Question 1

Proof.

$$
\begin{aligned}
& L \coloneqq \sqrt{2} \\
& \exists \varepsilon > 0 : \left| \frac{\sqrt{2n}}{\sqrt{n+1}} - L \right| < \varepsilon \\
& \varepsilon \text{ is arbitrary} \\
& \left| \frac{\sqrt{2n}}{\sqrt{n+1}} - L \right| \\
=& \left| \frac{\sqrt{2n}}{\sqrt{n+1}} - \sqrt{2} \right| \\
=& \left| \frac{\sqrt{2n} - \sqrt{2n+2}}{\sqrt{n+1}} \right| \\
=& \left| \frac{(\sqrt{2n} - \sqrt{2n+2})(\sqrt{2n} + \sqrt{2n+2})}{\sqrt{n+1}(\sqrt{2n} + \sqrt{2n+2})} \right| \\
=& \left| \frac{2n - (2n+2)}{\sqrt{n+1}\sqrt{2n} + 2(n+1)} \right| \\
=& \left| \frac{-2}{\sqrt{n+1}\sqrt{2n} + 2(n+1)} \right| \\
=& \frac{2}{\sqrt{n+1}\sqrt{2n} + 2(n+1)} \\
& \varepsilon > 0 \\
\Rightarrow & \exists N \in \mathbb{Z}_+ : N + 1 > \frac{1}{\varepsilon} \\
\Rightarrow & \varepsilon > \frac{1}{N+1} \\
& \frac{1}{N+1} > \frac{1}{(N+1)+1} \\
\Rightarrow & \forall n > N : \varepsilon > \frac{1}{n+1} \\
& \frac{2}{\sqrt{n+1}\sqrt{2n} + 2(n+1)} \\
<& \frac{2}{2(n+1)} = \frac{1}{n+1} \leqslant \frac{1}{N+1} < \varepsilon \\
\Rightarrow & \forall \varepsilon > 0, \exists N > 0 : \forall n > N : \left| \frac{\sqrt{2n}}{\sqrt{n+1}} - \sqrt{2} \right| < \varepsilon
\end{aligned}
$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sqrt{2n}}{\sqrt{n+1}} = \sqrt{2}$$

$\square$

Question 2

Proof.

$$
\begin{aligned}
&L \coloneqq 0 \\
&\exists\varepsilon > 0 : \left|\frac{\cos(n)}{n^2} - L\right| < \varepsilon \\
&\left|\frac{\cos(n)}{n^2} - L\right| = \frac{|\cos(n)|}{n^2} \\
&|\cos(n)| \leqslant 1 \\
\Rightarrow &\frac{|\cos(n)|}{n^2} \leqslant \frac{1}{n^2} \\
&\varepsilon > 0 \\
\Rightarrow &\exists N \in \mathbb{Z}_+ : N^2 > \frac{1}{\varepsilon} \\
\Rightarrow &\frac{1}{N^2} < \varepsilon \\
&\frac{1}{N^2} > \frac{1}{(N+1)^2} \\
\Rightarrow &\forall n \geqslant N, \frac{1}{n^2} < \varepsilon \\
&\frac{|\cos(n)|}{n^2} \leqslant \frac{1}{n^2} \leqslant \frac{1}{N^2} < \varepsilon \\
\Rightarrow &\forall\varepsilon > 0, \exists N > 0 : \forall n > N : \left|\frac{\cos(n)}{n^2} - 0\right| < \varepsilon \\
\Rightarrow &\lim_{n\to\infty} \frac{\cos(n)}{n^2} = 0
\end{aligned}
$$

□

Question 3

Proof.

$$
\begin{aligned}
& 2^n > 0 \wedge n! > 0 \\
\Rightarrow & \frac{2^n}{n!} > 0 \\
& L \coloneqq \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} \\
& L = \lim_{n\to\infty} \frac{2}{n+1} \\
& n \in \mathbb{Z}_+ \\
\Rightarrow & \forall n, n+1 \geqslant 2, n+1 = 2 \Leftrightarrow n = 1 \\
& \frac{2}{n+1} \leqslant 1, \frac{2}{n+1} = 1 \Leftrightarrow n = 1 \\
\Rightarrow & \lim_{n\to\infty} \frac{2}{n+1} < 1 \\
& L < 1 \\
\Rightarrow & \lim_{n\to\infty} \frac{2^n}{n!} = 0
\end{aligned}
$$

□

Question 4

a

$\{x_n\}_{n=1}^{\infty} \coloneqq \{1\}_{n=1}^{\infty} \implies$ constant sequence

It is easy to see that the sequence is convergent

$x_n = 1 \wedge x_{n+1} = 1$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1}{1} = 1$$

b

$\{x_n\}_{n=1}^{\infty} \coloneqq \{n\}_{n=1}^{\infty}$

$n \to \infty : x_n \to \infty \implies$ divergent

$x_{n+1} = n + 1 \wedge x_n = n$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} (1 + \frac{1}{n}) = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n} = 1 + 0 = 1$$

$$\Rightarrow \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1 \wedge \{x_n\}_{n=1}^{\infty} \text{ diverges}$$