Question 1

Proof.

$$
\begin{aligned}
&B_n \coloneqq \bigcup_{i=1}^{n} A_i \\
\Rightarrow &B_1 \subset B_2 \subset B_3 ... \\
&B \coloneqq \bigcup_{n>1} B_n \\
&\text{Let } U \cap V = \emptyset : B = U \cup V \\
&\text{Suppose } B_1 \in U \\
\Rightarrow &B_n \in U \forall n \\
\Rightarrow &U = B, V = \emptyset \\
\Rightarrow &B \text{ is connected} \\
\Rightarrow &\bigcup A_n \text{ is connected}
\end{aligned}
$$

□

Question 2

Proof.

$$
\begin{aligned}
&x_0 \in U \\
&A \coloneqq \{y \in U | \exists f : [0,1] \to U : f(0) = x_0, f(1) = y\} \\
&\text{non-empty :} \\
&\text{constant map } f \\
\Rightarrow &x_0 \in A \\
&\text{open :} \\
&y \in U \\
&B_d(y, \varepsilon) \in U, \varepsilon > 0 \\
&\forall z \in B_d(y, \varepsilon) \\
\Rightarrow &\exists g \text{ maps from } y \text{ to } z \\
&g \text{ is continuous} \\
&\text{combine } f \wedge g \text{ is a path from } x \text{ to } z \\
\Rightarrow &z \in A \\
\Rightarrow &B_d(y, \varepsilon) \in A \\
\Rightarrow &A \text{ is open} \\
&\text{close :} \\
&\exists y \in U : y \in \overline{A} \\
&B_d(y, \varepsilon) \in U, \varepsilon > 0 \\
\Rightarrow &B_d(y, \varepsilon) \cap A \neq \emptyset \\
&\exists p \in A \cap B_d(y, \varepsilon) \\
&\exists g \text{ maps from } x \text{ to } p \\
&g \text{ is continuous} \\
&\text{combine } f \wedge g \text{ is a path from } p \text{ to } y \\
\Rightarrow &y \in A \\
\Rightarrow &\forall y \in \overline{A}, y \in A \\
\Rightarrow &A \text{ is closed} \\
&A \text{ is non empty and clopen} \\
\Rightarrow &U \text{ is path connected}
\end{aligned}
$$

□

Question 3

Proof.

$$
\begin{aligned}
&g : [0,1] \to [-1,1] \\
&g(x) \coloneqq f(x) - x \\
\Rightarrow &g \text{ is continuous} \\
&f(0) \in [0,1], x = 0 \\
\Rightarrow &g(0) \in [0,1] \\
&f(1) \in [0,1], x = 1 \\
\Rightarrow &g(1) \in [-1,0] \\
\Rightarrow &g(1) \leqslant 0 \leqslant g(0) \\
&\text{By IVT :} \\
&\exists p \in [0,1] : g(p) = 0 \\
\Rightarrow &f(p) - p = 0 \implies f(p) = p \\
\Rightarrow &p \in [0,1] \text{ is a fixed point}
\end{aligned}
$$

□