

## Question 1

**a**

*Proof.*

Let  $f : S \rightarrow \mathbb{R}$

$f$  is uniformly continuous

$$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0 : \forall x, y \in S, |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

Given a specific  $\varepsilon$

$$\exists (x_i)_0^N : \bigcup_i U_{x_i} = S$$

$$\Rightarrow x_i \in S, \exists \delta > 0 : \forall x \in S, |x - x_i| < \delta \implies |f(x) - f(x_i)| < \varepsilon$$

$$|f(x)| - |f(x_i)| \leq ||f(x)| - |f(x_i)|| \leq |f(x) - f(x_i)|$$

$$\Rightarrow |f(x)| - |f(x_i)| < \varepsilon$$

$$|f(x)| < |f(x_i)| + \varepsilon$$

$$\Rightarrow f(x) < |f(x_i)| + \varepsilon$$

$$\Rightarrow \forall x \in (x_i - \delta, x_i + \delta), f(x) < |f(x_i)| + \varepsilon$$

Same for all the others

$$\Rightarrow \forall x \in S, f(x) < \max(|f(x_i)|) + \varepsilon$$

$$M := \max(f(x_i)) + \varepsilon \in \mathbb{R}$$

$$\Rightarrow \forall x \in S, f(x) < M$$

$f$  is bounded

□

*Proof.*

Suppose  $f$  is uniformly continuous on  $(0, 1)$

$(0, 1)$  is bounded

$\Rightarrow f$  is bounded on  $(0, 1)$

$$x \rightarrow 0^+ : f(x) \rightarrow \infty$$

$\Rightarrow f(x)$  is not bounded

$\nLeftrightarrow$

$\Rightarrow f$  is not uniformly continuous on  $(0, 1)$

□

## Question 2

*Proof.*

$f$  is periodic

Suppose  $p : f(x + p) = f(x)$

$\Rightarrow$  Consider interval  $[0, p]$  :

$f$  is continuous on  $[0, p]$

$\Rightarrow \exists m, n : \forall x \in [0, p] m \leq f(x) \leq n$

$\forall y \in \mathbb{R}, \exists n \in \mathbb{Z}, x \in [0, p] : y = x + np$

$\Rightarrow f(y) = f(x)$

$\forall x \in [0, p], m \leq f(x) \leq n$

$\Rightarrow \forall y \in \mathbb{R} m \leq f(y) \leq n$

$\Rightarrow f$  is bounded

$f$  is continuous on  $[0, p]$

$\Rightarrow f$  is uniformly continuous on  $[0, p]$

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0 : \forall x, y \in [0, p], |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$

$\forall x', y' \in \mathbb{R}, \exists m, n \in \mathbb{Z}, x \in [0, p] : x' = x + mp, y' = y + np$

$\Rightarrow f(x') = f(x), f(y') = f(y)$

$\Rightarrow |f(x') - f(y')| < \varepsilon$

$|x' - y'| = |x - y + (m - n)p| \leq |x - y| + |(m - n)p| < \delta + |(m - n)p|$

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0 : \forall x, y \in \mathbb{R}, |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$

$f$  is uniformly continuous on  $\mathbb{R}$

□

### Question 3

*Proof.*

$f$  is Lipschitz continuous

$$\Rightarrow \forall x, y \in [a, \infty), \exists K : |f(x) - f(y)| \leq K|x - y|$$

$$\begin{aligned} \left| \frac{f(x)}{x} - \frac{f(y)}{y} \right| &= \left| \frac{yf(x) - xf(y)}{xy} \right| \\ &= \left| \frac{yf(x) - xf(y) + xf(x) - xf(x)}{xy} \right| \\ &= \left| \frac{(y-x)f(x) + x(f(x) - f(y))}{xy} \right| \\ &\leq \left| \frac{(y-x)f(x)}{xy} \right| + \left| \frac{x(f(x) - f(y))}{xy} \right| = \frac{|f(x)|}{x} \cdot \frac{|y-x|}{y} + \frac{|f(x) - f(y)|}{y} \\ \Rightarrow \left| \frac{f(x)}{x} - \frac{f(y)}{y} \right| &\leq \frac{|f(x)|}{x} \cdot \frac{|y-x|}{y} + \frac{K|x-y|}{y} = |x-y| \left( \frac{|f(x)|}{xy} + \frac{K}{y} \right) \\ |f(x) - f(y)| &\leq K|x-y| \end{aligned}$$

$$y := a$$

$$\begin{aligned} \Rightarrow |f(x) - f(a)| &\leq K|x-a| \\ \left| \frac{f(x) - f(a)}{x} \right| &\leq \frac{K|x-a|}{x} \\ \left| \frac{f(x)}{x} - \frac{f(a)}{x} \right| &\leq \left| \frac{f(x) - f(a)}{x} \right| \\ \Rightarrow \left| \frac{f(x)}{x} - \frac{f(a)}{x} \right| &\leq \frac{K|x-a|}{x} \\ \frac{f(x)}{x} &\leq \frac{K(x-a)}{x} + \left| \frac{f(a)}{x} \right| = K - \frac{Ka}{x} + \frac{|f(a)|}{x} \end{aligned}$$

$$g(x) := K + \frac{|f(a)| - Ka}{x}$$

$\frac{1}{x}$  is monotonic on  $[a, \infty)$

$$\Rightarrow g(x) \text{ is monotonic on } [a, \infty)$$

$$x \rightarrow a : g(x) \rightarrow \frac{|f(a)|}{a}$$

$$x \rightarrow \infty : g(x) \rightarrow K$$

$$\Rightarrow K - \frac{Ka}{x} + \frac{|f(a)|}{x} \text{ is bounded}$$

$$\Rightarrow M \in \mathbb{R} : \forall x \in [a, \infty), K - \frac{Ka}{x} + \frac{|f(a)|}{x} < M$$

$$\Rightarrow \left| \frac{f(x)}{x} - \frac{f(y)}{y} \right| < |x - y| \left( \frac{M}{y} + \frac{K}{y} \right) < |x - y| \left( \frac{M}{a} + \frac{K}{a} \right)$$

$$\delta := \frac{a\varepsilon}{M + K}$$

$$\Rightarrow \forall \varepsilon > 0, \exists \delta = \frac{a\varepsilon}{M + K} > 0 :$$

$$\forall x, y \in [a, \infty), |x - y| < \delta \implies \left| \frac{f(x)}{x} - \frac{f(y)}{y} \right| < |x - y| \left( \frac{M}{a} + \frac{K}{a} \right) < \varepsilon$$

$$\Rightarrow f \text{ is uniformly continuous on } [a, \infty)$$

□

## Question 4

**a**

$$f(x) = x^2$$

Local Lipschitz continuous :

$$I := [a, b]$$

$$\forall x, y \in [a, b], |x + y| \leq |2b|$$

$$|x + y||x - y| \leq |x - y||2b|$$

$$|x^2 - y^2| = |f(x) - f(y)| \leq |x - y||2b|$$

$$K := |2b|$$

$$\Rightarrow \forall x, y \in [a, b], |f(x) - f(y)| \leq K|x - y|$$

Not Lipschitz continuous :

Suppose  $f$  is Lipschitz continuous

$$\Rightarrow \forall x, y \in \mathbb{R}, \exists K > 0 : |f(x) - f(y)| \leq K|x - y|$$

$$|x^2 - y^2| \leq K|x - y|$$

$$|x + y| \leq K$$

$$x := \frac{K}{2} + 1, y := \frac{K}{2}$$

$$\Rightarrow x, y \in \mathbb{R} : |x + y| > K$$

$\Leftrightarrow$

$\Rightarrow f$  is not Lipschitz continuous

**b**

$$f(x) = \sqrt{x}$$

Hölder continuous :

$$\begin{aligned}
& \text{Consider } 4|x - y| - (x + y - 2\sqrt{xy}) \\
& \text{Suppose } 0 < y < x < 1 \\
& 4x - 4y - x - y + 2\sqrt{xy} \\
& = 3x - 5y + 2\sqrt{xy} \\
& = (3x - 3y) + (2\sqrt{xy} - 2y) \\
& 3x - 3y > 0 \wedge 2\sqrt{xy} - 2y > 0 \\
& \Rightarrow (3x - 3y) + (2\sqrt{xy} - 2y) > 0 \\
& \text{Same for } 0 < x < y < 1 \\
& \text{For } x = y : (3x - 3y) + (2\sqrt{xy} - 2y) = 0 \\
& \Rightarrow 4|x - y| - (x + y - 2\sqrt{xy}) \geq 0 \\
& x + y - 2\sqrt{xy} \leq 4|x - y| \\
& |\sqrt{x} - \sqrt{y}|^2 \leq (2|x - y|^{\frac{1}{2}})^2 \\
& |\sqrt{x} - \sqrt{y}| \leq 2|x - y|^{\frac{1}{2}} \\
& K = 2, \alpha = \frac{1}{2} \\
& \Rightarrow \forall x, y \in [0, 1], \exists K : |g(x) - g(y)| \leq K|x - y|^\alpha, 0 < \alpha < 1
\end{aligned}$$

Not Lipschitz continuous :

Suppose  $g$  is Lipschitz continuous

$$\Rightarrow \forall x, y \in [0, 1], \exists K : |g(x) - g(y)| \leq K|x - y|$$

Consider  $y = 0 \in [0, 1] : |g(x)| \leq K|x|$

$$\sqrt{x} \leq Kx$$

$$x^{-\frac{1}{2}} \leq K$$

$$x := \left(\frac{1}{K+1}\right)^2 \in [0, 1]$$

$$x^{-\frac{1}{2}} = K+1 > K$$

$\Rightarrow \nleftrightarrow$

$\Rightarrow g$  is not Lipschitz continuous

**c**

*Proof.*

Suppose arbitrary  $0 < x < y < 1$

$$\begin{aligned}
x_k &:= x + \frac{k}{n}(y - x), k \in [0, n] \cap \mathbb{Z} \\
|x_{k+1} - x_k| &= \frac{y - x}{n} \\
\Rightarrow \exists K > 0, |h(x_k) - h(x_{k+1})| &\leq K \left(\frac{y - x}{n}\right)^\alpha \\
|h(x) - h(y)| &= |h(x_0) - h(x_n)| \leq \sum_{i=0}^{n-1} |h(x_i) - h(x_{i+1})| \\
\Rightarrow |h(x) - h(y)| &\leq \sum_{i=0}^{n-1} K \left(\frac{y - x}{n}\right)^\alpha \\
|h(x) - h(y)| &\leq nK \left(\frac{y - x}{n}\right)^\alpha \\
nK \left(\frac{y - x}{n}\right)^\alpha &= K(y - x)^\alpha n^{1-\alpha} \\
n \rightarrow \infty : n^{1-\alpha} &\rightarrow 0 \\
\Rightarrow nK \left(\frac{y - x}{n}\right)^\alpha &= 0 \\
|h(x) - h(y)| &\leq 0 \\
\Rightarrow h(x) &= h(y) \\
x, y \text{ are arbitrary} \\
\Rightarrow \forall x, y \in [0, 1], h(x) &= h(y) \\
h \text{ is a constant function}
\end{aligned}$$

□