# Question 1

## a

*Proof.*

$$
\begin{aligned}
& f(x+y) = f(x) + f(y) \\
\Rightarrow & \lim_{x\to 0} f(x) = \lim_{x\to 0}\left(f\left(\frac{x}{2}\right)\right) + \lim_{x\to 0}\left(f\left(\frac{x}{2}\right)\right) = 2\lim_{x\to 0}\left(f\left(\frac{x}{2}\right)\right) \\
& y \coloneqq \frac{x}{2} \\
& y \to 0 \\
\Rightarrow & \lim_{y\to 0} f(y) = \lim_{x\to 0} f(x) = L \\
\Rightarrow & L = 2L \\
& L = 0
\end{aligned}
$$

□

## b

*Proof.*

$$
\begin{aligned}
& \text{Suppose } \varepsilon \text{ small enough and } c \text{ any number } \in \mathbb{R} \\
& f(c+\varepsilon) = f(c) + f(\varepsilon) \\
& \lim_{\varepsilon\to 0} f(c+\varepsilon) = \lim_{\varepsilon\to 0}(f(c) + f(\varepsilon)) = f(c) + \lim_{\varepsilon\to 0} f(\varepsilon) \\
& \text{from a, } \lim_{\varepsilon\to 0} f(\varepsilon) = 0 \\
\Rightarrow & \lim_{\varepsilon\to 0} f(c+\varepsilon) = f(c) \\
& \varepsilon \to 0 = c + \varepsilon \to c \\
& x \coloneqq c + \varepsilon \\
\Rightarrow & \lim_{x\to c} f(x) = f(c) \\
\Rightarrow & \forall c \in \mathbb{R}, \lim_{x\to c} f(x) = f(c)
\end{aligned}
$$

□

### c

*Proof.*

$$
\begin{aligned}
& x \in \mathbb{Q}: \\
\Rightarrow & x := \frac{p}{q} \\
& f\left(\frac{p}{q}\right) = f\left(\frac{1}{q} + ...\frac{1}{q}\right) = p \cdot f\left(\frac{1}{q}\right) \\
& textBythesamemethod : f(1) = f\left(\frac{q}{q}\right) = q \cdot f\left(\frac{1}{q}\right) \\
\Rightarrow & f\left(\frac{1}{q}\right) = \frac{1}{q} f(1) \\
\Rightarrow & f\left(\frac{p}{q}\right) = \frac{p}{q} f\left(1\right) \\
\Rightarrow & \forall x \in \mathbb{Q} : f(x) = f(1)x \\
\\
& x \in \mathbb{R} \setminus \mathbb{Q}: \\
& \exists (x_n) \in \mathbb{Q} : n \to \infty, (x_n) \to x \\
\Rightarrow & \lim_{n\to\infty} x_n = x \\
& \text{By b} : \lim_{n\to\infty} f(x_n) = f(x) \\
& x_n \text{ is rational} \\
& \text{By previous case} : f(x_n) = f(1)x_n \\
\Rightarrow & \lim_{n\to\infty} f(x_n) = f(1) \lim_{n\to\infty} x_n \\
& \lim_{n\to\infty} x_n = x \\
\Rightarrow & \lim_{n\to\infty} f(x_n) = f(1)x \\
& \lim_{n\to\infty} f(x_n) = f(x) \\
\Rightarrow & f(x) = f(1)x \\
\\
& \forall x \in \mathbb{R}, f(x) = f(1)x
\end{aligned}
$$

□

## Question 2

*Proof.*

$$
\begin{aligned}
&x \in S \\
\Rightarrow &f(x) \leqslant g(x) \\
&\text{Since } f, g \text{ are continuous and } c \text{ is a cluster point} \\
\Rightarrow &\lim_{x \to c} f(x) = f(c) \\
&\lim_{x \to c} g(x) = g(c) \\
&f(x) \leqslant g(x) \\
&f, g \text{ are continuous on } S, c \text{ is a cluster point} \\
\Rightarrow &\lim_{x \to c} f(x) \leqslant \lim_{x \to c} g(x) \\
\Rightarrow &f(c) \leqslant g(c) \\
&c \in S
\end{aligned}
$$

□

## Question 3

*Proof.*

$$
\begin{aligned}
&\text{Suppose } f \text{ is not continuous}\\
\Rightarrow &\exists \varepsilon > 0, \forall \delta > 0, |x-c| < \delta : |f(x) - f(c)| \geqslant \varepsilon\\
\\
&x > c :\\
&f(x) \geqslant f(c)\\
&|f(x) - f(c)| \geqslant \varepsilon\\
\Rightarrow &f(x) \geqslant f(c) + \varepsilon\\
&x - c < \delta\\
\Rightarrow &x < c + \delta\\
&f(x) \leqslant f(c+\delta)\\
&IVP : \exists z \in [c, x] : f(z) = f(c) + \frac{\varepsilon}{2}\\
\Rightarrow &f(c) + \frac{\varepsilon}{2} < f(x) \leqslant f(c+\delta)\\
&\text{take } \delta \text{ small enough :}\\
&\exists \varepsilon > 0, \exists \delta > 0 : f(c+\delta) - f(c) < \frac{\varepsilon}{2} < \varepsilon\\
&\not\Leftrightarrow\\
\\
&x < c :\\
&f(x) \leqslant f(c)\\
&|f(x) - f(c)| \geqslant \varepsilon\\
\Rightarrow &f(x) \leqslant f(c) - \varepsilon\\
&c - x < \delta\\
\Rightarrow &x > c - \delta\\
&f(x) \geqslant f(c-\delta)\\
&IVP : \exists z \in [x, c] : f(z) = f(c) - \frac{\varepsilon}{2}\\
\Rightarrow &f(c-\delta) \leqslant f(x) < f(c) - \frac{\varepsilon}{2}\\
&\text{take } \delta \text{ small enough :}\\
&\exists \varepsilon > 0, \exists \delta > 0 : f(c) - f(c-\delta) < \frac{\varepsilon}{2} < \varepsilon\\
&\not\Leftrightarrow\\
\\
&f \text{ is continuous}
\end{aligned}
$$

$\square$