Question 1

a

Proof.

$$
\begin{aligned}
&\text{Let } f : S \to \mathbb{R} \\
&f \text{ is uniformly continuous} \\
\Rightarrow&\forall \varepsilon > 0, \exists \delta > 0 : \forall x, y \in S, |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \\
&\text{Given a specific } \varepsilon \\
&\exists (x_i)_0^N : \bigcup_i U_{x_i} = S \\
\Rightarrow&x_i \in S, \exists \delta > 0 : \forall x \in S, |x - x_i| < \delta \implies |f(x) - f(x_i)| < \varepsilon \\
&|f(x)| - |f(x_i)| \leqslant ||f(x)| - |f(x_i)|| \leqslant |f(x) - f(x_i)| \\
\Rightarrow&|f(x)| - |f(x_i)| < \varepsilon \\
&|f(x)| < |f(x_i)| + \varepsilon \\
\Rightarrow&f(x) < |f(x_i) + \varepsilon \\
\Rightarrow&\forall x \in (x_i - \delta, x_i + \delta), f(x) < |f(x_i)| + \varepsilon \\
&\text{Same for all the others} \\
\Rightarrow&\forall x \in S, f(x) < \max(|f(x_i)|) + \varepsilon \\
&M \coloneqq \max(f(x_i)) + \varepsilon \in \mathbb{R} \\
\Rightarrow&\forall x \in S, f(x) < M \\
&f \text{ is bounded}
\end{aligned}
$$

□

b

Proof.

$$
\begin{aligned}
&\text{Suppose } f \text{ is uniformly continuous on } (0, 1) \\
&(0, 1) \text{ is bounded} \\
\Rightarrow&f \text{ is bounded on } (0, 1) \\
&x \to 0^+ : f(x) \to \infty
\end{aligned}
$$

$\Rightarrow f(x)$ is not bounded

$\not\Leftrightarrow$

$\Rightarrow f$ is not uniformly continunous on $(0, 1)$

$\square$

Question 2

Proof.

f is periodic

Suppose $p : f(x+p) = f(x)$

$\Rightarrow$Consider interval $[0,p]$:

f is continuous on $[0,p]$

$\Rightarrow \exists m, n : \forall x \in [0,p] m \leqslant f(x) \leqslant n$

$\forall y \in \mathbb{R}, \exists n \in \mathbb{Z}, x \in [0,p] : y = x + np$

$\Rightarrow f(y) = f(x)$

$\forall x \in [0,p], m \leqslant f(x) \leqslant n$

$\Rightarrow \forall y \in \mathbb{R} m \leqslant f(y) \leqslant n$

$\Rightarrow f$ is bounded

f is continuous on $[0,p]$

$\Rightarrow f$ is uniformly continuous on $[0,p]$

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0 : \forall x, y \in [0,p], |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon$

$\forall x', y' \in \mathbb{R}, \exists m, n \in \mathbb{Z}, x \in [0,p] : x' = x + mp, y' = y + np$

$\Rightarrow f(x') = f(x), f(y') = f(y)$

$\Rightarrow |f(x') - f(y')| < \varepsilon$

$|x' - y'| = |x - y + (m-n)p| \leqslant |x-y| + |(m-n)p| < \delta + |(m-n)p|$

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0 : \forall x, y \in \mathbb{R}, |x-y| < \delta \implies |f(x) - f(y)| < \varepsilon$

f is uniformly continuous on $\mathbb{R}$

□

Question 3

Proof.

$$
\begin{aligned}
& f \text{ is Lipschitz continuous} \\
\Rightarrow & \forall x, y \in [a, \infty), \exists K : |f(x) - f(y)| \leqslant K|x - y| \\
& \left|\frac{f(x)}{x} - \frac{f(y)}{y}\right| = \left|\frac{yf(x) - xf(y)}{xy}\right| \\
& = \left|\frac{yf(x) - xf(y) + xf(x) - xf(x)}{xy}\right| \\
& = \left|\frac{(y - x)f(x) + x(f(x) - f(y))}{xy}\right| \\
& \leqslant \left|\frac{(y - x)f(x)}{xy}\right| + \left|\frac{x(f(x) - f(y))}{xy}\right| = \frac{|f(x)|}{x} \cdot \frac{|y - x|}{y} + \frac{|f(x) - f(y)|}{y} \\
\Rightarrow & \left|\frac{f(x)}{x} - \frac{f(y)}{y}\right| \leqslant \frac{|f(x)|}{x} \cdot \frac{|y - x|}{y} + \frac{K|x - y|}{y} = |x - y| \left(\frac{|f(x)|}{xy} + \frac{K}{y}\right) \\
& |f(x) - f(y)| \leqslant K|x - y| \\
& y \coloneqq a \\
\Rightarrow & |f(x) - f(a)| \leqslant K|x - a| \\
& \left|\frac{f(x) - f(a)}{x}\right| \leqslant \frac{K|x - a|}{x} \\
& \left|\frac{f(x)}{x}\right| - \left|\frac{f(a)}{x}\right| \leqslant \left|\frac{f(x) - f(a)}{x}\right| \\
\Rightarrow & \left|\frac{f(x)}{x}\right| - \left|\frac{f(a)}{x}\right| \leqslant \frac{K|x - a|}{x} \\
& \frac{f(x)}{x} \leqslant \frac{K(x - a)}{x} + \left|\frac{f(a)}{x}\right| = K - \frac{Ka}{x} + \frac{|f(a)|}{x} \\
& g(x) \coloneqq K + \frac{|f(a)| - Ka}{x} \\
& \frac{1}{x} \text{ is monotonic on } [a, \infty) \\
\Rightarrow & g(x) \text{ is monotonic on } [a, \infty) \\
& x \to a : g(x) \to \frac{|f(a)|}{a} \\
& x \to \infty : g(x) \to K \\
\Rightarrow & K - \frac{Ka}{x} + \frac{|f(a)|}{x} \text{ is bounded} \\
\Rightarrow & M \in \mathbb{R} : \forall x \in [a, \infty), K - \frac{Ka}{x} + \frac{|f(a)|}{x} < M
\end{aligned}
$$

$$\Rightarrow \left|\frac{f(x)}{x}-\frac{f(y)}{y}\right| < |x-y|\left(\frac{M}{y}+\frac{K}{y}\right) < |x-y|\left(\frac{M}{a}+\frac{K}{a}\right)$$

$$\delta \coloneqq \frac{a\varepsilon}{M+K}$$

$$\Rightarrow \forall \varepsilon > 0, \exists \delta = \frac{a\varepsilon}{M+K} > 0 :$$

$$\forall x, y \in [a,\infty), |x-y| < \delta \implies \left|\frac{f(x)}{x}-\frac{f(y)}{y}\right| < |x-y|\left(\frac{M}{a}+\frac{K}{a}\right) < \varepsilon$$

$\Rightarrow f$ is uniformly continuous on $[a,\infty)$

$\square$

Question 4

a

$$
\begin{aligned}
&f(x) = x^2\\
\\
&\text{Local Lipschitz continuous :}\\
&I \coloneqq [a, b]\\
&\forall x, y \in [a, b], |x + y| \leqslant |2b|\\
&|x + y||x - y| \leqslant |x - y||2b|\\
&|x^2 - y^2| = |f(x) - f(y)| \leqslant |x - y||2b|\\
&K \coloneqq |2b|\\
\Rightarrow &\forall x, y \in [a, b], |f(x) - f(y)| \leqslant K|x - y|\\
\\
&\text{Not Lipschitz continuous :}\\
&\text{Suppose } f \text{ is Lipschitz continuous}\\
\Rightarrow &\forall x, y \in \mathbb{R}, \exists K > 0 : |f(x) - f(y)| \leqslant K|x - y|\\
&|x^2 - y^2| \leqslant K|x - y|\\
&|x + y| \leqslant K\\
&x \coloneqq \frac{K}{2} + 1, y \coloneqq \frac{K}{2}\\
\Rightarrow &x, y \in \mathbb{R} : |x + y| > K\\
&\not\Leftrightarrow\\
\Rightarrow &f \text{ is not Lipschitz continuous}
\end{aligned}
$$

b

$$
\begin{aligned}
&f(x) = \sqrt{x}\\
\\
&\text{Hölder continuous :}
\end{aligned}
$$

Consider $4|x-y|-(x+y-2\sqrt{xy})$
Suppose $0<y<x<1$
$4x-4y-x-y+2\sqrt{xy}$
$=3x-5y+2\sqrt{xy}$
$=(3x-3y)+(2\sqrt{xy}-2y)$
$3x-3y>0 \wedge 2\sqrt{xy}-2y>0$
$\Rightarrow (3x-3y)+(2\sqrt{xy}-2y)>0$
Same for $0<x<y<1$
For $x=y:(3x-3y)+(2\sqrt{xy}-2y)=0$
$\Rightarrow 4|x-y|-(x+y-2\sqrt{xy})\geqslant 0$
$x+y-2\sqrt{xy}\leqslant 4|x-y|$
$|\sqrt{x}-\sqrt{y}|^2\leqslant(2|x-y|^{\frac{1}{2}})^2$
$|\sqrt{x}-\sqrt{y}|\leqslant 2|x-y|^{\frac{1}{2}}$
$K=2,\alpha=\dfrac{1}{2}$
$\Rightarrow \forall x,y\in[0,1],\exists K:|g(x)-g(y)|\leqslant K|x-y|^{\alpha},0<\alpha<1$

Not Lipschitz continuous :
Suppose g is Lipschitz continuous
$\Rightarrow \forall x,,y\in[0,1],\exists K:|g(x)-g(y)|\leqslant K|x-y|$
Consider $y=0\in[0,1]:|g(x)|\leqslant K|x|$
$\sqrt{x}\leqslant Kx$
$x^{-\frac{1}{2}}\leqslant K$
$x:=(\dfrac{1}{K+1})^2\in[0,1]$
$x^{-\frac{1}{2}}=K+1>K$
$\Rightarrow \not\Leftrightarrow$
$\Rightarrow g$ is not Lipschitz continuous

c

Proof.

Suppose arbitrary $0<x<y<1$

$x_k \coloneqq x + \frac{k}{n}(y-x), k \in [0,n] \cap \mathbb{Z}$

$|x_{k+1} - x_k| = \frac{y-x}{n}$

$\Rightarrow \exists K > 0, |h(x_k) - h(x_{k+1})| \leqslant K(\frac{y-x}{n})^\alpha$

$|h(x) - h(y)| = |h(x_0) - h(x_n)| \leqslant \sum_{i=0}^{n-1} |h(x_i) - h(x_{i+1})|$

$\Rightarrow |h(x) - h(y)| \leqslant \sum_{i=0}^{n-1} K(\frac{y-x}{n})^\alpha$

$|h(x) - h(y)| \leqslant nK(\frac{y-x}{n})^\alpha$

$nK(\frac{y-x}{n})^\alpha = K(y-x)^\alpha n^{1-\alpha}$

$n \to \infty : n^{1-\alpha} \to 0$

$\Rightarrow nK(\frac{y-x}{n})^\alpha = 0$

$|h(x) - h(y)| \leqslant 0$

$\Rightarrow h(x) = h(y)$

x, y are arbitrary

$\Rightarrow \forall x, y \in [0,1], h(x) = h(y)$

h is a constant function

$\square$