Question 1

Proof.

$$
\begin{aligned}
&\text{Suppose: } b > a \\
&\varepsilon := \frac{b-a}{2} \\
&b > a \implies \frac{b-a}{2} > 0 \\
\Rightarrow &\varepsilon > 0 \\
&b - \varepsilon = b - \frac{b-a}{2} = \frac{a+b}{2} \\
&\frac{a+b}{2} > a \\
\Rightarrow &b - \varepsilon > a \\
&b - \varepsilon > a \not\Leftrightarrow a > b - \varepsilon \\
\Rightarrow &a \geqslant b
\end{aligned}
$$

$\square$

Question 2

Proof.

$$
\begin{aligned}
& a^2 - b^2 = a^2 + ab - b^2 - ab \\
& = a(a+b) - b(a+b) \\
& = (a+b)(a-b) \\
& b \coloneqq -a \\
\Rightarrow & a^2 - (-a)^2 = (a + (-a))(a - (-a)) \\
& = 0 \times (a+a) \\
& = 0 \\
\Rightarrow & a^2 - (-a)^2 = 0 \\
\Rightarrow & (-a)^2 = a^2
\end{aligned}
$$

□

Question 3

Proof.

$$
\begin{aligned}
&\forall a, b \in \mathbb{R}: \\
&\frac{1}{2}a, \frac{1}{2}b \in \mathbb{R} \\
\Rightarrow &(\frac{1}{2}a - \frac{1}{2}b)^2 \geqslant 0 \\
&\frac{1}{4}a^2 - \frac{1}{2}ab + \frac{1}{4}b^2 \geqslant 0 \\
&\frac{1}{2}a^2 + \frac{1}{2}b^2 \geqslant \frac{1}{4}a^2 + \frac{1}{2}ab + \frac{1}{4}b^2 \\
&\frac{1}{2}(a^2 + b^2) \geqslant [\frac{1}{2}(a + b)]^2 \\
\\
&\frac{1}{2}(a^2 + b^2) = [\frac{1}{2}(a + b)]^2 \implies a = b: \\
&\frac{1}{2}a^2 + \frac{1}{2}b^2 - [\frac{1}{2}(a + b)]^2 = 0 \\
&\frac{1}{2}a^2 + \frac{1}{2}b^2 - (\frac{1}{4}a^2 + \frac{1}{2}ab + \frac{1}{4}b^2) \\
&\frac{1}{4}a^2 - \frac{1}{2}ab + \frac{1}{4}b^2 = 0 \\
&[\frac{1}{2}(a - b)]^2 = 0 \\
&a - b = 0 \\
&a = b \\
\Rightarrow &\frac{1}{2}(a^2 + b^2) = [\frac{1}{2}(a + b)]^2 \implies a = b \\
\\
&a = b \implies \frac{1}{2}(a^2 + b^2) = [\frac{1}{2}(a + b)]^2: \\
&LHS = \frac{1}{2}(a^2 + a^2) = a^2 \\
&RHS = [\frac{1}{2}(a + a)]^2 = a^2 \\
&LHS = RHS \\
\Rightarrow &\frac{1}{2}(a^2 + b^2) = [\frac{1}{2}(a + b)]^2 \\
\Rightarrow &a = b \implies \frac{1}{2}(a^2 + b^2) = [\frac{1}{2}(a + b)]^2 \\
\Rightarrow &\frac{1}{2}(a^2 + b^2) = [\frac{1}{2}(a + b)]^2 \Leftrightarrow a = b
\end{aligned}
$$

$\square$