Question 1

Proof.

$$
\begin{aligned}
&\omega \text{ is a lower bound of } S \\
\Rightarrow &\forall x \in S, \omega \leqslant x \\
&\forall x > \omega \in \mathbb{R} : \omega \in S \\
\Rightarrow &x \nleqslant n, \forall n \in S \\
\Rightarrow &\forall x > \omega \in \mathbb{R} : x \text{ is not a lower bound of } S \\
\Rightarrow &\omega \text{ is the greatest lower bound of } S \\
\Rightarrow &\omega = \inf S
\end{aligned}
$$

□

Question 2

Proof.

$$
\begin{aligned}
&\inf A \coloneqq \omega \\
&\forall x \in -A, -x \in A \\
\Rightarrow &\omega \leqslant -x \\
&x \leqslant -\omega \\
\Rightarrow &\omega \text{ is an upper bound of } -A \\
&\exists \beta \neq \omega \text{ is any upper bound of } -A \\
\Rightarrow &\forall x \in -A, x \leqslant \beta \\
&-\beta \leqslant -x \\
\Rightarrow &\forall -x \in A, -\beta \leqslant -x \\
&-\beta \text{ is a lower bound of } A \\
&\omega = \inf A \\
\Rightarrow &-\beta \leqslant \omega \\
&-\omega \leqslant \beta \\
&\beta \text{ is arbitrary } \implies \forall \beta \text{ is an upper bound of } -A, -\omega \leqslant \beta \\
\Rightarrow &-\omega = \sup(-A) \\
\Rightarrow &-\inf A = \sup(-A)
\end{aligned}
$$

□

Question 3

a

Proof.

$$
\begin{aligned}
&\text{Suppose: } \sup(A \cap B) > \sup A, \sup B \\
\Rightarrow &\exists m \in A \cap B : \forall a \in A, b \in B, m > a, b \\
\Rightarrow &m \notin A \land m \notin B \\
&\forall n \in A \cap B, n \in A \land n \in B \\
\Rightarrow &m \notin A \cap B \\
&m \notin A \cap B \not\Leftrightarrow m \in A \cap B \\
\Rightarrow &\sup(A \cap B) \leqslant \sup A, \sup B
\end{aligned}
$$

□

b

$$
\begin{aligned}
&A \coloneqq [0, 1) \\
&B \coloneqq \{1\} \\
&\sup A = 1 \\
&\sup B = 1 \\
&A \cap B = \emptyset \\
\Rightarrow &\nexists \sup(A \cap B) \\
\Rightarrow &\sup A = \sup B \neq \sup(A \cap B)
\end{aligned}
$$