Question 1

Problem a

$$
\begin{aligned}
&i \text{ is continuous} \implies \mathcal{T} \subseteq \mathcal{T}' : \\
&\exists U \in \mathcal{T} \\
&U = i^{-1}(U) \in \mathcal{T}' \\
\Rightarrow &\forall U \in \mathcal{T}, U \in \mathcal{T}' \\
\Rightarrow &\mathcal{T} \subseteq \mathcal{T}' \\
\\
&\mathcal{T} \subseteq \mathcal{T}' \implies i \text{ is continuous} : \\
&\forall U \in \mathcal{T}, \exists i^{-1} : i^{-1}(U) = U \in \mathcal{T}' \\
\Rightarrow &i \text{ is continuous}
\end{aligned}
$$

Problem b

$$
\begin{aligned}
&i \text{ is the identity function} \\
\Rightarrow &i \text{ is bijective} \\
&i \text{ is continuous} \Leftrightarrow \mathcal{T} \subseteq \mathcal{T}' \\
\Rightarrow &i^{-1} \text{ is continuous} \Leftrightarrow \mathcal{T}' \subseteq \mathcal{T} \\
&\mathcal{T}' \subseteq \mathcal{T} \wedge \mathcal{T} \subseteq \mathcal{T}' \implies \mathcal{T}' = \mathcal{T} \\
\Rightarrow &i \text{ is homeomorphism} \Leftrightarrow \mathcal{T} = \mathcal{T}'
\end{aligned}
$$

Question 2

$$\begin{aligned}
&U \setminus A \\
=&U \cap (X \setminus A) \\
&A \text{ is closed} \\
\Rightarrow &X \setminus A \text{ is open} \\
\Rightarrow &U \setminus A = U \cap (X \setminus A) \text{ is open} \\
\\
&A \setminus U \\
=&A \cap (X \setminus U) \\
&U \text{ is open} \\
\Rightarrow &X \setminus U \text{ is closed} \\
\Rightarrow &A \setminus U = A \cap (X \setminus U) \text{ is closed}
\end{aligned}$$

Question 3

Problem a

$$
\begin{aligned}
&\text{Suppose} : \exists x \in Bd(A) \cap Int(A) \\
\Rightarrow &x \in Bd(A) \wedge x \in \bigcup_{U \subseteq A \text{ open}} U \\
&\exists U_x \text{ is nbhd} : U_x \subseteq A \\
&Bd(A) = \overline{A} \cap \overline{X \setminus A} \\
\Rightarrow &x \in \overline{X \setminus A} \\
\Rightarrow &U_x \cap \overline{X \setminus A} \neq \emptyset \\
&U_x \cap \overline{X \setminus A} \neq \emptyset \not\Leftrightarrow U_x \subseteq A \\
\Rightarrow &Bd(A) \cap Int(A) = \emptyset
\end{aligned}
$$

$$
\begin{aligned}
&\overline{A} \subseteq Int(A) \cup Bd(A) \\
&\exists x \in \overline{A} \\
&\text{If } \exists U_x \subseteq A \\
\Rightarrow &x \in Int(A) \\
&\text{If } \nexists U_x \subseteq A \\
&\exists m \in U_x : m \notin A \\
\Rightarrow &m \in U_x \cap (X \setminus A) \\
\Rightarrow &U_x \cap (X \setminus A) \neq \emptyset \\
\Rightarrow &x \in \overline{X \setminus A} \\
\Rightarrow &x \in Bd(A) \\
\Rightarrow &x \in Int(A) \cup Bd(A)
\end{aligned}
$$

$$
\begin{aligned}
&Int(A) \cup Bd(A) \subseteq \overline{A} \\
&\text{If } x \in Int(A) \\
&Int(A) \subseteq A \subseteq \overline{A} \\
\Rightarrow &x \in \overline{A} \\
&\text{If } x \in Bd(A) \\
&Bd(A) = \overline{A} \cap \overline{X \setminus A} \\
\Rightarrow &x \in \overline{A} \\
\Rightarrow &x \in \overline{A}
\end{aligned}
$$

Problem b

$$\begin{aligned}
&Bd(A) = \emptyset \implies A \text{ is clopen}:\\
&\overline{A} = Int(A) \cup Bd(A)\\
\Rightarrow &\overline{A} = Int(A)\\
&Int(A) \subseteq A \subseteq \overline{A}\\
\Rightarrow &A = Int(A) = \overline{A}\\
\Rightarrow &A \text{ is clopen}\\
\\
&A \text{ is clopen} \implies Bd(A) = \emptyset:\\
&A = \overline{A}\\
\Rightarrow &\overline{X \setminus A} = X \setminus A\\
\Rightarrow &Bd(A) = \overline{A} \cap \overline{X \setminus A} = A \cap (X \setminus A) = \emptyset
\end{aligned}$$

0.1 Problem c

U is open $\implies Bd(U) = \overline{U} \setminus U$:

U is open

$\Rightarrow X \setminus U$ is closed

$\Rightarrow X \setminus U = \overline{X \setminus U}$

$Bd(U) = \overline{U} \cap \overline{X \setminus U}$

$\Rightarrow Bd(U) = \overline{U} \cap (X \setminus U) = \overline{U} \setminus U$

$Bd(U) = \overline{U} \setminus U \implies U$ is open

Suppose$\exists x \in U \setminus Int(U)$

$\exists U_x$ nbhd of $x : x \in U_x \cap U$

$\Rightarrow U_x \cap U \neq \emptyset$

$x \notin Int(U) \implies U_x \notin U$

$\Rightarrow U_x \setminus U \neq \emptyset$

$\Rightarrow U_x \cap (X \setminus U) \neq \emptyset$

$\Rightarrow x \in \overline{U} \cap \overline{X \setminus U}$

$\Rightarrow x \in Bd(U)$

$\Rightarrow x \in \overline{U} \setminus U \not\Leftrightarrow x \in U$

$\Rightarrow U \subseteq Int(U)$

$\Rightarrow U = Int(U)$

U is open

$\Rightarrow U$ is open $\Leftrightarrow Bd(U) = \overline{U} \setminus U$

Problem d

Consider standard topology on $\mathbb{R}$

$U \coloneqq (-1, 0) \cup (0, 1)$

$\Rightarrow \overline{U} = [-1, 1]$

$\Rightarrow Int(\overline{U}) = (-1, 1) \neq U$