# Question 1

## a

*Proof.*

$$
\begin{aligned}
&\forall n \in \mathbb{Z}_+, \frac{1}{n} \geqslant 0 \\
\Rightarrow &1 - \frac{1}{n} \leqslant 1 \\
&2|n+1 \implies (-1)^n = -1 \\
\Rightarrow &(-1)^n(1 - \frac{1}{n}) < 1 \\
&2|n \implies (-1)^n = 1 \\
\Rightarrow &(-1)^n(1 - \frac{1}{n}) \leqslant 1 \\
\\
\Rightarrow &(-1)^n(1 - \frac{1}{n}) \leqslant 1 \\
\Rightarrow &1 \text{ is an upper bound of } S
\end{aligned}
$$

□

## b

*Proof.*

$$
\begin{aligned}
&\text{Suppose} : \exists m < 1 \text{ is an upper bound of } S \\
\Rightarrow &\exists n \in \mathbb{R}_+ : m = 1 - \frac{1}{n} \\
\Rightarrow &\exists a, b \in \mathbb{Z}_+ : a \leqslant n < b \\
\Rightarrow &1 - \frac{1}{a} \leqslant 1 - \frac{1}{n} < 1 - \frac{1}{b} \\
&1 - \frac{1}{b} \in S \\
\Rightarrow &\exists x \in S : m < x \\
\Rightarrow &m \text{ is not an upper bound} \not\Leftrightarrow m \text{ is an upper bound}
\end{aligned}
$$

$$\Rightarrow m \text{ is an upper bound of } S \implies m \geqslant 1$$

□

**c**

*Proof.*

$$\begin{aligned}
&\text{b} : \forall m \text{ is an upper bound of } S \implies m \geqslant 1\\
&\text{a} : 1 \text{ is an upper bound of } S\\
\Rightarrow &1 \text{ is the least upper bound of } S\\
\Rightarrow &\sup S = 1
\end{aligned}$$

□

## Question 2

*Proof.*

$$\begin{aligned}
&\text{case 1} : \sup(A+B) \leqslant \sup(A) + \sup(B) \\
&A + B = \{a, b | a \in A, b \in B\} \\
&c \in A + B \coloneqq a + b, a \in A, b \in B \\
&\forall a \in A, a \leqslant \sup A, \forall b \in B, b \leqslant \sup B \\
\Rightarrow &\forall a \in A, b \in B, c \leqslant \sup A + \sup B \\
\Rightarrow &\sup A + \sup B \text{ is an upper bound of } A + B \\
\Rightarrow &\sup(A+B) \leqslant \sup A + \sup B \\
\\
&\text{case 2} : \sup A + \sup B \leqslant \sup(A+B) \\
&\text{arbitrary } a + b \in A + B \\
\Rightarrow &a + b \leqslant \sup(A+B) \\
&a \leqslant \sup(A+B) - b \\
\Rightarrow &\forall a \in A, a \leqslant \sup(A+B) - b \\
\Rightarrow &\sup(A+B) - b \text{ is an upper bound of } A \\
\Rightarrow &\sup A \leqslant \sup(A+B) - b \\
&b \leqslant \sup(A+B) - \sup A \\
\Rightarrow &\forall b \in B, b \leqslant \sup(A+B) - \sup A \\
\Rightarrow &\sup(A+B) - \sup A \text{ is an upper bound of } B \\
\Rightarrow &\sup B \leqslant \sup(A+B) - \sup A \\
&\sup A + \sup B \leqslant \sup(A+B) \\
\\
&\sup(A+B) \leqslant \sup A + \sup B \wedge \sup A + \sup B \leqslant \sup(A+B) \\
\Rightarrow &\sup(A+B) = \sup A + \sup B
\end{aligned}$$

$\square$

## Question 3

*Proof.*

$$
\begin{aligned}
&S \coloneqq \{x^n, n \in \mathbb{Z}_+\}\\
&\text{Supppose}: \exists m : m \text{ is the upper bound of } S\\
\Rightarrow &\forall n \in \mathbb{Z}_+, m \geqslant x^n\\
&\log_x(m) \geqslant n\\
&p \coloneqq \lceil \log_x(m) \rceil + 1\\
\Rightarrow &p > \log_x(m)\\
\Rightarrow &x^p > x^{\log_x(m)} = m\\
&p = \lceil \log_x(m) \rceil + 1\\
\Rightarrow &p \in \mathbb{Z}_+\\
\Rightarrow &x^p \in S\\
\Rightarrow &\exists x^p \in S : x^p > m \nleftrightarrow m \text{ is the upper bound of } S\\
\Rightarrow &S \text{ is not bounded above}
\end{aligned}
$$

□

## Question 4

### Approach 1

*Proof.*

$$
\begin{aligned}
&A \coloneqq \{a_n, n \in \mathbb{Z}_+\}\\
&B \coloneqq \{b_n, n \in \mathbb{Z}_+\}\\
&(1,4) = \bigcup_{n=1}^{\infty} I_n, [2,3] = \bigcap_{n=1}^{\infty} I_n\\
\Leftrightarrow &\exists I_n = [a_n, b_n) : \inf A = 1 \notin A, \inf B = 3 \notin B, \sup A = 2 \in A, \sup B = 4 \in B\\
&I_n \coloneqq [1 + \frac{1}{n}, 4 - \frac{1}{n})\\
&A \coloneqq \{1 + \frac{1}{n}, n \in \mathbb{Z}_+\}\\
&B \coloneqq \{3 + \frac{1}{n}, n \in \mathbb{Z}_+\}
\end{aligned}
$$

$$
\begin{aligned}
&\inf A = 1 \notin A :\\
&\forall n \in \mathbb{Z}_+, \frac{1}{n} > 0\\
\Rightarrow &1 \notin A, \forall a_n \in A, a_n > 1\\
&1 \text{ is a lower bound of } A\\
&\text{Suppose} : \exists p > 1 : p \text{ is a lower bound of } A\\
&\exists \varepsilon > 0 : p = 1 + \varepsilon\\
&\varepsilon > 0\\
\Rightarrow &\exists n_1 \in \mathbb{Z}_+ : \varepsilon < n_1\\
\Rightarrow &1 + \frac{1}{\varepsilon} > 1 + \frac{1}{n_1}\\
&1 + \frac{1}{n_1} \in A\\
\Rightarrow &\exists a_i \in A : a_i < p \not\Leftrightarrow p \text{ is a lower bound of } A\\
\Rightarrow &\nexists p > 1 : p \text{ is a lower bound of } A\\
\Rightarrow &1 \text{ is the most lower bound of } A\\
&\inf A = 1\\
\Rightarrow &\inf A = 1 \notin A
\end{aligned}
$$

$\inf B = 3 \notin B:$

$\forall n \in \mathbb{Z}_+, \dfrac{1}{n} > 0$

$\Rightarrow 3 \notin B, \forall b_n \in B, b_n > 3$

$\Rightarrow$3 is a lower bound of $B$

Suppose : $\exists s > 3$ is a lower bound of $B$

$\exists \varepsilon > 0 : s = 3 + \varepsilon$

$\varepsilon > 0$

$\Rightarrow \exists n_4 \in \mathbb{Z}_+ : \varepsilon < n_4$

$\Rightarrow 3 + \dfrac{1}{\varepsilon} > 3 + \dfrac{1}{n_4}$

$3 + \dfrac{1}{n_4} \in B$

$\Rightarrow \exists b_i \in B : s > b_i \not\Leftrightarrow s$ is a lower bound of $B$

$\Rightarrow \nexists s > 3$ is a lower bound of $B$

$\Rightarrow$3 is the most lower bound of $B$

$\inf B = 3$

$\Rightarrow \inf B = 3 \notin B$

$\sup A = 2 \in A:$

$n = 1 : 1 + \dfrac{1}{1} = 2 \in A$

$\forall b \in \mathbb{Z}_+, \dfrac{1}{n} \leqslant 1$

$\Rightarrow 1 + \dfrac{1}{n} \leqslant 2$

2 is an upper bound of $A$

Suppose : $\exists m < 2$ is a upper bound of $A$

$\exists \varepsilon < 1 : m = 1 + \varepsilon$

$\varepsilon < 1$

$\exists n_3 \in \mathbb{Z}_+ : \dfrac{1}{\varepsilon} > n_3$

$1 + \varepsilon < 1 + \dfrac{1}{n_3}$

$1 + \dfrac{1}{n_3} \in A$

$\Rightarrow \exists a_i \in A : m < a_i \not\Leftrightarrow m$ is a upper bound of $A$

$\Rightarrow \nexists m < 2 : m$ is a upper bound of $A$

$\Rightarrow$2 is the least upper bound of $A$

$$
\begin{aligned}
& \sup A = 2 \\
\Rightarrow & \sup A = 2 \in A \\
\\
& \sup B = 4 \in B \\
& n = 1 : 3 + \frac{1}{1} = 3 \in B \\
& \forall n \in \mathbb{Z}_+, \frac{1}{n} \leqslant 1 \\
\Rightarrow & 3 + \frac{1}{n} \leqslant 4 \\
& 4 \text{ is a lower bound of } B \\
& \text{Suppose} : \exists q < 4 : q \text{ is an upper bound of } B \\
& \exists \varepsilon < 1 : q = 3 + \varepsilon \\
& \varepsilon < 1 \\
\Rightarrow & \exists n_2 \in \mathbb{Z}_+ : \frac{1}{\varepsilon} > n_2 \\
\Rightarrow & 3 + \varepsilon < 3 + \frac{1}{n_2} \\
& 3 + \frac{1}{n_2} \in B \\
\Rightarrow & \exists b_i \in B : q < b_i \not\Leftrightarrow q \text{ is an upper bound of } B \\
\Rightarrow & \nexists q > 3 : q \text{ is an upper bound of } B \\
\Rightarrow & 4 \text{ is the least upper bound of } B \\
& \sup B = 4 \\
\Rightarrow & \sup B = 4 \in B \\
\\
\Rightarrow & I_n = [a_n, b_n) = [1 + \frac{1}{n}, 4 - \frac{1}{n}) : \\
& (1, 4) = \bigcup_{n=1}^{\infty} I_n, [2, 3] = \bigcap_{n=1}^{\infty} I_n
\end{aligned}
$$

□

## Approach 2

*Proof.*

$$A \coloneqq \{a_n, n \in \mathbb{Z}_+\}$$

$$B \coloneqq \{b_n, n \in \mathbb{Z}_+\}$$

$$(1,4) = \bigcup_{n=1}^{\infty} I_n, [2,3] = \bigcap_{n=1}^{\infty} I_n$$

$$I_n \coloneqq [1 + \frac{1}{n}, 4 - \frac{1}{n})$$

$$A \coloneqq \{1 + \frac{1}{n}, n \in \mathbb{Z}_+\}$$

$$B \coloneqq \{3 + \frac{1}{n}, n \in \mathbb{Z}_+\}$$

$$\{1 + \frac{1}{n}\}_{n=1}^{\infty} \to 1$$

$$n = 1 : 3 + \frac{1}{n} = 4$$

$$\text{Since } \{3 + \frac{1}{n}\}_{n=1}^{\infty} \to 3$$

$$\Rightarrow \max(3 + \frac{1}{n}) = 4$$

$$\Rightarrow (1,4) = \bigcup_{n=1}^{\infty} I_n$$

$$n = 1 : 1 + \frac{1}{n} = 2$$

$$\text{Since } \{1 + \frac{1}{n}\}_{n=1}^{\infty} \to 1$$

$$\Rightarrow \max(1 + \frac{1}{n}) = 2$$

$$\{3 + \frac{1}{n}\}_{n=1}^{\infty} \to 3$$

$$\Rightarrow [2,3] = \bigcap_{n=1}^{\infty} I_n$$

$$\Rightarrow (1,4) = \bigcup_{n=1}^{\infty} I_n, [2,3] = \bigcap_{n=1}^{\infty} I_n$$

$\square$