2015 Algebra Prelim

September 14, 2015

INSTRUCTIONS: Do as many of the eight problems as you can. Four completely correct solutions will be a pass; a few complete solutions will count more than many partial solutions. Always carefully justify your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in

1. Let $\mathbb{Q}^\times$ be the nonzero elements of $\mathbb{Q}$, a group under multiplication.
(a) Prove that the additive group of $\mathbb{Q}$ has no maximal proper subgroups.
(b) Is the same statement true for the multiplicative group $\mathbb{Q}^\times$?

2. Let V be a finite-dimensional vector space over a field F of characteristic 0. Let $B : V \times V \to F$ be a non-degenerate, skew-symmetric bilinear form. (In particular, we have $B(x, y) = B(y, x)$ for all $x, y \in V$.) If U is a subset of V , let

$$U^\perp = \{v \in V \mid B(u, v) = 0 \text{ for all } u \in U\}.$$

(a) Let U be a subspace of V . Prove that $U^\perp$ is a subspace of V and that

$$\dim_F(U) + \dim_F(U^\perp) = \dim_F(V).$$

(b) Prove that there exists a subspace W of V such that $W^\perp = W$.

3. (a) Suppose that G is a finitely-generated group. Let n be a positive integer. Prove that G has only finitely many subgroups of index n.
(b) Let p be a prime number. If G is any finitely-generated abelian group, let $t_p(G)$ denote the number of subgroups of G of index p. Determine the possible values of $t_p(G)$ as G varies over all finitely-generated abelian groups.

4. Suppose that G is a finite group of order 2013. Prove that G has a normal subgroup N of index 3 and that N is a cyclic group. Furthermore, prove that the center of G has order divisible by 11. (You will need the factorization $2013 = 3 \cdot 11 \cdot 61$.)

5. Let V be a finite dimensional vector space over $\mathbb{C}$. Let $n = \dim_{\mathbb{C}}(V)$. Let $T : V \to V$ be a linear map. Suppose that the following statement is true.

For every $c \in \mathbb{C}$, the subspace $\{v \in V \mid T(v) = cv\}$ of V has dimension 0 or 1.

Prove that there exists a vector $w \in V$ such that $\{w, T(w), \ldots, T^{n-1}(w)\}$ is a linearly independent set.

Solution:

The condition on T implies that T has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ with associated eigenvectors $v_1, \ldots, v_n$ which form a basis for V. We claim that choosing $w = v_1 + \cdots + v_n$ makes $\{w, T(w), \ldots, T^{n-1}(w)\}$ a linearly independent set. Note that

$$T^i(w) = \lambda_1^i v_1 + \cdots + \lambda_n^i v_n$$

and so to argue that $\{w, T(w), \ldots, T^{n-1}(w)\}$ is linearly independent it suffices to argue that the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & & \ddots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}$$

has linearly independent rows, i.e. that it is invertible. This is a Vandermonde matrix, and its determinant is given by

$$\prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i).$$

Since $\lambda_j \neq \lambda_i$ when $j \neq i$, we see that this determinant is nonzero and so the matrix is invertible. Hence $\{w, T(w), \ldots, T^{n-1}(w)\}$ forms a linearly independent set.

6. This question concerns an extension K of $\mathbb{Q}$ such that $[K : \mathbb{Q}] = 8$. Assume that $K/\mathbb{Q}$ is Galois and let $G = \mathrm{Gal}(K/\mathbb{Q})$. Furthermore, assume that G is nonabelian.

(a) Prove that K has a unique subfield F such that $F/\mathbb{Q}$ is Galois and $[F : \mathbb{Q}] = 4$.

(b) Prove that F has the form $F = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ where d_1 and d_2 are nonzero integers.

(c) Suppose that G is the quaternionic group. Prove that d_1 and d_2 are positive integers.

7. Let $R = \mathbb{C}[x_1, ..., x_n]$ be the polynomial ring over $\mathbb{C}$ in n indeterminates $x_1, ..., x_n$. Let S_n be the n-th symmetric group. If $\sigma \in S_n$, then we can identify σ with the automorphism of R defined as follows: $\sigma(c) = c$ for all $c \in \mathbb{C}$, and $\sigma(x_i) = x_{\sigma(i)}$ for all i, $1 \leq i \leq n$. Suppose that G is any subgroup of S_n. Let

$$S = R^G = \{r \in R \mid \sigma(r) = r \text{ for all } \sigma \in G\}.$$

Prove that S is a finitely-generated $\mathbb{C}$-algebra.

8. This question concerns the polynomial ring $R = \mathbb{Z}[x, y]$ and the ideal $I = (5, x^2 + 2)$ in R.

(a) Prove that I is a prime ideal of R and that R/I is a PID.

(b) Give an explicit example of a maximal ideal of R which contains I. (Give a set of generators for such an ideal.)

(c) Show that there are infinitely many distinct maximal ideals in R which contain I.

Solution:

(a)

Note that

$$R/I = \mathbb{Z}[x, y]/(5, x^2 + 2) \cong (\mathbb{Z}/5\mathbb{Z})[x, y]/(x^2 + 2).$$

The polynomial x^2+2 is irreducible over $\mathbb{Z}/5\mathbb{Z}$ (having no roots) and so $\mathbb{Z}/5\mathbb{Z}[x]/(x^2+2)$ is a field. Let F denote this field and observe that

$$R/I \cong F[y].$$

This is a polynomial ring over a field, and so is a PID. Since R/I is an integral domain we conclude that I must be prime.

(b)

The ideal $J = (5, x^2+2, y)$ is maximal since the quotient by this ideal is just $F[y]/(y) \cong F$, a field.

(c)

For any prime $p \in \mathbb{Z}$ not equal to 5 let $J_p = (5, x^2 + 2, py)$. Clearly the various J_p are distinct since the smallest positive integer n for which $ny \in J_p$ is always p and the various p are distinct. Moreover, since $p \neq 5$ for J_p we have p and 5 relatively prime so that p is invertible mod 5. Thus $(py) = (y)$ as ideals in $F[y]$. We then have that $R/J_p \cong F[y]/(yp) = F[y]/(y) \cong F$, so R/J_p is a field and J_p is maximal.