Question 1

a

Proof.

$$\begin{aligned}
&\forall n \in \mathbb{Z}_+, \frac{1}{n} \geqslant 0\\
\Rightarrow &1 - \frac{1}{n} \leqslant 1\\
&(-1)^n = 1 \text{ or } -1\\
\Rightarrow &(-1)^n = 1(1 - \frac{1}{n}) \leqslant 1\\
\Rightarrow &1 \text{ is an upper bound of } S
\end{aligned}$$

□

b

Proof.

$$\begin{aligned}
&\text{Suppose} : \exists m < 1 \text{ is an upper bound of } S\\
\Rightarrow &\exists n \in \mathbb{R}_+ : m = 1 - \frac{1}{n}\\
\Rightarrow &\exists a, b \in \mathbb{Z}_+ : a \leqslant n < b\\
\Rightarrow &1 - \frac{1}{a} \leqslant 1 - \frac{1}{n} < 1 - \frac{1}{b}\\
&1 - \frac{1}{b} \in S\\
\Rightarrow &\exists x \in S : m < x\\
\Rightarrow &m \text{ is not an upper bound} \not\Leftrightarrow m \text{ is an upper bound}\\
\Rightarrow &m \text{ is an upper bound of } S \implies m \geqslant 1
\end{aligned}$$

□

c

Proof.

$$\begin{aligned} &\text{b} : \forall m \text{ is an upper bound of } S \implies m \geqslant 1 \\ &\text{a} : 1 \text{ is an upper bound of } S \\ \Rightarrow &1 \text{ is the least upper bound of } S \\ \Rightarrow &\sup S = 1 \end{aligned}$$

$\square$

Question 2

Proof.

$$
\begin{aligned}
&\text{case 1} : \sup(A+B) \leqslant \sup(A) + \sup(B)\\
&A+B = \{a, b | a \in A, b \in B\}\\
&c \in A+B \coloneqq a+b, a \in A, b \in B\\
&\forall a \in A, a \leqslant \sup A, \forall b \in B, b \leqslant \sup B\\
\Rightarrow &\forall a \in A, b \in B, c \leqslant \sup A + \sup B\\
\Rightarrow &\sup A + \sup B \text{ is an upper bound of } A+B\\
\Rightarrow &\sup(A+B) \leqslant \sup A + \sup B\\
\\
&\text{case 2} : \sup A + \sup B \leqslant \sup(A+B)\\
&\text{arbitrary } a+b \in A+B\\
\Rightarrow &a+b \leqslant \sup(A+B)\\
&a \leqslant \sup(A+B) - b\\
\Rightarrow &\forall a \in A, a \leqslant \sup(A+B) - b\\
\Rightarrow &\sup(A+B) - b \text{ is an upper bound of } A\\
\Rightarrow &\sup A \leqslant \sup(A+B) - b\\
&b \leqslant \sup(A+B) - \sup A\\
\Rightarrow &\forall b \in B, b \leqslant \sup(A+B) - \sup A\\
\Rightarrow &\sup(A+B) - \sup A \text{ is an upper bound of } B\\
\Rightarrow &\sup B \leqslant \sup(A+B) - \sup A\\
&\sup A + \sup B \leqslant \sup(A+B)\\
\\
&\sup(A+B) \leqslant \sup A + \sup B \land \sup A + \sup B \leqslant \sup(A+B)\\
\Rightarrow &\sup(A+B) = \sup A + \sup B
\end{aligned}
$$

$\square$

Question 3

Proof.

$$
\begin{aligned}
&S \coloneqq \{x^n, n \in \mathbb{Z}_+\} \\
&\text{Supppose} : \exists m : m \text{ is the upper bound of } S \\
\Rightarrow &\forall n \in \mathbb{Z}_+, m \geqslant x^n \\
&\log_x(m) \geqslant n \\
&p \coloneqq \lceil \log_x(m) \rceil + 1 \\
\Rightarrow &p > \log_x(m) \\
\Rightarrow &x^p > x^{\log_x(m)} = m \\
&p = \lceil \log_x(m) \rceil + 1 \\
\Rightarrow &p \in \mathbb{Z}_+ \\
\Rightarrow &x^p \in S \\
\Rightarrow &\exists x^p \in S : x^p > m \nleftrightarrow m \text{ is the upper bound of } S \\
\Rightarrow &S \text{ is not bounded above}
\end{aligned}
$$

□

Question 4

Proof.

$$A \coloneqq \{a_n, n \in \mathbb{Z}_+\}$$
$$B \coloneqq \{b_n, n \in \mathbb{Z}_+\}$$
$$(1,4) = \bigcup_{n=1}^{\infty} I_n, [2,3] = \bigcap_{n=1}^{\infty} I_n$$
$$\Leftrightarrow \exists I_n = [a_n, b_n) : \inf A = 1 \notin A, \inf B = 3 \in B, \sup A = 2 \in A, \sup B = 4 \notin B$$
$$I_n \coloneqq [1 + \frac{1}{n}, 4 - \frac{1}{n})$$
$$A \coloneqq \{1 + \frac{1}{n}, n \in \mathbb{Z}_+\}$$
$$B \coloneqq \{4 - \frac{1}{n}, n \in \mathbb{Z}_+\}$$

$$\inf A = 1 \notin A :$$
$$\forall n \in \mathbb{Z}_+, \frac{1}{n} > 0$$
$$\Rightarrow 1 \notin A, \forall a_n \in A, a_n > 1$$
$$1 \text{ is a lower bound of } A$$
$$\text{Suppose} : \exists p > 1 : p \text{ is a lower bound of } A$$
$$\exists \varepsilon > 0 : p = 1 + \varepsilon$$
$$\varepsilon > 0$$
$$\Rightarrow \exists n_1 \in \mathbb{Z}_+ : \varepsilon < n_1$$
$$\Rightarrow 1 + \frac{1}{\varepsilon} > 1 + \frac{1}{n_1}$$
$$1 + \frac{1}{n_1} \in A$$
$$\Rightarrow \exists a_i \in A : a_i < p \not\Leftrightarrow p \text{ is a lower bound of } A$$
$$\Rightarrow \nexists p > 1 : p \text{ is a lower bound of } A$$
$$\Rightarrow 1 \text{ is the most lower bound of } A$$
$$\inf A = 1$$
$$\Rightarrow \inf A = 1 \notin A$$

$$\inf B = 3 \in B :$$
$$n = 1 : 4 - \frac{1}{1} = 3 \in B$$

$\forall n \in \mathbb{Z}_+, \dfrac{1}{n} \leqslant 1$

$\Rightarrow 4 - \dfrac{1}{n} \geqslant 3$

3 is a lower bound of B

Suppose : $\exists q > 3 : q$ is a lower bound of B

$\exists \varepsilon < 1 : q = 4 - \varepsilon$

$\varepsilon < 1$

$\Rightarrow \exists n_2 \in \mathbb{Z}_+ : \dfrac{1}{\varepsilon} > n_2$

$\Rightarrow 4 - \varepsilon > 4 - \dfrac{1}{n_2}$

$4 - \dfrac{1}{n_2} \in B$

$\Rightarrow \exists b_i \in B : b_i < q \not\Leftrightarrow q$ is a lower bound of B

$\Rightarrow \nexists q > 3 : q$ is a lower bound of B

$\Rightarrow$3 is the most lower bound of B

$\inf B = 3$

$\Rightarrow \inf B = 3 \in B$

$\sup A = 2 \in A :$

$n = 1 : 1 + \dfrac{1}{1} = 2 \in A$

$\forall b \in \mathbb{Z}_+, \dfrac{1}{n} \leqslant 1$

$\Rightarrow 1 + \dfrac{1}{n} \leqslant 2$

2 is an upper bound of A

Suppose : $\exists m < 2$ is a upper bound of A

$\exists \varepsilon < 1 : m = 1 + \varepsilon$

$\varepsilon < 1$

$\exists n_3 \in \mathbb{Z}_+ : \dfrac{1}{\varepsilon} > n_3$

$1 + \varepsilon < 1 + \dfrac{1}{n_3}$

$1 + \dfrac{1}{n_3} \in A$

$\Rightarrow \exists a_i \in A : m < a_i \not\Leftrightarrow m$ is a upper bound of A

$\Rightarrow \nexists m < 2 : m$ is a upper bound of A

$\Rightarrow$2 is the least upper bound of A

$$
\begin{aligned}
&\sup A = 2\\
\Rightarrow &\sup A = 2 \in A
\end{aligned}
$$

$$
\begin{aligned}
&\sup B = 4 \notin B :\\
&\forall n \in \mathbb{Z}_+, \frac{1}{n} > 0\\
\Rightarrow &4 \notin B, \forall b_n \in B, b_n < 4\\
\Rightarrow &4 \text{ is a upper bound of } B\\
&\text{Suppose} : \exists s < 4 \text{ is an upper bound of } B\\
&\exists \varepsilon > 0 : s = 4 - \varepsilon\\
&\varepsilon > 0\\
\Rightarrow &\exists n_4 \in \mathbb{Z}_+ : \varepsilon < n_4\\
\Rightarrow &4 - \frac{1}{\varepsilon} < 4 - \frac{1}{n_4}\\
&4 - \frac{1}{n_4} \in B\\
\Rightarrow &\exists b_i \in B : s < b_i \not\Leftrightarrow s \text{ is an upper bound of } B\\
\Rightarrow &\nexists s < 4 \text{ is an upper bound of } B\\
\Rightarrow &4 \text{ is the least upper bound of } B\\
&\sup B = 4\\
\Rightarrow &\sup B = 4 \notin B
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow &I_n = [a_n, b_n) = [1 + \frac{1}{n}, 4 - \frac{1}{n}) :\\
&(1, 4) = \bigcup_{n=1}^{\infty} I_n, [2, 3] = \bigcap_{n=1}^{\infty} I_n
\end{aligned}
$$

$\square$