Dummit and Foote Exercises

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Chapter 10

Introduction to Module Theory

10.1 Basic Definitions and Examples

In these exercises R is a ring with 1 and M is a left R-module.

Exercise 10.1.1. Prove that $0m = 0$ and $(-1)m = -m$ for all $m \in M$.

Solution: We have via straightforward application of the module axioms that

$$0m = (0-0)m = 0m - 0m = 0.$$

Likewise, we can compute that

$$(-1)m = -m + m + (-1)m = -m + (1)m + (-1)m = -m + (1-1)m = -m - 0m = -m.$$

□

Exercise 10.1.2. Prove that $R^\times$ and M satisfy the two axioms in Section 1.7 for a *group action* of the multiplicative group $R^\times$ on the set M.

Solution: We know that $R^\times$ is a group, and by the module axioms we know $1 \cdot m = m$ for all $m \in M$ and hence the identity acts on M in accordance with a group action. We also have via the module axioms that $uv \cdot m = u \cdot (v \cdot m)$ for all $u, v \in R^\times$, and so the action of $R^\times$ satisfies both axioms of a group action.

□

Exercise 10.1.3. Assume that $rm = 0$ for some $r \in R$ and some $m \in M$ with $m \neq 0$. Prove that r does not have a left inverse (i.e., there is no $s \in R$ such that $sr = 1$).

Solution: Suppose otherwise, so that there exists $s \in R$ so that $sr = 1$. Then we have that

$$m = (sr)m = s(rm) = s0 = 0$$

a contradiction. □

Exercise 10.1.4. Let M be the module R^n described in Example 3 and let $I_1, I_2, \dots, I_n$ be left ideals of R. Prove that the following are submodules of M:

(a) $\{(x_1, x_2, \dots, x_n) \mid x_i \in I_i\}$

(b) $\{(x_1, x_2, \dots, x_n) \mid x_i \in R \text{ and } x_1 + x_2 + \dots + x_n = 0\}$.

Solution: (a)
The set is clearly nonempty since $(0, 0, \dots, 0)$ is in it. The second condition of the submodule criterion is also satisfied since

$$(x_1, x_2, \dots, x_n) + r(x_1', x_2', \dots, x_n') = (x_1 + rx_1', x_2 + rx_2', \dots, x_n + rx_n')$$

for any $r \in R$ and $x_i + rx_i' \in I$ by virtue of I being an ideal. Thus the set is a submodule.

(b)
As in (a) we notice that $(0, 0, \dots, 0)$ is in the set, and so it is nonempty. Letting $x = (x_1, \dots, x_n)$ and $y = (x_1', \dots, x_n')$ be two elements of the set we have that $x + ry$ is in the set since

$$\begin{aligned}(x_1 + rx_1') + (x_2 + rx_2') + \cdots + (x_n + rx_n') &= (x_1 + x_2 + \cdots + x_n) + r(x_1' + x_2' + \cdots + x_n') \\ &= 0 + r0 \\ &= 0.\end{aligned}$$

Thus the set satisfies the submodule criterion and is a submodule.

□

Exercise 10.1.5. For any left ideal I of R define

$$IM = \{\sum_{\text{finite}} a_i m_i \mid a_i \in I, m_i \in M\}$$

to be the collection of all finite sums of elements of the form am where $a \in I$ and $m \in M$. Prove that IM is a submodule of M.

Solution: Note that $0_M \in IM$ since $0_R \in I$ and $0_M \in M$ so $0_M = 0_R \cdot 0_M \in IM$. Now let $x = \sum a_i m_i$ and $y = \sum b_j m_j$ be two elements of IM. Then notice for any $r \in R$ that

$$x + ry = \sum a_i m_i + \sum rb_j m_j$$

which is again in IM since both sums are finite and $rb_j \in I$ by virtue of I being a left ideal. Thus IM satisfies the submodule criterion and is a submodule. □

Exercise 10.1.6. Show that the intersection of any nonempty collection of submodules of an R-module is a submodule.

Solution: Let M be an R-module and let $\{N_\alpha\}$ be an arbitrary collection of submodules of M. Let $N = \cap_\alpha N_\alpha$. Notice that N is nonempty since each N_α must contain zero by virtue of being a subgroup over the overall module. Then let $x, y \in N$. Since each N_α is a submodule we have $x + ry \in N_\alpha$ for all $r \in R$ and all α. We conclude that $x + ry \in N$ and so N satisfies the submodule criterion. This proves the result. □

Exercise 10.1.7. Let $N_1 \subseteq N_2 \subseteq \cdots$ be an ascending chain of submodules of M. Prove that $\cup_{i=1}^{\infty} N_i$ is a submodule of M.

Solution: Let $N = \cup_{i=1}^{\infty} N_i$. Note that $0 \in N$ so N is nonempty. Then let $x, y \in N$. There must exist N_i so that $x, y \in N_i$ and by virtue of N_i being a submodule we will have $x + ry \in N_i$ for all $r \in R$ and hence $x + ry \in N$. This proves that N is a submodule.

□

Exercise 10.1.8. An element m of the R-module M is called a *torsion element* if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted

$$\mathrm{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

(a) Prove that if R is an integral domain then $\mathrm{Tor}(M)$ is a submodule of M (called the *torsion* submodule of M).

(b) Give an example of a ring R and an R-module M such that $\mathrm{Tor}(M)$ is not a submodule. [Consider the torsion elements in the R-module R.]

(c) If R has zero divisors show that every nonzero R-module has nonzero torsion elements.

Solution: (a)
Let R be an integral domain and observe that $\mathrm{Tor}(M)$ is nonempty since it contains zero. Then let $x, y \in \mathrm{Tor}(M)$ and let $r_1, r_2 \in R$ be nonzero so that $r_1 x = 0$ and $r_2 y = 0$. For an arbitrary $r \in R$ we can notice that

$$r_1 r_2(x + ry) = r_1 r_2 x + r_1 r_2 r y = r_2 r_1 x + r_1 r r_2 y = r_2 \cdot 0 + r_1 r \cdot 0 = 0 + 0 = 0$$

where above we have used the commutativity of R. Furthermore observe that $r_1 r_2$ is nonzero since R is an integral domain, and so $x + ry \in \mathrm{Tor}(M)$. This proves that $\mathrm{Tor}(M)$ is a submodule by the submodule criterion.

(b)
Consider $\mathbb{Z}/6\mathbb{Z}$. The torsion elements of this ring as a module over itself are $\{0, 2, 3, 4\}$ which do not even form an additive subgroup, much less a submodule.

(c)
Suppose R has zero divisors and let $x, y \in R$ be nonzero so that $xy = 0$. Then for some nonzero $m \in M$ consider ym. If $ym = 0$ then m is a nonzero torsion element. Otherwise ym is a nonzero torsion element since $x(ym) = (xy)m = 0m = 0$.

□

Exercise 10.1.9. If N is a submodule of M, the *annihilator of N in R* is defined to be $\{r \in R \mid rn = 0 \text{ for all } n \in N\}$. Prove that the annihilator of N in R is a 2-sided ideal of R.

Solution: Let N be a submodule and let I be its annihilator. Clearly I contains 0 and so is nonempty. Furthermore if $a, b \in I$ then $a - b \in I$ since for any $n \in N$ we have

$$(a - b)n = an + (-b)n = 0 - (bn) = 0 - 0 = 0$$

where above we have used the fact that $(-b)n = -(bn)$ which can be proved analogously to property 2 in Problem 1. Thus I is an additive subgroup of R.

Finally let $r \in R$ be arbitrary and let $a \in I$. Clearly $ra \in I$ since

$$ran = r(an) = r0 = 0$$

for any $n \in N$. We also have $ar \in I$ since

$$arn = a(rn) = 0$$

for any $n \in N$, where above we have used that $an \in N$. This proves that I is a 2-sided ideal in R. □

Exercise 10.1.10. If I is a right ideal of R, the *annihilator of I in M* is defined to be $\{m \in M \mid am = 0 \text{ for all } a \in I\}$. Prove that the annihilator of I in M is a submodule of M.

Solution: Let I be a right ideal of R and let N be its annihilator. Notice immediately that $0 \in N$ since $an = 0$ for all $a \in I$. Then let $n, n' \in N$ and $r \in R$. We have that

$$\begin{aligned} a(n + rn') &= an + arn' \\ &= 0 + (ar)n' \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

where above we have used that $ar \in I$ by virtue of I being a right ideal. This proves that N satisfies the submodule criterion, and so it is a submodule. □

Exercise 10.1.11. Let M be the abelian group (i.e., $\mathbb{Z}$-module) $\mathbb{Z}/24\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z} \times \mathbb{Z}/50\mathbb{Z}$.

(a) Find the annihilator of M in $\mathbb{Z}$ (i.e. a generator for this principal ideal).

(b) Let $I = 2\mathbb{Z}$. Describe the annihilator of I in M as a direct product of cyclic groups.

Solution: (a)
Notice that if $r \in \mathbb{Z}$ annihilates M it must annihilate each coordinate. In particular, it must be a multiple of 24, of 15, and of 50. This condition is both necessary and sufficient and so the annihilator of M is $600\mathbb{Z}$, the ideal generated by the least common multiple of 24, 15, and 50.

(b)
The ideal $2\mathbb{Z}$ annihilates 0 and 12 in the first coordinate, 0 in the second coordinate, and 0 and 25 in the third coordinate. Hence the annihilator of $2\mathbb{Z}$ is the set

$$\{(0, 0, 0), (12, 0, 0), (0, 0, 25), (12, 0, 25)\}$$

which as a direct product of cyclic groups is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

□

Exercise 10.1.12. In the notation of the preceding exercises prove the following facts about annihilators.

(a) Let N be a submodule of M and let I be its annihilator in R. Prove that the annihilator of I in M contains N. Give an example where the annihilator of I in M does not equal N.

(b) Let I be a right ideal of R and let N be its annihilator in M. Prove that the annihilator of N in R contains I. Give an example where the annihilator of N in R does not equal I.

Solution: (a)
Let A be the annihilator of I in M and let $n \in N$. Then $an = 0$ for all $a \in I$ by definition. But this means that $n \in A$. This proves that $N \subseteq A$ as desired. As an example where containment is strict let $M = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ be a $\mathbb{Z}$-module and let N be the subgroup $\{(0, 0), (1, 0)\}$. Notice that $2\mathbb{Z}$ is the annihilator of N, but the annihilator of $2\mathbb{Z}$ is all of M.

(b)
Let J be the annihilator of N in R and let $a \in I$. Then $an = 0$ for all $n \in N$. But then by definition

$a \in J$, and so $I \subseteq J$ as desired. An example where containment is strict occurs when considering the annihilator of $6\mathbb{Z}$ in the $\mathbb{Z}$-module $M = N = \mathbb{Z}/2\mathbb{Z}$. This ideal annihilates all of M, but the annihilator of M is $2\mathbb{Z}$ which strictly contains $6\mathbb{Z}$.

$\square$

Exercise 10.1.13. Let I be an ideal of R. Let M' be the subset of elements a of M that are annihilatored by some power, I^k of the ideal I, where the power may depend on a. Prove that M' is a submodule of M. [Use Excercise 7.]

Solution: Let N_k be the annihilator of I^k. Elements of I^k are of the form $\sum a_i^k$ where the sum is finite and each a_i is an element of I. We thus notice that $N_k \subseteq N_{k+1}$ since if n is annihilated by all finite sums $\sum a_i^k$ with $a_i \in I$ then

$$\left(\sum a_i^{k+1}\right) n = \sum(a_i^{k+1}n) = \sum(a_i a_i^k n) = \sum(a_i 0) = 0$$

and so it is also annihilated by elements of I^{k+1}. Thus the union of all N_k is a submodule by Exercise 7. This union is exactly M', proving the desired result. $\square$

Exercise 10.1.14. Let z be an element of the center of R, i.e. $zr = rz$ for all $r \in R$. Prove that zM is a submodule of M, where $zM = \{zm \mid m \in M\}$. Show that if R is the ring of 2×2 matrices over a field and e is the matrix with a 1 in position 1, 1 and zeros elsewhere then eR is *not* a left R-submodule (where $M = R$ is considered as a left R-module as in Example 1)—in this case the matrix e is not in the center of R.

Solution: Note that $0 = z0 \in zM$ and so zM is nonempty. Letting $zx, zy \in zM$ where $x, y \in M$ are abitrary and letting $r \in R$ we have that

$$zx + rzy = zx + zry = z(x + ry) \in zM$$

and so zM satisfies the submodule criterion.

Notice that

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$$

and so in the example eM is the set of matrices with zero entries in the bottom row and arbitrary entries in the top row. This collection is not a submodule since as a set it is not invariant under the left action of R on it. In particular,

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

which is not a matrix with zero entries in the bottom row. We conclude that e is indeed not in the center of R.

$\square$

Exercise 10.1.15. If M is a finite abelian group then M is naturally a $\mathbb{Z}$-module. Can this action be extended to make M into a $\mathbb{Q}$-module?

Solution: No, not always. Consider the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$. If this were naturally a $\mathbb{Q}$-module then it would have some element $\frac{1}{2} \cdot 1$. This element would satisfy

$$\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = \left(\frac{1}{2} + \frac{1}{2}\right) \cdot 1 = 1 \cdot 1 = 1$$

and in particular it would have order at least three as an element of the group $\mathbb{Z}/2\mathbb{Z}$. This is not possible. More generally, for any finite abelian group G one can consider the action of $\frac{1}{|G|}$ to derive a contradiction. Thus finite abelian group never has a $\mathbb{Q}$ action compatible with the natural $\mathbb{Z}$ action.

However, if an abelian group is divisible then we can extend its natural $\mathbb{Z}$ action to a $\mathbb{Q}$ action. Of course nonzero divisible abelian groups are necessarily infinite, so this falls outside the scope of the problem. □

Exercise 10.1.16. Prove that the submodules U_k describe in the example of $F[x]$-modules are all of the $F[x]$-submodules for the shift operator.

Solution: Let $V = F^n$ be a $F[x]$ module where x acts as the shift operator and F acts as normal. Let $U \subseteq V$ be a submodule of V. Let k be the largest index such that there exists a vector in U whose k-th coordinate is nonzero. Then we claim $U = U_k$. The inclusion $U \subseteq U_k$ is trivial since U_k is all vectors in V where coordinates following the k-th are zero. Hence we only have to show $U_k \subseteq U$.

To show that $U_k \subseteq U$ we will show straightforwardly that e_i is in U for $1 \leq i \leq k$. The set of these e_i forms a basis for U_k and so it will follow that $U_k \subseteq U$. Notice that we really only need to construct e_k, since all e_i for $i < k$ can be obtained by the action of x, which will still be in U since U is a submodule. To construct e_k, let $v = (v_1, v_2, \ldots, v_k, 0, 0, \ldots, 0)$ be a vector in U where $v_k \neq 0$. Then we can construct the basis vector e_k by repeatedly zeroing out smaller coordinates in v_k: first consider

$$v - \left(\frac{v_{k-1}}{v_k} x\right) v \in U.$$

The $(k-1)$-th coordinate of this vector will be $v_k - v_k = 0$. We can repeat this process, acting on our new vector by x^2 multiplied by an appropriate scalar, subtracting the result, and so on. This eventually leads to a vector $(0, 0, \ldots, 0, v_k, 0, 0, \ldots, 0)$ which can be transformed to e_k via multiplication by the scalar $\frac{1}{v_k}$. This proves that $e_k \in U$, and as previously discussed this implies that $e_i \in U$ for all $1 \leq i \leq k$. Hence $U_k \subseteq U$ and we are done. □

Exercise 10.1.17. Let T be the shift operator on the vector space V and let $e_1, \ldots, e_n$ be the usual basis vector described in the example of $F[x]$-modules. If $m \geq n$ find $(a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0)e_n$.

Solution: For convenience let $p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0$. We compute directly that

$$\begin{aligned}
p(x) \cdot e_n &= \left(\sum_{i=0}^{m} a_i x^i\right) \cdot e_n \\
&= \sum_{i=0}^{m} a_i (x^i \cdot e_n) && \text{Via module axioms} \\
&= \sum_{i=0}^{n} a_i (x^i \cdot e_n) && \text{Since } x^i \cdot e_n = 0 \text{ for } i > n \\
&= \sum_{i=0}^{n} a_i (e_{n-i}) && \text{Since } x \text{ acts as shift operator} \\
&= (a_n, a_{n-1}, \ldots, a_1, a_0).
\end{aligned}$$

Thus $p(x) \cdot e_n$ gives us the first $n+1$ coefficients in $p(x)$ in a vector in reverse order. □

Exercise 10.1.18. Let $F = \mathbb{R}$. Let $V = \mathbb{R}^2$ and let T be the linear transformation from V to V which is rotation clockwise about the origin by $\pi/2$ radians. Show that V and 0 are the only $F[x]$-submodules for this T.

Solution: It suffices to show that every nontrivial submodule is equal to V. Given a nontrivial submodule U, let v be a nonzero vector in U. Then notice that $x \cdot v \in U$ is linearly independent from v. Since U must also be a subspace of the vector space V, we see that U contains $\text{span}\{v, x \cdot v\} = V$. Hence U is all of V. □

Exercise 10.1.19. Let $F = \mathbb{R}$, let $V = \mathbb{R}^2$ and let T be the linear transformation from V to V which is projection onto the y-axis. Show that $V, 0$, the x-axis and the y-axis are the only $F[x]$-submodules for this T.

Solution: We know that 0 and V are always submodules. It remains to characterize the nontrivial proper submodules. Notice that such submodules are necessarily 1-dimensional subspaces of $V = \mathbb{R}^2$ since submodules under the action of $F[x]$ are always subspaces and 0- and 2-dimensional subspaces are trivial and non-proper submodules respectively.

Let $U = \text{span}\{v\}$ be some nontrivial proper submodule. Since U is 1-dimensional we must have that $x \cdot v = ax$ for some scalar a. In particular v is an eigenvector of T and so U is an eigenspace of T. The only eigenspaces are clearly the x and y axes. One can verify quickly that these are submodules: they both are subspaces (in particular subgroups) of V and are invariant under the action of $F[x]$ since the y-axis is only scaled and the x-axis is annihilated by any nonunits in $F[x]$. □

Exercise 10.1.20. Let $F = \mathbb{R}$, let $V = \mathbb{R}^2$ and let T be the linear transformation from V to V which is rotation clockwise about the origin by π radians. Show that *every* subspace of V is an $F[x]$ submodule for this T.

Solution: Rotating by π radians is the same as additive negation. Hence we have $x \cdot v = -v$ for all vectors v. Being invariant under the action of F and x is enough to be a submodule, and subspaces are invariant under both by the definition of being a subspace (and hence an additive subgroup). Thus all subspaces are submodules. □

Exercise 10.1.21. Let $n \in \mathbb{Z}^+$, $n > 1$ and let R be the ring of $n \times n$ matrices with entries from a field F. Let M be the set of $n \times n$ matrices with arbitrary elments of F in the first column and zeros elsewhere. Show that M is a submodule of R when R is considered as a left module over itself, but M is not a submodule of R when R is considered as a right R-module.

Solution: It is clear that M is an additive subgroup of the module R. When R acts on M from the left M is invariant since the i-th column of rm for $r \in R$ and $m \in M$ is just the product of r with the i-th column in m. For $i > 1$ this column is zero and so must be r's product with it. Hence $rm \in M$.

On the other hand when R acts from the right the columns in mr beyond the first may nonzero, as illustrated by the small example below.

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \notin M.$$

□

Exercise 10.1.22. Suppose that A is a ring with identity 1_A that is a (unital) left R-module satisfying $r \cdot (ab) = (r \cdot a)b = a(r \cdot b)$ for all $r \in R$ and $a, b \in A$. Prove that the map $f : R \to A$ defined by $f(r) = r \cdot 1_A$ is a ring homomorphism mapping 1_R to 1_A and $f(R)$ is contained in the center of A. Conclude that A is an R-algebra and that the R-module structure on A induced by its algebra structure is precisely the original R-module structure.

Solution: That f maps 1_R to 1_A follows from the fact that $f(1_R) = 1_R \cdot 1_A = 1_A$. Given $r, s \in R$ we have that

$$f(r+s) = (r+s) \cdot 1_S = r \cdot 1_S + s \cdot 1_S = f(r) + f(s)$$

and

$$f(rs) = rs \cdot 1_A = r \cdot (s \cdot 1_A) = r \cdot (s \cdot 1_A 1_A) = r \cdot (1_A(s \cdot 1_A)) = (r \cdot 1_A)(s \cdot 1_A) = f(r)f(s)$$

so f is a ring homomorphism. Let $r \cdot 1_A \in f(R)$ and $a \in A$. Then we have that

$$(r \cdot 1_A)a = r \cdot (1_A a) = r \cdot a = r \cdot (a 1_A) = a(r \cdot 1_A)$$

and so $f(R)$ is in the center of A. This proves that A is an R-algebra. The R-module structure on A as an algebra is the same as its original structure since $r \cdot a = r \cdot (1_A a) = (r \cdot 1_A)a$.

□

Exercise 10.1.23. Let A be the direct product ring $\mathbb{C} \times \mathbb{C}$ (cf Section 7.6). Let τ_1 denote the identity map on $\mathbb{C}$ and let τ_2 denote complex conjugation. For any pair $p, q \in \{1, 2\}$ (not necessarily distinct) define

$$f_{p,q} : \mathbb{C} \to \mathbb{C} \times \mathbb{C} \quad \text{by} \quad f_{p,q}(z) = (\tau_p(z), \tau_q(z)).$$

So, for example $f_{2,1} : z \mapsto (\overline{z}, z)$ where $\overline{z}$ is the complex conjugate of z, i.e. $\tau_2(z)$.

(a) Prove that each $f_{p,q}$ is an injective ring homomorphism, and that they all agree on the subfield $\mathbb{R}$ of $\mathbb{C}$. Deduce that A has four distinct $\mathbb{C}$-algebra structures. Explicitly give the action $z \cdot (u, v)$ of a complex number z on an ordered pair in A in each case.

(b) Prove that if $f_{p,q} \neq f_{p',q'}$ then the identity map on A is *not* a $\mathbb{C}$-algebra homomorphism from A considered as a $\mathbb{C}$-algebra via $f_{p,q}$ to A considered a $\mathbb{C}$ algebra via $f_{p',q'}$ (although the identity is an $\mathbb{R}$ algebra isomorphism).

(c) Prove that for any pair p, q there is some ring isomorphism from A to itself such that A is isomorphic as a $\mathbb{C}$ algebra via $f_{p,q}$ to A considered as a $\mathbb{C}$ algebra via $f_{1,1}$ (the "natural" $\mathbb{C}$-algebra structure on A).

Remark: In the preceding exercise $A = \mathbb{C} \times \mathbb{C}$ is not a $\mathbb{C}$-algebra over either of the direct factor component copies of $\mathbb{C}$ (for example the subring $\mathbb{C} \times 0 \cong \mathbb{C}$) since it is not a unital module over these copies of $\mathbb{C}$ (the 1 of these subrings is not the same as the 1 of A).

Solution: (a)
That each $f_{p,q}$ agrees on $\mathbb{R}$ is trivial since complex conjugation fixes $\mathbb{R}$. Also recall that complex conjugation is an automorphism of $\mathbb{C}$ and so each τ_p is an automorphism. Hence $f_{p,q}$ behaves as a ring homomorphism in each coordinate and overall will be a homomorphism. It is a proper ring homomorphism since it maps $1_{\mathbb{C}} = 1$ to $1_{\mathbb{C} \times \mathbb{C}} = (1, 1)$. That each $f_{p,q}$ is injective follows from the injectivity of τ_p for $p = 1, 2$. In particular if z is nonzero then $f_{p,q}(z)$ is nonzero for all p, q and hence the kernel of $f_{p,q}$ is trivial.

The explicit action induced by $f_{p,q}$ is just

$$z \cdot (u, v) = (\tau_p(z)u, \tau_q(z)v).$$

In particular, $f_{1,1}$ acts via natural scalar multiplication.

(b)
If $f_{p,q} \neq f_{p',q'}$ then we notice that

$$f_{p,q}(i) \neq f_{p',q'}(i)$$

since there must be a coordinate in which one map conjugates and the other does not. Hence the action of $i \in \mathbb{C}$ induced by $f_{p,q}$ differs from that induced by $f_{p',q'}$ and in particular there exists $(z_1, z_2) \in \mathbb{C} \times \mathbb{C}$ so that the action of i on (z_1, z_2) induced by each is a different element of $\mathbb{C} \times \mathbb{C}$. Denote by $\cdot$ the action induced by $f_{p,q}$ and by $\circ$ the action induced by $f_{p',q'}$. If the identity map Id on $\mathbb{C} \times \mathbb{C}$ were a $\mathbb{C}$-algebra homomorphism we would have that

$$i \cdot (z_1, z_2) = \mathrm{Id}(i \cdot (z_1, z_2)) = i \circ \mathrm{Id}((z_1, z_2)) = i \circ (z_1, z_2)$$

which is a contradiction. Hence the identity is not a $\mathbb{C}$-algebra homomorphism.

(c)
For $f_{p,q}$ the isomorphism of $\mathbb{C} \times \mathbb{C}$ which makes it isomorphic to the natural action is the isomorphism which acts as τ_p in the first coordinate and τ_q in the second. Let ϕ denote this map. The map ϕ is clearly a ring isomorphism since τ_p and τ_q are ring isomorphisms of each coordinate. To see that this gives $\mathbb{C} \times \mathbb{C}$ the natural $\mathbb{C}$-algebra structure, let $\cdot$ denote the natural action and $\circ$ denote the action induced by $f_{p,q}$. Then we have that ϕ is a $\mathbb{C}$-algebra isomorphism since

$$\begin{aligned}
\phi(z \circ (z_1, z_2)) &= \phi((\tau_p(z)z_1, \tau_q(z)z_2)) \\
&= \big(\tau_p(\tau_p(z)z_1), \tau_q(\tau_q(z)z_2)\big) \\
&= (z\tau_p(z_1), z\tau_q(z_2)) && \text{Since } \tau_p(\tau_p(z)) = z \text{ for all } \tau_p \\
&= z \cdot (\tau_p(z_1), \tau_q(z_2)) \\
&= z \cdot \phi((z_1, z_2)).
\end{aligned}$$

Hence $\mathbb{C} \times \mathbb{C}$ with the $f_{p,q}$ action is $\mathbb{C}$-algebra isomorphic to $\mathbb{C} \times \mathbb{C}$ with the natural action, as desired.

□

10.2 Quotient Modules and Module Homomorphisms

In these exercises R is a ring with 1 and M is a left R-module.

Exercise 10.2.1. Use the submodule criterion to show that kernels and images of R-module homomorphisms are submodules.

Solution: Kernels and images of R-module homomorphisms always contain zero by virtue of being kernels and images of the underlying group homomorphisms. Thus they are nonempty. Let $\phi : N \to M$ be an R-module homomorphism. We will check the second condition of the submodule criterion for $\ker \phi$ and $\phi(N)$. Letting $x_1, x_2 \in \ker \phi$ and $r \in R$ we notice that

$$\phi(x_1 + rx_2) = \phi(x_1) + r\phi(x_2) = 0 + r0 = 0$$

and so $x_1 + rx_2 \in \ker \phi$. This proves that $\ker \phi$ is a submodule of N. Letting $\phi(n_1)$ and $\phi(n_2)$ be arbitrary elements of $\phi(N)$ and letting $r \in R$ we have

$$\phi(n_1) + r\phi(n_2) = \phi(n_1 + rn_2) \in \phi(N).$$

Hence $\phi(N)$ also satisfies the second condition of the submodule criterion and is a submodule.

□

Exercise 10.2.2. Show that the relation "is R-module isomorphic to" is an equivalence relation on any set of R-modules.

Solution: We verify each property of an equivalence relation directly.

- *Reflexivity:* Any R-module is isomorphic to itself via the identity map.
- *Symmetry:* Let $\phi : N \to M$ be an isomorphism of R-modules. We claim that the map ϕ^{-1} is also an R-module isomorphism. We know it is a group isomorphism since ϕ is a group isomorphism, and so all we have to verify is that it preserves the action of R. Let $m \in M$ and $r \in R$. We know $m = \phi(n)$ for some $n \in N$ and since ϕ is an R-module isomorphism we also have $\phi(rn) = r\phi(n) = rm$. Putting this together, we have

  $$\phi^{-1}(rm) = \phi^{-1}(\phi(rn)) = rn = r\phi^{-1}(m)$$

  and so ϕ^{-1} is a homomorphism of R-modules. This proves that M is R-module isomorphic to N.
- *Transitivity:* Let

  $$N \xrightarrow{\phi} M \xrightarrow{\psi} L$$

  be a sequence of R-module isomorphisms. We claim that $\psi \circ \phi$ is an R-module isomorphism from N to L. It is a group isomorphism by virtue of ϕ and ψ being group isomorphisms, so we need only verify that the action of R is preserved. Given $r \in R$ and $n \in N$ we have directly that

  $$\psi(\phi(rn)) = \psi(r\phi(n)) = r\psi(\phi(n))$$

  by virtue of ϕ and ψ being R-module isomorphisms. This proves that N is R-module isomorphic to L, as desired. We conclude that "is R-module isomorphic to" is an equivalence relation.

□

Exercise 10.2.3. Give an explicit example of a map from one R-module to another which is a group homomorphism but not an R-module homomorphism.

Solution: Natural examples occur whenever a module M has two distinct R-module structures on it. In this case the identity map from M to M is a group homomorphism, but not an R-module homomorphism. Some examples of modules M which can have distinct structures are described below.

- The algebra $A = \mathbb{C} \times \mathbb{C}$ described in 10.1.23 as a module over $\mathbb{C}$.
- A vector space as an $F[x]$ module, where the action of x can be various linear transformations.

- Example 2 on page 346 also works: the map $x \mapsto x^2$ in $M = F[x]$ is never an $F[x]$-module homomorphism. Indeed, one can generalize this by sending $\phi : x \mapsto f(x)$ for any $f(x) \neq x$. This is a group homomorphism but not an $F[x]$ module homomorphism since we would have $f(x) = \phi(x) = \phi(x \cdot 1) = x\phi(1) = x$. Perhaps most generally one can consider a ring with unity and a nontrivial endomorphism. This endomorphism serves as a group homomorphism that is not an R-module homomorphism.

$\square$

Exercise 10.2.4. Let A be any $\mathbb{Z}$-module, let a be any element of A and let n be a positive integer. Prove that the map $\phi_a : \mathbb{Z}/n\mathbb{Z} \to A$ given by $\phi_a(\overline{k}) = ka$ is a well defined $\mathbb{Z}$-module homomorphism if and only if $na = 0$. Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong A_n$ where $A_n = \{a \in A \mid na = 0\}$ (so A_n is the annihilator in A of the ideal (n) of $\mathbb{Z}$ — cf. Exercise 10, Section 1).

Solution: We begin by proving that ϕ_a is a well defined $\mathbb{Z}$-module homomorphism if and only if $na = 0$.

($\Rightarrow$) Suppose ϕ_a is a well defined $\mathbb{Z}$-mdoule homomorphism. Then we have that $na = \phi_a(\overline{n}) = \phi_a(0)$ which must be zero since ϕ_a is a homomorphism of groups.

($\Leftarrow$) Suppose $na = 0$. To show ϕ_a is well defined we need to show that $\phi_a(\overline{k})$ does not depend on our choice of representative for $\overline{k}$. Letting $k + bn$ be an arbitrary representative of $\overline{k}$ we have that

$$\phi_a(\overline{k + bn}) = (k + bn)a = ka + bna = ka + b(na) = ka + b0 = ka$$

and so the map is well defined. To prove it is a group homomorphism let $\overline{k_1}, \overline{k_2} \in \mathbb{Z}/n\mathbb{Z}$. Then we have

$$\phi_a(\overline{k_1} + \overline{k_2}) = (\overline{k_1} + \overline{k_2})a = \overline{k_1}a + \overline{k_2}a = \phi(\overline{k_1}) + \phi(\overline{k_2}).$$

To see it is a $\mathbb{Z}$-module homomorphism, let $z \in \mathbb{Z}$ and observe that

$$\phi_a(z\overline{k}) = \phi_a(\overline{zk}) = \overline{zk}a = z\overline{k}a = z\phi_a(\overline{k})$$

where the second to last equality follows from the fact that z acts the same on multiples of a as any z' congruent to z mod n. This shows that ϕ_a is a homomorphism of $\mathbb{Z}$-modules.

To prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A) \cong A_n$ we show that each homomorphism ϕ from $\mathbb{Z}/n\mathbb{Z}$ to A is uniquely determined by $\phi(1)$ and $\phi(1) \in A_n$. In fact, we show that all $\phi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A)$ are of the form ϕ_a for some $a \in A_n$. Given an homomorphism $\phi : \mathbb{Z}/n\mathbb{Z} \to A$ consider $\phi(1) = a$. We know that $\phi(1) \in A_n$ since

$$na = n\phi(1) = \phi(n) = \phi(0) = 0.$$

Extending ϕ to the rest of $\mathbb{Z}/n\mathbb{Z}$ we see that necessarily $\phi = \phi_a$. By the result proven earlier in the problem, we conclude that every homomorphism in $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A)$ is of the form ϕ_a for $a \in A_n$. To prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, A)$ is isomorphic to A_n as a module, notice that by the properties of homomorphisms we have $\phi_a + \phi_b = \phi_{a+b}$ and $z\phi_a = \phi_z a$ and also $\phi_a = \phi_b$ if and only if $a = b$. Hence the map $\phi_a \mapsto a$ is an isomorphism of $\mathbb{Z}$-modules and we conclude the desired result. $\square$

Exercise 10.2.5. Exhibit all $\mathbb{Z}$-module homomorphisms from $\mathbb{Z}/30\mathbb{Z}$ to $\mathbb{Z}/21\mathbb{Z}$.

Solution: By the previous exercise we know that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/30\mathbb{Z}, \mathbb{Z}/21\mathbb{Z})$ consists of maps ϕ_a where $a \in \mathbb{Z}/21\mathbb{Z}$ is annihilated by $30\mathbb{Z}$. The elements in $\mathbb{Z}/21\mathbb{Z}$ annihilated by 30 are exactly those which are multiples of 7. Hence the only maps are the zero map, $a \mapsto 7a$ and $a \mapsto 14a$. $\square$

Exercise 10.2.6. Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/(n, m)\mathbb{Z}$.

Solution: By 10.2.4 we have that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ is isomorphic to the annihilator of $n\mathbb{Z}$ in $\mathbb{Z}/m\mathbb{Z}$. This annihilator will consist of exactly the $a \in \mathbb{Z}/m\mathbb{Z}$ for which na is a multiple of m. Let d be the greatest common divisor of n and m. Then this annihilator can be easily described as the cyclic module generated by m/d in $\mathbb{Z}/m\mathbb{Z}$. Indeed, na is a multiple of m if and only if a is a multiple of m/d. The cyclic module generated by m/d has d elements, and hence is isomorphic to $\mathbb{Z}/d\mathbb{Z}$. This proves the result.

□

Exercise 10.2.7. Let z be a fixed element of the center of R. Prove that the map $m \mapsto zm$ is an R-module homomorphism from M to itself. Show that for a commutative ring R the map from R to $\mathrm{End}_R(M)$ given by $r \mapsto rI$ is a ring homomorphism (where I is the identity endomorphism).

Solution: This is a group homomorphism since $z(m_1 + m_2) = zm_1 + zm_2$ by the module axioms. Since z is in the center of r we also have $r(zm) = z(rm)$ for all $r \in R$ and so this map also respects the R-module structure.

Let ϕ denote the map $r \mapsto rI$. Then the ring homomorphism conditions are easily verified: $\phi(r_1 + r_2) = (r_1 + r_2)I = r_1 I + r_2 I = \phi(r_1) + \phi(r_2)$, and $\phi(r_1 r_2) = r_1 r_2 I = r_1 I r_2 I = \phi(r_1)\phi(r_2)$. This proves the result. □

Exercise 10.2.8. Let $\phi : M \to N$ be an R-module homomorphism. Prove that $\phi(\mathrm{Tor}(M)) \subseteq \mathrm{Tor}(N)$ (cf. Exercise 8 in Section 1).

Solution: Let $m \in \mathrm{Tor}(M)$ and $r \in R$ be nonzero so that $rm = 0$. Then $r\phi(m) = \phi(rm) = \phi(0) = 0$ and so $\phi(m) \in \mathrm{Tor}(N)$. This proves the result. □

Exercise 10.2.9. Let R be a commutative ring. Prove that $\mathrm{Hom}_R(R, M)$ and M are isomorphic as left R-modules. [Show that each element of $\mathrm{Hom}_R(R, M)$ is determined by its value on the identity of R.]

Solution: Let $\phi \in \mathrm{Hom}_R(R, M)$ and let $r \in R$. We will show that $\phi(r)$ can be expressed in terms of $\phi(1)$. Notice that

$$\phi(r) = \phi(r \cdot 1) = r\phi(1)$$

by definition of being an R-module homomorphism. Hence each ϕ can be expressed as ϕ_m for $m \in M$ where $\phi_m(r) = rm$. We claim that the map $m \mapsto \phi_m$ is a homomorphism of the R-modules M and $\mathrm{Hom}_R(R, M)$.

First, note that this map is injective since $\phi_{m_1} = \phi_{m_2}$ means that $m_1 = \phi_{m_1}(1) = \phi_{m_2}(1) = m_2$. Furthermore it is surjective since every homomorphism is uniquely determined by its value on 1 and can be written as ϕ_m. This map is also a group homomorphism since

$$\phi_{m_1+m_2}(s) = s(m_1 + m_2) = sm_1 + sm_2 = \phi_{m_1}(s) + \phi_{m_2}(s)$$

for all $s \in R$ and hence $\phi_{m_1+m_2} = \phi_{m_1} + \phi_{m_2}$. To show this map respects the R-module structure, let $r \in R$ and observe that

$$r\phi_m(s) = rsm = s(rm) = \phi_{rm}(s)$$

for all $s \in R$, and so $r\phi_m = \phi_{rm}$. We conclude that $m \mapsto \phi_m$ is an R-module isomorphism as desired. □

Exercise 10.2.10. Let R be a commutative ring. Prove that $\mathrm{Hom}_R(R, R)$ and R are isomorphic as rings.

Solution: Consider the map $r \mapsto rI$ where I is the identity map on R. By 10.2.7 this is a homomorphism from R to $\mathrm{End}_R(R) = \mathrm{Hom}_R(R, R)$. But this is also the exact map described in the proof of 10.2.9. In particular, this is an isomorphism of the R-module $\mathrm{Hom}_R(R, R)$ with the R-module R. We conclude that this map is bijective, and by virtue of being a ring homomorphism it must be a ring isomorphism. This proves the result. $\square$

Exercise 10.2.11. Let $A_1, A_2, \dots, A_n$ be R-modules and let B_i be a submodule of A_i for each $i = 1, 2, \dots, n$. Prove that

$$(A_1 \times \cdots \times A_n)/(B_1 \times \cdots \times B_n) \cong (A_1/B_1) \times \cdots \times (A_n/B_n).$$

[Recall Exercise 14 in Section 5.1.]

Solution: Consider the map $\phi : A_1 \times \cdots \times A_n \to (A_1/B_1) \times \cdots \times (A_n/B_n)$ defined by

$$\phi(a_1, a_2, \dots, a_n) = (a_1 + B_1, a_2 + B_2, \dots, a_n + B_n).$$

Note that this is a homomorphism of R-modules since it is R-linear in each coordinate. Indeed,

$$a_i + ra_i' + B_i = (a_i + B_i) + r(a_i' + B_i)$$

by definition of the quotient module A_i/B_i. Then consider the kernel of this map. If $(a_1, \dots, a_n) \in \ker\phi$ we must have $a_i + B_i = 0 + B_i$ for all i. That is, we must have $a_i \in B_i$ and in particular $(a_1, \dots, a_n) \in B_1 \times \cdots \times B_n$. This condition is obviously necessary and sufficient to be in the kernel, and so the kernel is $B_1 \times \cdots \times B_n$. Also note that the map is surjective, with a preimage of $(a_1 + B_1, \dots, a_n + B_n)$ being simply $(a_1, \dots, a_n)$. By the first isomorphism theorem we conclude that

$$\begin{aligned}(A_1 \times \cdots \times A_n)/(B_1 \times \cdots \times B_n) &= (A_1 \times \cdots \times A_n)/\ker\phi \\ &\cong \phi(A_1 \times \cdots \times A_n) \\ &= (A_1/B_1) \times \cdots \times (A_n/B_n)\end{aligned}$$

which proves the result. $\square$

Exercise 10.2.12. Let I be a left ideal of R and let n be a positive integer. Prove

$$R^n/IR^n \cong R/IR \times \cdots \times R/IR \quad (n \text{ times})$$

where IR^n is defined as in Exercise 5 of Section 1. [Use the preceding exercise.]

Solution: By definition $R^n = R \times \cdots \times R$ where the product is taken n times. Thus we only need to show that $IR^n = (IR)^n$, and the result will follow immediately from the previous problem. To prove this we show containment in both directions. Elements of IR^n are of the form $a(r_1, \dots, r_n) = (ar_1, \dots, ar_n)$ where $a \in I$. Such elements are clearly in $(IR)^n$ since elements in $(IR)^n$ have the form $(a_1r_1, \dots, a_nr_n)$ for $a_i \in I$. Thus we have $IR^n \subseteq (IR)^n$ immediately.

To show that $(IR)^n \subseteq IR^n$ consider an arbitrary element $(a_1r_1, \dots, a_nr_n) \in (IR)^n$. Notice that the tuple $v_i = (0, \dots, a_ir_i, \dots, 0)$ which is zero in all coordinates but the i-th is in IR^n since it is just $a_i(0, \dots, a_i, \dots, 0)$. But IR^n is closed under finite sums, and so we can write

$$(a_1r_1, \dots, a_nr_n) = \sum_{i=1}^{n} v_i \in IR^n.$$

This proves that $(IR)^n \subseteq IR^n$, and so we conclude the desired result. As an interesting aside, I believe this also holds when the product is infinite since we only allow finitely many nonzero coordinates. $\square$

Exercise 10.2.13. Let I be a nilpotent ideal in a commutative ring R (cf. Exercise 37, 7.3), let M and N be R-modules and let $\phi : M \to N$ be an R-module homomorphism. Show that if the induced map $\overline{\phi} : M/IM \to N/IN$ is surjective, then ϕ is surjective.

Solution: *Note: I refered to* `https://crazyproject.wordpress.com/aadf/\#df-10` *for the solution to this problem. Wrote my own version of the solution however.*

We will first prove that $N = \phi(M) + I^k N$ for all k, independent of the fact that I is nilpotent. Consider the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \pi_M \big\downarrow & & \big\downarrow \pi_N \\ M/IM & \xrightarrow{\overline{\phi}} & N/IN \end{array}$$

Above we have π_M and π_N as projection mod IM and IN respectively. This diagram commutes by virtue of $\overline{\phi}$ being the induced map. We begin by showing that $N = \phi(M) + IN$. Notice that N is clearly the preimage of N/IN under π_N. Also $N/IN = \overline{\phi}(M/IM)$ and so any $n + IN \in N/IN$ can be written as $\phi(m) + IN$ for some $m \in M$. This implies that the preimage of N/IN under π_N will be $\phi(M) + IN$. Indeed, $\pi_N(n) = \phi(m) + IN$ implies that n is the sum of something in $\phi(M)$ and the kernel of π_N which is IN. So far we have shown that $N = \phi(M) + IN$.

To prove that $N = \phi(M) + I^k N$ we use induction on k, where we have just proven the base case. For the inductive step, we have

$$N = \phi(M) + I^k N = \phi(M) + I^k(\phi(M) + IN) = \phi(M) + I^k\phi(M) + I^{k+1}N = \phi(M) + I^{k+1}N$$

where the last equality follows from the fact that $I^k\phi(M) \subseteq \phi(M)$. By induction we conclude that $N = \phi(M) + I^k N$ for all k. Taking k large enough we have $I^k = 0$ and so $\phi(M) = N$ as desired.

It is illustrative to see the equality $N = \phi(M) + I^k N$ for some non-nilpotent ideal. For an example, we take $R = M = N = \mathbb{Z}$. Let $\phi : \mathbb{Z} \to \mathbb{Z}$ be the doubling map (i.e. $\phi(z) = 2z$), which is indeed a homomorphism of $\mathbb{Z}$ modules since it is a homomorphism of abelian groups. Notice that it is not surjective. For our ideal I we choose $3\mathbb{Z}$. Then our diagram of modules becomes

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\phi \text{ (doubling)}} & \mathbb{Z} \\ \pi \big\downarrow & & \big\downarrow \pi \\ \mathbb{Z}/3\mathbb{Z} & \xrightarrow{\overline{\phi} \text{ (doubling)}} & \mathbb{Z}/3\mathbb{Z} \end{array}$$

Now, the induced map is surjective since we have $0 \mapsto 0$, $1 \mapsto 2$ and $2 \mapsto 1$. Our result states that $\mathbb{Z} = \phi(\mathbb{Z}) + 3^k\mathbb{Z}$ for all k. Since $\phi(\mathbb{Z}) = 2\mathbb{Z}$ and $2\mathbb{Z}$ and $3^k\mathbb{Z}$ are always comaximal ideals, we see that the result holds.

$\square$

Exercise 10.2.14. Let $R = \mathbb{Z}[x]$ be the ring of polynomials in x and let $A = \mathbb{Z}[t_1, t_2, \ldots]$ be the ring of polynomials in the independent indeterminates $r_1, r_2, \ldots$. Define an action of R on A as follows: 1) let $1 \in R$ act on A as the identity, 2) for $n \geq 1$ let $x^n \circ 1 = t_n$, let $x^n \circ t_i = t_{n+i}$ for $i = 1, 2 \ldots$, and let x^n act as 0 on monomials in A of (total) degree at least two, and 3) extend $\mathbb{Z}$-linearly, i.e., so that the module axioms 2(a) and 2(c) are satisfied.

(a) Show that $x^{p+q} \circ t_i = x^p \circ (x^q \circ t_i) = t_{p+q+i}$ and use this show that under this action the ring A is a (unital) R-module.

(b) Show that the map $\phi : R \to A$ defined by $\phi(r) = r \circ 1_A$ is an R-module homomorphism of the ring R into the ring A mapping 1_R to 1_A, but not a ring homomorphism from R to A.

Solution: (a)
We can compute directly that

$$x^{p+q} \circ t_i = t_{p+q+i} = x^p \circ t_{q+i} = x^p \circ (x^q \circ t_i)$$

as desired. We can use this to show that A is an R-module by considering arbitrary polynomials $f = \sum_{i=0}^n a_i x^i$ and $g = \sum_{j=0}^m b_j x^j$ in $\mathbb{Z}[x]$. To prove that $fg \circ T = f \circ g \circ T$ for all $T \in A$ it suffices to consider $T = t_k$ since the action is by definition extended linearly and acts as zero on monomials of higher degree. We have that

$$\begin{aligned}
fg \circ t_k &= \left(\sum_{i=0}^n a_i x^i\right)\left(\sum_{j=0}^m b_j x^j\right) \circ t_k \\
&= \left(\sum_{i=0}^{n+m}\left(\sum_{j=0}^i a_j b_{i-j}\right) x^i\right) \circ t_k \\
&= \sum_{i=0}^{n+m}\left(\sum_{j=0}^i a_j b_{i-j}\right)(x^i \circ t_k) && \text{By } R\text{-linearity} \\
&= \sum_{i=0}^{n+m}\left(\sum_{j=0}^i a_j b_{i-j}\right) t_{k+i} && \text{By definition of the action}
\end{aligned}$$

Now, we can change the indices in this sum as follows. The various coefficients $a_j b_{j-i}$ are all of the form $a_{i'} b_{j'}$ where $0 \le i' \le n$ and $0 \le j' \le m$ (there are some additional pairs but for these we have $a_j = 0$ or $b_{j-i} = 0$). The coefficient $a_{i'} b_{j'}$ appears as the coefficient of $t_{k+i'+j'}$. Hence this all simplifies as

$$\begin{aligned}
fg \circ t_k &= \sum_{i=0}^n \sum_{j=0}^m a_i b_j t_{k+i+j} \\
&= \sum_{i=0}^n \sum_{j=0}^m a_i b_j x^i \circ (x^j \circ t_k) \\
&= \sum_{i=0}^n \sum_{j=0}^m a_i x_i \circ b_j \left(x^j \circ t_k\right) \\
&= \sum_{i=0}^n a_i x^i \circ \left(\sum_{j=0}^m b_j (x^j \circ t_k)\right) \\
&= \sum_{i=0}^n a_i x^i \circ (g \circ t_k) \\
&= f \circ (g \circ t_k).
\end{aligned}$$

This shows that the action obeys axiom 2(b) for modules. We already know it satisfies the other axioms so A is indeed an R-module. That the action is unital follows directly from the definition since $1 \in R$ acts as identity. Thus A is a unital R-module as desired.

(b)
This map is naturally a homomorphism of the abelian groups since

$$\phi(r_1 + r_2) = (r_1 + r_2) \circ 1_A = r_1 \circ 1_A + r_2 \circ 1_A = \phi(r_1) + \phi(r_2).$$

Indeed this is an example of the maps ϕ_a described in the solution to Problem 10.2.9. It maps 1_R to 1_A since the module action is unital.

To see that this is not a ring homomorphism, consider the image of x^2. We have that $\phi(x^2) = t_2$. But $\phi(x)\phi(x) = t_1^2 \neq t_2$ so the map is not a ring homomorphism.

□

10.3 Generation of Modules, Direct Sums, and Free Modules

In these exercises R is a ring with 1 and M is a left R-module.

Exercise 10.3.1. Prove that if A and B are sets of the same cardinality, then the free modules $F(A)$ and $F(B)$ are isomorphic.

Solution: Let $\phi : A \to B$ be a bijection and ϕ^{-1} be its inverse. Then Theorem 6 tells us that there exist unique R-module homomorphisms $\Phi : F(A) \to F(B)$ and $\Phi^{-1} : F(B) \to F(A)$ so that Φ agrees with ϕ on A and Φ^{-1} agrees with ϕ^{-1} on B. We claim that Φ is an R-module isomorphism. It is clear that $\Phi^{-1} \circ \Phi$ is identity on $F(A)$ and so Φ must be injective. On the other hand $\Phi \circ \Phi^{-1}$ is identity on $F(B)$ which means that Φ must be surjective. Hence Φ is an isomorphism, proving the result. □

Exercise 10.3.2. Assume R is commutative. Prove that $R^n \cong R^m$ if and only if $n = m$, i.e., two free R-modules of finite rank are isomorphic if and only if they have the same rank. [Apply Exercise 12 of Section 2 with I a maximal ideal of R. You may assume that if F is a field, then $F^n \cong F^m$ if and only if $n = m$, i.e. two finite dimensional vector spaces over F are isomorphic if and only if they have the same dimension — this will be proved later in Section 11.1]

Solution: ($\Leftarrow$) If the modules have the same rank then they are isomorphic by the previous problem together with the result that any free module of rank n is free over its basis.

($\Rightarrow$) We begin by proving the following general fact. If $M \cong N$ as R-modules and I is an ideal of R, then $M/IM \cong N/IN$. To prove this let $\phi : M \to N$ be an R-module isomorphism and consider its induced map $\overline{\phi} : M/IM \to N/IN$ which maps $m + IM \mapsto \phi(m) + IN$. This map is well defined since we are taking a quotient of each module by the action of the same ideal. This map is surjective since if $n + IN \in N/IN$ it has as a preimage $\phi^{-1}(n) + IM \in M/IM$. On the other hand it is well defined to talk about the induced inverse $\overline{\phi^{-1}} : N/IN \to M/IM$. One can observe that $\overline{\phi^{-1}} \circ \overline{\phi}$ acts as identity on M/IM since

$$\overline{\phi^{-1}}(\overline{\phi}(m + IM)) = \overline{\phi^{-1}}(\phi(m) + IN) = \phi^{-1}(\phi(m)) + IM = m + IM.$$

Hence $\overline{\phi}$ must be injective. We conclude that $\overline{\phi}$ is an isomorphism.

Now suppose that $R^n \cong R^m$. Letting I be a maximal ideal, we have from 10.2.12 that

$$(R/IR)^n \cong R^n/IR^n \cong R^m/IR^m \cong (R/IR)^m$$

where the middle isomorphism is the one induced from $R^n \cong R^m$ when modding out by the action of I. But this says that two vectors spaces of dimension m and n respectively are isomorphic, and hence $m = n$. This proves the result. □

Exercise 10.3.3. Show that the $F[x]$-modules in Exercises 18 and 19 of Section 1 are both cyclic.

Solution: **Exercise 18:** This module is $V = \mathbb{R}^2$ with the action of x being given by the linear transformation that rotates by $\pi/2$. We notice that V is generated by $(1,0)$ since we have $x \cdot (1,0) = (0,1)$ and $\{(1,0),(0,1\}$ spans $\mathbb{R}^2$ over $\mathbb{R}$, which is a subring of $\mathbb{R}[x]$. In fact we could choose any nonzero vector and V would be cyclicly generated by it.

Exercise 19: Again the module if $V = \mathbb{R}^2$, but now the action of x is given by projection onto the y-axis. In this case we see that V is not cyclicly generated by $(1,0)$ since the projection of this is just the zero vector. However, V is generated by $(1,1)$ since $x \cdot (1,1) = (0,1)$ which is linearly independent from $(1,1)$. Hence together $(1,1)$ and $x \cdot (1,1)$ span V over $\mathbb{R}$ and since $\mathbb{R} \subseteq \mathbb{R}[x]$ we see that $(1,1)$ generated V.

□

Exercise 10.3.4. An R-module M is called a *torsion* module if for each $m \in M$ there is a nonzero element of $r \in R$ such that $rm = 0$, where r may depend on m (i.e., $M = \mathrm{Tor}(M)$ in the notation of Exercise 8 of Section 1).Prove that every finite abelian group is a torsion $\mathbb{Z}$-module. Give an example of an infinite abelian group that is a torsion $\mathbb{Z}$-module.

Solution: Let G be an abelian group. Then the nonzero element $|G| \in \mathbb{Z}$ annihilates G and we conclude that G is a torsion module.

For an example of an infinite abelian group one can consider $\mathbb{Q}/\mathbb{Z}$. Every element has finite order and hence is annihilated by some integer. A less interesting example is any infinite product of finite abelian groups. □

Exercise 10.3.5. Let R be an integral domain. Prove that every finitely generated torsion R-module has a nonzero annihilator i.e., there is a nonzero element $r \in R$ such that $rm = 0$ for all $m \in M$ — here r does not depend on m (the annihilator of a module was defined in Exercise 9 of Section 1). Give an example of a torsion R-module whose annihilator is the zero ideal.

Solution: Let $M = RA$ for a finite set $\{a_1, \ldots, a_n\}$. For each a_i let $r_i \neq 0$ be such that $r_i a_i = 0$. We claim that $r = r_1 r_2 \cdots r_n$ is a nonzero element of the annihilator of M in R. That $r \neq 0$ follows from the fact that R is an integral domain. To see that r is in the annihilator of M notice that r annihilates each a_i by the commutativity of R. Since r annihilates a generating set for M it must annihilate M, proving the result.

For an example where the annihilator is zero and the module is still torsion, consider the group G which is the product of $\mathbb{Z}/n\mathbb{Z}$ for all $n \geq 2$, considered as a $\mathbb{Z}$-module. Every element is annihilated by the least common multiple of its nonzero components, but no nonzero integer can annihilate every element of G simultaneously. □

Exercise 10.3.6. Prove that if M is a finitely generated R-module that is generated by n elements then every quotient of M may be generated by n (or fewer) elements. Deduce that quotients of cyclic modules are cyclic.

Solution: Let $\{a_1, \ldots, a_n\}$ be a generating set for an R-module M. Then we claim that $\{a_1 + N, \ldots, a_n + N\}$ generates M/N as an R-module. Indeed, notice that any $m + N \in M/N$ can be written as

$$m + N = \left(\sum r_i a_i\right) + N = \sum r_i (a_i + N)$$

proving the result. Hence the quotient of a cyclic module can be generated by 1 or 0 elements and is again cyclic.

□

Exercise 10.3.7. Let N be a submodule of M. Prove that if both M/N and N are finitely generated then so is M.

Solution: Let $\{a_1, \ldots, a_n\}$ be a finite generating set for N and let $\{b_1 + N, \ldots, b_m + N\}$ be a finite generating set for M/N. We claim that the set

$$A = \{a_1, \ldots, a_n, b_1 \ldots, b_m\}$$

generates M as an R-module. Let $\pi : M \to M/N$ be the natural projection map. For an arbitrary $m \in M$, let $r_1, \ldots, r_m$ be such that

$$m + N = \sum r_i(b_i + N) = \left(\sum r_i b_i\right) + N$$

Now notice that $m - \sum r_i b_i$ must be in the kernel of π, i.e. in N. Then there must exist $s_1, \ldots, s_n$ so that

$$m - \sum r_i b_i = \sum s_j a_j$$

which implies

$$m = \sum r_i b_i + \sum s_j a_j \in RA.$$

Hence $RA = M$ and A is a finite generating set for M. This proves the result. □

Exercise 10.3.8. Let S be the collection of sequences $(a_1, a_2, a_3, \ldots)$ of integers $a_1, a_2, a_3, \ldots$ where all but finitely many of the a_i are 0 (called the *direct sum* of infinitely many copies of $\mathbb{Z}$). Recall taht S is a ring under componentwise addition and multiplication and S does not have a multiplicative identity — cf. Exercise 20, Section 7.1. Prove that S is not finitely generated as a module over itself.

Solution: Given any finite set $A = \{a_1, \ldots, a_n\}$ let n_i be an integer such that $N > n_i$ implies that the N-th component of a_i is zero. Taking the maximum M of all n_i we see that every a_i is zero past index M. Hence the set A does not generate any list which is nonzero after M and A does not generate S as amodule. □

Exercise 10.3.9. An R-module M is called *irreducible* if $M \neq 0$ and if 0 and M are the only submodules of M. Show that M is irreducible if and only if $M \neq 0$ and M is a cyclic module with any nonzero element as a generator. Determine all the irreducible $\mathbb{Z}$-modules.

Solution: ($\Rightarrow$) Suppose that M is irreducible. We know by definition $M \neq 0$. Taking some nonzero $m \in M$, we see that Rm is a nonzero submodule of M, and so $Rm = M$. This proves that M is generated by any nonzero element.

($\Leftarrow$) Suppose $M \neq 0$ and M is cyclic with any nonzero element as a generator. Then let $N \subseteq M$ be any nonzero submodule of M. Let $n \in N$ be nonzero and notice then that $M = Rn \subseteq N$ and so $N = M$. This proves that the only nonzero submodule of M is M itself and so M is irreducible.

To classify all irreducible $\mathbb{Z}$-modules, we need only consider cyclic modules. If a cyclic module is not a torsion module it is isomorphic to $\mathbb{Z}$. But this is not irreducible since it contains a submodule isomorphic to $2\mathbb{Z}$. This leaves cyclic torsion modules of $\mathbb{Z}$. These are simply finite cyclic groups. Among these we see that the only irreducible ones are $\mathbb{Z}/p\mathbb{Z}$ for some prime p.

□

Exercise 10.3.10. Assume R is commutative. Show that an R-module M is irreducible if and only if M is isomorphic (as an R-module) to R/I where I is a maximal ideal of R. [By the previous exercise, if M is irreducible then there is a natural map $R \to M$ defined by $r \mapsto rm$ where m is any fixed nonzero element of M.]

Solution: ($\Rightarrow$) Suppose that M is an irreducible R-module and fix some nonzero $m \in M$. We know that $Rm = M$. Let $\phi : R \to M$ be the map $r \mapsto rm$. This is certainly a homomorphism of R-modules and furthermore it is surjective. Thus we have $M \cong R/\ker\phi$. If we can show $\ker\phi$ (as a submodule of R) is a maximal ideal of R then we are done. First by virtue of being a submodule of the commutative ring R we know that $\ker\phi$ is an ideal.

Next notice that $\ker\phi$ is exactly the annihilator of m (and hence M) in R. Thus any ideal J strictly containing $\ker\phi$ must contain some r so that $rm \neq 0$. But then we have that $JM \neq 0$ and so $JM = M$. This means that J contains some element s so that $sm = m$, or equivalently $(s-1)m = 0$. We conclude that $s - 1 \in \ker\phi$. But $\ker\phi \subseteq J$ and so we have $s, s-1 \in J$ which means $1 \in J$. We conclude that $J = R$ and so I is maximal. This proves the result.

($\Leftarrow$) Suppose $M \cong R/I$ for a maximal ideal I. We aim to show that $Rm = M$ for all nonzero $m \in M$. We can write any nonzero $m \in M$ as $a+I$ via the isomorphism between M and R/I where $a \notin I$. But then $R(a+I) = Ra + RI = Ra + I$. Notice that $Ra + I$ is an ideal strictly containing I since it contains a, and so $Ra + I = R$ since I is maximal. We conclude that $R(a+I) = R/I$ in the module R/I, proving the result. □

Exercise 10.3.11. Show that if M_1 and M_2 are irreducible R-modules, then any nonzero R-module homomorphism from M_1 to M_2 is an isomorphism. Deduce that if M is irreducible then $\mathrm{End}_R(M)$ is a division ring (this result is called *Schur's Lemma*). [Consider the kernel and the image.]

Solution: Let $\phi : M_1 \to M_2$ be a nonzero R-module homomorphism. We know that $\ker\phi$ is not all of M_1, and hence $\ker\phi = \{0\}$. On the other hand we know $\phi(M_1) \neq \{0\}$ and so it is all of M_2. This tells us that ϕ is injective and surjective, and so we conclude that ϕ is an isomorphism. □

Exercise 10.3.12. Let R be a commutative ring and let A, B and M be R-modules. Prove the following isomorphisms of R-modules:

(a) $\mathrm{Hom}_R(A \times B, M) \cong \mathrm{Hom}_R(A, M) \times \mathrm{Hom}_R(B, M)$

(b) $\mathrm{Hom}_R(M, A \times B) \cong \mathrm{Hom}_R(M, A) \times \mathrm{Hom}_R(M, B)$.

Solution: (a)
Let $(\phi, \psi) \in \mathrm{Hom}_R(A, M) \times \mathrm{Hom}_R(B, M)$. We claim that the map sending $(\phi, \psi) \mapsto \Phi$ where Φ acts as ϕ in the first coordinate and ψ in the second is an isomorphism of R-modules. More specifically, we define $\Phi(a, b) = \phi(a) + \psi(b)$. First we show that it is even a well defined map between the hom-sets of concern, namely that $\Phi \in \mathrm{Hom}_R(A \times B, M)$. This is straightforward since if $(a, b), (a', b') \in A \times B$ and $r \in R$ then we have

$$\begin{aligned}
\Phi((a,b) + r(a',b')) &= \Phi(a + ra', b + rb') \\
&= \phi(a + ra') + \psi(b + rb') \\
&= \phi(a) + \psi(b) + r\phi(a') + r\psi(b') \\
&= \Phi(a + b) + r\Phi(a' + b').
\end{aligned}$$

Now we need to show that $(\phi, \psi) \mapsto \Phi$ is a homomorphism of R-modules. Suppose we have $(\phi, \psi) \mapsto \Phi$ and $(\phi', \psi') \mapsto \Phi'$. It is clear that $(\phi + r\phi', \psi + r\psi')$ maps to $\Phi + r\Phi'$ and so we see that the map is an R-module homomorphism.

Injectivity is straightforward since the only map Φ that can act as zero in both coordinates comes from $\phi = \psi = 0$. For surjectivity, notice that any Φ acts as an R-module homomorphism in each coordinate. In particular, if we define $\Phi_A : A \to M$ by $\Phi_A(a) = \Phi(a, 0)$ and Φ_B symmetrically then we see that $(\Phi_A, \Phi_B) \mapsto \Phi$. Hence the map is surjective and we conclude the desired isomorphism.

(b)
The proof here is essentially the same as (a): the isomorphism is given by decomposing any homomorphism in $\text{Hom}_R(M, A \times B)$ into its coordinate pieces on A and B. In particular, we associate $\Phi \in \text{Hom}_R(M, A \times B)$ with the pair (ϕ, ψ) where $\phi(a)$ is the first coordinate of $\Phi(a)$ and $\psi(b)$ is the second coordinate of $\Phi(b)$. $\square$

Exercise 10.3.13. Let R be a commutative ring and let F be a free R-module of finite rank. Prove the following isomorphism of R-modules: $\text{Hom}_R(F, R) \cong F$.

Solution: Write $F \cong R^n$. Applying the result of the previous exercise we have that

$$\begin{aligned}\text{Hom}_R(F, R) &\cong \text{Hom}_R(R^n, R)\\ &\cong \text{Hom}_R(R, R)^n\\ &\cong R^n\\ &\cong F.\end{aligned}$$

Note that above we have used the fact that $\text{Hom}_R(R, R) \cong R$, which was proven in another exercise. $\square$

Exercise 10.3.14. Let R be a commutative ring and let F be the free R-module of rank n. Prove that $\text{Hom}_R(F, M) \cong M \times \cdots \times M$ (n times). [Use Exercise 9 in Section 2 and Exercise 12.]

Solution: Recall from 10.2.9 that $\text{Hom}_R(R, M) \cong M$ since every homomorphism is determined by its value on $1 \in R$. Hence we have (similar to the previous exercise) that

$$\begin{aligned}\text{Hom}_R(F, M) &\cong \text{Hom}_R(R^n, M)\\ &\cong \text{Hom}_R(R, M)^n\\ &\cong M^n.\end{aligned}$$

This is exactly what we hoped to show. Note that the previous exercise is in fact a special case of this. $\square$

Exercise 10.3.15. An element $e \in R$ is called a *central idempotent* if $e^2 = e$ and $er = re$ for all $r \in R$. If e is a central idempotent in R, prove that $M = eM \oplus (1 - e)M$. [Recall Exercise 14 in Section 1.]

Solution: We must show two things. First, that M is generated by eM together with $(1 - e)M$. Second, that $eM \cap (1 - e)M = \{0\}$.

For the first statement, notice that since e is in the center of R the subsets $eM = \{em \mid m \in M\}$ and $(1 - e)M = \{(1 - e)m \mid m \in M\}$ are indeed submodules of M. Then let $m \in M$ be arbitrary and notice that

$$m = 1 \cdot m = (e + (1 - e)) \cdot m = e \cdot m + (1 - e) \cdot m$$

and so eM together with $(1 - e)M$ generates M. Note this is independent of e being central idempotent.

To prove that the sum of eM and $(1 - e)M$ is direct, suppose that $m \in eM \cap (1 - e)M$. Then there exist $m_1, m_2 \in M$ so that

$$em_1 = m = (1 - e)m_2.$$

But acting on the left and right quantities above by e yields

$$e^2 m_1 = (e - e^2) m_2$$

which simplifies to $em_1 = 0$ since $e - e^2 = 0$. But this tells us immediately that $m = 0$, so the sum of eM and $(1-e)M = 0$. $\square$

The next two exercises establish the Chinese Remainder Theorem for modules (cf. Section 7.6).

Exercise 10.3.16. For any ideal I of R let IM be the submodule defined in Exercise 5 of Section 1. Let $A_1, \ldots, A_k$ be any ideals in the ring R. Prove that the map

$$M \to M/A_1M \times \cdots M/A_kM \quad \text{defined by} \quad m \mapsto (m + A_1M, \ldots, m + A_kM)$$

is an R-module homomorphism with kernel $A_1M \cap A_2M \cap \cdots \cap A_kM$.

Solution: TODO

$\square$

Exercise 10.3.17. In the notation of the preceding exercise, assume further that the ideals $A_1, \ldots A_k$ are pairwise comaximal (i.e. $A_i + A_j = R$ for al $i \neq j$). Prove that

$$M/(A_1 \cdots A_k)M \cong M/A_1M \times \cdots MA_kM.$$

[See the proof of the Chinese Remainder Theorem for rings in Section 7.6.]

Solution: TODO

$\square$

Exercise 10.3.18. Let R be a Principal Ideal Domain and let M be an R-module that is annihilated by the nonzero, proper ideal (a). Let $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the unique factorization of a into distinct prime powers in R. Let M_i be the annihilator of $p_i^{\alpha_i}$ in M, i.e. M_i is the set $\{m \in M \mid p_i^{\alpha_i} m = 0\}$—called the *$p_i$-primary component of M*. Prove that

$$M = M_1 \oplus M_2 \oplus \cdots \oplus M_k.$$

Solution: TODO

$\square$

Exercise 10.3.19. Show that if M is a finite abelian group of order $a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ then, considered as a $\mathbb{Z}$-module, M is annihilated by (a), the p_i-primary component of M is the unique Sylow p_i-subgroup of M and M is isomorphic to the direct product of its Sylow subgroups.

Solution: TODO

$\square$

Exercise 10.3.20. Let I be a nonempty index set and for each $i \in I$ let M_i be an R-module. The *direct product* of the modules M_i is defined to be their direct product as abelian groups (cf. Exercise 15 in Section 5.1) with the action of R componentwise multiplication. The *direct sum* of the modules M_i is defined to be the restricted direct product of the abelian groups M_i (cf. Exercise 17 in Section 5.1) with the action of R componentwise multiplication. In other words, the direct sum of the M_i's is the subset of the direct product $\prod_{i \in I} M_i$, which consists of all elements $\prod_{i \in I} m_i$ such that only finitely many of the components m_i are nonzero; the action of R on the direct product or direct sum is given by $r \prod_{i \in I} m_i = \prod_{i \in I} rm_i$ (cf. Appendix I for the definition of the Cartesian products of infinitely many sets). The direct sum will be denoted by $\oplus_{i \in I} M_i$.

(a) Prove that the direct product of the M_i's is an R-module and the direct sum of the M_i's is a submodule of their direct product.

(b) Show that if $R = \mathbb{Z}$, $I = \mathbb{Z}^+$ and M_i is the cyclic group of order i for each i, then the direct sum of the M_i's is not isomorphic to their direct product. [Look at torsion.]

Solution: TODO

□

Exercise 10.3.21. let I be a nonempty index set and for each $i \in I$ let N_i be a submodule of M. Prove that the following are equivalent:

(i) the submodule of M generated by all the N_i's i isomorphic to the direct sum of the N_i's

(ii) if $\{i_1, i_2, \ldots, i_k\}$ is any finite subset of I then $N_{i_1} \cap (N_{i_2} + \cdots + N_{i_k}) = 0$

(iii) if $\{i_1, i_2, \ldots, i_k\}$ is any finite subset of I then $N_1 + \cdots + N_k = N_1 \oplus \cdots \oplus N_k$

(iv) for every element x of the submodule of M generated by the N_i's there are unique elements $a_i \in N_i$ for all $i \in I$ such that all but a finite number of the a_i are zero and x is the (finite) sum of the a_i.

Solution: TODO

□

Exercise 10.3.22. Let R be a Principal Ideal Domain, let M be a torsion R-module (cf. Exercise 4) and let p be a prime in R (do not assume M is finitely generated, hence it need not have a nonzero annihilator — cf. Exercise 5). The *p-primary component* of M is the set of all elements of M that are annihilated by some positive power of p.

(a) Prove that the p-primary component is a submodule. [See Exercise 13 in Section 1.]

(b) Prove that this definition of p-primary component agrees with the one given in Exercise 18 when M has a nonzero annihilator.

(c) Prove that M is the (possible infinite) direct sum of its p-primary components, as p runs over all primes of R.

Solution: TODO

□

Exercise 10.3.23. Show that any direct sum of free R-modules is free.

Solution: TODO

□

Exercise 10.3.24. *(An arbitrary direct product of free modules need not be free)* For each positive integer i let M_i be the free $\mathbb{Z}$-module $\mathbb{Z}$, and let M be the direct product $\prod_{i \in \mathbb{Z}^+} M_i$ (cf. Exercise 20). Each element of M can be written uniquely in the form $(a_1, a_2, a_3, \ldots)$ with $a_i \in \mathbb{Z}$ for all i. Let N be the submodule of M consisting of all such tuples with only finitely many nonzero a_i. Assume M is a free $\mathbb{Z}$ module with basis $\mathcal{B}$.

(a) Show that N is countable.

(b) Show that there is some countable subset $\mathcal{B}_1$ of $\mathcal{B}$ such that N is contained in the submodule, N_1, generated by $\mathcal{B}_1$. Show also that N_1 is countable.

(c) Let $\overline{M} = M/N_1$. Show that $\overline{M}$ is a free $\mathbb{Z}$-module. Deduce that if $\overline{x}$ is any nonzero element of $\overline{M}$ then there are only finitely many distinct positive integers k such that $\overline{x} = k\overline{m}$ for some $m \in M$ (depending on k).

(d) Let $\mathcal{S} = \{(b_1, b_2, b_3, \ldots) \mid b_i = \pm i! \text{ for all } i\}$. Prove that $\mathcal{S}$ is uncountable. Deduce that there is some $s \in \mathcal{S}$ with $s \notin N_1$.

(e) Show that the assumption M is free leads to a contradiction: By (d) we may choose $s \in \mathcal{S}$ with $s \notin N_1$. Show that for each positive integer k there is some $m \in M$ with $\overline{s} = k\overline{m}$, contrary to (c). [Use the fact that $N \subseteq N_1$.]

Solution: TODO

□

Exercise 10.3.25. In the construction of direct limits, Exercise 8 of Section 7.6, show that if all A_i are R-modules and the maps ρ_{ij} are R-module homomorphisms, then the direct limit $A = \varinjlim A_i$ may be given the structure of an R-module in a natural way such that the maps $\rho_i : A_i \to A$ are all R-module homomorphisms. Verify the corresponding universal property (part (e)) for R-module homomorphism $\phi_i : A_i \to C$ commuting with the ρ_{ij}.

Solution: TODO

□

Exercise 10.3.26. Carry out the analysis of the preceding exercise corresponding to the inverse limits to show that the invese limit of R-modules is an R-module satisfying the appropriate universal property (cf. Exercise 10 of Section 7.6).

Solution: TODO

□

Exercise 10.3.27. *(Free modules over noncommutative rings need not have a unique rank)* Let M be the $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z} \times \cdots$ of Exercise 24 and let R be its endomorphism ring, $R = \mathrm{End}_{\mathbb{Z}}(M)$ (cf. Exercises 29 and 30 in Section 7.1). Define $\phi_1, \phi_2 \in R$ by

$$\phi_1(a_1, a_2, a_3, \ldots) = (a_1, a_3, a_5, \ldots)$$
$$\phi_2(a_1, a_2, a_3, \ldots) = (a_2, a_4, a_6, \ldots)$$

(a) Prove that $\{\phi_1, \phi_2\}$ is a free basis of the left R-module R. [Define the maps ψ_1 and ψ_2 by $\psi_1(a_1, a_2, \ldots) = (a_1, 0, a_2, 0, \ldots)$ and $\psi_2(a_1, a_2, \ldots) = (0a_1, 0, a_2, \ldots)$. Verify that $\phi_i\psi_i = 1$, $\phi_1\psi_2 = 0 = \phi_2\psi_1$ and $\psi_1\phi + \psi_2\phi_2 = 1$. use these relations to prove that ψ_2, ϕ_2 are independent and gereate R as a left R-module.]

(b) Use (a) to prove that $R \cong R^2$ and deduce that $R \cong R^n$ for all $n \in \mathbb{Z}^+$.

Solution: TODO

□

10.4 Tensor Products of Modules

Let R be a ring with 1.

Exercise 10.4.1. Let $f : R \to S$ be a ring homomorphism from the ring R to the ring S with $f(1_R) = 1_S$. Verify the details that $sr = sf(r)$ deefines a right R-action on S under which S is an (S, R)-bimodule.

Solution: TODO

□

Exercise 10.4.2. Show that the element "$2 \otimes 1$" is 0 in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ but is nonzero in $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$.

Solution: TODO

□

Exercise 10.4.3. Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $C \otimes_{\mathbb{C}} \mathbb{C}$ are both left $\mathbb{R}$-modules but are not isomorphic as $\mathbb{R}$-modules.

Solution: TODO

□

Exercise 10.4.4. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $Q \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic left $\mathbb{Q}$-modules. [Show they are both 1-dimensional vector spaces over $\mathbb{Q}$.]

Solution: TODO

□

Exercise 10.4.5. Let A be a finite abelian group of order n and let p^k be the largest power of the prime p dividing n. Prove that $\mathbb{Z}/p^k\mathbb{Z} \otimes \mathbb{Z}A$ i sisomorphic to the Sylow p-subgroup of A.

Solution: TODO

□

Exercise 10.4.6. If R is any integral domain with a quotient field Q, prove that $(Q/R) \otimes_R (Q/R) = 0$.

Solution: TODO

□

Exercise 10.4.7. If R is any integral domain with quotient field Q and N is a left R-module, prove that every element of the tensor product $Q \otimes_R N$ can be written as a simple tensor of the form $(1/d) \otimes n$ for some nonzero $d \in R$ and some $n \in N$.

Solution: TODO

□

Exercise 10.4.8. Suppose R is an integral domain with quotient field Q and let N be any R-module. Let $U = R^\times$ be the set of nonzero elements in R and define $U^{-1}N$ to be the set of equivalence classes of ordered pairs of elements (u, n) with $u \in U$ and $n \in N$ under the equivalence relation $(u, n) \sim (u', n)$ if and only if $u'n = un'$ in N.

(a) Prove that $U^{-1}N$ is an abelian group under the addition defined by $\overline{(u_1, n_1)} + \overline{(u_2, n_2)} = \overline{(u_1u_2, u_2n_1 + u_1n_2)}$. Prove that $r\overline{(u, n)} = \overline{(u, rn)}$ defines an action of R on $U^{-1}N$ making it into an R-module. [This is an example of *localization* considered in general in Section 4 of Chapter 15, cf. also Section 6 in Chapter 7.]

(b) Show that the map from $Q \times N$ to $U^{-1}N$ defined by sending $(a/b, n)$ to $\overline{(b, an)}$ for $a \in R, b \in U, n \in N$, is an R-balanced map, so induces a homomorphism f from $Q \otimes_R N$ to $U^{-1}N$. Show that the map g from $U^{-1}N$ to $Q \otimes_R N$ defined by $g(\overline{(u,n)}) = (1/u) \otimes n$ is well defined and is an inverse homomorphism to f. Conclude that $Q \otimes_R N \cong U^{-1}N$ as R-modules.

(c) Conclude from (b) that $(1/d) \otimes n$ is 0 in $Q \otimes_R N$ if and only if $rn = 0$ for some nonzero $r \in R$.

(d) If A is an abelian group show that $\mathbb{Q} \otimes_Z A = 0$ if and only if A is a torsion abelian group (i.e., every element of A has finite order).

Solution: TODO

□

Exercise 10.4.9. Suppose R is an integral domain with the quotient field Q and let N be any R-module. Let $Q \otimes_R N$ be the module obtained from N by extension of scalars from R to Q. Prove that the kernel of the R-module homomorphism $\iota : N \to Q \otimes_R N$ is the torsion submodule of N (cf. Exercise 8 in Section 1). [Use the previous exercise.]

Solution: TODO

□

Exercise 10.4.10. Suppose R is commutative and $N \cong R^n$ is a free R-module of rank n with R-module basis $e_1, \ldots, e_n$.

(a) For any nonzero R-module M show that every element of $M \otimes N$ can be written uniquely in the form $\sum_{i=1}^n m_i \otimes e_1$ where $m_i \in M$. Deduce that if $\sum_{i=1}^n m_i \otimes e_i = 0$ in $M \otimes N$ then $m_i = 0$ for $i = 1, \ldots, n$.

(b) Show that if $\sum m_i \otimes n_i = 0$ in $M \otimes N$ where n_i are merely assumed to be R-linearly independent then it is not necessarily true that all m_i are 0. [Consider $R = \mathbb{Z}, n = 1, M = \mathbb{Z}/2\mathbb{Z}$ and the element $1 \otimes 2$.]

Solution: TODO

□

Exercise 10.4.11. Let $\{e_1, e_2\}$ be a basis of $V = \mathbb{R}^2$. Show that the element $e_1 \otimes e_2 + e_2 \otimes e_1$ in $V \otimes_{\mathbb{R}} V$ cannot be written as a simple tensor $v \otimes w$ for any $v, w \in \mathbb{R}^2$.

Solution: TODO

□

Exercise 10.4.12. Let V be a vector space over the field F and let v, v' be nonzero elements of V. Prove that $v \otimes v' = v' \otimes v$ in $V \otimes_F V$ if and only if $v = av'$ for some $a \in F$.

Solution: TODO

□

Exercise 10.4.13. Prove that the usual dot product of vectors defined by letting $(a_1, \ldots, a_n) \cdots (b_1, \ldots, b_n)$ be $a_1 b_1 + \cdots + a_n b_n$ is a bilinear map from $R^n \times R^n$ to $\mathbb{R}$.

Solution: TODO

□

Exercise 10.4.14. Let I be an arbitrary nonempty index set and for each $i \in I$ let N_i be a left R-modules. Let M be a right R-module. Prove the group isomorphism: $M \otimes (\oplus_{i \in I} N_i) \cong \oplus_{i \in I}(M \otimes N_i)$, where the direct sum of an arbitrary collection of modules is defined in Exercise 20, Section 3. [Use the same argument as for the direct sum of two modules, taking care to note where the direct *sum* hypothesis is needed — cf. the next exercise.]

Solution: TODO

□

Exercise 10.4.15. Show that tensor products do not commute with direct products in general. [Consider the extension of scalars from $\mathbb{Z}$ to $\mathbb{Q}$ of the direct product of the modules $M_i = \mathbb{Z}/2^i\mathbb{Z}$, $i = 1, 2, \ldots$].

Solution: TODO

□

Exercise 10.4.16. Suppose R is commutative and let I and J be ideals of R, so R/I and R/J are naturally R-modules.

(a) Prove that every element of $R/I \otimes_R R/J$ can be written as a simple tensor of the form $(1 \bmod I) \otimes (r \bmod J)$.

(b) Prove that there is an R-module isomorphism $R/I \otimes_R R/J \cong R/(I+J)$ mapping $(r \bmod I) \otimes (r' \bmod J)$ to $rr' \bmod (I+J)$.

Solution: TODO

□

Exercise 10.4.17. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$. The ring $\mathbb{Z}/2\mathbb{Z} = R/I$ is naturally an R-module annihilated by both 2 and x.

(a) Show that the map $\phi : I \times I \to \mathbb{Z}/2\mathbb{Z}$ defined by

$$\phi(a_0 + a_1 x + \cdots a_n x^n, b_0 + b_1 x + \cdots b_m x^m) = \frac{a_0}{2} b_1 \bmod 2$$

is R-bilinear.

(b) Show that there is an R-module homomorphism from $I \otimes_R I \to \mathbb{Z}/2\mathbb{Z}$ mapping $p(x) \otimes q(x)$ to $\frac{p(0)}{2} q'(0)$ where q' denotes the usual polynomial derivative of q.

(c) Show that $2 \otimes x \neq x \otimes 2$ in $I \otimes_R I$.

Solution: TODO

□

Exercise 10.4.18. Suppose I is a principal ideal in the integral domain R. Prove that the R-modules $I \otimes_R I$ has no nonzero torsion elements (i.e. $rm = 0$ with $0 \neq r \in R$ and $m \in I \otimes_R I$ implies $m = 0$).

Solution: TODO

□

Exercise 10.4.19. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R - \mathbb{Z}[x]$ as in Exercise 17. Show that the nonzero element $2 \otimes x - x \otimes 2$ in $I \otimes_R I$ is a torsion element. Show in fact that $2 \otimes x - x \otimes 2$ is annihilated by both 2 and x and that the submodule of $I \otimes_R I$ generated by $2 \otimes x - x \otimes 2$ is isomorphic to R/I.

Solution: TODO

$\square$

Exercise 10.4.20. Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$. Show that the element $2 \otimes 2 + x \otimes x$ in $I \otimes_R I$ is not a simple tensor, i.e. cannot be written as $a \otimes b$ for some $a, b \in I$.

Solution: TODO

$\square$

Exercise 10.4.21. Suppose R is commutative and let I and J be ideals of R.

(a) Show that there is a surjective R-module homomorphism from $I \otimes_R J$ to the product ideal IJ mapping $I \otimes J$ to the element ij.

(b) Give an example to show that the map in (a) need not be injective (cf. Exercise 17).

Solution: TODO

$\square$

Exercise 10.4.22. Suppose that m is a left and a right R-module such that $rm = mr$ for all $r \in R$ and $m \in M$. Show that the elements ${}_1r_2$ and r_2r_1 act the same on M for every $r_1, r_2 \in R$. (This explains why the assumption that R is commutative in the definition of an R-algebra is a fairly natural one.)

Solution: TODO

$\square$

Exercise 10.4.23. Verify the details that the multiplication in Proposition 19 makes $A \otimes_R B$ into an R-algebra.

Solution: TODO

$\square$

Exercise 10.4.24. Prove that the extension of scalars from $\mathbb{Z}$ to the Gaussian integers $\mathbb{Z}[i]$ of the ring $\mathbb{R}$ is isomorphic to $\mathbb{C}$ as a ring:$\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$ as rings.

Solution: TODO

$\square$

Exercise 10.4.25. Let R be a subring of the commutative ring S and let x be an indeterminate over S. Prove that $S[x]$ and $S \otimes_R R[x]$ are isomorphic as S-algebras.

Solution: TODO

$\square$

Exercise 10.4.26. Let S be a commutative ring containing R (with $1_s = 1_R$) and let $x_1, \ldots, x_n$ be independent indeterminates over the ring S. Show that for every ideal I in the polynomial ring $R[x_1, \ldots, x_n]$ that $S \otimes_R (R[x_1, \ldots, x_n]/I) \cong S[x_1, \ldots, x_n]/IS[x_1, \ldots, x_n]$.

Solution: TODO

□

Exercise 10.4.27. The next exercise shows the ring $C \otimes_R \mathbb{C}$ introduced at the end of this section is isomorphic to $\mathbb{C} \times \mathbb{C}$. One may also prove this via Exercise 26 and Proposition 16 in Section 9.5, since $\mathbb{C} \cong \mathbb{R}[x]/(x^2+1)$. The ring $C \times \mathbb{C}$ is also discussed in Exercise 23 of Section 1.

(a) Write down a formula for the multiplication of two elements $a \cdot 1 + b \cdot e_2 + c \cdot e_3 + d \cdot e_4$ and $a' \cdot 1 + b' \cdot e_2 + c' \cdot e_3 + d. \cdot e_4$ in the example $A = \mathbb{C} \otimes \mathbb{R}\mathbb{C}$ following Proposition 21 (where $1 = 1 \otimes 1$ is the identity of A).

(b) Let $\epsilon_1 = \frac{1}{2}(1 \otimes 1 + i \otimes i)$ and let $\epsilon_2 = \frac{1}{2}(1 \otimes 1 - i \otimes i)$. Show that $\epsilon_1\epsilon_2 = 0$, $\epsilon_1 + \epsilon_2 = 1$ and $\epsilon_j^2 = \epsilon_j$ for $j = 1, 2$ (ϵ_1 and ϵ_2 are called *orthogonal idempotents* in A). Deduce that A is isomorphic as a ring to the direct product of two principal ideals:$A \cong A\epsilon_1 \times A\epsilon_2$ (cf. Exercise 1, Section 7.6).

(c) Prove that the map $\phi : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ by $\phi(z_1, z_2) = (z_1z_2, z_1\overline{z_2})$, where $\overline{z_2}$ denotes the complex conjugate of z_2, is an $\mathbb{R}$-bilinear map.

(d) Let Φ be the $\mathbb{R}$-module homomorphism from A to $\mathbb{C} \times \mathbb{C}$ obtained from ϕ in (c). Show that $\Phi(\epsilon_1) = (0, 1)$ and $\Phi(\epsilon_2) = (1, 0)$. Show also that Φ is $\mathbb{C}$-linear, where the action of $\mathbb{C}$ is on the left tensor factor in A and on both factors in $\mathbb{C} \times \mathbb{C}$. Deduce that Φ is surjective. Show that Φ is a $\mathbb{C}$-algebra isomorphism.

Solution: TODO

□

10.5 Exact Sequences—Projective, Injective, and Flat Modules

Exercise 10.5.1. Suppose that

$$\begin{array}{ccccc} A & \xrightarrow{\psi} & B & \xrightarrow{\phi} & C \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow \\ A' & \xrightarrow{\psi'} & B' & \xrightarrow{\phi'} & C' \end{array}$$

is a commutative diagram of groups and that the rows are exact. Prove that

(a) If ϕ and α are surjective, and β is injective then γ is injective. [If $c \in \ker\gamma$, show there is a $b \in B$ with $\phi(b) = c$. Show that $\phi'(\beta(b)) = 0$ and deduce that $\beta(b) = \phi'(a')$ for some $a' \in A'$. Show that there is an $a \in A$ with $\alpha(a) = a'$ and that $\beta(\psi(a)) = \beta(b)$. Conclude that $b = \psi(a)$ and hence $c = \phi(b) = 0$.]

(b) If ϕ', α and γ are injective, then β is injective.

(c) If ϕ, α and γ are surjective, then β is surjective.

(d) If β is injective, α and ϕ are surjective, then γ is injective.

(e) If β is surjective, γ and ψ' are injective, then α is surjective.

Solution: TODO

□

Exercise 10.5.2. Suppose that

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[d, "\delta"'] \\
A' \arrow[r, "'"] & B' \arrow[r] & C' \arrow[r] & D'
\end{tikzcd}$$

is a commutative diagram of groups and that the rows are exact. Prove that

(a) If α is surjective and β, δ are injective, then γ is injective.

(b) If δ is injective, and α, γ are surjective, then β is surjective.

Solution: TODO

$\square$

Exercise 10.5.3. Let P_1 and P_2 be R-modules. Prove that $P_1 \oplus P_2$ is a projective R-module if and only if both P_1 and P_2 are projective.

Solution: TODO

$\square$

Exercise 10.5.4. Let Q_1 and Q_2 be R-modules. Prove that $Q_1 \oplus Q_2$ is an injective R-modules if and only if both Q_1 and Q_2 are injective.

Solution: TODO

$\square$

Exercise 10.5.5. Let A_1 and A_2 be R-modules. Prove that $A_1 \oplus A_2$is a flat R-modules if and only if both A_1 and A_2 are flat. More generally, prove that an arbitrary direct sum $\sum A_i$ of R-modules is flat if and only if each A_i is flat. [Use the fact that tensor product sommutes with arbitrary direct sums.]

Solution: TODO

$\square$

Exercise 10.5.6. Prove that the following are equivalent for a ring R:

(i) Every R-module is projective.

(ii) Every R-module is injective.

Solution: TODO

$\square$

Exercise 10.5.7. Let A be a nonzero finite abelian group.

(a) Prove that A is not a projective $\mathbb{Z}$-module.

(b) Prove that A is not an injective $\mathbb{Z}$-module.

Solution: TODO

$\square$

Exercise 10.5.8. Let Q be a nonzero divisible $\mathbb{Z}$-module. Prove that Q is not a projective $\mathbb{Z}$-module. Deduce that the rational numbers $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module. [Show first that if F is any free module then $\cap_{n=1}^{\infty} nF = 0$ (use a basis of F to prove this). Now suppose to the contrary that Q is projective and derive a contradiction from Proposition 30(4).]

Solution: TODO

$\square$

Exercise 10.5.9. Assume R is commutative with 1.

(a) Prove that the tensor product of two free R-modules is free. [Use the fact that tensor products commute with arbitrary direct sums.]

(b) Use (a) to prove that the tensor product of two projective R-modules is projective.

Solution: TODO

$\square$

Exercise 10.5.10. Let R and W be rings with 1 and let M and N be left R-modules. Assume also that M is an (R, S)-bimodule.

(a) For $s \in S$ and for $\phi \in \mathrm{Hom}_R(M, N)$ define $(s\phi) : M \to N$ by $(s\phi)(m) = \phi(ms)$. Prove that $s\phi$ is a homomorphism of left R-modules, and that this action of S on $\mathrm{Hom}_R(M, N)$ makes it into a *left* S-module.

(b) Let $S = R$ and let $M = R$ (considered as an (R, R)-bimodule by left and right ring multiplication on itself). For each $n \in N$ define $\phi_n : R \to N$ by $\phi_n(r) = rn$, i.e. ϕ_n is the unique R-module homomorphism mapping 1_R to n. Show that $\phi_n \in \mathrm{Hom}_R(R, N)$. Use part (a) to show that the map $n \mapsto \phi_n$ is an isomorphism of left R-modules:$N \cong \mathrm{Hom}_R(R, N)$.

(c) Deduce that if N is a free (respective, projective, injective, flat) left R-module, then $\mathrm{Hom}_R(R, N)$ is also a free (respective, projective, injective, flat) left R-module.

Solution: TODO

$\square$

Exercise 10.5.11. Let R and W be rings with 1 and let M and N be left R-modules. Assume also that M is an (R, S)-bimodule.

(a) For $s \in S$ and for $\phi \in \mathrm{Hom}_R(M, N)$ define $(\phi s) : M \to N$ by $(\phi s)(m) = \phi(m)s$. Prove that $s\phi$ is a homomorphism of left R-modules, and that this action of S on $\mathrm{Hom}_R(M, N)$ makes it into a *right* S-module. Deduce that $\mathrm{Hom}_R(M, R)$ is a right R-module, for any R-module M — called the *dual module* to M.

(b) Let $N = R$ be considered as an (R, R) bimodule as usual. Under the action defined in part (a) show that the map $r \mapsto \phi_r$ is an isomorphism of right R-modules: $\mathrm{Hom}_R(R, R) \cong R$, where ϕ_r is the homomorphism that maps 1_R to r. Deduce that if M is a finitely generated free left R-module, then $\mathrm{Hom}_R(M, R)$ is a free right R-module of the same rank. (cf. also Exercise 13).

(c) Show that if M is a finitely generated projective R-module then its dual module $\mathrm{Hom}_R(M, R)$ is also projective.

Solution: TODO

□

Exercise 10.5.12. Let A be an R-module, let I be any nonempty index set and for each $i \in I$ let B_i be an R-module. Prove the following isomorphisms of abelian groups; when R is commutative prove also that these are R-module isomorphisms. (Arbitrary direct sums and direct products of modules are introduced in Exercise 20 of Section 3.)

(a) $\operatorname{Hom}_R(\bigoplus_{i\in I} B_i, A) \cong \prod_{i\in I} \operatorname{Hom}_R(B_i, A)$

(b) $\operatorname{Hom}_R(A, \prod_{i\in I} B_i) \cong \prod_{i\in I} \operatorname{Hom}_R(A, B_i)$.

Solution: TODO

□

Exercise 10.5.13. (a) Show that the dual of the free $\mathbb{Z}$-module with countable basis is not free. [Use the preceding exercise and Exercise 24, Section 3.] (See also Exercise 5 in Section 11.3.)

(b) Show that the dual of the free $\mathbb{Z}$-module with countable basis is not projective. [You may use the fact that any submodule of a free $\mathbb{Z}$-module is free.]

Solution: TODO

□

Exercise 10.5.14. Let $0 \longrightarrow L \xrightarrow{\psi} M \xrightarrow{\phi} N \longrightarrow 0$ be a sequence of R-modules.

(a) Prove that the associated sequence

$$0 \longrightarrow \operatorname{Hom}_R(D, L) \xrightarrow{\psi'} \operatorname{Hom}_R(D, M) \xrightarrow{\phi'} \operatorname{Hom}_R(D, N) \longrightarrow 0$$

is a short exact sequence of abelian groups for all R-modules D if and only if the original sequence is a split short exact sequence. [To show the sequence splits, take $D = N$ and show the lift of the identity map in $\operatorname{Hom}_R(N, N)$ to $\operatorname{Hom}_R(N, M)$ is a splitting homomorphism for ϕ.]

(b) Prove that the associated sequence

$$0 \longrightarrow \operatorname{Hom}_R(N, D) \xrightarrow{\phi'} \operatorname{Hom}_R(M, D) \xrightarrow{\psi'} \operatorname{Hom}_R(L, D) \longrightarrow 0$$

is a short exact sequence of abelian groups for all R-modules D if and only if the original sequence is a split short exact sequence.

Solution: TODO

□

Exercise 10.5.15. Let M be a left $\mathbb{Z}$-module and let R be a ring with 1.

(a) Show that $\operatorname{Hom}_{\mathbb{Z}}(R, M)$ is a left R-module under the action $(r\phi)(r') = \phi(r'r)$ (see Exercise 10).

(b) Suppose that $0 \longrightarrow A \xrightarrow{\psi} B$ is an exact sequence of R-modules. Prove that if every $\mathbb{Z}$-module homomorphism f from A to M lifts to a $\mathbb{Z}$-module homomorphism F from B to M with $f = F \circ \psi$, then every R-module homomorphism f' from A to $\operatorname{Hom}_Z(R, M)$ lifts to an R-module homomorphism F' from B to $\operatorname{Hom}_{\mathbb{Z}}(R, M)$ with $f' = F' \circ \psi$. [Given f', show that $f(a) = f'(a)(1_R)$ defines a $\mathbb{Z}$-module homomorphism of A to M. If F is the associated lift of f to B, show that $F'(b)(r) = F(rb)$ defines an R-modules homomorphism from B to $\operatorname{Hom}_Z(R, M)$ that lifts f'.]

(c) Prove that if Q is an injective $\mathbb{Z}$-module then $\operatorname{Hom}_{\mathbb{Z}}(R, Q)$ is an injective R-module.

Solution: TODO

□

Exercise 10.5.16. This exercise proves Theorem 38 that every left R-module M is contained in an injective left R-module.

(a) Show that M is contained in an injective $\mathbb{Z}$-module Q. [M is a $\mathbb{Z}$-module—use Corollary 37.]

(b) Show that $\operatorname{Hom}_R(R, M) \subseteq \operatorname{Hom}_{\mathbb{Z}}(R, M) \subseteq \operatorname{Hom}_{\mathbb{Z}}(R, Q)$.

(c) Use the R-module isomorphism $M \cong \operatorname{Hom}_R(R, M)$ (Exercise 10) and the previous exercise to conclude that M is contained in an injective R-module.

Solution: TODO

□

Exercise 10.5.17. This exercise completes the proof of Proposition 34. Suppose that Q is an R-module with the property that every short exact sequence $0 \longrightarrow Q \longrightarrow M_1 \longrightarrow N \longrightarrow 0$ splits and suppose that the sequence $0@>>> L@>\psi>> M$ is exact. Prove that every R-module homomorphism f from L to Q can be lifted to an R-module homomorphism F from M to Q with $f = F \circ \psi$. [By the previous exercise, Q is contained in an injective R-module. Use the splitting property together with Exercise 4 (noting that Exercise 4 can be proved using (2) in Proposition 34 as the definition of an injective module).]

Solution: TODO

□

Exercise 10.5.18. Prove that the injective hull of the $\mathbb{Z}$-module $\mathbb{Z}$ is $\mathbb{Q}$ [Let H be the injective hull of $\mathbb{Z}$ and argue that $\mathbb{Q}$ contains an isomorphic copy of H. Use the divisibility of H to show that $1/n \in H$ for all nonzero integers n, and deduce that $H = \mathbb{Q}$.]

Solution: TODO

□

Exercise 10.5.19. If F is a field, prove that the injective hull of F is F.

Solution: TODO

□

Exercise 10.5.20. Prove that the polynomial ring $R[x]$ with indeterminate x over the commutative ring R is a flat R-module.

Solution: TODO

□

Exercise 10.5.21. Let R and S be rings with 1 and suppose M is a right R-module, and N is an (R, S)-bimodule. If M is flat over R and N is flat as an S-module prove that $M \otimes_R N$ is flat as a right S-module.

Solution: TODO

□

Exercise 10.5.22. Suppose that R is a commutative ring and that M and N are flat R-modules. Prove that $M \otimes_R N$ is a flat R-module. [Use the previous exercise.]

Solution: TODO

$\square$

Exercise 10.5.23. Prove that the (right) module $M \otimes_R S$ obtained by changing the base from the ring R to the ring S (by some homomorphism $f : R \to S$ with $f(1_R) = 1_S$ cf. Example 6 following Corollary 12 in Section 4) of the flat (right) R-module M is a flat S-module.

Solution: TODO

$\square$

Exercise 10.5.24. Prove that A is a flat R-module if and only if for any left R-modules L and M where L is *finitely egenerated*, then $\psi : L \to M$ is injective implies that laso $1 \otimes \psi : A \otimes_R L \to A \otimes_R M$ is injective. [Use the techniques if the proof of corollary 42.]

Solution: TODO

$\square$

Exercise 10.5.25. *(*A Flatness Criterion) Parts (a)-(c) of this exercise prove that A is a flat R-module if and only if for every finitely generated ideal I of R, the map from $A \otimes_R I \to A \otimes_R R \cong A$ induced by the inclusion $I \subseteq R$ is again injective (or equivalently, $A \otimes_R I \cong AI \subseteq A$).

(a) Prove that if A is flat then $A \otimes_R I \to A \otimes_R R$ is injective.

(b) If $A \otimes_R I \to A \otimes_R R$ is injective for every finitely generated ideal I, prove that $A \otimes_R I \to A \otimes_R R$ is injective for every ideal I. Show that if K is any submodule of a finitely generated free module F then $A \otimes_R K \to A \otimes_R F$ is injective. Show that the same is true for any free module F. [Cf. the proof of Corollary 42.]

(c) Under the assumption in (b), suppose L and M are R-modules and $L \xrightarrow{\psi} M$ is injective. Prove that $A \otimes_R L \xrightarrow{1 \otimes \psi} A \otimes_R M$ is injective and conclude that A is flat. [Write M as a quotient of the free module F, giving a short exact sequence

$$0 \longrightarrow K \longrightarrow F \xrightarrow{f} M \longrightarrow 0.$$

Show that if $J = f^{-1}(\psi(L))$ and $\iota : J \to F$ is the natural injection, then the diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "id"'] & J \arrow[r] \arrow[d, "\iota"'] & L \arrow[r] \arrow[d, "\psi"'] & 0 \\ 0 \arrow[r] & K \arrow[r] & F \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

is commutative with exact rows. Show that the induced diagram

$$\begin{tikzcd} A \otimes_R K \arrow[r] \arrow[d, "id"'] & A \otimes_R J \arrow[r] \arrow[d, "1\otimes\iota"'] & A \otimes_R L \arrow[r] \arrow[d, "1\otimes\psi"'] & 0 \\ A \otimes_R K \arrow[r] & A \otimes_R F \arrow[r] & A \otimes_R M \arrow[r] & 0 \end{tikzcd}$$

is commutative with exact rows. Use (b) to show that $1 \otimes \iota$ is injective, then use Exercise 1 to conclude that $1 \otimes \psi$ is injective.]

(d) (A Flatness Criterion for quotients) Suppose $A = F/K$ where F is flat (e.g., if F is free) and K is an R-submodule of F. Prove that A is flat if and only if $FI \cap K = KI$ for every finitely generated ideal I of R. [Use (a) to prove $F \otimes_R I \cong FI$ and observe the image of $K \otimes_R I$ is KI; tensor the exact sequence $0 \to K \to F \to A \to 0$ with I to prove that $A \otimes_R I \cong FI/KI$, and apply the flatness criterion.]

Solution: TODO

□

Exercise 10.5.26. Suppose R is a PID. This exercise proves that A is a flat R-module if and only if A is a torsion free R-module (i.e., if $a \in A$ is nonzero and $r \in R$, then $ra = 0$ implies $r = 0$).

(a) Suppose that A is flat and for fixed $r \in R$ consider the map $\psi_r : R \to R$ defined by multiplication by r: $\psi_r(x) = rx$. If r is nonzero show that ψ_r is an injection. Conclude from the flatness of A that the map from A to A defined by mapping a to ra is injective and that A is torsion free.

(b) Suppose that A is torsion free. If I is a nonzero ideal of R, then $I = rR$ for some nonzero $r \in R$. Show that the map ψ_r in (a) induces an isomorphism $R \cong I$ of R-modules and that the composite $R \xrightarrow{\psi} I \xrightarrow{\iota} R$ of ψ_r with the inclusion $\iota : I \subseteq R$ is multiplication by r. Prove that the composite $A \otimes_R R \xrightarrow{1 \otimes \psi_r} A \otimes_R I \xrightarrow{1 \otimes \iota} A \otimes_R R$ corresponds to the map $a \mapsto ra$ under the identification $A \otimes_R R = A$ and that this composite is injective since A is torsion free. Show that $1 \otimes \psi_r$ is an isomorphism and deduce that $1 \otimes i$ is injective. Use the previous exercise to conclude that A is flat.

Solution: TODO

□

Exercise 10.5.27. Let M, A and B be R-modules.

(a) Suppose $f : A \to M$ and $g : B \to M$ are R-module homomorphisms. Prove that $X = \{(a, b) \mid a \in A, b \in B \text{ with } f(a) = g(b)\}$ is an R-submodule of the direct sum $A \oplus B$ (called the *pullback* or *fiber product* of f and g) and that there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & M \end{tikzcd}$$

where π_1 and π_2 are the natural projections onto the first and second components.

(b) Suppose $f' : M \to A$ and $g' : M \to B$ are R-module homomorphisms. Prove that the quotient Y of $A \oplus B$ by $\{(f'(m), -g'(m)) \mid m \in M\}$ is an R-module (called the *pushout* or *fiber sum* of f' and g') and that there is a commutative diagram

$$\begin{tikzcd} M \arrow[r, "g'"] \arrow[d, "f'"'] & B \arrow[d, "\pi_2'"] \\ A \arrow[r, "\pi_1'"] & X \end{tikzcd}$$

where ϕ_1' and ϕ_2' are the natural maps to the quotient induced by the maps into the first and second components.

Solution: TODO

$\square$

Exercise 10.5.28. (a) *(Schanuel's Lemma)* If $0 \longrightarrow K \longrightarrow P \xrightarrow{\phi} M \longrightarrow 0$ and $0 \longrightarrow K' \longrightarrow P' \xrightarrow{\phi'} M \longrightarrow 0$ are exact sequences of R-modules where P and p' are projective, prove that $P \oplus K' \cong P' \oplus K$ as R-modules. [Show that there is an exact sequence $0 \longrightarrow \ker\phi \longrightarrow X \xrightarrow{\pi} P \longrightarrow 0$ with $\ker\pi \cong K'$, where X is the fiber product of ϕ and ϕ' as in the previous exercise. Deduce that $X \cong P \oplus K'$. Show similarly that $X \cong P' \oplus K$.]

(b) If $0 \longrightarrow M \longrightarrow Q \xrightarrow{\psi} L \longrightarrow 0$ and $0 \longrightarrow M \longrightarrow Q' \xrightarrow{\psi'} L' \longrightarrow 0$ are exact sequences of R-modules where Q and Q' are injective, prove that $Q \oplus L' \cong Q' \oplus L$ as R-modules.

The R modules M and N are said to be *projectively equivalent* if $M \oplus P \cong N \oplus P'$ for some projective modules P, P'. Simiarly, M and N are injective equivalent if $M \oplus Q \cong N \oplus Q'$ for some injective modules Q, Q'. The previous exercise shows K and K' are projectively equivalent and L and L' are injectively equivalent.

Solution: TODO

$\square$