Lecture 3   9/1/16   Math 281 $P(A)=\frac{|A|}{|\Omega|}$ if $\forall \omega\ P(\{\omega\})=\frac{1}{|\Omega|}$ equally likely $|\Omega|<\infty$ 1

$$P(A^c) = \frac{|A^c|}{|\Omega|} = \frac{|\Omega|-|A|}{|\Omega|} = \frac{|\Omega|}{|\Omega|} - \frac{|A|}{|\Omega|} = 1 - P(A) \quad \text{(complement rule)}$$

What if $P(\{H\}) \neq P(\{T\})$ def doesn't work ... for now ... $P(\{\omega_i\}) = \frac{1}{|\Omega|}$ (all outcomes equally likely)

What does $|2^{\Omega}|$ mean? The # of possible events $\neq$ possible outcomes.

All things you can ask "what is the prob of ... ?"

Two coin flips:

$\Omega' := \Omega^2$

Ω'

$\langle H,H\rangle$	$\langle H,T\rangle$
$\langle T,H\rangle$	$\langle T,T\rangle$

$|\Omega'| = |\Omega|^2 = 2^2 = 4$

a bit of notation ...

$P(\{\langle H,H\rangle\})$ colloquially, $P(H,H)$

$$= \frac{|\{\langle H,H\rangle\}|}{|\Omega|} = \frac{1}{4}$$

A: at least one Head

$$P(A) = P(\{\langle H,H\rangle,\langle H,T\rangle,\langle T,H\rangle\}) = \frac{3}{4}$$

$$2^{\Omega} = \{\emptyset, \{\langle H,H\rangle\}, \{\langle H,T\rangle\}, \{\langle T,H\rangle\}, \{\langle T,T\rangle\},$$
$$\{\langle H,H\rangle, \langle H,T\rangle\}, \ldots$$
$$\{\langle H,H\rangle, \langle H,T\rangle, \langle T,H\rangle\}, \ldots,$$
$$\Omega\}$$

$$|2^{\Omega}| = 2^{|\Omega|} = 2^4 = 16$$

B: at least one tail

$$P(B) = \frac{|\{\langle H,T\rangle, \langle T,T\rangle, \langle T,H\rangle\}|}{|\Omega|} = \frac{3}{4}$$

$P(A \cup B) \stackrel{?}{=} P(A) + P(B) = \frac{3}{4} + \frac{3}{4} = 1.5$? No...

$$= \frac{|\{\langle H,H\rangle, \langle H,T\rangle, \langle T,H\rangle, \langle T,T\rangle\}|}{|\Omega|} = \frac{4}{4} = 1$$

$\Rightarrow$ "you will ~~always~~ get at least one H or at least one T

$P(A \cap B) := P(A \text{ and } B) := P(A \,\&\, B),\ P(A,B) = P(AB)$

all the same thing

$$= \frac{|\{\langle H,T\rangle, \langle T,H\rangle\}|}{|\Omega|} = \frac{1}{2} \neq \underbrace{\frac{3}{4} + \frac{3}{4}}_{P(A)+P(B)} \neq \underbrace{\frac{3}{4} \cdot \frac{3}{4}}_{P(A)\cdot P(B)} \neq \underbrace{\frac{3}{4} - \frac{3}{4}}_{P(A)-P(B)} \neq \underbrace{\frac{3/4}{3/4}}_{\frac{P(A)}{P(B)}}$$

Not simple... gotta figure out the outcomes in the event!

4 coin tosses

$\Omega' = \Omega^4$ $\quad |\Omega'| = 16$ ← # of outcomes. Unique outcomes of a single experiment.

$$P(\underset{\downarrow}{\overset{\omega_1}{}}HHHH) = \frac{1}{16}$$ due to the equally likely assumption

$P(HHHH) = P(HTHT)$?

Yes $P(\langle H,H,H,H\rangle) = P(\langle H,T,H,T\rangle)$

$\stackrel{?}{=} P(2H \text{ and } 2T)$ No...

$= \dfrac{|\{\langle H,H,T,T\rangle, \langle H,T,H,T\rangle, \langle T,H,T,H\rangle, \langle T,H,H,T\rangle, \langle T,T,H,H\rangle, \langle H,T,T,H\rangle\}|}{|\Omega|}$

$= \dfrac{6}{16} = .375$ was this trivial?

$> \dfrac{1}{16}$!

$P(\text{at least one } H) = \dfrac{|\{\langle H,T,T,T\rangle, \{T,H,T,T\}, \ldots|}{|\Omega|}$ long time... and we make one mistake $\Rightarrow$ cooked!

$\{\geq 1H\}^c = \{<1H\} = \{0H\} = \{\langle T,T,T,T\rangle\}$

$= 1 - P(\text{no } H) = 1 - \dfrac{1}{16} = \dfrac{15}{16}$

Flip 10 coins!

$\Omega' = \Omega^{10}$, $|\Omega'| = 2^{10} \approx 1000$ size of our space?

CS: $2^x \approx 1000^{\frac{x}{10}}$ $|2^{\Omega'}| = 2^{|\Omega'|} = 2^{2^{10}} \approx \infty$

$\approx 10^{301}$

lots of questions can be posed...

$\Rightarrow 2^{20} \approx 1m$

$2^{30} \approx 1b$

why?

$\vdots$

$\ln 2^x \approx \ln 1000^{\frac{x}{10}}$

.2460938

$\Rightarrow \cancel{x} \ln 2 \approx \frac{\cancel{x}}{10} \ln 1000$

$P(5H \text{ and } 5T) = \dfrac{|\{\ldots ??\}|}{|\Omega|}$

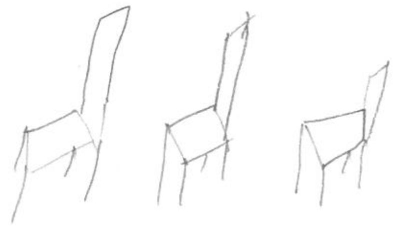
$\Rightarrow \ln_{10} 2 \approx 1000^{\frac{1}{10}} = 1.9953$

difficult!! Need a way to count sets better than enumeration!

Consider Jane, Mary, Susan standing in front of you.

$\Omega = \{J, M, S\}$

Consider 3 chairs



How many ways to seat J, M, S?

$$\frac{3}{\text{Seat 1}} \cdot \frac{2}{\text{Seat 2}} \cdot \frac{1}{\text{Seat 3}} = 6$$

As a tree.. different events (not events like we defined above)

Seat 1	Seat 2	Seat 3
J	M	S
	S	M
M	J	S
	S	M
S	J	M
	M	J

all 6 possibilities

fork in road... possible directions to take

$$\Omega' = \{\langle J, M, S\rangle, \langle J, S, M\rangle, \ldots, \{S, M, J\}\}$$

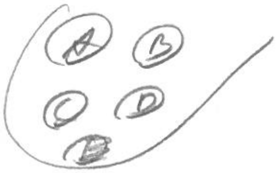
$|\Omega'| = 6 \neq 3^3 = 27$ Why?

$\langle J, J, J\rangle \in \Omega^3$ but not allowed!

Sampling 3 objects without replacement.

3 coin flips...

Sampling 2 objects <u>with</u> replacement.



Take a ball out, do you put it back?

Yes → with replacement

No → without replacement

# of ways to sample n objects without replacement?

$$\underset{\text{seat 1}}{\underline{n}} \quad \underset{\text{seat 2}}{\underline{n-1}} \quad \underset{\text{seat 3}}{\underline{n-2}} \quad \underset{\text{seat 4}}{\underline{n-3}} \cdots \underset{\text{seat } n-1}{\underline{2}} \quad \underset{\text{seat } n}{\underline{1}} = n!$$

$$n! := \prod_{i=1}^{n} i$$

# of ways to sample n objects with replacement?

$$\underline{n}\ \underline{n}\ \underline{n} \cdots \underline{n} = n^n \qquad n^n \overset{?}{>} n!$$

But factorial #'s are still big. # ways to seat 5 people $5! = 120$

10 people $10!$ 3.6 m

$20!$ 2.7×10^{32}

diam(universe) in ft.

10 people 3 chairs

$$\underset{\text{seat 1}}{\underline{10}} \cdot \underset{\text{seat 2}}{\underline{9}} \cdot \underset{\text{seat 3}}{\underline{8}} = \frac{10!}{7!} = \frac{10 \cdot 9 \cdot 8 \cdot \cancel{7!}}{\cancel{7!}} = \frac{10!}{(10-3)!} = {}_{10}P_3$$

permutations (# of orderings)

10 people 10 chairs $\frac{10!}{(10-10)!} = \frac{10!}{0!} \Rightarrow \boxed{0! := 1}$

In general n people k chairs ${}_nP_k := \frac{n!}{(n-k)!}$

$\Rightarrow {}_4P_4 = \frac{4!}{(4-4)!} = \frac{4!}{0!} = 4!$

Various 6 seats Bob-Jane, Richard-Susan, Max-Alice - must sit next to each other.
What is prob? (Assume all choose a seat equally likely).
How many ways?

$\boxed{\text{(A)}} \; \Omega$

$P(\overset{A}{\text{all sit next to each other}}) = \frac{|A|}{|\Omega|} = \frac{|A|}{6!}$ ← numerator usually difficult to compute

two techniques

(a) Seat #

$\underset{1}{6} \; \underset{2}{1} \; \underset{3}{4} \; \underset{4}{1} \; \underset{5}{2} \; \underset{6}{1} = 6 \cdot 4 \cdot 2 = 48$

(b)

$\underset{\text{Seat 1\&2}}{3} \quad \underset{\text{Seat 3\&4}}{2} \quad \underset{\text{Seat 5\&6}}{1} = 3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 48$

2 2 2

$\frac{48}{6!} = \frac{\cancel{48}^{24}}{\cancel{6} \cdot 5 \cdot \cancel{4} \cdot 3 \cdot \cancel{2}} = \boxed{\frac{1}{15}}$

What is the $P(\text{alternating gender}) = \frac{|A|}{|\Omega| = 6!}$

$\underset{1(B)}{3} \; \underset{2(G)}{3} \; \underset{3(B)}{2} \; \underset{4(G)}{2} \; \underset{5(B)}{1} \; \underset{6(G)}{1} = 3 \cdot 3 \cdot 2 \cdot 2 = 36$

— or — +

$\underset{1(G)}{3} \; \underset{2(B)}{3} \; \underset{3(G)}{2} \; \underset{4(G)}{2} \; \underset{5(B)}{1} \; \underset{6(G)}{1} = 3 \cdot 3 \cdot 2 \cdot 2 = 36$

72

$\Rightarrow \frac{\cancel{72}}{\cancel{6} \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}} = \boxed{\frac{1}{10}}$

Why were we able to add?

$A = A_1 \cup A_2$ $\qquad$ $A_1 \cap A_2 = \phi$

BGBGBG GBGBGB $\qquad$ why?

$\Rightarrow P(A) = P(A_1 \cup A_2) = P(A_1) + P(A_2)$

probabilities of disjoint events can be added (we will see this again)

$7 = 4+3$

$\Rightarrow f(7) = f(4+3)$

How about prob just Bob-Jane sit together?

Hmm. look at one way... see pattern

B J R S M A $\qquad$ $\underline{B\text{-}J}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}$ or $\underline{J\text{-}B}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}$

R B J S M A $\qquad$ $\underline{4}\ \underline{BJ}\ \underline{3}\ \underline{2}\ \underline{1}$ or " " " "

same as one unit

$\underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1} + \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1} = 2 \cdot 5!$

$P(A) = \dfrac{2 \cdot \cancel{5!}}{\cancel{6!}_{\,6}} = \boxed{\dfrac{1}{3}}$

~~All 6 in a circle... with no starting position. How many ways?~~

100 balls, 3 selections (without replacement). How many unique orders?

$\underline{100}\ \underline{99}\ \underline{98} = {}_{100}P_3$

100 balls 3 selections with replacement....

$\underline{100}\ \underline{100}\ \underline{100} = n^k$

$\dfrac{{}_{100}P_3}{100^3} = \dfrac{100 \cdot 99 \cdot 98}{100 \cdot 100 \cdot 100} = .9702$

10,000 balls

$\underline{10{,}000}\ \ \underline{9999}\ \ \underline{9998}$

ratio = .9997

$\underline{10000}\ \ \underline{10000}\ \ \underline{10000}$

Calculus fact:

$\lim g(x) f(x) = \lim g(x) \lim f(x)$ if g, f cont.

$\lim\limits_{n\to\infty} \dfrac{{}_nP_k}{n^k} = 1$ does k matter?

$$\lim \frac{\frac{n!}{(n-k)!}}{n^k} = \lim \frac{\overbrace{n(n-1)(n-2)\cdots(n-k+1)}^{k\text{ terms}}}{\underbrace{n\cdot n\cdot n\cdot\ \cdots\ \cdot n}_{k\text{ terms}}} = \underbrace{\lim\frac{n}{n}}_{1}\ \underbrace{\lim\frac{n-1}{n}}_{1}$$

$$\cdot \underbrace{\lim\frac{n-2}{n}}_{\underbrace{\frac{n}{n}}_{1}-\underbrace{\frac{2}{n}}_{0}} \cdots \underbrace{\lim\frac{n-k+1}{n}}_{1-0+0} = 1$$

Back to BJRSMA. They now sit in a ring where you don't care about where the ring begins. How many ways?

$\dfrac{6!}{6}$ ← you get this ($6!$), ← what's this? (6)

Lesson: dividing out nuisance features

$$\left.\begin{array}{c} \text{B J R S M A} \\ \text{J R S M A B} \\ \text{R S M A B J} \\ \text{S M A B J R} \\ \text{M A B J R S} \\ \text{A B J R S M} \end{array}\right\}$$

all these permutations are considered the "same" or "indistinguishable" or "non-unique" or "non-distinct" or "invariant"

9

5 flowers 3 Orchids (O), 2 Chrysanthemums (X)

Initially all Orchids distinct, all X's distinct. How many ways to arrange? If all are distinct how many ways

↓

they are all completely unique flowers ... it doesn't even matter the species!

$O_1\ O_2\ O_3\ X_1\ X_2$ ... $\rightarrow 5! = 120$

$\underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}$

But let's say... Orchids not distinct... all orchids are the same! How many ways?

In the list of 120... we find...

OOO $X_1 X_2$ =

don't care

$$\left.\begin{array}{l} O_1\ O_2\ O_3\ X_1\ X_2 \\ O_1\ O_3\ O_2\ X_1\ X_2 \\ O_2\ O_1\ O_3\ \_\_\_ \\ O_2\ O_3\ O_1\ \_\_\_ \\ O_3\ O_1\ O_2\ \_\_\_ \\ O_3\ O_2\ O_1\ \_\_\_ \end{array}\right\}$$

but all these 6 are now the same!

so all these groups of 6 are only counted as 1

Why $6 = 3! = {}_3P_3$

so $\frac{120}{6}$ is the answer! $\frac{5!}{3!}$

divide out redundance