Question 1

Suppose g, h extend from $f \wedge$ defined on $\overline{A}$

$\exists x \in \overline{A} : g(x) \neq h(x)$

Y is Hausdorff

$\Rightarrow g(x) \in U, h(x) \in V, U \cap V = \emptyset$

$\Rightarrow x \in g^{-1}(U) \cap h^{-1}(V)$

$g^{-1}(U)$ and $h^{-1}(V)$ are open because of continuity

$\Rightarrow \exists x' \in A \cap g^{-1}(U) \cap h^{-1}(V)$

Since g, h extend f

$\Rightarrow f(x') = g(x') = h(x'), x' \in A$

$\Rightarrow U, V$ are not disjoint $\not\Leftrightarrow U \cap V = \emptyset$

$\Rightarrow$uniqueness

Question 2

$$
\begin{aligned}
&\mathcal{B} \text{ is a basis of } B\\
&\mathcal{D} \text{ is a basis of } D\\
\Rightarrow&\mathcal{B} \times \mathcal{D} \text{ is a basis of } B \times D\\
&\mathcal{A} \coloneqq f^{-1}(\mathcal{B})\\
\Rightarrow&\mathcal{A} \text{ is open in } A\\
&\mathcal{B} = f(\mathcal{A})\\
&\mathcal{C} \coloneqq f^{-1}(\mathcal{D})\\
\Rightarrow&\mathcal{C} \text{ is open in } C\\
&\mathcal{D} = g(\mathcal{C})\\
\Rightarrow&\mathcal{A} \times \mathcal{C} \text{ is open in } A \times C\\
&(f \times g)^{-1}(\mathcal{B} \times \mathcal{D})\\
=&(f \times g)^{-1}(f(\mathcal{A}) \times g(\mathcal{C}))\\
=&(f \times g)^{-1}((f \times g)(\mathcal{A} \times \mathcal{C}))\\
=&\mathcal{A} \times \mathcal{C}\\
&\mathcal{A} \times \mathcal{C} \text{ is open in } A \times C \text{ for } \mathcal{B} \times \mathcal{D} \text{ is a basis of } B \times D\\
\Rightarrow&f \times g \text{ is continuous}
\end{aligned}
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Question 3

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\begin{aligned}
&d(x,y)=0 \Leftrightarrow x=y\\
&d(x,y)=0 \vee d(x,y)=1\\
\Rightarrow &d(x,y)\geqslant 0\\
&\text{Case } d(x,y)=0:\\
&\text{subcase } 1: x=y=z\\
\Rightarrow &d(x,y)=d(x,z)+d(y,z)=0\\
&\text{subcase } 2: x=y\neq z\\
\Rightarrow &d(x,y)=0<d(y,z)+d(x,z)=2\\
&\text{Case } d(x,y)=1:\\
&\text{subcase } 1: x\neq y=z\\
&d(x,y)=d(x,z)+d(y,z)=1\\
&\text{subcase } 2: x\neq y\neq z\\
&d(x,y)=1<d(x,z)+d(y,z)=2\\
\Rightarrow &d(x,y)\leqslant d(x,z)+d(y,z)\\
&x=y \implies d(x,y)=d(y,x)=0\\
&x\neq y \implies d(x.y)=d(y,x)=1\\
\Rightarrow &d(x,y)=d(y,x)\\
\Rightarrow &d(x,y) \text{ is a metric}
\end{aligned}
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