Question 1

Problem a

Proof.

$$
\begin{aligned}
&\forall (x,y) \in X \times X :\\
&V = (d(x,y) - \varepsilon, d(x,y) + \varepsilon) \text{ is open in } \mathbb{R}\\
&(x,y) \in d^{-1}(V)\\
\Rightarrow &V \text{ is a nbhd of } d((x,y))\\
&U_{(x,y)} \coloneqq B_d(x, \frac{1}{2}\varepsilon) \times B_d(y, \frac{1}{2}\varepsilon)\\
&\forall (x',y') \neq (x,y) \in U_{(x,y)} :\\
&|d(x,y) - d(x',y')| = |d(x,y) - d(x',y) + d(x',y) - d(x',y')|\\
&|d(x,y) - d(x',y) + d(x',y) - d(x',y')| \leqslant |d(x,y) - d(x',y)| + |d(x',y) - d(x',y')|\\
&|d(x,y) - d(x',y)| + |d(x',y) - d(x',y')| \leqslant d(x,x') + d(y,y')\\
\Rightarrow &d(x,y) - d(x',y') \leqslant d(x,x') + d(y,y')\\
\Rightarrow &d((x',y')) \in V\\
&(x',y') \in d^{-1}(V)\\
&\text{Since } (x',y') \text{ is arbitrary :}\\
\Rightarrow &U_{(x,y)} \subseteq d^{-1}(V)\\
&d^{-1}(V) = \bigcup_{(x,y) \in d^{-1}(V)} \quad \text{open in } X \times X\\
\Rightarrow &d \text{ is continuous}
\end{aligned}
$$

□

Problem b

Proof.

$$
\begin{aligned}
&\text{Suppose } \{x_n\}_{n=1}^{\infty} \to x \\
&\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{Z}_+ : d(x_n, x) < \varepsilon, \forall n \geqslant n_\epsilon \\
&\forall m, n \geqslant n_\varepsilon \\
&d(x_n, x_m) \leqslant d(x_n, x) + d(x_m, x) < 2\varepsilon \\
\Rightarrow &\{x_n\}_{n=1}^{\infty} \text{ is cauchy}
\end{aligned}
$$

□

Question 2

Problem a

Proof.

$$
\begin{aligned}
&F : X \to \mathbb{R}\\
&F(x) = d(f(x), g(x))\\
&G : X \to Y \times Y\\
&G(x) = (f(x), g(x))\\
&H(x) : Y \times Y \to \mathbb{R}\\
&H((a, b)) = d(a, b)\\
\Rightarrow &F(x) = H \circ G(x)\\
&f(x), g(x) \text{ are continuous}\\
\Rightarrow &G(x) = (f(x), g(x)) \text{ is continuous}\\
&Y \text{ is metric space}\\
&\text{ From Problem 1a :}\\
&H((a, b)) = d(a, b) \text{ is continuous}\\
\Rightarrow &F(x) = H \circ G(x) \text{ is continuous}
\end{aligned}
$$

□

Problem b

i

Proof.

$$\begin{aligned}
&X \text{ is compact}\\
&F : X \to \mathbb{R}\\
&F(x) = d(f(x), g(x)) \text{ is continuous}\\
\Rightarrow &F(x) \text{ is bounded}\\
&\{d(f(x), g(x))|x \in X\} \text{ is bounded}\\
\Rightarrow &\rho \text{ is well defined}
\end{aligned}$$

□

ii

Proof.

$$\begin{aligned}
&F(x) = d(f(x), g(x)) \text{ is continuous}\\
&X \text{ is closed and bounded}\\
\Rightarrow &\forall x \in X, \exists x_{max} : F(x) \leqslant F(x_{max})\\
&F(x_{max}) = \max(\{F(x)|x \in X\}) = \max(\{d(f(x), g(x))|x \in X\})\\
&F(x_{max}) \text{ is the upper bound of } F(x)\\
\Rightarrow &F(x_{max}) = \sup(\{d(f(x), g(x))|x \in X\})\\
\Rightarrow &\max(\{d(f(x), g(x))|x \in X\}) = \rho(f, g)
\end{aligned}$$

□

iii

Proof.

$$\begin{aligned}
&\rho(f,g) = \max(\{d(f(x),g(x))|x \in X\})\\
&\rho(g,f) = \max(\{d(g(x),f(x))|x \in X\})\\
\Rightarrow &\rho(g,f) = \rho(f,g)\\
\\
&\text{Suppose } \rho(f,g) = 0\\
&\max(\{d(g(x),f(x))|x \in X\}) = 0\\
&\forall x \in X, d(f(x),g(x)) = 0\\
\Rightarrow &f(x) = g(x)\\
\\
&\rho(f,g) = \max(\{d(f(x),g(x))|x \in X\})\\
&\forall f,g \in C(X,Y), d(f(x),g(x)) \geqslant 0\\
\Rightarrow &\max(\{d(f(x),g(x))|x \in X\}) \geqslant 0\\
&\rho(f,g) \geqslant 0\\
\\
&\rho(f,g) = \max(\{d(f(x),g(x))|x \in X\})\\
&\rho(f,h) + \rho(g,h) = \max(\{d(f(x),h(x))|x \in X\}) + \max(\{d(h(x),g(x))|x \in X\})\\
&\rho(f,h) + \rho(g,h) = \max(\{d(f(x),h(x)) + d(h(x),g(x))|x \in X\})\\
&d(f(x),g(x)) \leqslant d(f(x),h(x)) + d(h(x),g(x))\\
\Rightarrow &\max(\{d(f(x),g(x))|x \in X\}) \leqslant \max(\{d(f(x),h(x)) + d(h(x),g(x))|x \in X\})\\
&\rho(f,g) \leqslant \rho(f,h) + \rho(g,h)\\
\Rightarrow &\rho \text{ is a metric}
\end{aligned}$$

□

Problem c

$$
\begin{aligned}
&\{f_n\}_{n=1}^{\infty} \text{ is Cauchy} \\
\Rightarrow &\{f_n\}_{n=1}^{\infty} \text{ converges} \\
&\forall x \in X, d(f_n(x), f_m(x)) \leqslant \rho(f_n, f_m) \\
\Rightarrow &\forall \varepsilon > 0, \exists n_\varepsilon : d(f_n(x), f_m(x)) \leqslant \rho(f_n, f_m) < \varepsilon \\
\Rightarrow &\{f_n(x)\}_{x=1}^{\infty} \text{ converges} \\
&\forall \varepsilon > 0, \exists n_\varepsilon : d(f_n(x), f_m(x)) < \varepsilon \\
&\{f_n(x)\}_{x=1}^{\infty} \text{ is Cauchy}
\end{aligned}
$$