TAMING THE COULOMB TERM

May 2020
PART II

THE COULOMB POTENTIAL BETWEEN THE ELECTRON AND A NUCLEUS WITH A SPHERICALLY SYMMETRIC CHARGE DISTRIBUTION IS GIVEN BY

$$V(r) = -Ze^2 \int \frac{\rho_{ch}(r')}{|\vec{r}-\vec{r}\,'|}\, d^3r'$$

$$= -Ze^2 \int_0^\infty r'^2 dr'\, d\Omega'\, \rho_{ch}(r') \left[\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^{l}}{r_>^{l+1}} Y_{lm}(\theta,\varphi)\, Y^*_{lm}(\theta',\varphi') \right]$$

$$= -Ze^2 \sum_{l,m} \frac{4\pi}{2l+1} Y_{lm}(\theta,\varphi) \int_0^\infty r'^2 dr'\, \rho_{ch}(r') \frac{r_<^{l}}{r_>^{l+1}}$$

$$\int d\Omega'\, Y^*_{lm}(\theta',\varphi')\, \underbrace{Y_{00}(\theta',\varphi')\sqrt{4\pi}}_{=1}$$

$$\to \delta_{l0}\,\delta_{m0}$$

$$= -Ze^2 \cdot 4\pi \int_0^\infty \frac{r'^2}{r_>} \rho_{ch}(r')\, dr'$$

$$= -4\pi Z e^2 \left[\int_0^r \frac{r'^2}{r} \rho_{ch}(r')\, dr' + \int_r^\infty \frac{r'^2}{r'} \rho_{ch}(r')\, dr' \right]$$

OR

$$\boxed{V(r) = -4\pi Z e^2 \left[\frac{1}{r} \int_0^r r'^2 \rho_{ch}(r')\, dr' + \int_r^\infty r' \rho_{ch}(r')\, dr' \right]}$$

COULOMB POTENTIAL FROM A SPHERICAL NUCLEAR CHARGE DISTRIBUTION.

NOTE THAT:

- $F_{ch}(q=0) = \int d^3r\, \rho_{ch}(r) = 4\pi \int_0^\infty r^2 \rho_{ch}(r)\, dr = 1$

- $\lim_{r\to\infty} V(r) = -4\pi \frac{Ze^2}{r} \int_0^\infty r^2 \rho_{ch}(r)\, dr = -\frac{Ze^2}{r}$

FOR LARGE DISTANCES THE ELECTRON SEES A POINT CHARGE OF MAGNITUDE Ze.

IT IS PRECISELY THIS BEHAVIOR THAT CAUSES PROBLEMS. RECALL THAT THE CENTRAL POTENTIAL DEFINED IN THE NOTES IS GIVEN BY

$$V_c(r) = \frac{E}{m} V(r) - \frac{V^2(r)}{2m}$$

HENCE,

$$\chi_c(b) = i\frac{m}{k} \int_{-\infty}^{\infty} V_c(r)\, dz \quad ; \quad r = \sqrt{b^2+z^2}, \quad dr = \frac{z}{r} dz$$

$$= 2i\frac{m}{k} \int_0^\infty V_c(r)\, dz$$

$$= 2i\frac{m}{k} \int_b^\infty V_c(r) \frac{r}{\sqrt{r^2-b^2}}\, dr$$

THEN WE CAN WRITE:

$$\chi_c(b) = 2i\frac{m}{k} \int_b^\infty \left(\frac{E}{m} V(r) - \frac{V^2(r)}{2m} \right) \frac{r}{\sqrt{r^2-b^2}}\, dr$$

$$\equiv \chi_c^{(1)}(b) + \chi_c^{(2)}(b)$$

WHERE

$$\chi_c^{(1)}(b) \equiv 2i\frac{E}{k}\int_b^\infty V(r)\,\frac{r}{\sqrt{r^2-b^2}}\,dr$$

$$\chi_c^{(2)}(b) \equiv -\frac{i}{k}\int_b^\infty V^2(r)\,\frac{r}{\sqrt{r^2-b^2}}\,dr$$

NOTE THAT THERE IS NO PROBLEM WITH THE LOWER LIMIT AS THE SQUARE-ROOT SINGULARITY IS INTEGRABLE; THAT IS,

$$\int_0 \frac{dx}{\sqrt{x}} = \int_0 x^{-1/2}dx \sim x^{1/2} \to 0$$

AS EXPECTED, HOWEVER, THE PROBLEM IS WITH THE LONG-RANGE BEHAVIOR OF THE COULOMB POTENTIAL. INDEED, STUDYING THE UPPER LIMIT ONE GETS:

$$\chi_c^{(1)}(b) \sim 2i\frac{E}{k}\int^\infty \left[-\frac{Ze^2}{r}\right]dr \longrightarrow \text{logarithmic divergence}$$

$$\chi_c^{(2)}(b) \sim -\frac{i}{k}\int^\infty \frac{(-Ze^2)^2}{r^2}\,dr \longrightarrow \text{CONVERGES}$$

TAMING THE LOGARITHMIC DIVERGENCE.

WE WRITE THE COULOMB POTENTIAL AS FOLLOWS

$$V(r) = \left[-Ze^2\int\frac{\rho_{ch}(r')}{|\vec{r}-\vec{r}\,'|} + \frac{Ze^2}{r}\right] - \frac{Ze^2}{r}$$

WHERE WE HAVE ADDED/SUBTRACTED THE ELECTROSTATIC POTENTIAL OF A "POINT" NUCLEUS OF CHARGE Ze.

THE PROFILE FUNCTION ASSOCIATED WITH THE EXPRESSION IN BRACKETS IS NO LONGER DIVERGENT (BY CONSTRUCTION). WHAT REMAINS IS TO COMPUTE THE PROFILE FUNCTION ASSOCIATED WITH THE "POINT" NUCLEUS CONTRIBUTION WHICH WHILE IT ALSO DIVERGES, WILL BE COMPUTED EXACTLY RATHER THAN IN THE EIKONAL APPROXIMATION

TO DO SO, I FOLLOW CLOSELY THE NOTES OF CARLOS BERTULANI "PHYSICS OF RADIOACTIVE BEAMS" CHAPTER 5. IN HIS NOTES THE CLAIM IS THAT THE FACTOR MULTIPLYING $J_0(qb)$ IS GIVEN BY

$$\left[e^{i\chi_c(b)} - 1\right] = \left[e^{2i\eta \ln(kb)} - 1\right]$$

WHERE

$$\eta \equiv Z_1 Z_2 \frac{e^2}{\hbar v} = Z_1 Z_2 e^2 \frac{m}{k}$$

WHAT IS THE PROPER DEFINITION OF THE SOMMERFELD PARAMETER η FOR OUR CASE. TO ANSWER THIS QUESTION WE MUST GO BACK TO PAGE 4 OF THE PREVIOUS SET OF NOTES WHERE WE WROTE THE DIFFERENTIAL EQUATION FOR $\phi_{\vec{k}}^{(+)}(\vec{r})$. KEEPING ONLY THE "OFFENDING" TERM WE HAVE:

$$\left[\frac{\hat{p}^2}{2m} + \frac{E}{m}\left(\frac{-Ze^2}{r}\right)\right]\phi_{\vec{k}}^{(+)}(\vec{r}) = \frac{k^2}{2m}\phi_{\vec{k}}^{(+)}(\vec{r})$$

OR USING THE FACT THAT THE POTENTIAL IS SPHERICALLY SYMMETRIC, THIS EQUATION REDUCES TO

$$\left(\frac{d^2}{dr^2} - \frac{2m}{\hbar^2}V_{EFF}(r) + k^2\right) u_{El}(r) = 0$$

$$V_{EFF}(r) = V(r) + \ell(\ell+1)\hbar^2/2mr^2$$

HENCE, SCHRÖDINGER'S EQUATION READS:

$$\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - 2m\left[\frac{E}{m}\left(-\frac{Ze^2}{r}\right)\right] + k^2 \right) u_{E\ell}(r) = 0$$

OR

$$\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - \frac{2mZe^2}{r}\left(-\frac{E}{m}\right) + k^2 \right) u_{E\ell}(r) = 0$$

NOW INTRODUCE THE DIMENSIONLESS VARIABLE $x \equiv kr$.
THEN,

$$\left(k^2 \frac{d^2}{dx^2} - k^2 \frac{\ell(\ell+1)}{x^2} - 2mZe^2 \frac{k}{x}\left(-\frac{E}{m}\right) + k^2 \right) u_{E\ell}(x) = 0$$

OR

$$\left[\frac{d^2}{dx^2} - \frac{\ell(\ell+1)}{x^2} - 2\left(\frac{mZe^2}{k}\right)\left(\frac{-E}{m}\right) + 1 \right] u_{E\ell}(x) = 0$$

IN BERTULANI (OR IN TAYLOR PAGE 261) THE SOMMERFELD PARAMETER IS DEFINED AS

$$\eta = Z_1 Z_2 e^2 \frac{\mu}{k}$$ APPROPRIATE FOR A REPULSIVE COULOMB POTENTIAL BETWEEN TWO NUCLEI WITH CHARGES Z_1 AND Z_2

SO IN OUR CASE WE MUST IMPLEMENT TWO CHANGES:

- INSERT A MINUS SIGN TO ACCOUNT FOR THE <u>ATTRACTIVE</u> e-NUCLEUS INTERACTION
- INSERT THE RELATIVISTIC CORRECTION E/m

THIS IMPLIES THAT THE EXPRESSION MULTIPLYING THE $J_0(qb)$ TERM SHOULD BECOME:

$$\left[e^{2i\eta \ln(kb)} - 1\right] = \left[e^{2i\left(Z_1 Z_2 e^2 \frac{m}{k}\right)\ln(kb)} - 1\right]$$

$$\longrightarrow \left[e^{2i\left(Ze^2 \frac{m}{k}\right)\left(\frac{-E}{m}\right)\ln(kb)} - 1\right]$$

$$= \left[e^{-2i\left(Ze^2 \frac{E}{k}\right)\ln(kb)} - 1\right]$$

$$\equiv \left[e^{-\chi_0(b)} - 1\right]$$

THAT IS, FOR US THE PROFILE FUNCTION FOR THE "POINT" COULOMB INTERACTION IS GIVEN BY

$$\boxed{\begin{aligned} \chi_0(b) &= 2i\eta \ln(kb) \\ \eta &= Ze^2E/k > 0 \end{aligned}}$$

NOTE THE LOGARITHMIC SINGULARITY AS $b \to \infty$!

ALL THAT REMAINS IS TO SHOW HOW THIS PROFILE AMPLITUDE REPRODUCES THE COULOMB SCATTERING AMPLITUDE. THAT IS, WE DEFINE

$$f_0(\vec{k}\to\vec{k}') = -ik\int_0^\infty b\,db\, J_0(qb)\left[e^{-\chi_0(b)} - 1\right]$$

$$= -ik\int_0^\infty b\,db\, J_0(qb)\left[e^{-2i\eta \ln(kb)} - 1\right]$$

$$= -ik\int_0^\infty b\,dk\, J_0(qb)\left[e^{\ln(kb)^{-2i\eta}} - 1\right]$$

$$= -ik\int_0^\infty b\,db\, J_0(qb)\left[(kb)^{-2i\eta} - 1\right]$$

NOW CHANGE VARIABLES TO $x \equiv qb$. THEN,

$$f_0(\vec{k}\to\vec{k}') = -\frac{ik}{q^2}\int_0^\infty x\,dx\,J_0(x)\left[\left(\frac{k}{q}x\right)^{-2i\eta}-1\right]$$

WE NOW USE THE FACT THAT $x J_0(x) = \frac{d}{dx}\left(x J_1(x)\right)$ TO INTEGRATE BY PARTS:

$$f_0(\vec{k}\to\vec{k}') = -\frac{ik}{q^2}\int_0^\infty \left[\frac{d}{dx}\left(xJ_1(x)\right)\right]\left[\left(\frac{k}{q}x\right)^{-2i\eta}-1\right]dx$$

$$= -\frac{ik}{q^2}\int_0^\infty \left\{\frac{d}{dx}\left[xJ_1(x)\left(\left(\frac{kx}{q}\right)^{-2i\eta}-1\right)\right] - xJ_1(x)\frac{d}{dx}\left[\left(\frac{k}{q}x\right)^{-2i\eta}-1\right]\right\}dx$$

$$= -\frac{ik}{q^2}\left\{xJ_1(x)\left[\left(\frac{kx}{q}\right)^{-2i\eta}-1\right]\right\}_0^\infty$$

IGNORE SURFACE TERM (WHY?)

$$+\frac{ik}{q^2}\int_0^\infty xJ_1(x)\left(\frac{k}{q}\right)^{-2i\eta}\left[(-2i\eta)x^{-2i\eta-1}\right]dx$$

$$= 2\eta\frac{k}{q^2}\left(\frac{k}{q}\right)^{-2i\eta}\int_0^\infty x^{-2i\eta}J_1(x)\,dx$$

$$= 2\eta\left(\frac{k}{q^2}\right)\left(\frac{k}{q}\right)^{-2i\eta}\left[(2)^{-2i\eta}\frac{\Gamma(1-i\eta)}{\Gamma(1+i\eta)}\right]$$

$$= 2\eta\frac{k}{q^2}\cdot\left(\frac{q}{2k}\right)^{2i\eta}\frac{\Gamma(1-i\eta)}{\Gamma(1+i\eta)}$$

WE NOW USE Eq. 6.1.31, page 256 IN ABRAMOWITZ-STEGUN THAT READS:

$$\Gamma(1+iy)\Gamma(1-iy) = |\Gamma(1+iy)|^2 = \frac{\pi y}{\sinh(\pi y)}$$

HENCE,

$$\frac{\Gamma(1-i\eta)}{\Gamma(1+i\eta)} = \frac{\Gamma(1-i\eta)\Gamma(1+i\eta)}{\Gamma^2(1+i\eta)} = \frac{|\Gamma(1+i\eta)|^2}{\Gamma^2(1+i\eta)}$$

SO IF WE WRITE $\Gamma(1+i\eta) \equiv |\Gamma(1+i\eta)| e^{i\sigma_0}$, WE OBTAIN

$$\frac{\Gamma(1-i\eta)}{\Gamma(1+i\eta)} = e^{-2i\sigma_0}$$

THE SCATTERING AMPLITUDE NOW BECOMES

$$\boxed{\begin{array}{l} f_0(\vec{k}\to\vec{k}') = 2\eta\left(\frac{k}{q^2}\right)\left(\frac{q}{2k}\right)^{2i\eta} e^{-2i\sigma_0} \\ \text{WITH } \sigma_0 = \arg\Gamma(1+i\eta);\ \eta = \frac{Ze^2E}{k} \end{array}}$$

ALTHOUGH THE ABOVE EXPRESSION IS ENOUGH, WE NOW CAST THE ANSWER IN A MORE CONVENTIONAL FORM BY USING

$$q = 2k\sin(\theta/2)$$

$$f_0(\vec{k}\to\vec{k}') = 2\eta\frac{k}{4k^2\sin^2(\theta/2)} e^{i\eta\ln(\sin^2(\frac{\theta}{2})-2i\sigma_0}$$

$$= \frac{\eta}{2k\sin^2(\theta/2)}\exp\left[i\left(\eta\ln\sin^2(\theta/2)-2\sigma_0\right)\right]$$

OR

$$f_0(\vec{k}\to\vec{k}') = \left(\frac{Ze^2 E}{2k^2 \sin^2(\theta/2)}\right) \exp\left[i\left(\eta \ln \sin^2\left(\tfrac{\theta}{2}\right) - 2\sigma_0\right]\right.$$

IN COMPUTING THE CROSS SECTION THE PHASE FACTOR BECOMES IRRELEVANT AND WE OBTAIN

$$\boxed{\frac{d\sigma}{d\Omega} = \underbrace{\frac{(Ze^2)^2 E^2}{4k^4 \sin^4(\theta/2)}}_{\text{NON RELATIVISTIC LIMIT}} \longrightarrow \frac{(Ze^2)^2 m^2}{4k^4 \sin^4(\theta/2)} = \text{RUTHERFORD FORMULA}}$$

ALTHOUGH THE ABOVE RESULT FOR THE AMPLITUDE IS EXACT, IT IS REMARKABLE THAT THE CROSS SECTION MAY BE OBTAINED IN THE FIRST BORN APPROXIMATION. THAT IS,

$$f_{BORN}(\vec{k}\to\vec{k}') = -\frac{m}{2\pi}\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\, V(r)$$

$$= -\frac{m}{2\pi}\int_0^\infty r^2 dr\, V(r)\, 4\pi \frac{\sin(qr)}{qr}$$

$$= -\frac{2m}{q}\int_0^\infty V(r) \sin(qr)\, r\,dr$$

$$= -\frac{2m}{q}\left[\int_0^\infty -\left(\frac{Ze^2}{r}\right)\left(\frac{E}{m}\right)\sin(qr)\, r\,dr\right]$$

$$= \frac{2E}{q} Ze^2 \int_0^\infty \frac{1}{r}\sin(qr)\, r\,dr$$

$$= \left(\frac{2E}{q}Ze^2\right) \lim_{\mu\to 0}\int_0^\infty \frac{e^{-\mu r}}{r}\sin(qr)\, r\,dr$$

$$= \left(\frac{2E}{q}Ze^2\right) \lim_{\mu\to 0}\int_0^\infty e^{-\mu r}\sin(qr)\,dr$$

OR $$f_{BORN}(\vec{k}\to\vec{k}') = \left(\frac{2E}{q} Ze^2\right) \lim_{\mu\to 0}\left[\frac{q}{q^2+\mu^2}\right] = \frac{2EZe^2}{q^2}$$

OR

$$\boxed{f_{BORN}(\vec{k}\to\vec{k}') = \frac{Ze^2 E}{2k^2 \sin^2(\theta/2)}}$$

WHICH IS EXACTLY THE EXACT EXPRESSION EXCEPT FOR THE PHASE FACTOR. AND WHEREAS THE COULOMB CROSS SECTION IS INDEPENDENT OF THE PHASE FACTOR, IN OUR CALCULATION IT BECOMES IMPORTANT BECAUSE OF INTERFERENCE EFFECTS.

SO HOW DO THINGS CHANGE RELATIVE TO THE EXPRESSION WRITTEN IN PAGE 12; THAT IS,

$$\hat{\chi}(b) = \chi_c(b) + \chi_{so}(b)\vec{\sigma}\cdot(\hat{b}\times\hat{k}) + \chi_A(b)\,\vec{\sigma}\cdot\hat{k}$$

THE TERM THAT HAS TO BE UPDATED IS $\chi_c(b)$ WHICH WE WROTE AS

$$\chi_c(b) = i\frac{m}{k}\int_{-\infty}^{\infty} V_c(r)\,dz\,;\quad V_c(r) = \frac{E}{m}V(r) - \frac{V^2(r)}{2m}$$

IN AN ATTEMPT TO ENSURE CONVERGENCE WE WROTE

$$\chi_c(b) = \chi_0(b) + i\frac{m}{k}\int_{-\infty}^{\infty}\left(\frac{E}{m}\left[V(r)+\frac{Ze^2}{r}\right] - \frac{V^2(r)}{2m}\right)dz$$

OR

$$\boxed{\begin{aligned}
&\chi_c(b) = \chi_0(b) + \chi_c'(b)\\
&\chi_0(b) = 2i\eta \ln(kb)\,;\quad \eta = Ze^2E/k\\
&\chi_c'(b) = i\frac{m}{k}\int_{-\infty}^{\infty}\left(\frac{E}{m}\left[V(r)+\frac{Ze^2}{r}\right] - \frac{V^2(r)}{2m}\right)dz
\end{aligned}}$$

THAT IS,

$$\hat{\chi}(b) = \chi_0(b) + \chi_C'(b) + \chi_{SO}(b)\,\vec{\sigma}\cdot(\hat{b}\times\hat{K}) + \chi_A(b)\,\vec{\sigma}\cdot\hat{K}$$

AND

$$e^{-\hat{\chi}(\vec{b})} \equiv e^{-\chi_0(b)}\, e^{-\hat{\chi}'(\vec{b})}$$

THE SCATTERING AMPLITUDE IS THEN GIVEN BY

$$\begin{aligned}
\mathcal{F}(\vec{k}\to\vec{k}') &= -ik\int \frac{d^2b}{2\pi}\, e^{i\vec{q}\cdot\vec{b}}\left[e^{-\hat{\chi}(\vec{b})} - 1\right] \\
&= -ik\int \frac{d^2b}{2\pi}\, e^{i\vec{q}\cdot\vec{b}}\left[e^{-\chi_0(b)} e^{-\hat{\chi}'(b)} - 1\right] \\
&= -ik\int \frac{d^2b}{2\pi}\, e^{i\vec{q}\cdot\vec{b}}\left[e^{-\chi_0(b)} e^{-\hat{\chi}'(\vec{b})} - e^{-\chi_0(b)} \right. \\
&\qquad \left. + e^{-\chi_0(b)} - 1\right] \\
&= -ik\int \frac{d^2b}{2\pi}\, e^{i\vec{q}\cdot\vec{b}}\, e^{-\chi_0(b)}\left[e^{-\hat{\chi}'(\vec{b})} - 1\right] \\
&\quad - ik\int \frac{d^2b}{2\pi}\, e^{i\vec{q}\cdot\vec{b}}\left[e^{-\chi_0(b)} - 1\right] \\
&= \mathcal{F}'(\vec{k}\to\vec{k}') + \underbrace{f_0(\vec{k}\to\vec{k}')}_{\text{EXACT COULOMB SCATTERING AMPLITUDE}}
\end{aligned}$$

WE NOW DEAL WITH THE FIRST TERM EXACTLY AS WE DID BEFORE. THAT IS,

$$\begin{aligned}
\mathcal{F}'(\vec{k}\to\vec{k}') = ik\int \frac{d^2b}{2\pi}\, e^{i\vec{q}\cdot\vec{b}}\, e^{-\chi_0(b)} \Big[1 - e^{-\chi_C'(b)}\big(1 - \chi_{SO}(b)\,\vec{\sigma}\cdot(\hat{b}\times\hat{K}) \\
- \chi_A(b)\,\vec{\sigma}\cdot\hat{K}\big)\Big]
\end{aligned}$$

$$\mathcal{F}'(\vec{k}\to\vec{k}') = ik\int \frac{d^2b}{2\pi}\, e^{i\vec{q}\cdot\vec{b}}\, e^{-\chi_0(b)}\Big[1 - e^{-\chi_c'(b)}$$
$$+ e^{-\chi_c'(b)}\,\chi_{SO}(b)\,\vec{\sigma}\cdot(\hat{b}\times\hat{k})$$
$$+ e^{-\chi_c'(b)}\,\chi_A(b)\,\vec{\sigma}\cdot\hat{k}\Big]$$

$$= ik\int \frac{d^2b}{2\pi}\, e^{i\vec{q}\cdot\vec{b}}\, e^{-\chi_0(b)}\left[1 - e^{-\chi_c'(b)}\right]$$
$$+ ik\int \frac{d^2b}{2\pi}\, e^{i\vec{q}\cdot\vec{b}}\, e^{-[\chi_0(b)+\chi_c'(b)]}\,\chi_{SO}(b)\,\vec{\sigma}\cdot(\hat{b}\times\hat{k})$$
$$+ ik\int \frac{d^2b}{2\pi}\, e^{i\vec{q}\cdot\vec{b}}\, e^{-[\chi_0(b)+\chi_c'(b)]}\,\chi_A(b)\,\vec{\sigma}\cdot\hat{k}$$

$$= ik\int_0^\infty b\,db\, J_0(qb)\, e^{-\chi_0(b)}\left[1 - e^{-\chi_c'(b)}\right]$$
$$- k\int_0^\infty b\,db\, J_1(qb)\, e^{-[\chi_0(b)+\chi_c'(b)]}\,\chi_{SO}(b)\,\vec{\sigma}\cdot\hat{n}$$
$$+ ik\int_0^\infty b\,db\, J_0(qb)\, e^{-[\chi_0(b)+\chi_c'(b)]}\,\chi_A(b)\,\vec{\sigma}\cdot\hat{k}$$

FINALLY, WE OBTAIN – AFTER – REMOVING "PRIMES":

$$\hat{\mathcal{F}}(\vec{k}\to\vec{k}') = \mathcal{F}_0(q) + \mathcal{F}_n(q)\,\vec{\sigma}\cdot\hat{n} + \mathcal{F}_k(q)\,\vec{\sigma}\cdot\hat{k}$$

WHERE

$$\mathcal{F}_0(q) = ik\int_0^\infty b\,db\, J_0(qb)\, e^{-\chi_0(b)}\left[1 - e^{-\chi_c(b)}\right] + f_0(\vec{k}\to\vec{k}')$$

$$\mathcal{F}_n(q) = -k\int_0^\infty b\,db\, J_1(qb)\, e^{-[\chi_0(b)+\chi_c(b)]}\,\chi_{SO}(b)$$

$$\mathcal{F}_k(q) = ik\int_0^\infty b\,db\, J_0(qb)\, e^{-[\chi_0(b)+\chi_c(b)]}\,\chi_A(b)$$

LET US SUMMARIZE THE VARIOUS PROFILE FUNCTIONS:

$$\boxed{\begin{aligned} \chi_0(b) &= 2i\eta \ln(kb); \quad \eta = Ze^2E/k \\ \chi_c(b) &= \frac{im}{K}\int_{-\infty}^{\infty}\left(\frac{E}{m}\left[V(r)+\frac{Ze^2}{r}\right]-\frac{V^2(r)}{2m}\right)dz \\ \chi_{SO}(b) &= imb\int_{-\infty}^{\infty} V_{SO}(r)\,dz \\ \chi_A(b) &= i\int_{-\infty}^{\infty} A(r)\,dz \end{aligned}}$$

THE HOPE IS THAT BY TREATING EXACTLY THE CONTRIBUTION FROM A POINT NUCLEUS OF CHARGE Z, ALL INTEGRALS (BOTH OVER dz AND db) WILL BE CONVERGENT.

A SMALL ADDENDUM: DEALING WITH SHORT- AND LONG RANGE FORCES. JULY 2020

A SERIOUS CHALLENGE TO THE ACCURATE COMPUTATION OF THE SCATTERING AMPLITUDES IS THE LONG-RANGE NATURE OF THE COULOMB INTERACTION. FORTUNATELY WE FOUND THE OGATA METHOD. NOTE, HOWEVER, THAT AS GIVEN (SAY IN THE PYTHON ROUTINE; N=120 POINTS AND h=0.03), THE OGATA METHOD IS NOT OPTIMAL IN DEALING WITH LONGITUDINAL-SPIN AMPLITUDE, WHICH IS SHORT RANGED – AND THUS BETTER DEALT WITH A SIMPLE SIMPSON ROUTINE. HOW, THEN, SHOULD WE DEAL WITH THE CENTRAL AMPLITUDE THAT CONTAINS BOTH SHORT AND LONG RANGE PIECES. RECALL THAT THE CENTRAL, (SPIN-INDEPENDENT) AMPLITUDE IS GIVEN BY

$$\mathcal{F}_0(q) = ik\int_0^\infty b\,db\,J_0(qb)\,e^{-\chi_0(b)}\left[1-e^{-\chi_c(b)}\right]$$

WHERE

$$\chi_c(b) = i\frac{m}{k}\int_{-\infty}^{\infty}\left[\underbrace{\frac{E}{m}\left(V(r)+\frac{Ze^2}{r}\right)}_{\text{SHORT-RANGE}} - \underbrace{\frac{V^2(r)}{2m}}_{\text{LONG-RANGE}}\right]dz$$

$$\equiv \chi_{c1}(b) + \chi_{c2}(b)$$

LET'S DEFINE

$$\mathcal{F}_{c1}(q) = ik\int_0^\infty b\,db\,J_0(qb)\,e^{-\chi_0(b)}\left[1-e^{-\chi_{c1}(b)}\right]$$

AND

$$\mathcal{F}_{c2}(q) \equiv \mathcal{F}_0(q) - \mathcal{F}_{c1}(q)$$

GIVEN THAT $\chi_{c1}(b)$ IS SHORT RANGED (INDEED, IT VANISHES FOR $b>R$) $\mathcal{F}_{c1}(q)$ MAY BE COMPUTED USING SIMPSON'S RULE, JUST AS $\mathcal{F}_K(q)$.

IN TURN, $\mathcal{F}_{c2}(q)$ CAN BE COMPUTED WITH OGATA AS IT IS LONG RANGED. THAT IS,

$$\mathcal{F}_{c2}(q) = ik\int_0^\infty b\,db\,J_0(qb)\,e^{-\chi_0(b)} \left\{\left[1-e^{-\chi_c(b)}\right] - \left[1-e^{-\chi_{c1}(b)}\right]\right\}$$

$$= -ik\int_0^\infty b\,db\,J_0(qb)\,e^{-\chi_0(b)}\left[e^{-\chi_c(b)} - e^{-\chi_{c1}(b)}\right]$$

$$= -ik\int_0^\infty b\,db\,J_0(qb)\,e^{-\chi_0(b)}\left[e^{-\chi_{c1}(b)}\,e^{-\chi_{c2}(b)} - e^{-\chi_{c1}(b)}\right]$$

$$= +ik\int_0^\infty b\,db\,J_0(qb)\,e^{-[\chi_0(b)+\chi_{c1}(b)]}\underbrace{\left[1-e^{-\chi_{c2}(b)}\right]}_{\text{LONG RANGE}}$$

OR

$$\boxed{\begin{aligned} \mathcal{F}_0(q) &= \mathcal{F}_{c1}(q) + \mathcal{F}_{c2}(q) \\ \mathcal{F}_{c1}(q) &= ik\int_0^\infty b\,db\,J_0(qb)\,e^{-\chi_0(b)}\left[1-e^{-\chi_{c1}(b)}\right] \\ \mathcal{F}_{c2}(q) &= ik\int_0^\infty b\,db\,J_0(qb)\,e^{-(\chi_0+\chi_{c1}(b))}\left[1-e^{-\chi_{c2}(b)}\right] \end{aligned}}$$

WITH $\mathcal{F}_{c1}(q)$ EVALUATED WITH SIMPSON AND $\mathcal{F}_{c2}(q)$ EVALUATED WITH OGATA.