Question 1

a

Proof.

$$\begin{aligned}
&\forall x^t \in E[x,y], t < y \implies x^t < x^y \\
\Rightarrow &x^y \text{ is an upper bound for } E[x,y] \\
&\text{Suppose } \exists x^m : x^m < x^y \text{ is an upper bound of } E[x,y] \\
\Rightarrow &m < y \\
\Rightarrow &x^m \in E[x,y] \nLeftrightarrow x^m \text{ is an upper bound of } E[x,y] \\
\Rightarrow &x^y \text{ is the smallest upper bound} \\
\Rightarrow &x^y = \sup E[x,y]
\end{aligned}$$

□

b

Proof.

$$\begin{aligned}
&\exists y' > y \in \mathbb{R} \\
\Rightarrow &x^{y'} > x^t \forall x^t \in E[x,y] \\
\Rightarrow &x^{y'} \text{ is an upper bound of } E[x,y] \\
\Rightarrow &E[x,y] \text{ is bounded}
\end{aligned}$$

□

c

Proof.

$$
\begin{aligned}
&E[x, y+z] \coloneqq \{x^t | t < y+z, t \in \mathbb{R}\}\\
&t \coloneqq t_1 + t_2, t_1 \leqslant y, t_2 \leqslant z\\
&t \leqslant y+z\\
\Rightarrow &x^t \in E[x, y+z]\\
\\
&x^{y+z} \leqslant x^y x^z :\\
&x^y x^z = \sup E[x, y] \sup E[x, z]\\
&x^t = x^{t_1} x^{t_2} \leqslant x^y x^z\\
&t \text{ is arbitrary}\\
\Rightarrow &x^y x^z \text{ is an upper bound of } E[x, y+z]\\
\Rightarrow &x^{y+z} = \sup E[x, y+z] \leqslant x^y x^z\\
\\
&x^y x^z \leqslant x^{y+z} :\\
&x^t \leqslant \sup E[x, y+z] = x^{y+z}\\
&x^t = x^{t_1} x^{t_2}\\
&t_1 \text{ and } t_2 \text{ are arbitrary}\\
\Rightarrow &t_1 = y, t_2 = z\\
&x^y x^z \leqslant x^{y+z}\\
\\
\Rightarrow &x^y x^z = x^{y+z}\\
\\
&\text{Injective :}\\
&t_1 \leqslant t_2\\
&x^{t_1} = x^{t_2}\\
\Rightarrow &x^{t_2 - t_1} = 1\\
&\text{Suppose } t_2 - t_1 \neq 0\\
\Rightarrow &\exists t : t \in (0, t_2 - t_1)\\
&x^{t_2 - t_1} = \sup E[x, t_2 - t_1] \neq 1 \nLeftrightarrow x^{t_2 - t_1} = 1\\
\Rightarrow &t_2 - t_1 = 0\\
&t_1 = t_2
\end{aligned}
$$

□

Question 2

Proof.

$$((x^{\frac{1}{n}}-1)+1)^n \geqslant 1+n(x^{\frac{1}{n}}-1)$$
$$x \geqslant 1+n(x^{\frac{1}{n}}-1)$$
$$x-1 \geqslant n(x^{\frac{1}{n}}-1)$$
$$x^{\frac{1}{n}} \leqslant \frac{x-1}{n}-1$$
$$\text{set } t>0 \wedge n>\frac{x-1}{t-1}$$
$$t>\frac{x-1}{n}+1$$
$$\Rightarrow x^{\frac{1}{n}}<t$$

$$A(z) \coloneqq \{w \in \mathbb{R} | x^w < z\}, y \coloneqq \sup A(z)$$
$$\text{case 1}:$$
$$z>x^y$$
$$t \coloneqq \frac{z}{x^y}>1$$
$$\exists n: n>\frac{x-1}{t-1}$$
$$x^{\frac{1}{n}}<t=\frac{z}{x^y}$$
$$x^{y+\frac{1}{n}}<z$$
$$\Rightarrow y+\frac{1}{n}>y \in A(z) \not\Leftrightarrow y=\sup A(z)$$

$$\text{case 2}:$$
$$z<x^y$$
$$t \coloneqq \frac{x^y}{z}>1$$
$$\exists n: n>\frac{x-1}{t-1}$$
$$x^{\frac{1}{n}}<t=\frac{x^y}{z}$$
$$x^{y-\frac{1}{n}}>z$$
$$y=\sup A(z)$$
$$\Rightarrow \exists w: t-\frac{1}{n}<w$$

$$\Rightarrow x^{y-\frac{1}{n}} < x^w < z \not\Leftrightarrow x^{y-\frac{1}{n}} > z$$

$$\Rightarrow x^y = z$$

$\square$

Question 3

Proof.

$$
\begin{aligned}
&(x_n) \coloneqq \sqrt{2}, \sqrt{2\sqrt{2}}, ... \\
&x_n = 2^{\sum_{k=1}^{n}(\frac{1}{2})^k} \\
\\
&n = 1: \\
&x_1 = 2^{\frac{1}{2}} = \sqrt{2} \\
&IH : x_{n+1} = \sqrt{2x_n} \\
&LHS = 2^{\sum_{k=1}^{n+1}(\frac{1}{2})^k} \\
&RHS = \sqrt{2\sqrt{2^{\sum_{k=1}^{n}(\frac{1}{2})^k}}} \\
&\ = 2^{\frac{1}{2}+\frac{1}{2}\sum_{k=1}^{n}(\frac{1}{2})^k} \\
&\ = 2^{\sum_{k=1}^{n+1}(\frac{1}{2})^k} \\
&LHS = RHS \\
\Rightarrow &x_n = 2^{\sum_{k=1}^{n}(\frac{1}{2})^k} \\
&\{\sum_{k=1}^{n}(\frac{1}{2})^k\}_{n\to\infty} \to 1 \\
\to &\{2^{\sum_{k=1}^{n}(\frac{1}{2})^k}\}_{n\to\infty} \to 2 \\
\Rightarrow &(x_n) \text{ is convergent and has limit 2}
\end{aligned}
$$

□

Question 4

Proof.

(x_n) is convergent $\to$ (x_n) has a convergent subsequence :
trivial

(x_{n_k}) is an increasing subsequence of (x_n)
$(x_{n_k})_{k\to\infty} \to x$
$\Rightarrow \forall \varepsilon > 0, \exists K : \forall k \geqslant K, x - \varepsilon < x_{n_k} < x + \varepsilon$

case 1 :
$k \geqslant K$
$x_k \leqslant x_{n_k} < x + \varepsilon$

case 2 :
$k \geqslant n_K$
$x_k \geqslant x_{n_K} > x - \varepsilon$

$\Rightarrow k \geqslant \max\{K, n_K\}, x - \varepsilon < x_k < x + \varepsilon$
$\Rightarrow$convergent

$\Rightarrow (x_n)$ is convergent $\Leftrightarrow$ (x_n) has a convergent subsequence

$\square$

Question 5

Proof.

convergent $\rightarrow$ bounded
trivial : every convergent sequence is bounded

bounded $\rightarrow$ convergent
(x_n) is monotone and bounded
$\Rightarrow \exists (x_{n_k})$ is a subsequence and convergent
$\Rightarrow (x_n)$ is convergent

$\Rightarrow$a monotone sequence is convergent if and only if it is bounded

$\square$