Question 1

Proof.

$$
\begin{aligned}
&\forall U \text{ is closed } \in X\\
&X \text{ is compact}\\
\Rightarrow &U \text{ is compact}\\
\Rightarrow &f(U) \text{ is compact}\\
&Y \text{ is Hausdorff}\\
\Rightarrow &f(U) \text{ is closed}\\
&\forall U \text{ is closed } \in X : f(U) \text{ is closed}\\
\Rightarrow &f \text{ is a closed map}
\end{aligned}
$$

□

Question 2

Proof.

$$
\begin{aligned}
&M = (S, d) \text{ is metric space} \\
&C \subset M \\
&m \in M \\
&n \in \mathbb{N} \\
\Rightarrow &B_n(m) \text{ is an open ball of } m \\
&\forall x \in C, \exists n : d(x, m) < n \\
\Rightarrow &C \subseteq \bigcup_{n=1}^{\infty} B_n(m) \\
\Rightarrow &\{B_n(m) | n \in \mathbb{N}\} \text{ is an open cover of } C \\
&C \text{ is compact} \\
\Rightarrow &C \text{ has a finite subcover} \\
&\{B_{n_1}(m), B_{n_2}(m), ..., B_{n_p}(m)\} \\
&n \coloneqq \max(n_1, n_2, ..., n_p) \\
&C \subseteq \bigcup_{n=1}^{p} B_{n_p}(m) = B_n(m) \\
\Rightarrow &C \text{ is bounded}
\end{aligned}
$$

□

Proof.

$$\begin{aligned}
&M = (S, d) \text{ is metric space}\\
&C \subset M\\
&m \in C\\
&n \in M \setminus C\\
&U_m \coloneqq B(m, \frac{d(m,n)}{2})\\
&U_n \coloneqq B(n, \frac{d(m,n)}{2})\\
\Rightarrow &U_m \cap U_n = \emptyset\\
&\{U_m, m \in C\} \text{ is a cover of } C\\
&C \text{ is compact}\\
\Rightarrow &m \text{ has finitely many neighborhoods} : \{U_{m_1}, U_{m_2}, ..., U_{m_p}\}\\
&\forall i \geqslant p \in \mathbb{N}, U_{m_i} \cap U_{n_i} = \emptyset\\
\Rightarrow &\bigcap_{i=1}^{p} U_{n_i} \cap A = \emptyset\\
&\bigcap_{i=1}^{p} U_{n_i} \text{ is a neighborhood of } n\\
\Rightarrow &A \text{ is closed}
\end{aligned}$$

□

$$\begin{aligned}
&X \text{ is discrete topolgy}\\
\Rightarrow &C \subseteq X \text{ is closed}\\
&\forall x \in C, \exists x_0 \in C : d(x, x_0) < 1\\
\Rightarrow &C \text{ is bounded}\\
&\{\{x\}, x \in C\} \text{ is an open cover of } C\\
&x \text{ is infinite}\\
\Rightarrow &C \text{ is not compact}
\end{aligned}$$

Question 3

Proof.

$$
\begin{aligned}
&\forall a \in A \\
&U_a \in A \text{ is a neighborhood} \\
&\forall b \in B \\
&U_b \in B \text{ is a neighborhood} \\
\Rightarrow &U_a \cap U_b = \emptyset \\
&\{U_{a_1}, U_{a_2}, ..., U_{a_n}\} \text{ is an open cover of } A \text{ since } A \text{ is compact} \\
&V_a \coloneqq \bigcup_{i=1}^{n} U_{a_i} \\
&\forall i, A \subseteq V_{a_i} \\
&V_A \coloneqq \bigcap_{i=1}^{n} V_{a_i} \\
&V_A \text{ is open} \\
&A \subseteq V_A \\
&V_b \coloneqq \bigcap_{i=1}^{n} U_{b_i} \\
&\{V_{b_n}, n \in \mathbb{N}\} \text{ is an open cover of } B \\
&B \text{ is compact} \\
\Rightarrow &\{V_{b_1}, V_{b_2}, ..., V_{b_n}\} \text{ is an open cover of } B \\
&V_B \coloneqq \bigcup_{i=1}^{n} V_{b_i} \\
&B \subseteq V_B \\
&\forall i \geqslant n \in \mathbb{N}, V_{b_i} \cap V_A = \emptyset \\
&\bigcup_{i=1}^{n} V_{b_i} \cap V_A = \bigcup_{i=1}^{n} (V_{b_i} \cap V_A) \\
\Rightarrow &\bigcup_{i=1}^{n} V_{b_i} \cap V_A = \emptyset \\
\Rightarrow &V_B \cap V_A = \emptyset
\end{aligned}
$$

□