

2015 Algebra Prelim

September 14, 2015

INSTRUCTIONS: Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in

1. Let \mathbb{Q}^\times be the nonzero elements of \mathbb{Q} , a group under multiplication.
 - (a) Prove that the additive group of \mathbb{Q} has no maximal proper subgroups.
 - (b) Is the same statement true for the multiplicative group \mathbb{Q}^\times ?

2. Let V be a finite-dimensional vector space over a field F of characteristic 0. Let $B : V \times V \rightarrow F$ be a non-degenerate, skew-symmetric bilinear form. (In particular, we have $B(x, y) = -B(y, x)$ for all $x, y \in V$.) If U is a subset of V , let

$$U^\perp = \{v \in V \mid B(u, v) = 0 \text{ for all } u \in U\}.$$

- (a) Let U be a subspace of V . Prove that U^\perp is a subspace of V and that

$$\dim_F(U) + \dim_F(U^\perp) = \dim_F(V).$$

- (b) Prove that there exists a subspace W of V such that $W^\perp = W$.

3. (a) Suppose that G is a finitely-generated group. Let n be a positive integer. Prove that G has only finitely many subgroups of index n .

(b) Let p be a prime number. If G is any finitely-generated abelian group, let $t_p(G)$ denote the number of subgroups of G of index p . Determine the possible values of $t_p(G)$ as G varies over all finitely-generated abelian groups.

4. Suppose that G is a finite group of order 2013. Prove that G has a normal subgroup N of index 3 and that N is a cyclic group. Furthermore, prove that the center of G has order divisible by 11. (You will need the factorization $2013 = 3 \cdot 11 \cdot 61$.)

5. Let V be a finite dimensional vector space over \mathbb{C} . Let $n = \dim_{\mathbb{C}}(V)$. Let $T : V \rightarrow V$ be a linear map. Suppose that the following statement is true.

For every $c \in \mathbb{C}$, the subspace $\{v \in V \mid T(v) = cv\}$ of V has dimension 0 or 1.

Prove that there exists a vector $w \in V$ such that $\{w, T(w), \dots, T^{n-1}(w)\}$ is a linearly independent set.

6. This question concerns an extension K of \mathbb{Q} such that $[K : \mathbb{Q}] = 8$. Assume that K/\mathbb{Q} is Galois and let $G = \text{Gal}(K/\mathbb{Q})$. Furthermore, assume that G is nonabelian.

- (a) Prove that K has a unique subfield F such that F/\mathbb{Q} is Galois and $[F : \mathbb{Q}] = 4$.
- (b) Prove that F has the form $F = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ where d_1 and d_2 are nonzero integers.
- (c) Suppose that G is the quaternionic group. Prove that d_1 and d_2 are positive integers.

7. Let $R = \mathbb{C}[x_1, \dots, x_n]$ be the polynomial ring over \mathbb{C} in n indeterminates x_1, \dots, x_n . Let S_n be the n -th symmetric group. If $\sigma \in S_n$, then we can identify σ with the automorphism of R defined

as follows: $\sigma(c) = c$ for all $c \in \mathbb{C}$, and $\sigma(x_i) = x_{\sigma(i)}$ for all i , $1 \leq i \leq n$. Suppose that G is any subgroup of S_n . Let

$$S = R^G = \{r \in R \mid \sigma(r) = r \text{ for all } \sigma \in G\}.$$

Prove that S is a finitely-generated \mathbb{C} -algebra.

8. This question concerns the polynomial ring $R = \mathbb{Z}[x, y]$ and the ideal $I = (5, x^2 + 2)$ in R .
- (a) Prove that I is a prime ideal of R and that R/I is a *PID*.
 - (b) Give an explicit example of a maximal ideal of R which contains I . (Give a set of generators for such an ideal.)
 - (c) Show that there are infinitely many distinct maximal ideals in R which contain I .