Question 1

$$
\begin{aligned}
&\mathcal{T} \text{ is defined as the topology on } X\\
&\mathcal{T}_Y \coloneqq \{U \cap Y | U \in \mathcal{T}\}\\
&\mathcal{T}_{A\in Y} \coloneqq \{V \cap A|, V \in \mathcal{T}_Y\}\\
&\mathcal{T}_{A\in X} \coloneqq \{U \cap A | U \in \mathcal{T}\}\\
&\mathcal{T}_{A\in Y} \subset \mathcal{T}_{A\in X}:\\
&\exists V \in \mathcal{T}_Y : V \cap A \in \mathcal{T}_{A\in Y}\\
\Rightarrow &\exists U \in \mathcal{T} : V = U \cap Y\\
&V \cap A = (U \cap Y) \cap A = U \cap (Y \cap A) = U \cap A\\
\Rightarrow &V \cap A = \mathcal{T}_{A\in X}\\
\Rightarrow &\mathcal{T}_{A\in Y} \subset \mathcal{T}_{A\in X}\\
&\mathcal{T}_{A\in X} \subset \mathcal{T}_{A\in Y}:\\
&\exists U \cap A \in \mathcal{T}_{A\in X}\\
\Rightarrow &U \in \mathcal{T}\\
&U \cap Y \in \mathcal{T}_Y\\
\Rightarrow &(U \cap Y) \cap A \in \mathcal{T}_{A\in Y}\\
&(U \cap Y) \cap A = U \cap (Y \cap A) = U \cap A\\
\Rightarrow &U \cap A \in \mathcal{T}_{A\in Y}\\
\Rightarrow &\mathcal{T}_{A\in X} \subset \mathcal{T}_{A\in Y}\\
\Rightarrow &\mathcal{T}_{A\in X} = \mathcal{T}_{A\in Y}
\end{aligned}
$$

Question 2

Consider a basis for $X \times Y$:

$\{(U, V) | U$ is open in X, V is open in $Y\}$

$\pi_1((U, V)) \coloneqq U \wedge \pi_1((U, V)) \coloneqq V$

$(U, V), U, V$ are open by the definition of the basis

$\pi_1((U, V)) = U$ is open for (U, V) is open

$\Rightarrow \pi_1 : X \times Y \to X$ is open

$\pi_2((U, V)) = V$ is open for (U, V) is open

$\Rightarrow \pi_2 : X \times Y \to Y$ is open

Question 3

Problem a

$$\begin{aligned}
&\text{Suppose } U \text{ is open in } \mathbb{R}_d \times \mathbb{R}\\
&\exists (a,b) \in U\\
&\text{Consider} : \{a\} \times (c,d), a,c,d \in \mathbb{R} \text{ as a basis}\\
&\{a\} \times (c,d) = ((a,c),(a,d))\\
&(a,b) \text{ is open } \in \{a\} \times (c,d) = ((a,c),(a,d)) \subseteq U\\
\Rightarrow &(a,b) \text{ is an interior}\\
&\text{Suppose } U \text{ is open in } \mathbb{R} \times \mathbb{R}\\
&\exists (a,b) \in U\\
&\exists (a_1,b_1) \times (a_2,b_2), a_1,a_2,b_1,b_2 \in \mathbb{R} : (a,b) \in (a_1,b_1) \times (a_2,b_2)\\
&m := \begin{cases} b-1, a > a_1 \\ b, a = a_1 \end{cases}\\
\Rightarrow &\text{By definition} : (a_1,b_1) < (a,m) < (a,b)\\
&n := \begin{cases} b+1, a < a_2 \\ b, a = a_2 \end{cases}\\
\Rightarrow &\text{By definition} : (a,b) < (a,n) < (a_2,b_2)\\
\Rightarrow &(a,b) \in \{a\} \times (m,n) \subseteq ((a_1,b_1),(a_2,b_2)) \subseteq U\\
\Rightarrow &(a,b) \text{ is an interior}\\
&\text{They are the same}
\end{aligned}$$

Problem b

$$\begin{aligned}
&\mathbb{R}^2 \text{ generates from } (a,b) \times (c,d) : \text{ open}\\
&\mathbb{R}^2 \text{ takes horizontal and vertical lines into consideration simultaneously}\\
&\text{Dictionary order topology takes vertical lines into consideration first}\\
&\text{It disregards horizontal direction when defining open sets}\\
\Rightarrow &\mathbb{R}^2 \text{ is finer}
\end{aligned}$$