2015 Algebra Prelim

September 14, 2015

INSTRUCTIONS: Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in

1. Classify all groups of order 2012 up to isomorphism. (Hint: 503 is prime.)

2. For any positive integer n, let G_n be the group generated by a and b subject to the following three relations:

$$a^2 = 1, \quad b^2 = 1, \quad \text{and} \quad (ab)^n = 1.$$

(a) Find the order of the group G_n.
(b) Classify all irreducible complex representations of G_4 up to isomorphism.

3. Let R be a (commutative) principal ideal domain, let M and N be finitely generated free R-modules, and let $\phi : M \to N$ be an R-module homomorphism.
(a) Let K be the kernel of ϕ. Prove that K is a direct summand of M.
(b) Let C be the image of ϕ. Show by example (specifying R, M, N and ϕ) that C need not be a direct summand of N.

4. Let G be an abelian group. Prove that the group ring $\mathbb{Z}[G]$ is noetherian if and only if G is finitely generated.

5. Let $M_3(\mathbb{R})$ be the 3×3-matrix algebra over the real numbers $\mathbb{R}$. For any $b \in \mathbb{R}$ let $B \in M_3(\mathbb{R})$ be the matrix $\begin{pmatrix} 1 & b & 0 \\ b & 1 & b \\ 0 & b & 1 \end{pmatrix}$. Find the set of numbers b so that the matrix equation $X^2 = B$ has at least one, and only finitely many, solutions in $M_3(\mathbb{R})$.

6. Determine the Galois groups of the following polynomials over $\mathbb{Q}$.
(a) $f(x) = x^4 + 4x^2 + 1$
(b) $f(x) = x^4 + 4x^2 - 5$

7. Prove that if A is a finite abelian group, then $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \cong A$. (Here $\mathrm{Ext}^1_{\mathbb{Z}}(-,-)$ is also sometimes written as $\mathrm{Ext}(-,-)$.

8. Let A be the $\mathbb{C}$-algebra $\mathbb{C}[x, y]$, and define algebra automorphisms σ and τ of A by

$$\sigma(x) = y, \quad \sigma(x) = y$$

and

$$\tau(x) = x, \quad \tau(y) = \zeta y,$$

where $\zeta \in \mathbb{C}$ is a primitive third root of unity (namely, $\zeta \neq 1$ and $\zeta^3 = 1$). Let G be the group of algebra automorphisms of A generated by σ and τ. Define

$$A^G = \{f \in A \mid \phi(f) = f \text{ for all } \phi \in G\}.$$

Then A^G is a subalgebra of A – you do not need to prove this. Describe the algebra A^G by finding a set of generators and a set of relations.