$C_3 \coloneqq [\max(a, c), \min(b, d))$

$\Rightarrow x \in C_3$

$C_1, C_2 \in \mathcal{C}$

$\Rightarrow a, b, c, d \in \mathbb{Q}$

$\Rightarrow \max(a, c), \min(b, d) \in \mathbb{Q}$

$\Rightarrow C_3 \in \mathcal{C}$

$\Rightarrow \forall x \in C_1 \cap C_2, \exists C_3 \in \mathcal{C} : x \in C_3$

$\Rightarrow \mathcal{C}$ is a basis of a topology on $\mathbb{R}$

Suppose : $\mathcal{C}$ is a basis for lower limit topology

$|\mathcal{C}| = |\mathbb{Q} \times \mathbb{Q}|$

$\mathbb{Q}$ is countable

$\Rightarrow \mathcal{C}$ is countable

$\Rightarrow \mathbb{Q} \times \mathbb{Q}$ is countable

$\forall r \in \mathbb{R}, \exists U_r \in \mathcal{C} : r \in U_r$

Define : $U_r \subset [r, +\infty)$

$f : \mathbb{R} \to \mathcal{C} : f(r) \coloneqq U_r$

$\exists r_1 \neq r_2 \in \mathbb{R}$

$f(r_1) \neq f(r_2) \leftarrow \min(f(r_1)) \neq \min(f(r_2))$

$\Rightarrow r_1 \neq r_2 \implies f(r_1) \neq f(r_2)$

$\Rightarrow f$ is one-to-one

$\mathbb{R}$ is uncountable

$\Rightarrow \mathcal{C}$ is uncountable

$\mathcal{C}$ is countable $\not\Leftrightarrow$ $\mathcal{C}$ is uncountable

$\Rightarrow \mathcal{C}$ is not a basis for lower limit topology