Question 1

Proof.

$$
\begin{aligned}
& n = 1 : \\
& \frac{1}{\sqrt[3]{1}} = 1 \geqslant 1^{\frac{2}{3}} \\
& \text{Suppose: } \sum_{n=1}^{k} \frac{1}{\sqrt[3]{n}} \geqslant k^{\frac{2}{3}} \\
& k + 1 : \\
& LHS = \sum_{n=1}^{k+1} \frac{1}{\sqrt[3]{n}} \\
& RHS = (k+1)^{\frac{2}{3}} \\
& \Delta LHS = \frac{1}{\sqrt[3]{k+1}} = (k+1)^{-\frac{1}{3}} \\
& \Delta RHS = (k+1)^{\frac{2}{3}} - k^{\frac{2}{3}} \\
& \frac{\Delta RHS}{\Delta LHS} = ((k+1)^{\frac{2}{3}} - k^{\frac{2}{3}})(k+1)^{\frac{1}{3}} \\
& = k + 1 - (k+1)^{\frac{1}{3}} k^{\frac{2}{3}} \\
& k < (k+1)^{\frac{1}{3}} k^{\frac{2}{3}} < k + 1 \\
\Rightarrow & 0 < k + 1 - (k+1)^{\frac{1}{3}} k^{\frac{2}{3}} < 1 \\
& 0 < \frac{\Delta RHS}{\Delta LHS} < 1 \\
\Rightarrow & \Delta LHS > \Delta RHS \\
& \sum_{n=1}^{k} \frac{1}{\sqrt[3]{n}} \geqslant k^{\frac{2}{3}} \\
\Rightarrow & \sum_{n=1}^{k} \frac{1}{\sqrt[3]{n}} + \Delta LHS \geqslant k^{\frac{2}{3}} + \Delta RHS \\
\Rightarrow & \sum_{n=1}^{k+1} \frac{1}{\sqrt[3]{n}} \geqslant (k+1)^{\frac{2}{3}} \\
& \frac{1}{\sqrt[3]{1}} \geqslant 1^{\frac{2}{3}} \wedge (\sum_{n=1}^{k} \frac{1}{\sqrt[3]{n}} \geqslant k^{\frac{2}{3}} \implies \sum_{n=1}^{k+1} \frac{1}{\sqrt[3]{n}} \geqslant (k+1)^{\frac{2}{3}}) \\
\Rightarrow & \forall n \in \mathbb{N} : \sum_{i=1}^{n} \frac{1}{\sqrt[3]{i}} \geqslant n^{\frac{2}{3}}
\end{aligned}
$$

□

Question 2

Proof.

$$
\begin{aligned}
&n = 0 : \\
&x_1 = \frac{1}{8}{x_0}^2 + 2 \\
&= \frac{9}{8} + 2 = 3\frac{1}{8} \\
&3 < 3\frac{1}{8} < 4 \\
&3 < x_1 < 4 \\
&\text{Suppose: } x_k < x_{k+1} < 4 \\
&k + 1 : \\
&x_{k+2} = \frac{1}{8}{x_{k+1}}^2 + 2 \\
&x_{k+2} - x_{k+1} = \frac{1}{8}{x_{k+1}}^2 - x_{k+1} + 2 = \frac{1}{8}(x_{k+1} - 4)^2 \\
&x_{k+1} < 4 \\
\Rightarrow &\frac{1}{8}(x_{k+1} - 4)^2 > 0 \\
\Rightarrow &x_{k+1} < x_{k+2} \\
&\text{Suppose: } x_{k+2} \geqslant 4 \\
&\frac{1}{8}{x_{k+1}}^2 + 2 \geqslant 4 \\
&\frac{1}{8}{x_{k+1}}^2 - 2 \geqslant 0 \\
&\frac{1}{8}(x_{k+1} + 4)(x_{k+1} - 4) \geqslant 0 \\
&x_{k+1} + 4 > 0 \\
\Rightarrow &x_{k+1} - 4 \geqslant 0 \\
&x_{k+1} \geqslant 4 \not\Leftrightarrow x_{k+1} < 4 \\
\Rightarrow &x_{k+2} < 4 \\
\Rightarrow &x_{k+1} < x_{k+2} < 4 \\
&3 < x_1 < 4 \wedge (x_k < x_{k+1} < 4 \implies x_{k+1} < x_{k+2} < 4) \\
\Rightarrow &\forall n \in \mathbb{N} \cup \{0\} : x_n < x_{n+1} < 4
\end{aligned}
$$

□

Question 3

Proof.

$$
\begin{aligned}
& m = 2 \wedge k = 1 : \\
& F_3 = 2 \\
& F_1F_1 + F_2F_2 = 1 + 1 = 2 \\
\Rightarrow & F_3 = F_1F_1 + F_2F_2 \\
& \text{Suppose: } F_{p+q} = F_{p-1}F_q + F_pF_{q+1} \\
& p + 1 : \\
& F_{p+1+q} = F_{p+q} + F_{p-1+q} \\
& F_{p+q} = F_{p-1}F_q + F_pF_{q+1} \\
& F_{p-1+q} = F_{p-2}F_q + F_{p-1}F_{q+1} \\
& F_{p+q} + F_{p-1+q} = F_{p-1}F_q + F_pF_{q+1} + F_{p-2}F_q + F_{p-1}F_{q+1} \\
& = (F_{p-1} + F_{p-2})F_q + (F_p + F_{p-1})F_{q+1} \\
& = F_pF_q + F_{p+1}F_{q+1} \\
\Rightarrow & F_{p+1+q} = F_{p+1-1}F_q + F_{p+1}F_{q+1} \\
& F_3 = F_1F_1 + F_2F_2 \\
& \wedge (F_{p+q} = F_{p-1}F_q + F_pF_{q+1} \implies F_{p+1+q} = F_{p+1-1}F_q + F_{p+1}F_{q+1}) \\
\Rightarrow & \forall m \in \mathbb{N}, m \geqslant 2 : F_{m+q} = F_{m-1}F_q + F_mF_{q+1} \\
& q + 1 : \\
& F_{p+q+1} = F_{p+q} + F_{p+q-1} \\
& F_{p+q} = F_{p-1}F_q + F_pF_{q+1} \\
& F_{p+q-1} = F_{p-1}F_{q-1} + F_pF_q \\
& F_{p+q} + F_{p+q-1} = F_{p-1}F_q + F_pF_{q+1} + F_{p-1}F_{q-1} + F_pF_q \\
& = F_{p-1}(F_q + F_{q-1}) + F_p(F_{q+1} + F_q) \\
& = F_{p-1}F_{q+1} + F_pF_{q+2} \\
\Rightarrow & F_{p+q+1} = F_{p-1}F_{q+1} + F_pF_{q+1+1} \\
& F_3 = F_1F_1 + F_2F_2 \\
& \wedge (F_{p+q} = F_{p-1}F_q + F_pF_{q+1} \implies F_{p+q+1} = F_{p-1}F_{q+1} + F_pF_{q+1+1}) \\
\Rightarrow & \forall k \in \mathbb{N}, F_{q+k} = F_{p-1}F_k + F_pF_{k+1} \\
\Rightarrow & \forall m, k \in \mathbb{N}, m \geqslant 2 : F_{m+k} = F_{m-1}F_k + F_mF_{k+1}
\end{aligned}
$$

□

Question 4

Proof.

$P_N \coloneqq \{a_k x^k + a_{k-1}x^{k-1} + ... + a_1 x + a_0 = 0 | k + |a_k| + ... + |a_0| = N\}$

$k, a_0, ..., a_k \in \mathbb{Z}^+$

$\Rightarrow P_N$ is finite

$Z_N \coloneqq \{z | \exists p(x) \in P_N : p(z) = 0\}$

k is finite

$\Rightarrow$There are k complex roots for polynomials of degree k

P_N is finite

$\Rightarrow$Only finite combinations of $p(x) \in P_N$

$\Rightarrow$Finite roots for $p(x) \in P_N$

$\Rightarrow Z_N$ is finite

$\Rightarrow \bigcup_{N \geqslant 0} Z_N$ is countable

$\bigcup_{N \geqslant 0} Z_N$ are all the algebraic numbers

$\Rightarrow$The set of algebraic numbers are countably infinite

$\square$