Question 1

Proof.

$$x \coloneqq (x_1, x_2, ..., x_n)$$
$$y \coloneqq (y_1, y_2, ..., y_n)$$
$$\rho(x, y) = \max(|x_i - y_i|) \coloneqq |x_k - y_k|$$
$$d(x, y) = \sqrt{\sum_i^n (x_i - y_i)^2}$$

$$\rho(x, y) \leqslant d(x, y) :$$
$$|x_k - y_k| = \sqrt{(x_k - y_k)^2}$$
$$k \in \{i, 0 < i \leqslant n\}$$
$$\Rightarrow (x_k - y_k)^2 \leqslant \sum_i^n (x_i - y_i)^2$$
$$\Rightarrow \rho(x, y) \leqslant d(x, y)$$
Equality is taken when $x = y : \rho(x, y) = d(x, y) = 0$

$$d(x, y) \leqslant \sqrt{n}\rho(x, y)$$
$$\sqrt{n}\rho(x, y) = \sqrt{n}|x_k - y_k| = \sqrt{n(x_k - y_k)^2}$$
$$\max(|x_i - y_i|) = |x_k - y_k|$$
$$\Rightarrow (x_k - y_k)^2 = \max((x_i - y_i)^2)$$
$$\Rightarrow \sum_i^n (x_i - y_i)^2 \leqslant n(x_k - y_k)^2$$
$$\Rightarrow d(x, y) \leqslant \sqrt{n}\rho(x, y)$$
Equality is taken when $x = y : d(x, y) = \sqrt{n}\rho(x, y) = 0$
$$\Rightarrow \rho(x, y) \leqslant d(x, y) \leqslant \sqrt{n}\rho(x, y)$$
Equality is taken when $x = y : \rho(x, y) = d(x, y) = \sqrt{n}\rho(x, y) = 0$

□

Question 2

Proof.

$$
\begin{aligned}
&\text{Suppose} : \exists \{x_n\}_{n=1}^{\infty} \subset A : \{x_n\}_{n=1}^{\infty} \to 100 \\
\Rightarrow &\forall U_{100}, \exists N \in \mathbb{Z}_+ : \forall i \geqslant N, x_i \in U_{100} \\
&U \coloneqq [100, 101) \\
\Rightarrow &U \text{ is a neighborhood of } 100 \\
&A \cap U = \emptyset \\
&\{x_n\}_{n=1}^{\infty} \subset A \\
\Rightarrow &\{x_n\}_{n=1}^{\infty} \cap U = \emptyset \\
\Rightarrow &\nexists N \in \mathbb{Z}_+ : \forall i \geqslant N, x_i \in U \\
\Rightarrow &\nexists \{x_n\}_{n=1}^{\infty} \subset A : \{x_n\}_{n=1}^{\infty} \to 100
\end{aligned}
$$

□

Question 3

Problem a

$$
\begin{aligned}
&B_d(x, \frac{1}{10}) = \{y \in \mathbb{R}^n | d(x, y) < \frac{1}{10}\} = \{y \in \mathbb{R}^n | d(x, y) = 0\} = \{x\} \\
&B_d(x, 0.9) = \{y \in \mathbb{R}^n | d(x, y) < 0.9\} = \{y \in \mathbb{R}^n | d(x, y) = 0\} = \{x\} \\
&B_d(x, 1) = \{y \in \mathbb{R}^n | d(x, y) < 1\} = \{y \in \mathbb{R}^n | d(x, y) = 0\} = \{x\} \\
&B_d(x, 2) = \{y \in \mathbb{R}^n | d(x, y) < 2\} = \{y \in \mathbb{R}^n | d(x, y) = 0 \text{ or } 1\} = \mathbb{R}^n \\
&B_d(x, 100) = \{y \in \mathbb{R}^n | d(x, y) < 100\} = \{y \in \mathbb{R}^n | d(x, y) = 0 \text{ or } 1\} = \mathbb{R}^n
\end{aligned}
$$

Problem b

Proof.

$$
\begin{aligned}
&y \coloneqq (y_1, y_2, ..., y_n) \\
&B_\rho(x, \varepsilon) = \{y \in \mathbb{R}^n | \rho(x, y) < \varepsilon\} \\
&\rho(x, y) = \max(|x_k - y_k|) \\
&\max(|x_k - y_k|) < \varepsilon \\
\Rightarrow &\forall i \in [1, n] \cap \mathbb{Z} : |x_i - y_i| < \varepsilon \\
\Rightarrow &\forall i \in [1, n] \cap \mathbb{Z} : y_i \in (x_i - \varepsilon, x_i + \varepsilon) \Leftrightarrow \rho(x, y) < \varepsilon \\
\Rightarrow &\forall y : \rho(x, y) < \varepsilon, y \in (x_1 - \varepsilon, x_1 + \varepsilon) \times ... \times (x_n - \varepsilon, x_n + \varepsilon) \\
&\forall y : \rho(x, y) < \varepsilon, y \in B_\rho(x, \varepsilon) \\
\Rightarrow &B_\rho(x, \varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times ... \times (x_n - \varepsilon, x_n + \varepsilon)
\end{aligned}
$$

□