AN EXAMPLE: A UNIFORM CHARGE AND WEAK-CHARGE DISTRIBUTION

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PART III

LET'S START BY COMPUTING THE COULOMB POTENTIAL DUE TO A UNIFORM CHARGE DISTRIBUTION OF RADIUS R. THAT IS

$$\rho_{ch}(\vec{r}) = \frac{\Theta(R-r)}{\frac{4}{3}\pi R^3} = \frac{3}{4\pi R^3}\begin{cases} 1 & \text{IF } 0 \le r \le R \\ 0 & \text{IF } r > R \end{cases}$$

THEN,

$$V(r) = -4\pi Z e^2 \left[\frac{1}{r}\int_0^r r'^2 \rho_{ch}(r')\,dr' + \int_r^\infty r' \rho_{ch}(r')\,dr' \right]$$

- LET $0 \le r \le R$, THEN

$$V(r) = -4\pi Z e^2 \left[\frac{1}{r}\int_0^r r'^2 \rho_{ch}(r')\,dr' + \int_r^R r' \rho_{ch}(r')\,dr' \right]$$

$$= -4\pi Z e^2 \rho_0 \left[\frac{1}{r}\int_0^r r'^2\,dr' + \int_r^R r'\,dr' \right]$$

$$= -4\pi Z e^2 \rho_0 \left[\frac{1}{r}\cdot\frac{r^3}{3} + \frac{1}{2}(R^2 - r^2) \right]$$

$$= -4\pi Z e^2 \rho_0 \left[\frac{R^2}{2} - \frac{r^2}{6} \right]$$

$$= -4\pi Z e^2 \rho_0 \frac{R^2}{2}\left[1 - \frac{r^2}{3R^2} \right]$$

$$= -\frac{3}{2}\frac{Ze^2}{R}\left[1 - \frac{r^2}{3R^2} \right]$$

- LET $r > R$, THEN

$$V(r) = -4\pi Z e^2 \left[\frac{1}{r} \int_0^R r'^2 \rho_{ch}(r')\, dr' \right]$$

$$= -4\pi Z e^2 \rho_0 \frac{1}{r} \frac{R^3}{3} = -\frac{Ze^2}{r}$$

HENCE,

$$\boxed{V(r) = -Ze^2 \begin{cases} \frac{3}{2R}\left(1 - \frac{r^2}{3R^2}\right) & \text{FOR } 0 \leq r \leq R \\ 1/r & \text{FOR } r \geq R \end{cases}}$$

$V(r)$ AND $V'(r)$ ARE BOTH CONTINUOUS AT $r = R$

TO COMPUTE THE CENTRAL PROFILE WE MUST PERFORM THE FOLLOWING INTEGRAL:

$$\chi_c(b) = i \frac{m}{K} \int_{-\infty}^{\infty} \left(\frac{E}{m} \left[V(r) + \frac{Ze^2}{r} \right] - \frac{V^2(r)}{2m} \right) dz$$

$$= i \frac{E}{K} \int_{-\infty}^{\infty} \left(V(r) + \frac{Ze^2}{r} \right) dz - \frac{i}{2K} \int_{-\infty}^{\infty} V^2(r)\, dz$$

$$\equiv i \frac{E}{K} t_1(b) - \frac{i}{2K} t_2(b)$$

WHERE

$$t_1(b) = \int_{-\infty}^{\infty} \left(V(r) + \frac{Ze^2}{r} \right) dz \equiv \int_{-\infty}^{\infty} V_{SR}(r)\, dz$$

SHORT-RANGE POTENTIAL.

$$t_2(b) = \int_{-\infty}^{\infty} V^2(r)\, dz$$

WE HAVE DEFINED THE SHORT RANGE POTENTIAL AS

$$V_{SR}(r) \equiv V(r) + \frac{Ze^2}{r} = -Ze^2 \begin{cases} \left[\frac{3}{2R}\left(1-\frac{r^2}{3R^2}\right) - \frac{1}{r}\right] & 0 \le r \le R \\ 0 & \text{OTHERWISE} \end{cases}$$

HENCE,

$$t_1(b) = 2\int_0^\infty V_{SR}(r)\,dz = 2\int_b^\infty V_{SR}(r)\frac{r}{\sqrt{r^2-b^2}}\,dr$$

$$= \begin{cases} 2\int_b^R V_{SR}(r)\frac{r}{\sqrt{r^2-b^2}} & \text{IF } 0 \le b \le R \\ 0 & \text{otherwise} \end{cases}$$

SO FOR $0 \le b \le R$ WE HAVE

$$t_1(b) = -2Ze^2 \int_b^R \left(\frac{3}{2R} - \frac{r^2}{2R^3} - \frac{1}{r}\right)\frac{r}{\sqrt{r^2-b^2}}\,dr$$

Forgot a factor of 2

OR CHANGING VARIABLES TO $r \equiv bx$, WE OBTAIN

$$t_1(b) = -2Ze^2 \int_1^{R/b} \left[\frac{3b}{2R} - \left(\frac{b^3}{2R^3}\right)x^2 - \frac{1}{x}\right]\frac{x\,dx}{\sqrt{x^2-1}}$$

$$= -2Ze^2 \left\{ \frac{3b}{2R}\int_1^{R/b}\frac{x\,dx}{\sqrt{x^2-1}} - \frac{b^3}{2R^3}\int_1^{R/b}\frac{x^3\,dx}{\sqrt{x^2-1}} - \int_1^{R/b}\frac{dx}{\sqrt{x^2-1}} \right\}$$

$$= -2Ze^2 \left\{ \frac{3}{2\alpha}\sqrt{\alpha^2-1} - \frac{1}{2\alpha^3}\cdot\frac{1}{3}\sqrt{\alpha^2-1}\,(\alpha^2+2) - \ln\left(\alpha+\sqrt{\alpha^2-1}\right) \right\} \qquad \alpha \equiv R/b$$

IT MAY BE BEST TO CAST THE ANSWER IN TERMS OF $\xi \equiv \alpha^{-1} = \frac{b}{R}$ WHERE $0 \le \xi \le 1$. THAT IS,

$$t_1(b) = -2Ze^2 \left\{ \frac{3}{2}\sqrt{1-\xi^2} - \frac{1}{6}\sqrt{1-\xi^2}\,(1+2\xi^2) - \ln\left(\frac{1+\sqrt{1-\xi^2}}{\xi}\right) \right\}$$

OR

$$\boxed{t_1(b) = -2Ze^2\left[\frac{4}{3}\left(1-\frac{\xi^2}{4}\right)\sqrt{1-\xi^2} - \ln\left(\frac{1+\sqrt{1-\xi^2}}{\xi}\right)\right]}$$

WHERE $\xi \equiv b/R$ AND $0 \le \xi \le 1$

$t_1(b)$ VANISHES FOR $b > R$.

logarithmic divergence as $\xi \to 0$

ON TO $t_2(b)$:

- FOR $0 \le b \le R$

$$t_2(b) = 2\int_b^\infty V^2(r)\,\frac{r}{\sqrt{r^2-b^2}}\,dr$$

$$= 2\left\{\int_b^R \left[\frac{3}{2R}\left(1-\frac{r^2}{3R^2}\right)\right]^2 \frac{r\,dr}{\sqrt{r^2-b^2}} + \int_R^\infty \frac{1}{r^2}\,\frac{r\,dr}{\sqrt{r^2-b^2}}\right\}$$

CHANGING VARIABLES TO $r = bx$,

$$\frac{t_2(b)}{(Ze^2)^2} = 2\left\{\int_1^{R/b}\left[\frac{3}{2R}\left(1-\frac{b^2}{3R^2}x^2\right)\right]^2 \frac{b^2\,x\,dx}{b\sqrt{x^2-1}} + \int_{R/b}^\infty \frac{b\,dx}{b^2 x\sqrt{x^2-1}}\right\}$$

Hence,

$$t_2(b) = 2(Ze^2)^2 \left\{ \frac{9b}{4R^2}\int_1^{R/b}\left(1-\frac{b^2}{3R^2}x^2\right)^2 \frac{x}{\sqrt{x^2-1}}\,dx + \frac{1}{b}\int_{R/b}^{\infty}\frac{dx}{x\sqrt{x^2-1}}\right\}$$

OR

$$t_2(b) = \frac{2}{R}(Ze^2)^2 \left\{ \frac{9}{4}\xi\int_1^{\alpha}\left(1-\frac{\xi^2}{3}x^2\right)^2 \frac{x}{\sqrt{x^2-1}}\,dx + \alpha\int_{\alpha}^{\infty}\frac{dx}{x\sqrt{x^2-1}}\right\}$$

OR

$$t_2(b) = \frac{2}{R}(Ze^2)^2 \left\{ \frac{9}{4}\xi\int_1^{\alpha}\frac{x}{\sqrt{x^2-1}}\,dx - \frac{3}{2}\xi^3\int_1^{\infty}\frac{x^3}{\sqrt{x^2-1}}\,dx + \frac{1}{4}\xi^5\int_1^{\alpha}\frac{x^5}{\sqrt{x^2-1}}\,dx + \alpha\int_{\alpha}^{\infty}\frac{dx}{x\sqrt{x^2-1}}\right\}$$

OR

$$t_2(b) = \frac{2}{R}(Ze^2)^2 \left\{ \frac{9}{4}\xi\sqrt{\alpha^2-1} - \frac{3}{2}\xi^3\cdot\frac{1}{3}\sqrt{\alpha^2-1}\,(\alpha^2+2) + \frac{1}{4}\xi^5\cdot\frac{1}{15}\sqrt{\alpha^2-1}\,(3\alpha^4+4\alpha^2+8) + \alpha\cdot\arctan\left(\frac{1}{\sqrt{\alpha^2-1}}\right)\right\}$$

$$= \frac{2}{R}(Ze^2)^2 \left\{ \frac{9}{4}\sqrt{1-\xi^2} - \frac{1}{2}\sqrt{1-\xi^2}\,(1+2\xi^2) + \frac{1}{60}\sqrt{1-\xi^2}\,(3+4\xi^2+8\xi^4) + \frac{1}{\xi}\arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right)\right\}$$

OR

$$t_2(b) = \frac{2}{R}(Ze^2)^2 \left[\sqrt{1-\xi^2}\left(\frac{9}{4} - \frac{1}{2}(1+2\xi^2) + \frac{(3+4\xi^2+8\xi^4)}{60} \right) + \frac{1}{\xi}\arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right) \right]$$

OR

$$\boxed{t_2(b) = \frac{2}{R}(Ze^2)^2 \left[\sqrt{1-\xi^2}\left(\frac{2\xi^4 - 14\xi^2 + 27}{15} \right) + \frac{1}{\xi}\arctan\left(\frac{\xi}{\sqrt{1-\xi^2}}\right) \right]}$$

FOR $0 \le b \le R$ AND WITH $\xi \equiv b/R$

NOW FOR $b > R$

$$t_2(b) = 2(Ze^2)^2 \int_b^\infty \frac{1}{r^2}\frac{r}{\sqrt{r^2-b^2}}\,dr \; ; \quad r = xb$$

$$= 2(Ze^2)^2 \int_1^\infty \frac{b\,dx}{b^2 x\sqrt{x^2-1}}$$

$$= \frac{2(Ze^2)^2}{b}\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \frac{2(Ze^2)^2}{b}\cdot\frac{\pi}{2}$$

OR

$$\boxed{t_2(b) = \frac{2}{R}(Ze^2)^2\left(\frac{\pi}{2\xi}\right)} = \boxed{\frac{\pi}{b}(Ze^2)^2}$$

FOR $b > R$ AND $\xi \equiv b/R$

SLOW FALLOFF!

WE NOW COMPUTE THE SPIN-ORBIT PROFILE FUNCTION. THAT IS,

$$\chi_{so}(b) = imb \int_{-\infty}^{\infty} V_{so}(r)\, dz$$

$$= imb \int_{-\infty}^{\infty} \frac{1}{2mr} \frac{V'(r)}{(E+m-V(r))}\, dz$$

FOR THE PURPOSE OF ANALYTIC EVALUATION LET'S IGNORE THE POTENTIAL IN THE DENOMINATOR. THAT IS,

$$\chi_{so}(b) \simeq \frac{ib}{2(E+m)} \int_{-\infty}^{\infty} \frac{V'(r)}{r}\, dz \equiv \frac{ib}{2(E+m)}\, t_{so}(b)$$

WHERE

$$t_{so}(b) = \int_{-\infty}^{\infty} \frac{V'(r)}{r}\, dz = 2\int_{b}^{\infty} \frac{V'(r)}{r} \cdot \frac{r}{\sqrt{r^2-b^2}}\, dr$$

$$= 2\int_{b}^{\infty} \frac{V'(r)}{\sqrt{r^2-b^2}}\, dr$$

AND

$$V'(r) = +Ze^2 \begin{cases} +r/R^3 & \text{FOR } 0 \le r \le R \\ +\dfrac{1}{r^2} & \text{FOR } r \ge R \end{cases}$$

HENCE,

- FOR $0 \le b \le R$

$$t_{so}(b) = 2(Ze^2)\left[\int_{b}^{R} \frac{r}{R^3} \cdot \frac{dr}{\sqrt{r^2-b^2}} + \int_{R}^{\infty} \frac{1}{r^2} \frac{dr}{\sqrt{r^2-b^2}}\right]$$

$$= 2(Ze^2)\left[\frac{1}{R^3}\int_{b}^{R} \frac{r\,dr}{\sqrt{r^2-b^2}} + \int_{R}^{\infty} \frac{dr}{r^2\sqrt{r^2-b^2}}\right]$$

NOW CHANGE VARIABLES TO $r \equiv bx$; THEN

$$t_{so}(b) = 2(Ze^2)\left[\frac{b}{R^3}\cdot\int_1^{R/b}\frac{x\,dx}{\sqrt{x^2-1}} + \frac{1}{b^2}\int_{R/b}^{\infty}\frac{dx}{x^2\sqrt{x^2-1}}\right]$$

$$= \frac{2}{R^2}(Ze)^2\left[\xi\int_1^{\alpha}\frac{x}{\sqrt{x^2-1}}\,dx + \frac{1}{\xi^2}\int_{\alpha}^{\infty}\frac{dx}{x^2\sqrt{x^2-1}}\right]$$

$$= \frac{2(Ze^2)}{R^2}\left[\sqrt{1-\xi^2} + \frac{1}{\xi^2}\left(1-\sqrt{1-\xi^2}\right)\right]$$

$$= \frac{2(Ze^2)}{R^2}\left[\frac{1+(\xi^2-1)\sqrt{1-\xi^2}}{\xi^2}\right]$$

OR

$$\boxed{t_{so}(b) = \frac{2}{b^2}(Ze^2)^2\left[1-(1-\xi^2)^{3/2}\right]}$$

FOR $0 \le b \le R$ AND $\xi \equiv b/R$

- NOW FOR $b > R$

$$t_{so}(b) = 2(Ze^2)\int_b^{\infty}\frac{dr}{r^2\sqrt{r^2-b^2}} \quad ; \quad r = bx$$

$$= \frac{2(Ze^2)}{b^2}\int_1^{\infty}\cancelto{1}{\frac{dx}{x^2\sqrt{x^2-1}}}$$

OR

$$\boxed{t_{so}(b) = \frac{2}{b^2}(Ze^2)}$$

FOR $b > R$

FINALLY, THE SHORT-RANGE AXIAL PROFILE FUNCTION.

$$\chi_A(b) = i\int_{-\infty}^{\infty} A(r)\,dz = i\,\frac{G_F}{2\sqrt{2}}\,Q_W \int_{-\infty}^{\infty} \rho_{WR}(r)\,dz$$

$$\equiv i\,\frac{G_F}{2\sqrt{2}}\,Q_W\,t_A(b)$$

WHERE

$$t_A(b) = \int_{-\infty}^{\infty} \rho_{WR}(r)\,dz = 2\int_b^{\infty} \rho_{WR}(r)\,\frac{r}{\sqrt{r^2-b^2}}\,dr$$

OR

$$t_A(b) = \begin{cases} 2\rho_0 \displaystyle\int_b^R \frac{r\,dr}{\sqrt{r^2-b^2}}, & \text{FOR } 0 \le b \le R \\ 0 & \text{FOR } b > R \end{cases}$$

NOW,

$$\int_b^R \frac{r\,dr}{\sqrt{r^2-b^2}} = b\int_1^{R/b} \frac{x\,dx}{\sqrt{x^2-1}} = b\,\frac{\sqrt{1-\xi^2}}{\xi}$$

HENCE, FOR $0 \le b \le R$

$$t_A(b) = 2\cdot\frac{3}{4\pi R^3}\cdot b\,\frac{\sqrt{1-\xi^2}}{\xi} = \frac{3}{2\pi R^2}\sqrt{1-\xi^2}$$

OR

$$\boxed{t_A(b) = \frac{3}{2\pi R^2}\begin{cases} \sqrt{1-\xi^2} & \text{FOR } 0 \le b \le R \\ 0 & \text{FOR } b > R \end{cases}}$$

WHERE $\xi \equiv b/R$