# Problem 1

## Question a

$$
\begin{aligned}
&g \circ f(x) = g(f(x)) \\
&g \circ f(x_1) = g \circ f(x_2) \\
\Rightarrow &f(x_1) = f(x_2) \leftarrow g \text{ is one-to-one} \\
\Rightarrow &x_1 = x_2 \leftarrow f \text{ is one-to-one} \\
&g \circ f(x_1) = g \circ f(x_2) \implies x_1 = x_2 \\
\Rightarrow &g \circ f \text{ is one-to-one}
\end{aligned}
$$

## Question b

$$
\begin{aligned}
&g \circ f(x) = g(f(x)) \\
&\forall u \in C, \exists v \in \{n | n = f(x), x \in A\} : u = g(v) \leftarrow g \text{ is onto} \\
&\forall v \in \{n | n = f(x), x \in A\}, \exists w \in A : v = f(w) \leftarrow f \text{ is onto} \\
\Rightarrow &\forall u \in C, \exists w \in A : u = g(f(w)) = g \circ f(w) \\
\Rightarrow &g \circ f \text{ is onto}
\end{aligned}
$$

# Problem 2

## Question a

$$\begin{aligned}
&F \coloneqq \{f | f : \{0,1\} \to \mathbb{Z}_+\} \\
&\text{There are only 2 elements in domain of } f \\
&\text{By permutation: } |F| = |\mathbb{Z}_+ \times \mathbb{Z}_+| \\
&|\mathbb{Z}_+ \times \mathbb{Z}_+| = |\mathbb{Z}_+| \\
\Rightarrow &F \text{ is countable}
\end{aligned}$$

## Question b

$$\begin{aligned}
&F \coloneqq \{f | f : \mathbb{Z}_+ \to \mathbb{Z}_+\} \\
&\text{Suppose } F \text{ is countable, We can construct a table for all the functions:} \\
&\begin{array}{c|cccc}
 & n=1 & n=2 & n=3 & \cdots \\
\hline
 & & & & \\
f_1(n) & f_1(1) & f_1(2) & f_1(3) & \cdots \\
f_2(n) & f_2(1) & f_2(2) & f_2(3) & \cdots \\
f_3(n) & f_3(1) & f_3(2) & f_3(3) & \cdots \\
\vdots & & & & \ddots
\end{array} \\
&g(x) \coloneqq \begin{cases} 0, f_a(a) = 1 \\ 1, f_a(a) \neq 1 \end{cases} \\
&\text{According to the definition, } g(x) \text{ cannot be any of the } f_n(x) \\
\Rightarrow &F \text{ is uncountable}
\end{aligned}$$

# Problem 3

## Question a

$$
\begin{aligned}
& f(x) \coloneqq 3x, x \in \mathbb{Z}_+ \\
\Rightarrow & f : \mathbb{Z}_+ \to 3\mathbb{Z}_+ \\
& f(x_1) = f(x_2) \\
\Rightarrow & 3x_1 = 3x_2 \\
\Rightarrow & x_1 = x_2 \\
\Rightarrow & f(x) \text{ is one-to-one} \\
& \forall y \in 3\mathbb{Z}_+, \exists x \in \mathbb{Z}_+ : x = \frac{y}{3} \\
\Rightarrow & f(x) \text{ is surjective} \\
\Rightarrow & f(x) \text{ is bijective} \\
\Rightarrow & |\mathbb{Z}_+| = |3\mathbb{Z}_+| \\
& \mathbb{Z} = \mathbb{Z}_+ \cup \{0\} \cup \mathbb{Z}_- \\
& \mathbb{Z} \text{ is countable} \\
\Rightarrow & \mathbb{Z}_+ \text{ is countable} \\
\Rightarrow & |\mathbb{Z}_+| = |\mathbb{Z}| \\
\Rightarrow & |\mathbb{Z}| = |3\mathbb{Z}_+|
\end{aligned}
$$

## Question b

$$
\begin{aligned}
& \text{Suppose: } \exists f : \wp(A) \to A \\
& \text{There is no one-to-one map for } \wp(A) \to A \\
\Rightarrow & f \text{ cannot be one-to-one} \\
\Rightarrow & f \text{ is not bijective} \\
\Rightarrow & |\wp(A)| \neq |A|
\end{aligned}
$$