Question 1

Proof.

$$
\begin{aligned}
& [0,1] \text{ is closed } \in \mathbb{R} \\
& \forall \varepsilon > 0, \exists \{x_n | n \in \mathbb{N}\} \in [0,1] : \\
& [0,1] \in \bigcup_n B_d(x_n, \varepsilon) \\
\Rightarrow & [0,1] \text{ is bounded} \\
\Rightarrow & [0,1] \text{ is compact} \\
& A_0 = [0,1] \\
& A_1 = A_0 \setminus \bigcup_{k=1}^{\infty} (\frac{1+3k}{3}, \frac{2+3k}{3}) \\
& A_1 = [0,1] \setminus (\frac{1}{3}, \frac{2}{3}) \\
\Rightarrow & A_1 \neq \emptyset \\
& (\frac{1+3k}{3}, \frac{2+3k}{3}) \text{ is open} \\
\Rightarrow & \bigcup_{k=1}^{\infty} (\frac{1+3k}{3}, \frac{2+3k}{3}) \text{ is open} \\
\Rightarrow & A_1 \text{ is closed} \\
& \text{By Induction :} \\
& \forall n \in \mathbb{N} : A_n = A_{n-1} \setminus \bigcup_{k=1}^{\infty} (\frac{1+3k}{3^n}, \frac{2+3k}{3^n}) \text{ is closed} \\
& \text{Self-similarity :} \\
& \forall n \in \mathbb{N}, A_n \neq \emptyset \\
& \forall n \in \mathbb{N}, A_n \subset A_{n-1} \\
\Rightarrow & \bigcap_{n \in \mathbb{N}} A_n \neq \emptyset
\end{aligned}
$$

□

Question 2

Proof.

$$
\begin{aligned}
&\forall m, n \in \mathbb{R}^2 \setminus A : \\
&F \coloneqq \{f(x) = ax + b | f(m_x) = m_y, f : \mathbb{R}^2 \to \mathbb{R}^2\} \\
&|F| = \infty \\
&G \coloneqq \{g(x) = ax + b | g(n_x) = n_y, g : \mathbb{R}^2 \to \mathbb{R}^2\} \\
&|G| = \infty \\
&F* \coloneqq F \setminus \{f(x) | f(x) \in F, \exists a \in A : f(a_x) = a_y\} \\
&\{f(x) | f(x) \in F, \exists a \in A : f(a_x) = a_y\} \text{ is countable} \\
\Rightarrow &|F*| = \infty \\
&G* \coloneqq G \setminus \{g(x) | g(x) \in G, \exists a \in A : g(a_x) = a_y\} \\
&\{g(x) | g(x) \in G, \exists a \in A : g(a_x) = a_y\} \text{ is countable} \\
\Rightarrow &|G*| = \infty \\
&\exists f \in F*, g \in G* : f, g \text{ intersects} \\
&p \coloneqq (a, b) : f(a) = g(a) = b \\
\Rightarrow &h(x) \coloneqq \begin{cases} f(x), h : m \mapsto p \\ g(x), h : p \mapsto n \end{cases} \\
&h(x) \text{ is a path in} \mathbb{R}^2 \setminus A \\
&h(x) \text{ stands for arbitrary } m, n \\
\Rightarrow &\text{There is a path for every two points in } \mathbb{R}^2 \setminus A \\
\Rightarrow &\mathbb{R}^2 \setminus A \text{ is path connected}
\end{aligned}
$$

□

Question 3

Proof.

$$
\begin{aligned}
&\text{Suppose} : C \text{ is connected intersecting} A, X \setminus A : C \cap Bd(A) = \emptyset \\
&\text{Bd}(A) = \overline{A} \cap \overline{X \setminus A} \\
&\text{Int}(A) \cap \overline{X \setminus A} = \emptyset \\
&\text{Int}(X \setminus A) \cap \overline{A} = \emptyset \\
\Rightarrow &\text{Int}(A), \text{Int}(X \setminus A), \text{Bd}(A) \text{ are disjoint} \\
&\text{Bd}(A) \cup \text{Int}(X \setminus A) = \overline{X \setminus A} \\
&\forall x \in X, x \in \text{Int}(A) \vee x \in \overline{X \setminus A} \\
\Rightarrow &X = \text{Int}(A) \cup \text{Bd}(A) \cup \text{Int}(X \setminus A) \\
&C = C \cap X \\
\Rightarrow &C = (C \cap \text{Int}(A)) \cup (C \cap \text{Int}(X \setminus A)) \\
&\text{Int}(A) \cap \text{Int}(X \setminus A) = \emptyset \\
\Rightarrow &\text{There is a seperation} \\
\Rightarrow &C \text{ is not connected} \not\Leftrightarrow C \text{ is connected} \\
\Rightarrow &C \text{ is connected intersecting} A, X \setminus A : C \cap Bd(A) \neq \emptyset
\end{aligned}
$$

□