JUNE 24, 2020

HANKEL TRANSFORMS OF POLYNOMIALS

I HAVE BEEN MESMERIZED BY THE CONVERGENCE OF CERTAIN HANKEL TRANSFORMS INVOLVING POLYNOMIALS; e.g. $\frac{1}{b}, b, b^3, \ldots$

SO LET US START BY WRITING

$$F_n(q) = \int_0^\infty b\,db\, J_0(qb)\, b^n$$

AND THEN USE AN INTEGRAL REPRESENTIAN OF $J_0(x)$; NAMELY,

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos\varphi}\,d\varphi$$

THEN WE CAN WRITE

$$F_n(q) = \int_0^\infty b^{n+1}\,db\left[\frac{1}{2\pi}\int_0^{2\pi} e^{iqb\cos\varphi}\,d\varphi\right]$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\varphi\left[\int_0^\infty b^{n+1} e^{iqb\cos\varphi}\,db\right]$$

WE THEN START BY COMPUTING THE EXPRESSION IN BRACKETS:

$$G_m(z) \equiv \int_0^\infty b^{m-1} e^{izb}\,db \qquad \text{WHERE } z \equiv q\cos\varphi + i\eta$$

WHERE WE HAVE ADDES A SMALL IMAGINART PART TO z TO ENSURE CONVERGENCE. THAT IS, WE CAN WRITE

$$G_m(z) = \int_0^\infty b^{m-1} e^{-(\eta+iz)b}\,db \;;$$ NOW CHANGE VARIABLES TO $x = (\eta+iz)b$

THEN

$$G_m(z) = \int_0^\infty \left[\frac{x}{(\eta+iz)}\right]^{m-1} e^{-x}\,\frac{dx}{(\eta+iz)}$$

HENCE,

$$G_m(z) = \frac{1}{(\eta+iz)^m}\int_0^\infty x^{m-1}e^{-x}\,dx = \frac{\Gamma(m)}{(\eta+iz)^m}$$

OR

$$\boxed{G_m(z) = \frac{\Gamma(m)}{(\eta+iz)^m} = \frac{(m-1)!}{i^m(z-i\eta)^m}}$$

NOW SETTING $m=n+2$ WE OBTAIN

$$F_n(q) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi\, G_{n+2}(z) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \frac{(n+1)!}{i^{n+2}(q\cos\varphi - i\eta)^{n+2}}$$

$$\boxed{F_n(q) = \frac{(n+1)!}{2\pi\, i^{n+2} q^{n+2}}\int_0^{2\pi}\frac{d\varphi}{(\cos\varphi - i\eta)^{n+2}}}$$

WITH

$$\lim_{\eta\to 0}\frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi}{(\cos\varphi - i\eta)^{n+2}} = i\begin{cases} 1 & \text{FOR } n=-1 \\ \frac{1}{2} & \text{FOR } n=+1 \\ \frac{3}{8} & \text{FOR } n=+3 \\ \vdots & \\ 0 & \text{FOR } n=\text{even} \end{cases}$$

HENCE,

$$\boxed{\begin{aligned} F_{-1}(q) &= \int_0^\infty J_0(qb)\,db = +1/q \\ F_{+1}(q) &= \int_0^\infty b^2 J_0(qb)\,db = -1/q^3 \\ F_3(q) &= \int_0^\infty b^4 J_0(qb)\,db = +9/q^5 \\ &\vdots \end{aligned}}$$

A FEW HANKEL TRANSFORMS...