Question 1

Problem a

$$
\begin{aligned}
& A \coloneqq [0 \times 0, 0 \times 1] \\
& X \setminus A = ((I \times I) \setminus A) \cup \{e \times 0\} \\
& X \setminus A = (\frac{1}{n} \times 0, \frac{1}{n} \times 1) \cup \{e \times 0\} \\
& B \coloneqq (\frac{1}{n} \times 1, e \times 0] \\
& B \cap X = (\frac{1}{n} \times 0, \frac{1}{n} \times 1) \cup \{e \times 0\} = X \setminus A \\
& B \text{ is not closed nor open in } \mathbb{R}^2 \\
\Rightarrow & X \setminus A \text{ is not closed nor open} \\
\Rightarrow & A \text{ is not closed nor open}
\end{aligned}
$$

Problem b

$$
\begin{aligned}
& I \times I = \{(a, b), 0 \leqslant a < b \leqslant 1\} \\
\Rightarrow & I \times I = \{(a, b)\} \text{ is open} \\
& \overline{I \times I} \text{ is the set containing all the limit points of } I \times I \\
\Rightarrow & \overline{I \times I} = \{[a, b], 0 \leqslant a < b \leqslant 1\}
\end{aligned}
$$

Question 2

Define the finite complement topology X

$\exists x \in X, U_x$ is the nbhd of x

$\Rightarrow {U_x}^{\complement}$ is finite

$\Rightarrow \exists m \in \mathbb{N} : \forall n > m, \dfrac{1}{n} \notin {U_x}^{\complement}$

$\Leftrightarrow \exists m \in \mathbb{N} : \forall n > m, \dfrac{1}{n} \in U_x$

$\Rightarrow U_x \cap (\left\{\dfrac{1}{n}\right\}_{n=1}^{\infty} \setminus \{x\}) \neq \emptyset$

$\Rightarrow \left\{\dfrac{1}{n}\right\}_{n=1}^{\infty}$ converges to x

x is arbitrary

$\Rightarrow \left\{\dfrac{1}{n}\right\}_{n=1}^{\infty}$ converges to all $x \in \mathbb{R}$

Question 3

Problem a

$$
\begin{aligned}
&A \subset B \\
&\exists x \in \overline{A}, U_x \text{ is nbhd of } x \\
\Rightarrow &\exists U_x : U_x \cap A \neq \emptyset \\
&A \subset B \\
\Rightarrow &U_x \cap B \neq \emptyset \\
\Rightarrow &x \in \overline{B} \\
&\forall x \in \overline{A}, x \in \overline{B} \\
\Rightarrow &\overline{A} \subset \overline{B} \\
\Rightarrow &A \subset B \implies \overline{A} \subset \overline{B}
\end{aligned}
$$

Problem b

$$
\begin{aligned}
&\overline{A} \cup \overline{B} \subseteq \overline{A \cup B} : \\
&A \subset A \cup B \\
\Rightarrow &\overline{A} \subset \overline{A \cup B} \\
&B \subset A \cup B \\
\Rightarrow &\overline{B} \subset \overline{A \cup B} \\
\Rightarrow &\overline{A} \cup \overline{B} \subseteq \overline{A \cup B} \\
&\overline{A \cup B} \subseteq \overline{A} \cup \overline{B} : \\
&A \subset \overline{A} \wedge B \subset \overline{B} \\
\Rightarrow &A \cup B \subset \overline{A} \cup \overline{B} \\
&\overline{A \cup B} \text{ is the smallest closed set containing } A \cup B \\
&\text{Any other closed set containing } A \cup B \text{ contains } \overline{A \cup B} \\
\Rightarrow &\overline{A \cup B} \subseteq \overline{A} \cup \overline{B} \\
\Rightarrow &\overline{A} \cup \overline{B} = \overline{A \cup B}
\end{aligned}
$$

Problem c

$$
\begin{aligned}
&\exists x \in \bigcup_{\alpha} \overline{A_{\alpha}}, U_x \text{ is the nbhd of } x \\
&\exists \alpha, x \in \overline{A_{\alpha}} \\
\Rightarrow &U_x \cap A_{\alpha} \neq \emptyset \\
\Rightarrow &(U_x \cap A_1) \cup (U_x \cap A_2) \cup ... \cup (U_x \cap A_{\alpha}) \cup ... \neq \emptyset \\
&U_x \cap (A_1 \cup A_2 \cup ... \cup A_{\alpha} \cup ...) \neq \emptyset \\
\Rightarrow &U_x \cap (\bigcup_{\alpha} A_{\alpha}) \neq \emptyset \\
\Rightarrow &x \in \overline{\bigcup_{\alpha} A_{\alpha}} \\
\Rightarrow &\bigcup_{\alpha} \overline{A_{\alpha}} \subset \overline{\bigcup_{\alpha} A_{\alpha}} \\
&\lim_{n \to \infty} \frac{1}{n} = 0 \\
\Rightarrow &\forall U_0, U_0 \cap \{\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \setminus \{0\}\} \neq \emptyset \\
\Rightarrow &\{\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \setminus \{0\}\} \cap (\bigcup U_0) \neq \emptyset \\
&\text{Suppose } m \text{ has } \{\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \setminus \{0\}\} \text{ as its nbhd} \\
\Rightarrow &m \in \overline{\bigcup U_0} \\
&\text{Suppose } m \in \bigcup \overline{U_0} \\
&\exists \overline{U_{0_a}}, \overline{U_{0_b}} \subset \bigcup U_0 : m \in \overline{U_{0_a}} \wedge m \notin \overline{U_{0_b}} \\
\Rightarrow &\{\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \setminus \{0\}\} \cap U_{0_a} \neq \emptyset \wedge \cap U_{0_b} = \emptyset \\
&\{\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \setminus \{0\}\} \cap U_{0_a} \neq \emptyset \\
\Rightarrow &0 \in \overline{\{\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \setminus \{0\}\}} \\
&\{\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \setminus \{0\}\} \cap U_{0_b} = \emptyset \\
\Rightarrow &0 \notin \overline{\{\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \setminus \{0\}\}}
\end{aligned}
$$

$$0 \in \overline{\{\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \setminus \{0\}\}} \not\Leftrightarrow 0 \notin \overline{\{\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \setminus \{0\}\}}$$
$$\Rightarrow m \notin \bigcup \overline{U_0} \wedge m \in m \in \overline{\bigcup U_0} \implies \text{Equality fails}$$