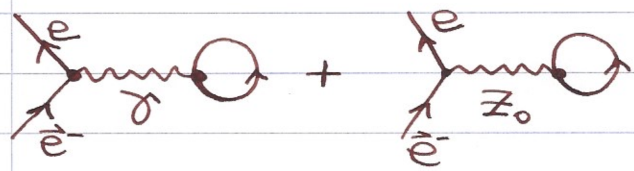
PARITY VIOLATING ELECTRON-NUCLEUS SCATTERING IN AN EIKONAL APPROXIMATION

May 2020

WE WANT TO COMPUTE BOTH THE NORMAL AND THE PARITY VIOLATING ASYMMETRY IN ELASTIC ELECTRON-NUCLEUS SCATTERING USING AN EIKONAL APPROXIMATION TO THE DIRAC EQUATION. IN PARTICULAR, THE PARITY-VIOLATING ASYMMETRY EMERGES FROM THE INTERFERENCE OF PHOTON AND Z_0 EXCHANGES:



WHICH IN A PLANE-WAVE IMPULSE APPROXIMATION IS GIVEN BY

$$A_{PV} = -\frac{G_F Q^2}{4\pi\alpha\sqrt{2}} \cdot \frac{Q_{wk} F_{wk}(Q^2)}{Z F_{ch}(Q^2)}\ ;$$ SEE PRC 100, 054301 (2019)

ALTOUGH QUALITATIVELY CORRECT, THE ABOVE RESULT HAS TO BE IMPROVED BY INCORPORATING DISTORTIONS TO THE PLANE-WAVE BEHAVIOR COMING FROM THE MULTIPLE INTERACTIONS BETWEEN THE ELECTRON AND THE NUCLEUS. THE DISTORTING POTENTIAL EXPERIENCED BY THE ELECTRON CONSISTS OF A TIMELIKE VECTOR (COULOMB) AND AXIAL-VECTOR TERMS; THAT IS

$$\hat{V}(\vec{r}) = V(r) + \gamma_5 A(r)$$

WHERE THE TIMELIKE COULOMB POTENTIAL IS GIVEN BY

$$V(r) = -Ze^2 \int \frac{\rho_{ch}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3r'$$

WHILE THE AXIAL VECTOR POTENTIAL BY

$$A(r) = \frac{G_F}{2\sqrt{2}} Q_{wk}\, \rho_{wk}(r)$$

WHERE

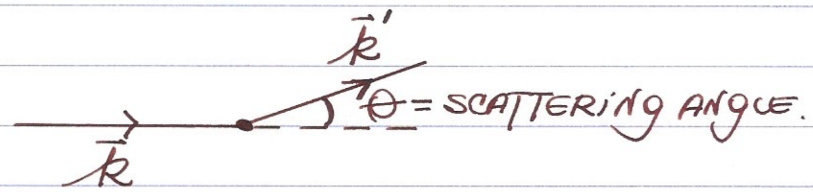
$$Q_{WK} = -N + Z\left(1 - 4\sin^2\theta_W\right)$$

AND BOTH THE CHARGE AND WEAK-CHARGE DENSITIES HAVE BEEN NORMALIZED TO ONE. THAT IS,

$$F_{ch}(q^2=0) = \int d^3r\, \rho_{ch}(\vec{r}) = 1$$

$$F_{WK}(q^2=0) = \int d^3r\, \rho_{WK}(\vec{r}) = 1.$$

KINEMATICS



WE DEFINE THE ORTHONORMAL COORDINATE SYSTEM IN TERMS OF $\vec{k}$ AND $\vec{k}'$ AS FOLLOWS:

$$\boxed{\begin{array}{ll} \vec{K} = \frac{1}{2}(\vec{k}+\vec{k}') \,; & \hat{z} = \hat{K} \\ \vec{q} = \vec{k}-\vec{k}' \,; & \hat{y} = \hat{q} \\ \vec{n} = \vec{q}\times\vec{K} = \vec{k}\times\vec{k}' \,; & \hat{x} = \hat{n} \end{array}}$$

NOTE THAT INDEED

$$\vec{q}\cdot\vec{K} = \frac{1}{2}(\vec{k}-\vec{k}')\cdot(\vec{k}+\vec{k}') = \frac{1}{2}\left(k^2 - k'^2\right) = 0$$

EIKONAL APPROXIMATION FOR A VECTOR PLUS AXIAL-VECTOR POTENTIAL | May 2020

WE NOW TRY TO MERGE BOTH APPROACHES. TO DO SO WE WRITE THE HAMILTONIAN AS

$$\hat{H} = \vec{\alpha}\cdot\vec{p} + \beta m_e + V(r) + \gamma_5 A(r) \,;\quad \hat{H}\Psi^{(+)}_{\vec{k}}(\vec{r}) = E\Psi^{(+)}_{\vec{k}}(\vec{r})$$

OR

$$0 = (E-\hat{H})\Psi^{(+)}_{\vec{k}}(\vec{r}) = \begin{pmatrix} E-V(r)-m_e & -\vec{\sigma}\cdot\vec{p}-A(r) \\ -\vec{\sigma}\cdot\vec{p}-A(r) & E-V(r)+m_e \end{pmatrix}\begin{pmatrix}\phi(\vec{r})\\ \chi(\vec{r})\end{pmatrix}$$

OR

$$(E-V(r)-m_e)\,\phi(\vec{r}) = (\vec{\sigma}\cdot\vec{p}+A(r))\,\chi(\vec{r})$$
$$(E-V(r)+m_e)\,\chi(\vec{r}) = (\vec{\sigma}\cdot\vec{p}+A(r))\,\phi(\vec{r})$$

AS BEFORE, "SOLVING" FOR THE LOWER COMPONENT IN FAVOR OF THE UPPER COMPONENT, i.e.,

$$\chi(\vec{r}) = \frac{1}{(\mathcal{E}(r)+m)}(\vec{\sigma}\cdot\vec{p}+A(r))\,\phi(\vec{r})$$

AND SUBSTITUTING THIS IN THE OTHER EQUATION YIELDS:

$$(\vec{\sigma}\cdot\vec{p}+A(r))\frac{1}{(\mathcal{E}(r)+m)}(\vec{\sigma}\cdot\vec{p}+A(r))\,\phi(\vec{r}) = (\mathcal{E}(r)-m)\,\phi(\vec{r})$$

$$= \left\{(\vec{\sigma}\cdot\vec{p})\frac{1}{(\mathcal{E}(r)+m)}(\vec{\sigma}\cdot\vec{p}) + \left[\vec{\sigma}\cdot\vec{p}\left(\frac{A(r)}{\mathcal{E}(r)+m}\right) + \left(\frac{A(r)}{\mathcal{E}(r)+m}\right)\vec{\sigma}\cdot\vec{p}\right] + \left[\frac{A^2(r)}{\mathcal{E}(r)+m}\right]\right\}\phi(\vec{r}) = (\mathcal{E}(r)-m)\,\phi(\vec{r})$$

NOW IMPLEMENT THE SIMPLIFICATIONS CARRIED OUT BEFORE.

THAT IS, IGNORE THE $A^2(r)$ TERM AND WRITE THE EXPRESSION IN BRACKETS AS:

$$\left[(\vec{\sigma}\cdot\vec{p})\left(\frac{A(r)}{\mathcal{E}(r)+m}\right)+\left(\frac{A(r)}{\mathcal{E}(r)+m}\right)(\vec{\sigma}\cdot\vec{p})\right]\rightarrow\frac{2A(r)}{\mathcal{E}(r)+m}(\vec{\sigma}\cdot\vec{k})$$

IN THIS WAY WE OBTAIN

$$\left[(\vec{\sigma}\cdot\vec{p})\frac{1}{(\mathcal{E}(r)+m)}\vec{\sigma}\cdot\vec{p}+\frac{2A(r)}{\mathcal{E}(r)+m}\vec{\sigma}\cdot\vec{k}\right]\phi_{\vec{k}}(\vec{r})=(\mathcal{E}(r)-m)\phi_{\vec{k}}(\vec{r})$$

THE FIRST TERM IS TREATED EXACTLY AS IN THE CASE OF $A(r)=0$. THAT IS,

$$(\vec{\sigma}\cdot\vec{p})\frac{1}{\mathcal{E}(r)+m}\vec{\sigma}\cdot\vec{p}=\frac{1}{(\mathcal{E}(r)+m)}\left\{(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p})+\left[\vec{\sigma}\cdot\vec{p},\frac{1}{\mathcal{E}+m}\right]\vec{\sigma}\cdot\vec{p}\right\}$$

$$=\frac{p^2}{(\mathcal{E}+m)}+\sigma_i\left(-i\partial_i(\mathcal{E}(r)+m)^{-1}\right)(\vec{\sigma}\cdot\vec{p})$$

$$=\frac{p^2}{(\mathcal{E}+m)}+i\sigma_i\frac{\left[-V'(r)\frac{x_i}{r}\right]}{(\mathcal{E}(r)+m)^2}(\vec{\sigma}\cdot\vec{p})$$

$$=\frac{1}{(\mathcal{E}+m)}\left[\hat{p}^2-i\frac{V'(r)}{r}\frac{(\vec{\sigma}\cdot\vec{r})(\vec{\sigma}\cdot\vec{p})}{(\mathcal{E}(r)+m)}\right]$$

$$=\frac{1}{(\mathcal{E}+m)}\left[\hat{p}^2-i\frac{V'(r)}{r}\left(\frac{\vec{r}\cdot\vec{p}+i\vec{\sigma}\cdot\vec{L}}{\mathcal{E}(r)+m}\right)\right]$$

$$=\frac{1}{(\mathcal{E}+m)}\left[\hat{p}^2+2mV_{SO}(r)\left(-i\vec{r}\cdot\vec{k}+\vec{\sigma}\cdot(\vec{r}\times\vec{k})\right)\right]$$

NOTE THAT IMMEDIATELY AFTER EQ. (9) OF MY 1983 PRC PAPER WE SAY "WHERE $\vec{k}$ HAS BEEN SUBSTITUDED INTO THE S.O. AND DARWIN TERMS.

THIS IS GOOD AS IT IS THE SAME SUBSTITION DONE FOR THE AXIAL TERM!

WHERE

$$\boxed{V_{SO}(r) \equiv \frac{1}{2mr}\frac{V'(r)}{\xi(r)+m}}$$ SPIN-ORBIT TERM

IN THIS MANNER WE OBTAIN:

$$\left(\hat{p}^2 + 2mV_{SO}(r)\left[-i\vec{r}\cdot\vec{k} + \vec{\sigma}\cdot(\vec{r}\times\vec{k})\right] + 2A(r)\,\vec{\sigma}\cdot\vec{k}\right)\phi_{\vec{k}}(\vec{r})$$

$$= (\xi(r)+m)(\xi(r)-m)\,\phi_{\vec{k}}(\vec{r}).$$

NOW, $(\xi(r)+m)(\xi(r)-m) = \xi^2(r) - m^2 = (E-V(r))^2 - m^2$
$= E^2 - 2EV(r) + V^2(r) - m^2$
$= (E^2-m^2) - 2mV_c(r)$

HENCE,

$$\left[\hat{p}^2 + 2mV_c(r) + 2mV_{SO}(r)\left[\vec{\sigma}\cdot(\vec{r}\times\vec{k}) - i\vec{r}\cdot\vec{k}\right] + 2A(r)\,\vec{\sigma}\cdot\vec{k}\right]\phi_{\vec{k}}$$

$$= (E^2-m^2)\,\phi_{\vec{k}}(\vec{r})$$

WHERE THE CENTRAL POTENTIAL IS GIVEN BY

$$\boxed{V_c(r) = \frac{E}{m}V(r) - \frac{V^2(r)}{2m}}$$

THE SCHRÖDINGER LIKE EQUATION FOR $\phi_{\vec{k}}(\vec{r})$ NOW BECOMES

$$\left(\frac{\hat{p}^2}{2m} + V_c(r) + V_{SO}(r)\left[\vec{\sigma}\cdot(\vec{r}\times\vec{k}) - i\vec{r}\cdot\vec{k}\right] + \frac{A(r)}{m}\,\vec{\sigma}\cdot\vec{k}\right)\phi_{\vec{k}}(\vec{r})$$

$$= \frac{k^2}{2m}\,\phi_{\vec{k}}(\vec{r})$$

OR
$$\left(\frac{\hat{P}^2}{2m}+\hat{V}_{EFF}(\vec{r})\right)\phi^{(+)}_{\vec{k}}(\vec{r})=\frac{k^2}{2m}\phi^{(+)}_{\vec{k}}(\vec{r})$$

WITH
$$\boxed{\hat{V}_{EFF}(\vec{r})\equiv V_C(r)+V_{SO}\left[\vec{\sigma}\cdot(\vec{r}\times\vec{K})-i\vec{r}\cdot\vec{K}\right]+\frac{A(r)}{m}\vec{\sigma}\cdot\vec{K}}$$

THE EIKONAL APPROXIMATION CONSISTS IN LINEARIZING THE $\hat{P}^2$ OPERATOR. THAT IS, WE WRITE

$$\begin{aligned}\hat{P}^2=\left[(\hat{P}-\vec{K})+\vec{K}\right]^2 &= (\vec{P}-\vec{K})^2+2\vec{K}\cdot(\vec{P}-\vec{K})+K^2\\ &=(\vec{P}-\vec{K})^2+2\vec{K}\cdot\vec{P}-K^2\\ &\simeq 2\vec{K}\cdot\vec{P}-K^2\end{aligned}$$

SO IN THE EIKONAL APPROXIMATION, SCHRÖDINGER'S EQUATION BECOMES

$$\frac{1}{2m}\left[2\vec{K}\cdot\vec{P}-K^2-k^2\right]\phi^{(+)}_{\vec{k}}(\vec{r})=-\hat{V}_{EFF}(\vec{r})\,\phi^{(+)}_{\vec{k}}(\vec{r})$$

WE NOW WRITE $\phi^{(+)}_{\vec{k}}(\vec{r})=e^{i\vec{k}\cdot\vec{r}}e^{iS(\vec{r})}$ ← THE EIKONAL PHASE

THEN,
$$\frac{1}{2m}\left[2\vec{K}\cdot(\vec{k}+\nabla S(\vec{r}))-K^2-k^2\right]\phi^{(+)}_{\vec{k}}(\vec{r})=-\hat{V}_{EFF}(\vec{r})\,\phi^{(+)}_{\vec{k}}(\vec{r})$$

$$=\frac{\vec{K}}{m}\cdot\nabla S(\vec{r})-\frac{1}{2m}\left[\underbrace{(\vec{K}-\vec{k})^2}_{Q^2/4}\right]=-\hat{V}_{EFF}(\vec{r})$$

← IGNORE IN THE EIKONAL (HIGH ENERGY/SMALL ANGLE) APPROXIMATION

SINCE WE DEFINE THE Z-AXIS ALONG THE DIRECTION OF $\vec{K}$ WE OBTAIN

$$\frac{\partial S}{\partial z}=-\frac{m}{K}\hat{V}_{EFF}(\vec{r})$$

HENCE,

$$\boxed{\begin{aligned} \hat{S}(\vec{r}) &= -\frac{m}{K}\int_{-\infty}^{z} \hat{V}_{EFF}(\vec{r}')\,dz' \\ \phi_{\vec{k}}^{(+)}(\vec{r}) &= e^{i\vec{k}\cdot\vec{r}}\, e^{i\hat{S}(\vec{r})} \end{aligned}}$$

WE NOW CONSTRUCT THE SCATTERING AMPLITUDE IN THE EIKONAL APPROXIMATION. THAT IS,

$$\mathcal{F}(\vec{k}\to\vec{k}') = -\frac{m}{2\pi}\int d^3r\, e^{-i\vec{k}'\cdot\vec{r}}\, u^{\dagger}(\vec{k}')\left[V(r)+\gamma^5 A(r)\right] \psi_{\vec{k}}^{(+)}(\vec{r})$$

$$= -\frac{m}{2\pi}\cdot\left(\frac{E+m}{2m}\right)\int d^3r\, e^{-i\vec{k}'\cdot\vec{r}}\left[1, \frac{\vec{\sigma}\cdot\vec{k}'}{E+m}\right]\begin{pmatrix} V(r) & A(r) \\ A(r) & V(r)\end{pmatrix}\psi_{\vec{k}}^{(+)}(\vec{r})$$

$$= -\frac{m}{2\pi}\left(\frac{E+m}{2m}\right)\int d^3r\, e^{-i\vec{k}'\cdot\vec{r}}\left(1, \frac{\vec{\sigma}\cdot\vec{k}'}{E+m}\right)\begin{pmatrix} V(r) & A(r) \\ A(r) & V(r)\end{pmatrix}\begin{pmatrix} 1 \\ \frac{1}{\mathcal{E}(r)+m}(\vec{\sigma}\cdot\vec{p}+A(r))\end{pmatrix} e^{i\vec{k}\cdot\vec{r}}\, e^{i\hat{S}(\vec{r})}$$

WORK OUT THE LINEAR ALGEBRA

$$\left(1, \frac{\vec{\sigma}\cdot\vec{k}'}{E+m}\right)\begin{pmatrix} V(r) & A(r) \\ A(r) & V(r)\end{pmatrix}\begin{pmatrix} 1 \\ \frac{1}{\mathcal{E}(r)+m}(\vec{\sigma}\cdot\vec{p}+A(r))\end{pmatrix}$$

$$= \left(1, \frac{\vec{\sigma}\cdot\vec{k}'}{E+m}\right)\begin{pmatrix} V(r) + \frac{A(r)}{\mathcal{E}(r)+m}(\vec{\sigma}\cdot\vec{p}+A(r)) \\ A(r) + \frac{V(r)}{\mathcal{E}(r)+m}(\vec{\sigma}\cdot\vec{p}+A(r))\end{pmatrix}$$

OR

$$\left[V(r) + \frac{A(r)}{\mathcal{E}(r)+m}\left(\vec{\sigma}\cdot\vec{p} + A(r)\right) + \frac{\vec{\sigma}\cdot\vec{k}'}{E+m} A(r) \right.$$

$$\left. + \frac{\vec{\sigma}\cdot\vec{k}'}{E+m}\,\frac{V(r)}{\mathcal{E}(r)+m}\left(\vec{\sigma}\cdot\vec{p} + A(r)\right) \right]$$

$$= \left\{ V(r) + \frac{A(r)}{\mathcal{E}(r)+m}\,\vec{\sigma}\cdot\vec{p} + (\vec{\sigma}\cdot\vec{k}')\frac{A(r)}{E+m} + \frac{V(r)}{\mathcal{E}(r)+m}\,\frac{\vec{\sigma}\cdot\vec{k}'}{E+m}\,\vec{\sigma}\cdot\vec{p} \right.$$

NEGLECT.

$$\left. + \cancel{\frac{A^2(r)}{\mathcal{E}(r)+m}} + \frac{V(r)A(r)}{\mathcal{E}(r)+m}\cdot\frac{\vec{\sigma}\cdot\vec{k}'}{E+m} \right\}$$

$$\simeq \left\{ V(r) + \frac{V(r)}{\mathcal{E}(r)+m}\,\frac{\vec{\sigma}\cdot\vec{k}'}{(E+m)}\,\vec{\sigma}\cdot\vec{p} + \right.$$

$$\left. A(r)\left\{ \frac{\vec{\sigma}\cdot\vec{K}}{\mathcal{E}(r)+m} + \frac{\vec{\sigma}\cdot\vec{K}}{E+m} + \frac{V(r)}{\mathcal{E}(r)+m}\,\frac{\vec{\sigma}\cdot\vec{K}}{E+m} \right\} \right\}$$

TERM LINEAR IN A:

$$A(r)\,\vec{\sigma}\cdot\vec{K}\;\frac{\left[(E+m) + (\mathcal{E}(r)+m) + V(r)\right]}{(\mathcal{E}(r)+m)(E+m)}$$

$$= \frac{A(r)\,\vec{\sigma}\cdot\vec{K}}{(\mathcal{E}(r)+m)(E+m)}\left[(E+m) + (E - \cancel{V(r)} + m) + \cancel{V(r)}\right]$$

$$= \frac{A(r)\,\vec{\sigma}\cdot\vec{K}}{(\mathcal{E}(r)+m)(E+m)}\,2(E+m) = \frac{2A(r)\,\vec{\sigma}\cdot\vec{K}}{\mathcal{E}(r)+m}$$

SO THE LINEAR-ALGEBRA EXERCISE YIELDS:

$$\mathcal{F}(\vec{k},\vec{k}') = -\frac{(E+m)}{4\pi}\int d^3r\, e^{-i\vec{k}'\cdot\vec{r}}\Big[V(r) + \frac{V(r)}{\xi(r)+m}\frac{\vec{\sigma}\cdot\vec{k}'}{E+m}\vec{\sigma}\cdot\vec{p} + \frac{2A(r)}{\xi(r)+m}\vec{\sigma}\cdot\vec{k}\Big]\phi^{(+)}_{\vec{k}}(\vec{r})$$

WE LEAVE EVERYTHINK INTACT AND CONCENTRATE ON THE SECOND TERM. THAT IS,

$$f_2(\vec{k}\to\vec{k}') \equiv \left(\frac{\vec{\sigma}\cdot\vec{k}'}{E+m}\right)\int d^3r\, e^{-i\vec{k}'\cdot\vec{r}}\frac{V(r)}{(\xi(r)+m)}(\vec{\sigma}\cdot\vec{p})\,\phi^{(+)}_{\vec{k}}(\vec{r})$$

$$= \left(\frac{\vec{\sigma}\cdot\vec{k}'}{E+m}\right)\int d^3r\, e^{-i\vec{k}'\cdot\vec{r}}\frac{V(r)}{(\xi(r)+m)}\left[\sigma_a(-i\partial_a)\right]\phi^{(+)}_{\vec{k}}$$

INTEGRATION BY PARTS

$$= \left(\frac{\vec{\sigma}\cdot\vec{k}'}{E+m}\right)[i\sigma_a]\int d^3r\left(\partial_a\left[e^{-i\vec{k}'\cdot\vec{r}}\frac{V(r)}{\xi(r)+m}\right]\right)\phi^{(+)}_{\vec{k}}$$

$$= \left(\frac{\vec{\sigma}\cdot\vec{k}'}{E+m}\right)[i\sigma_a]\int d^3r\, e^{-i\vec{k}'\cdot\vec{r}}\left\{-ik'_a\frac{V(r)}{\xi(r)+m} + \frac{V'(r)}{\xi(r)+m}\frac{x_a}{r} + \frac{V(r)V'(r)}{(\xi(r)+m)^2}\frac{x_a}{r}\right\}\phi^{(+)}_{\vec{k}}$$

$$= \int d^3r\, e^{-i\vec{k}'\cdot\vec{r}}\left[\frac{V(r)}{\xi(r)+m}\frac{(\vec{\sigma}\cdot\vec{k}')^2}{(E+m)} + \frac{iV'(r)}{r(\xi+m)}\frac{\vec{\sigma}\cdot\vec{k}'}{E+m}\vec{\sigma}\cdot\vec{r} + i\frac{V(r)V'(r)}{r(\xi+m)^2}\frac{\vec{\sigma}\cdot\vec{k}'}{E+m}\vec{\sigma}\cdot\vec{r}\right]\phi^{(+)}_{\vec{k}}(\vec{r})$$

LET'S CONTINUE BY SIMPLIFYING SOME OF THE EXPRESSIONS IN BRACKETS.

- $$\frac{V(r)}{\xi(r)+m} \cdot \frac{(\vec{\sigma}\cdot\vec{k}')^2}{(E+m)} = \frac{V(r)}{\xi(r)+m} \cdot \frac{(E^2-m^2)}{(E+m)} = \left(\frac{E-m}{\xi(r)+m}\right)V(r)$$

- $$\frac{i V'(r)(\vec{\sigma}\cdot\vec{k}')}{r(\xi+m)\,(E+m)}\vec{\sigma}\cdot\vec{r}\,\underbrace{\left[1+\frac{V(r)}{\xi+m}\right]}_{\frac{[E-\cancel{V(r)}+m+\cancel{V(r)}]}{\xi+m}}$$

$$= \frac{iV'(r)}{r(\xi+m)^2}(\vec{\sigma}\cdot\vec{k})(\vec{\sigma}\cdot\vec{r}) = 2mi\frac{V_{so}(r)}{(\xi+m)}\left[\vec{k}\cdot\vec{r}+i\vec{\sigma}\cdot(\vec{k}\times\vec{r})\right]$$

$$= \frac{2mV_{so}(r)}{(\xi+m)}\left[\vec{\sigma}\cdot(\vec{r}\times\vec{k})+i\vec{k}\cdot\vec{r}\right]$$

HENCE,

$$f_2(\vec{k}\to\vec{k}') = \int d^3r\, e^{-i\vec{k}'\cdot\vec{r}}\left[\left(\frac{E-m}{\xi+m}\right)V(r)+\frac{2mV_{so}(r)}{\xi+m}\times\left(\vec{\sigma}\cdot(\vec{r}\times\vec{k})+i\vec{k}\cdot\vec{r}\right)\right]\phi^{(+)}_{\vec{k}}$$

OR GOING BACK TO $\mathcal{F}$ WE OBTAIN

$$\mathcal{F}(\vec{k},\vec{k}') = -\frac{(E+m)}{4\pi}\int d^3r\, e^{-i\vec{k}'\cdot\vec{r}}\left[V(r)+\left(\frac{E-m}{\xi+m}\right)V(r)\right.$$

$$\left. +\frac{2m}{(\xi+m)}V_{so}(r)\left(\vec{\sigma}\cdot(\vec{r}\times\vec{k})+i\vec{k}\cdot\vec{r}\right)+\frac{2A(r)}{(\xi+m)}\vec{\sigma}\cdot\vec{k}\right]\phi^{(+)}_{\vec{k}}(\vec{r})$$

A FEW MORE STEPS. FIRST NOTE THAT

$$V(r)+\left(\frac{E-m}{\mathcal{E}+m}\right)V(r)=\left[\frac{(\mathcal{E}+m)+(E-m)}{\mathcal{E}+m}\right]V(r)$$

$$=\frac{[2E-V(r)]V(r)}{\mathcal{E}+m}$$

$$=\left(\frac{2m}{\mathcal{E}+m}\right)\left[\frac{E}{m}V(r)-\frac{V^2(r)}{2m}\right]$$

$$=\left(\frac{2m}{\mathcal{E}+m}\right)V_c(r) \longleftarrow \text{CENTRAL POTENTIAL}$$

HENCE,

$$\mathcal{F}(\vec{k}\to\vec{k}')=-\frac{m}{2\pi}\int d^3r\, e^{-i\vec{k}'\cdot\vec{r}}\left(\frac{E+m}{\mathcal{E}(r)+m}\right)\Big\{V_c(r)+ + V_{SO}(r)+V_{SO}(r)\left[\vec{\sigma}\cdot(\vec{r}\times\vec{k})+i\vec{k}\cdot\vec{r}\right] + \frac{A(r)}{m}\vec{\sigma}\cdot\vec{k}\Big\}\, e^{i\vec{k}\cdot\vec{r}}e^{i\hat{S}(\vec{r})}$$

WE ARE ALMOST THERE, ALTHOUGH NOT QUITE SINCE RECALL THAT THE EFFECTIVE POTENTIAL IS GIVEN BY

$$\hat{V}_{EFF}(\vec{r})=V_c(r)+V_{SO}(r)\left[\vec{\sigma}\cdot(\vec{r}\times\vec{k})-i\vec{k}\cdot\vec{r}\right]+\frac{A(r)}{m}\vec{\sigma}\cdot\vec{k}$$

THAT IS, THE SIGN OF THE DARWIN TERM IS DIFFENT. THIS IMPLIES THAT THE INTEGRAL IS GIVEN BY

$$\mathcal{F}(\vec{k}\to\vec{k}')=-\frac{m}{2\pi}\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\left(\frac{E+m}{\mathcal{E}(r)+m}\right)\left[\hat{V}_{EFF}(\vec{r})+2iV_{SO}(r)(\vec{k}\cdot\vec{r})\right]e^{i\hat{S}(\vec{r})}$$

NOW NOTE THAT

$$\hat{V}_{EFF}(\vec{r})\, e^{iS(\vec{r})} = \hat{V}_{EFF}(\vec{r})\, e^{-i\frac{m}{K}\int_{-\infty}^{z}\hat{V}_{EFF}(\vec{r}')dz'}$$

$$= \frac{iK}{m}\frac{\partial}{\partial z} e^{-i\frac{m}{K}\int_{-\infty}^{z} V_{EFF}(\vec{r}')dz'}$$

$$= \frac{iK}{m}\frac{\partial}{\partial z} e^{i\hat{S}(\vec{r})}$$

IN ADDITION, WE HAVE

$$\frac{d}{dr}\left(\mathcal{E}(r)+m\right)^{-1} = \frac{V'(r)}{(\mathcal{E}(r)+m)^2} = \frac{2mr}{(\mathcal{E}(r)+m)} V_{SO}(r)$$

OR

$$\frac{1}{r}\frac{d}{dr}\left(\mathcal{E}(r)+m\right)^{-1} = \frac{1}{z}\frac{\partial}{\partial z}\left(\mathcal{E}(r)+m\right)^{-1} = \left(\frac{2m}{\mathcal{E}+m}\right) V_{SO}(r)$$

HENCE, THE "OFFENDING" TERM MAY ALSO BE WRITTEN AS A DERIVATIVE WITH RESPECT TO z. THAT IS,

$$2iV_{SO}(r)\vec{K}\cdot\vec{r} = 2iKzV_{SO}(r) = 2iK\cdot\left(\frac{\mathcal{E}+m}{2m}\right)\frac{\partial}{\partial z}\left(\frac{1}{\mathcal{E}+m}\right)$$

$$= \frac{iK}{m}(\mathcal{E}+m)\frac{\partial}{\partial z}\left(\frac{1}{\mathcal{E}+m}\right)$$

IN THIS WAY WE CAN WRITE

$$\left(\frac{E+m}{\mathcal{E}+m}\right)\left[\hat{V}_{EFF}(\vec{r}) + 2iV_{SO}(r)\vec{K}\cdot\vec{r}\right]$$

$$= \frac{iK}{m}(E+m)\left[(\mathcal{E}+m)^{-1}\frac{\partial}{\partial z}e^{i\hat{S}(\vec{r})} + e^{i\hat{S}(\vec{r})}\frac{\partial}{\partial z}(\mathcal{E}+m)^{-1}\right]$$

$$= \frac{iK}{m}(E+m)\frac{\partial}{\partial z}\left[(\mathcal{E}+m)^{-1}e^{i\hat{S}(\vec{r})}\right] = \frac{iK}{m}\frac{\partial}{\partial z}\left[\frac{E+m}{\mathcal{E}+m}e^{i\hat{S}(\vec{r})}\right]$$

HENCE, THE SCATTERING AMPLITUDE MAY BE WRITTEN AS

$$\mathcal{F}(\vec{k}\to\vec{k}') = -\frac{ik}{2\pi}\int d^3r\, e^{i\vec{q}\cdot\vec{b}}\,\frac{\partial}{\partial z}\left[\left(\frac{E+m}{\mathcal{E}(r)+m}\right)e^{i\hat{S}(\vec{r})}\right]$$

$$= -\frac{ik}{2\pi}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}\left[\left(\frac{E+m}{\mathcal{E}(r)+m}\right)e^{i\hat{S}(\vec{r})}\right]_{-\infty}^{\infty}$$

OR

$$\mathcal{F}(\vec{k}\to\vec{k}') = ik\int \frac{d^2b}{2\pi}\, e^{i\vec{q}\cdot\vec{b}}\left(1-e^{-\hat{\chi}(\vec{b})}\right)$$

WHERE

$$\hat{\chi}(\vec{b}) = +\frac{im}{k}\int_{-\infty}^{\infty}\left(V_c(r) + V_{so}(r)\,\vec{\sigma}\cdot(\vec{b}\times\vec{k}) + \frac{A(r)}{m}\,\vec{\sigma}\cdot\vec{k}\right)dz$$

WITHOUT THE DARWIN TERM BECAUSE IT IS AN ODD FUNCTION OF Z.

AXIAL POTENTIAL.

AND WHERE

$$V_c(r) = \frac{E}{m}V(r) - \frac{V^2(r)}{2m}\,; \quad \text{CENTRAL POTENTIAL.}$$

$$V_{so}(r) = \frac{1}{2mr}\,\frac{V'(r)}{(E+m-V(r))}\,; \quad \text{SPIN-ORBIT POTENTIAL.}$$

EIKONAL APPROXIMATION FOR e^- SCATTERING FROM A VECTOR-PLUS-AXIAL POTENTIALS.

WE CONTINUE BY PERFORMING THE SPIN ALGEBRA THAT WILL EVENTUALLY YIELD THE SPIN-DEPENDENT SCATTERING AMPLITUDE IN A FORM THAT CAN BE EASILY MANIPULATED.

WE WRITE THE PROFILE OPERATOR AS FOLLOWS:

$$\hat{\chi}(\vec{b}) = \chi_c(b) + \chi_{SO}(b)\,\vec{\sigma}\cdot(\hat{b}\times\hat{k}) + \chi_A(b)\,\vec{\sigma}\cdot\hat{k}$$

WHERE,

$$\chi_c(b) = i\frac{m}{k}\int_{-\infty}^{\infty} V_c(r)\,dz$$

$$\chi_{SO}(b) = imb\int_{-\infty}^{\infty} V_{SO}(r)\,dz$$

$$\chi_A(b) = i\int_{-\infty}^{\infty} A(r)\,dz$$

THEN,

$$e^{-\hat{\chi}(\vec{b})} = e^{-[\chi_c(b) + \chi_{SO}(b)\,\vec{\sigma}\cdot(\hat{b}\times\hat{k}) + \chi_A(b)\,\vec{\sigma}\cdot\hat{k}]}$$
$$= e^{-\chi_c(b)}\,e^{-[\chi_{SO}(b)\,\vec{\sigma}\cdot(\hat{b}\times\hat{k}) + \chi_A(b)\,\vec{\sigma}\cdot\hat{k}]}$$

NOTE THAT WE CANNOT WRITE THE EXPONENTIAL INVOLVING THE PAULI MATRICES AS A PRODUCT OF TWO EXPONENTIALS. INDEED, ACCORDING TO THE BAKER-CAMPBELL-HAUSDORFF FORMULA FOR TWO OPERATORS $\hat{A}$ AND $\hat{B}$

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]+\cdots}$$

FOR OUR PROBLEM BOTH THE NORMAL ASYMMETRY (OF ORDER m/E) AND THE PV ASYMMETRY (OF ORDER $G_F Q^2$) ARE VERY SMALL SO WE WILL WORK TO FIRST ORDER IN BOTH $\chi_{SO}(b)$ AND $\chi_A(b)$. HOWEVER, IF NEEDED, WE CAN TURN OFF THE AXIAL-VECTOR CONTRIBUTION AND COMPUTE THE NORMAL ASYMMETRY TO ALL ORDERS IN $\chi_{SO}(b)$ AS IN MY DISSERTATION AND IN PRC 28, 1663 (1983).

HENCE, TO FIRST ORDER IN BOTH χ_{SO} AND χ_A WE OBTAIN

$$e^{-\hat{\chi}(b)} = e^{-\chi_c(b)}\left[1-\chi_{SO}(b)\vec{\sigma}\cdot(\hat{b}\times\hat{k})-\chi_A(b)\,\vec{\sigma}\cdot\hat{k}\right]$$

INSERTING THIS EXPRESSION INTO THE SCATTERING AMPLITUDE WE GET

$$\hat{F}(\vec{k}\to\vec{k}') = ik\int\frac{d^2b}{2\pi}\,e^{i\vec{q}\cdot\vec{b}}\left[1-e^{-\chi_c(b)}\left(1-\chi_{SO}(b)\,\vec{\sigma}\cdot(\hat{b}\times\hat{k}) - \chi_A(b)\,\vec{\sigma}\cdot\hat{k}\right)\right]$$

$$= ik\int\frac{d^2b}{2\pi}\,e^{i\vec{q}\cdot\vec{b}}\left[1-e^{-\chi_c(b)}+e^{-\chi_c(b)}\chi_{SO}(b)\,\vec{\sigma}\cdot(\hat{b}\times\hat{k}) + e^{-\chi_c(b)}\chi_A(b)\,\vec{\sigma}\cdot\hat{k}\right]$$

IN ALL CASES THE INTEGRAL OVER THE ASYMUTHAL ANGLE CAN BE DONE BECAUSE ALL THREE PROFILE FUNCTION DEPEND ONLY ON b AND ARE INDEPENDENT OF φ. WE HAVE,

$$I_1(b) \equiv \int_0^{2\pi}\frac{d\varphi}{2\pi}\,e^{i\vec{q}\cdot\vec{b}}\ ; \quad \vec{b} = b(\cos\varphi\,\hat{x}+\sin\varphi\,\hat{y}) = b(\cos\varphi\,\hat{n}+\sin\varphi\,\hat{q})$$

$$= \int_0^{2\pi}\frac{d\varphi}{2\pi}\,e^{iqb\sin\varphi} = J_0(qb)$$

BESSEL FUNCTION

$$\vec{I}_2(b) = \int_0^{2\pi}\frac{d\varphi}{2\pi}\,e^{iqb\sin\varphi}\,\hat{b}$$

$$= \left(\int_0^{2\pi}\frac{d\varphi}{2\pi}\,e^{iqb\sin\varphi}\cos\varphi\right)\hat{n} + \left(\int_0^{2\pi}\frac{d\varphi}{2\pi}\,e^{iqb\sin\varphi}\sin\varphi\right)\hat{q}$$

$$= 0 + i\,J_1(qb)\,\hat{q}$$ [FROM WOLFRAM ALPHA]

FINALLY, WE OBTAIN THE FOLLOWING EXPRESSION FOR THE SCATTERING AMPLITUDE

$$\boxed{\begin{aligned} &\hat{\mathcal{F}}(\vec{k}\to\vec{k}') = \mathcal{F}_0(q) + \mathcal{F}_n(q)\,\vec{\sigma}\cdot\hat{n} + \mathcal{F}_k(q)\,\vec{\sigma}\cdot\hat{k} \\ \text{WHERE} \quad &\mathcal{F}_0(q) = ik\int_0^\infty b\,db\,J_0(qb)\left[1 - e^{-\chi_c(b)}\right] \\ &\mathcal{F}_n(q) = -k\int_0^\infty b\,db\,J_1(qb)\,e^{-\chi_c(b)}\,\chi_{so}(b) \\ &\mathcal{F}_k(q) = ik\int_0^\infty b\,db\,J_0(qb)\,e^{-\chi_c(b)}\,\chi_A(b) \end{aligned}}$$

IN PRINCIPLE, ONE COULD HAVE ALSO OBTAINED A TERM OF THE FORM $\mathcal{F}_q(q)\,\vec{\sigma}\cdot\hat{q}$, BUT TO FIRST ORDER IN $\chi_{so}(b)$ AND $\chi_A(b)$ IS ABSENT.

SCATTERING OBSERVABLES

CONSIDER THE SCATTERING AMPLITUDE FOR AN e^- WITH INITIAL MOMENTUM $\vec{k}$ AND POLARIZATION $\hat{s}$ SCATTERING INTO AN ELECTRON WITH FINAL MOMENTUM $\vec{k}'$ AND POLARIZATION $\hat{s}'$. THEN, THE CROSS SECTION FOR SUCH A PROCESS IS GIVEN BY

$$\begin{aligned} \frac{d\sigma}{d\Omega}(\vec{k}\hat{s}\to\vec{k}'\hat{s}') &= \left|\langle\hat{s}'|\hat{\mathcal{F}}(\vec{k},\vec{k}')|\hat{s}\rangle\right|^2 \\ &= \langle\hat{s}'|\hat{\mathcal{F}}(\vec{k},\vec{k}')|\hat{s}\rangle\langle\hat{s}'|\mathcal{F}(\vec{k},\vec{k}')|\hat{s}\rangle^* \\ &= \langle\hat{s}'|\hat{\mathcal{F}}(\vec{k},\vec{k}')|\hat{s}\rangle\langle\hat{s}|\hat{\mathcal{F}}^\dagger(\vec{k},\vec{k}')|\hat{s}'\rangle \end{aligned}$$

GIVEN THAN IN THE EXPERIMENT THERE WILL BE NO MEASUREMENT OF THE OUTGOING POLARIZATION, WE SIMPLY SUM OVER $s' = \pm 1/2$.

THEN,

$$\frac{d\sigma_s(\vec{k}\to\vec{k}')}{d\Omega} = \sum_{s'} \langle \hat{s}' | \hat{\mathcal{F}}(\vec{k},\vec{k}') | \hat{s} \rangle \langle \hat{s} | \hat{\mathcal{F}}^{\dagger}(\vec{k},\vec{k}') | \hat{s}' \rangle$$

$$= Tr\left[\hat{\mathcal{F}}(\vec{k},\vec{k}') | \hat{s} \rangle \langle \hat{s} | \hat{\mathcal{F}}^{\dagger}(\vec{k},\vec{k}') \right]$$

AND USING

$$|\hat{s}\rangle\langle\hat{s}| = \frac{1}{2}(1+\vec{\sigma}\cdot\hat{s})$$

WE OBTAIN

$$\boxed{\frac{d\sigma_s(\vec{k}\to\vec{k}')}{d\Omega} = \frac{1}{2} Tr\left[\hat{\mathcal{F}}(\vec{k},\vec{k}')(1+\vec{\sigma}\cdot\hat{s})\hat{\mathcal{F}}^{\dagger}(\vec{k},\vec{k}') \right]}$$

WE ARE INTERESTED IN THREE SCATTERING OBSERVABLES

(a) THE UNPOLARIZED CROSS SECTION

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{d\sigma_\uparrow}{d\Omega} + \frac{d\sigma_\downarrow}{d\Omega}\right) = \frac{1}{2} Tr\left[\hat{\mathcal{F}}(\vec{k},\vec{k}')\hat{\mathcal{F}}^{\dagger}(\vec{k},\vec{k}') \right]$$

$$= |\mathcal{F}_0(q)|^2 + |\mathcal{F}_n(q)|^2 + |\mathcal{F}_k(q)|^2$$

(b) THE NORMAL POLARIZATION $\rightarrow \hat{s}=\hat{n}$ DEFINED AS

$$A_n = \frac{d\sigma_\uparrow - d\sigma_\downarrow}{d\sigma_\uparrow + d\sigma_\downarrow} = \frac{d\sigma_\uparrow - d\sigma_\downarrow}{2\left(\frac{d\sigma}{d\Omega}\right)}$$

WITH

$$d\sigma_\uparrow - d\sigma_\downarrow = Tr\left[\hat{\mathcal{F}}(\vec{k},\vec{k}')\, \vec{\sigma}\cdot\hat{n}\, \hat{\mathcal{F}}^{\dagger}(\vec{k},\vec{k}') \right]$$

$$= 2\left[\mathcal{F}_0(q)\mathcal{F}_n^*(q) + \mathcal{F}_0^*(q)\mathcal{F}_n(q) \right]$$

OR

$$A_n = \frac{\mathcal{F}_0\mathcal{F}_n^* + \mathcal{F}_0^*\mathcal{F}_n}{d\sigma/d\Omega} = \frac{2\,Re\left(\mathcal{F}_0(q)\mathcal{F}_n^*(q)\right)}{d\sigma/d\Omega}$$

(c) THE LONGITUDINAL PARITY-VIOLATING ASYMMETRY $\hat{S}=\hat{\tilde{k}}\simeq\hat{k}$. NOTE THAT WE CAN REPLACE $\hat{\tilde{k}}$ WITH $\hat{k}$ BECAUSE THE DIFFERENCE IS PROPORTIONAL TO THE PRODUCT $\chi_{SO}\cdot\chi_A$ WHICH WAS ALREADY NEGLECTED. HENCE, WITH $\hat{S}=\hat{k}$ WE OBTAIN

$$A_{PV}=\frac{d\sigma_R-d\sigma_L}{d\sigma_R+d\sigma_L}=\frac{d\sigma_R-d\sigma_L}{2\,d\sigma/d\Omega}$$

WITH

$$d\sigma_R-d\sigma_L=Tr\left[\hat{\mathcal{F}}(\vec{k},\vec{k}')\,\vec{\sigma}\cdot\hat{k}\,\hat{\mathcal{F}}^{\dagger}(\vec{k},\vec{k}')\right]=2\left[\mathcal{F}_0(q)\mathcal{F}_k^*(q)+\mathcal{F}_0^*(q)\mathcal{F}_k(q)\right]$$

OR

$$A_{PV}=\frac{\mathcal{F}_0(q)\mathcal{F}_k^*(q)+\mathcal{F}_0^*(q)\mathcal{F}_k(q)}{d\sigma/d\Omega}=\frac{2Re(\mathcal{F}_0\mathcal{F}_k^*)}{d\sigma/d\Omega}$$

IN SUMMARY:

$$\boxed{\begin{gathered}\frac{d\sigma}{d\Omega}=|\mathcal{F}_0(q)|^2+|\mathcal{F}_n(q)|^2+|\mathcal{F}_k(q)|^2\simeq|\mathcal{F}_0(q)|^2\\[1em] A_n=\frac{2Re(\mathcal{F}_0(q)\mathcal{F}_n^*(q))}{d\sigma/d\Omega};\quad A_{PV}=\frac{2Re(\mathcal{F}_0\mathcal{F}_k^*)}{d\sigma/d\Omega}\end{gathered}}$$

SCATTERING OBSERVABLES IN ELASTIC $\vec{e}^-$-NUCLEUS SCATTERING IN AN EIKONAL APPROXIMATION.