Question 1

Proof.

$$
\begin{aligned}
&\text{Suppose } m > n \\
&|x_m - x_n| \leqslant \sum_{i=n}^{m-1} |x_{i+1} - x_i| \\
&\sum_{i=n}^{m-1} |x_{i+1} - x_i| \leqslant \sum_{i=n}^{m-1} 2^{-i} \\
&\sum_{i=n}^{m-1} 2^{-i} = 2^{-n} \left(\frac{1 - 2 - 2^{-(m-n)}}{1 - 2^{-1}} \right) = 2^{-(n-1)} - 2^{-(m+1)} < 2^{-(n-1)} \\
&\{2^{-n}\}_{n \to \infty} \to 0 \\
\Rightarrow &\forall \varepsilon >, \exists n : 0 < 2^{n} < 2^{-(n-1)} < \varepsilon \\
&N \coloneqq n - 1 \\
\Rightarrow &m > n > N, |x_m - x_n| < 2^{N} < \varepsilon \\
&m, n, N \text{ are arbitrary} \\
\Rightarrow &\forall \varepsilon > 0, \exists N : \forall m, n > N \in \mathbb{N} : |x_m - x_n| < \varepsilon \\
&(x_n) \text{ is cauchy}
\end{aligned}
$$

$\square$

Question 2

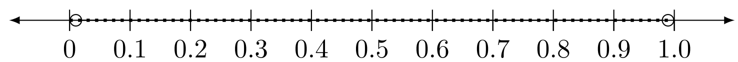
Proof.

$$
\begin{aligned}
&(S_k) \coloneqq \left(\sum_{i=0}^{k} \frac{x^n}{n!}\right) \\
&\text{Suppose } p > q \\
&|S_p - S_q| \\
=&\left|\sum_{i=0}^{p} \frac{x^i}{i!} - \sum_{i=0}^{q} \frac{x^i}{i!}\right| \\
=&\left|\sum_{i=q+1}^{p} \frac{x^i}{i!}\right| \\
&\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \frac{x}{n+1} \\
&\left\{\frac{x}{n+1}\right\}_{n\to\infty} \to 0 \\
\to&\exists n \text{ large enough} : \forall m > 0, \frac{x^n}{n!} > \frac{x^{m+n}}{(m+n)!} \\
\Rightarrow&\exists N : \forall p > q > N, \left|\sum_{i=q+1}^{p} \frac{x^i}{i!}\right| < (p-q)\left|\frac{x^q}{q!}\right| \\
&\left\{(p-q)\left|\frac{x^q}{q!}\right|\right\}_{q\to\infty} \to 0 \\
\Rightarrow&\forall \varepsilon > 0, \exists q \in \mathbb{N} : (p-q)\left|\frac{x^q}{q!}\right| < \varepsilon \\
&p, q \text{ are arbitrary} \\
\Rightarrow&\forall \varepsilon > 0, \exists N \in \mathbb{N} : p > q \geqslant N, |S_p - S_q| < \varepsilon \\
&x \text{ is arbitrary in this case} \\
\Rightarrow&\sum_{i=0}^{k} \frac{x^n}{n!} \text{ is convergent for any } x
\end{aligned}
$$

□

Question 3

(a)



(b)

Proof.

$$
\begin{aligned}
&(S_n) \coloneqq \left(\frac{1}{2} + 7 \times 10^{-(n+1)}\right)\\
\Rightarrow &S \subseteq E\\
&\left\{7 \times 10^{-(n+1)}\right\}_{n\to\infty} \to 0\\
\Rightarrow &\lim_{n\to\infty} 7 \times 10^{-(n+1)} = 0\\
&\lim_{n\to\infty} \frac{1}{2} + 7 \times 10^{-(n+1)} = \frac{1}{2}\\
&\forall n, \frac{1}{2} \neq S_n\\
\Rightarrow &\frac{1}{2} \text{ is a cluster point of } E
\end{aligned}
$$

$\square$

Question 4

Proof.

$$
\begin{aligned}
&\delta : 0 < |x+1| < \delta \\
&\left| \frac{x^4 - 2x^3 + x^2 - 1}{3x^6 + x^3 + 1} - 1 \right| \\
=&\left| \frac{-3x^6 + x^4 - 3x^3 + x^2 - 2}{3x^6 + x^3 + 1} \right| \\
=&\left| \frac{(x+1)(-3x^5 + 3x^4 - 2x^3 - x^2 + 2x - 2)}{3x^6 + x^3 + 1} \right| \\
=&|x+1| \left| \frac{-3x^5 + 3x^4 - 2x^3 - x^2 + 2x - 2}{3x^6 + x^3 + 1} \right| \\
&\text{set } |x+1| < \frac{1}{10} \\
\Rightarrow& x \in (-1.1, -0.9) \\
\Rightarrow& \frac{-3x^5 + 3x^4 - 2x^3 - x^2 + 2x - 2}{3x^6 + x^3 + 1} < 1.3 \\
&|x+1| \left| \frac{-3x^5 + 3x^4 - 2x^3 - x^2 + 2x - 2}{3x^6 + x^3 + 1} \right| < 1.3|x+1| \\
&\varepsilon > 0, \delta := \min\left\{ \frac{1}{10}, \frac{\varepsilon}{1.3} \right\} \\
\Rightarrow& \forall \varepsilon > 0, \exists \delta > 0 : 0 < |x+1| < \delta \implies \left| \frac{x^4 - 2x^3 + x^2 - 1}{3x^6 + x^3 + 1} - 1 \right| < \varepsilon
\end{aligned}
$$

$\square$