High-accuracy fast Hankel transform for optical beam propagation

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Received March 3, 1992; accepted May 6, 1992

We describe a new method for the numerical calculation of the zero-order Hankel (Fourier–Bessel) transform that has a high computational efficiency and an accuracy that can be 2 orders of magnitude greater than that of the standard quasi-fast Hankel procedure. The new method offers particular advantages in calculating optical beam propagation and resonator modes at high Fresnel numbers.

Within the paraxial scalar diffraction theory the propagation of an optical beam through a cylindrically symmetric system can be reduced to the calculation of a Hankel transform integral.[1] Since the numerical Hankel transformation often must be applied repeatedly, for example, in resonator-mode calculations, computational efficiency and accuracy are main issues. Various numerical procedures for the Hankel transform have been described[2–8]: in particular an efficient algorithm, called quasi-fast Hankel transform (QFHT), was independently developed by Siegman[2] and Talman[3] and was improved by Agrawal and Lax.[6]

In this paper we describe a new fast Hankel transform of high accuracy (FHATHA) algorithm that evaluates the zero-order Hankel (Fourier–Bessel) transform with the same speed as that of the QFHT with a substantially higher precision, especially in the case of optical propagation at large Fresnel numbers. The basic idea of the FHATHA algorithm is the following: the transforming function is sampled and approximated with a finite sum of rectangular functions so that the Hankel transform can be analytically calculated for each elementary term. After the sampling points in a geometrical progression are chosen, the resulting sum is written in the form of a discrete cross correlation that is then *exactly* evaluated, with high computational efficiency, by means of the fast Fourier transform (FFT).

The Hankel transform integral suitable for optical beam propagation in terms of normalized coordinates assumes the form

$$g(y) = 2\pi \int_0^1 f(x) J_0(2\pi N_f y x) x \mathrm{d}x\,, \qquad (1)$$

where J_0 is a Bessel function of order 0. The upper integration limit is set to 1 instead of to ∞, as in the standard Hankel transform, since from a physical and computational point of view the transforming field $f(x)$ and the transformed field $g(y)$ are limited by some finite aperture. Therefore, after a suitable variable normalization, one can consider functions defined in the intervals $0 \le x \le 1$ and $0 \le y \le 1$. In Eq. (1) the finite widths of the input and the output apertures can therefore be thought to be included in the factor N_f, which in terms of optical parameters represents the Fresnel number of the propagation.[9]

To derive the algorithm, we proceed as follows: The interval $0 \le x \le 1$ is divided into N subintervals through the $N + 1$ points ξ_n $(n = 0, 1, \ldots, N)$, where $0 = \xi_0 < \xi_1 < \cdots < \xi_N = 1$. The transforming function $f(x)$ is sampled in N points x_n $(n = 0, 1, \ldots, N - 1)$, such that $\xi_n \le x_n \le \xi_{n+1}$. The transforming function $f(x)$ is now approximated by $\hat{f}(x) = f(x_n)$ for $\xi_n < x < \xi_{n+1}$. The Hankel transform $\hat{g}(y)$ of $\hat{f}(x)$, which is the searched approximation of $g(y)$, can be calculated in closed form by applying Eq. (1) to each subinterval:

$$2\pi \int_{\xi_n}^{\xi_{n+1}} \hat{f}(x) J_0(2\pi N_f y x) x \mathrm{d}x = \frac{f(x_n)}{N_f y} [\xi_{n+1} J_1(2\pi N_f y \xi_{n+1}) - \xi_n J_1(2\pi N_f y \xi_n)], \qquad (2)$$

where J_1 is a Bessel function of order 1. Summing the contributions of all the subintervals gives

$$\hat{g}(y) = \frac{1}{N_f y} \sum_{0}^{N-1}{}_{n} [f(x_n) - f(x_{n+1})] \xi_{n+1} J_1(2\pi N_f y \xi_{n+1})\,, \qquad (3)$$

where, by definition, $f(x_N) \equiv 0$. The sum in Eq. (3) can be recast as a discrete cross correlation by setting the integration subintervals and the sampling points in a geometrical progression:

$$\begin{aligned} &\xi_0 = 0\,, \\ &\xi_n = \exp[\alpha(n - N)] \qquad \text{for } n = 1, 2, \ldots, N, \\ &x_n = y_n = x_0 \exp(\alpha n) \qquad \text{for } n = 0, 1, \ldots, N - 1\,, \end{aligned} \qquad (4)$$

where $\alpha > 0$. The sampling points y_n for the transformed function $\hat{g}(y)$ have been set equal to x_n to permit direct repeated application of the transformation. Although the point x_0 can be arbitrarily chosen, better results may be obtained when the sampling points are in the center of the relevant subintervals, as for

$$x_0 = [1 + \exp(\alpha)] \exp(-\alpha N)/2\,. \qquad (5)$$

0740-3232/92/112031-03$05.00

The first point, x_0, however, does not lie in the center of the first subinterval. To avoid the use of an extra sampling point in $x = \xi_1/2$, we evaluate the function $f(x)$ in $\xi_1/2$ by an interpolation through the values in x_0 and x_1 with a parabola with zero derivative in the origin. This interpolated value is assigned to $\hat{f}$ for $0 < x < \xi_1$. If we use the sampling sequence defined above, Eq. (3) becomes

$$\hat{g}(y_m) = \frac{1}{N_f y_m} \sum_{0}^{N-1}{}_n [f(x_n) - f(x_{n+1})]k_n \times \exp[\alpha(n + 1 - N)] \times J_1\{2\pi N_f x_0 \exp[\alpha(n + m + 1 - N)]\}, \quad (6)$$

where $f(x_N) \equiv 0$. The factor k_n is 1 for any n except $n = 0$, where

$$k_0 = \frac{2\exp(\alpha) + \exp(2\alpha)}{[1 + \exp(\alpha)]^2[1 - \exp(-2\alpha)]} \simeq \frac{3}{8\alpha} + \frac{1}{2} + \cdots. \quad (7)$$

This factor arises from the integral over the first subinterval. The parameter α can be chosen with a certain arbitrariness. A value of α that may give better results causes the first and the last integration subintervals to have the same width. In this case α is given by

$$\alpha = -\ln[1 - \exp(\alpha)]/(N - 1), \quad (8)$$

which can easily be solved numerically. The number of sampling points N depends on the accuracy required and can be determined by following the prescriptions of the sampling theory.[10] The discrete cross correlation in Eq. (6) can be calculated exactly with high computational efficiency by means of three applications of the FFT by using sequences of $2N$ values and choosing for N a power of 2.[11] In summary, the final recipe for the FHATHA is

$$\hat{g}(y_m) = 1/(N_f y_m)\mathrm{FFT}[\mathrm{FFT}(\varphi_n)\mathrm{IFFT}(j_{1n})], \quad (9)$$

where (a) the FFT and the inverse fast Fourier transform (IFFT), as defined in Ref. 12, are calculated on a sequence of $2N$ points; (b)

$$\varphi_n = \begin{cases} k_0[f(x_0) - f(x_1)]\exp[\alpha(1 - N)] & \text{for } n = 0 \\ [f(x_n) - f(x_{n+1})]\exp[\alpha(n + 1 - N)] & \text{for } n = 1, 2, \ldots, N - 1 \\ 0 & \text{for } n = N, N + 1, \ldots, 2N - 1 \end{cases},$$

with $f(x_N) \equiv 0$; (c) $j_{1n} = J_1\{2\pi N_f x_0 \exp[\alpha(n + 1 - N)]\}$ for $n = 0, 1, \ldots, 2N - 1$; (d) the two Fourier transforms in brackets are multiplied term by term before the external transform is performed; (e) the significant values for $\hat{g}(y_m)$ are the first N, while those from $m = N$ to $2N - 1$ must be discarded.

Since the new algorithm is similar to the well-assessed procedure of the QFHT described by Siegman[2] a comparison of the two methods is presented. Expressing the final formula for QFHT with the notations used in this paper, one obtains

$$\tilde{g}(y_m) = 2\pi\alpha \sum_{0}^{N-1}{}_n f(x_n)x_n^2 J_0\{2\pi N_f x_0^2 \exp[\alpha(n + m)]\} + \pi f(x_0)x_0^2, \quad (10)$$

where $\tilde{g}$ is the approximation to g and the last term is the end correction term.[6] Neglecting this term, one can prove by direct substitution into Eq. (1) that $\tilde{g}(y_m)$ is the exact analytical Hankel transform, sampled in $y = y_m = x_m$, of the function

$$\sum_{0}^{N-1}{}_n \alpha x_n f(x_n)\delta(x - x_n), \quad (11)$$

where δ is the Dirac delta function. The QFHT procedure can therefore be interpreted in a new way, as follows: First the transforming function $f(x)$ is approximated by a sum of δ functions of suitable amplitude placed along a sequence of points in geometrical progression; then the new function is analytically Hankel transformed to give Eq. (10) (the end correction is excluded); and finally the discrete cross correlation in Eq. (10) is evaluated exactly by means of three FFT's. Since the approximation of the original function with a sequence of rectangles appears to be much more accurate than the approximation with a sequence of δ, the result provided by FHATHA must be much closer to the exact transform than that provided by QFHT. As an example, the FHATHA algorithm provides

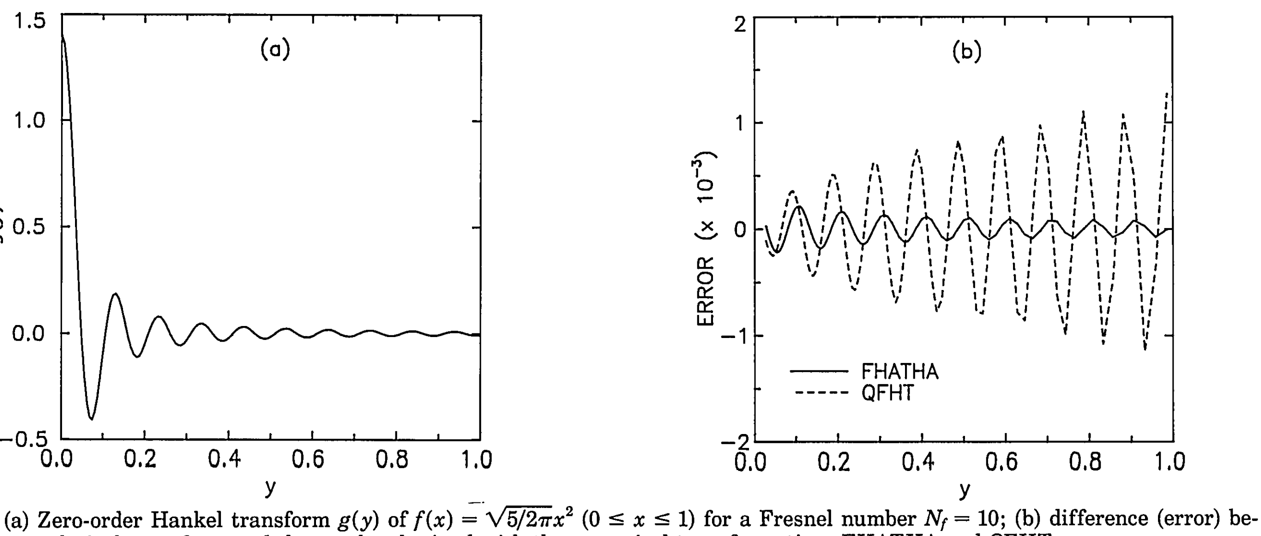

Fig. 1. (a) Zero-order Hankel transform $g(y)$ of $f(x) = \sqrt{5/2\pi}x^2$ $(0 \le x \le 1)$ for a Fresnel number $N_f = 10$; (b) difference (error) between the analytical transform and the results obtained with the numerical transformations FHATHA and QFHT.

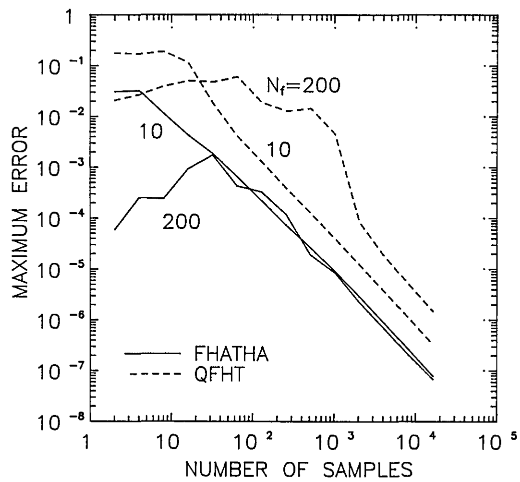

Fig. 2. Maximum error generated by the FHATHA and the QFHT algorithms in the numerical calculation of the Hankel transform of $f(x) = \sqrt{5/2\pi x^2}$ for two Fresnel numbers, $N_f = 10$ and $N_f = 200$.

the exact Hankel transform of a constant (i.e., the Airy function) independently of the number of sampling points, whereas the standard QFHT does not. The two procedures, however, require the same number of elementary computer operations and the same amount of memory.

As an illustration of the accuracy of the algorithm, we apply it to the parabola $f(x) = \sqrt{5/2\pi}\, x^2$, whose exact transform is

$$g(y) = \sqrt{10\pi}\, \eta^{-4}[2\eta^2 J_0(\eta) + (\eta^3 - 4\eta)J_1(\eta)], \tag{12}$$

where $\eta = 2\pi N_f y$. Figure 1(a) shows a plot of $g(y)$ for a Fresnel number $N_f = 10$. The computations were performed with double-precision arithmetic. The error, defined as the difference between the exact analytical transform and the values obtained with the numerical transformations FHATHA and QFHT, respectively, is plotted in Fig. 1(b) for 128 sampling points. The behavior of the maximum error is shown in Fig. 2 as a function of the number of samples (ranging from 2 to 2^{14}) used in the numerical transformation for two values of the Fresnel number, namely, $N_f = 10$ and $N = 200$. In this figure the results obtained by the FHATHA are compared with those obtained by the QFHT with end correction. It can be seen that for large values of N the error of the FHATHA is independent of the Fresnel number N_f, whereas the error of the standard QFHT increases with N_f. For $N_f = 200$ (or higher) the results obtained with the FHATHA can be approximately 2 orders of magnitude more precise than those produced by the standard QFHT. This circumstance can be explained by considering that the accuracy of the numerical integration depends both on the bandwidth of the integrating function and on the number of sampling points. In the case of the QFHT, the total bandwidth is determined by the transforming function and by the kernel J_0, and hence the total bandwidth increases as the Fresnel number increases. On the other hand for the FHATHA, only the bandwidth of the input function $f(x)$ has influence, since the kernel J_0 is integrated analytically and its bandwidth need not be taken into account. Since N_f does not affect the bandwidth of $f(x)$, the accuracy is almost independent of this parameter. For very low numbers of sampling points ($N = 2, 4, 8$) and large Fresnel numbers, the error generated by FHATHA is also very low, but this fact seems to be fortuitous.

ACKNOWLEDGMENTS

The research reported in this paper has been partially supported by the National Research Council of Italy, under the Progetto Finalizzato on Electro-optical Technologies.

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