Adversarial training protects the non-robust features. A trade-off emerges if those features are useful.

A High Dimensional Statistical Model for Adversarial Training: Geometry and Trade-Offs

Empirical Misk Minimization

$$\sum_{i=1}^{n} g\left(y_i \frac{\boldsymbol{\theta}^\top \boldsymbol{x}_i}{\sqrt{d}} - \varepsilon_t \frac{\sqrt{\boldsymbol{\theta}^\top \Sigma_{\boldsymbol{\delta}} \boldsymbol{\theta}}}{\sqrt{d}}\right) + r(\boldsymbol{\theta}) \,. \tag{1}$$

Block Features

$$\begin{aligned} \Sigma_{\boldsymbol{x}} &= \mathrm{blockdiag}\left(\psi_1 \mathbb{1}_{d_1}, \dots, \psi_k \mathbb{1}_{d_k}\right), \\ \Sigma_{\boldsymbol{\delta}} &= \mathrm{blockdiag}\left(\Delta_1 \mathbb{1}_{d_1}, \dots, \Delta_k \mathbb{1}_{d_k}\right), \\ \Sigma_{\boldsymbol{\upsilon}} &= \mathrm{blockdiag}\left(\Upsilon_1 \mathbb{1}_{d_1}, \dots, \Upsilon_k \mathbb{1}_{d_k}\right), \\ \Sigma_{\boldsymbol{\theta}} &= \mathrm{blockdiag}\left(t_1 \mathbb{1}_{d_1}, \dots, t_k \mathbb{1}_{d_k}\right), \end{aligned} \tag{2}$$

Usefulness and robustness

$$\mathcal{U}_{\boldsymbol{\theta}_0} = \frac{1}{\sqrt{d}} \mathbb{E}_{\boldsymbol{x},y}[y \boldsymbol{\theta}_0^\top \boldsymbol{x}] \,, \tag{3}$$

$$\mathcal{R}_{\boldsymbol{\theta}_0} = \frac{1}{\sqrt{d}} \mathbb{E}_{\boldsymbol{x},y}\left[\inf_{\|\boldsymbol{\delta}\|_{\Sigma_{\boldsymbol{\upsilon}}^{-1}} \le \varepsilon_g} y \boldsymbol{\theta}_0^\top (\boldsymbol{x} + \boldsymbol{\delta})\right]. \tag{4}$$

Main Theorem

For the ERM estimator of the risk function with ℓ_2 regularisation $r(\boldsymbol{\theta}) = \frac{\lambda}{2}\|\boldsymbol{\theta}\|_2^2$ and $\lambda \ge 0$, under the data model defined in ?? and in the high dimensional proportional limit, the generalisation error E_{gen} and the boundary error E_{bnd} concentrate to

$$E_{\mathrm{gen}} = \frac{1}{\pi} \arccos\left(m / \sqrt{(\rho + \tau^2) q}\right), \tag{5}$$

$$E_{\mathrm{bnd}} = \int_0^{\varepsilon_g \frac{\sqrt{A}}{\sqrt{q}}} \mathrm{erfc}\left(\frac{-\frac{m}{\sqrt{q}}\nu}{\sqrt{2(\rho + \tau^2 - m^2/q)}}\right) \frac{e^{-\frac{\nu^2}{2}}}{\sqrt{2\pi}} \, \mathrm{d}\nu \,, \tag{6}$$

and the adversarial generalisation error concentrates to $E_{\mathrm{adv}} = E_{\mathrm{gen}} + E_{\mathrm{bnd}}$.

The values of m and q are the solutions of a system of eight self-consistent equations for the unknowns $(m, q, V, P, \hat{m}, \hat{q}, \hat{V}, \hat{P})$. The first four equations are dependant on the loss function g and the adversarial training strength ε_t and read

$$\begin{cases} \hat{m} = \alpha \mathbb{E}_\xi\left[\int_{\mathbb{R}} \mathrm{d}y \, \partial_\omega \mathcal{Z}_0 f_g(y, \sqrt{q}\xi, P)\right] \\ \hat{q} = \alpha \mathbb{E}_\xi\left[\int_{\mathbb{R}} \mathrm{d}y \, \mathcal{Z}_0 f_g^2(y, \sqrt{q}\xi, P)\right] \\ \hat{V} = -\alpha \mathbb{E}_\xi\left[\int_{\mathbb{R}} \mathrm{d}y \, \mathcal{Z}_0 \partial_\omega f_g(y, \sqrt{q}\xi, P)\right] \\ \hat{P} = -\frac{\varepsilon_t}{2\sqrt{P}} \alpha \mathbb{E}_\xi\left[\int_{\mathbb{R}} \mathrm{d}y \, y \mathcal{Z}_0 f_g(y, \sqrt{q}\xi, P)\right] \end{cases} , \tag{7}$$

where $\xi \sim \mathcal{N}(0,1)$ and $\mathcal{Z}_0 = 1/2 \, \mathrm{erfc}(-y\omega/\sqrt{2(V+\tau^2)})$ and $f_g(y, \omega, V, P) = (\mathcal{P}(\omega) - \omega)/V$, where $\mathcal{P}$ is the following proximal operator

$$\mathcal{P}(\omega) = \min_x \left[\frac{(x-\omega)^2}{2V} + g(yx - \varepsilon_t \sqrt{P})\right]. \tag{8}$$

The second set of equation depend on the spectral distribution of the matrices $\Sigma_{\boldsymbol{x}}, \Sigma_{\boldsymbol{\delta}}$ and on the limiting distribution of the elements of $\bar{\boldsymbol{\theta}}$. The equations read

$$\begin{cases} m = \mathbb{E}_\mu\left[\frac{\hat{m}\bar{\theta}^2}{\lambda + \hat{V}\omega + \hat{P}\delta}\right] \\ q = \mathbb{E}_\mu\left[\frac{\hat{m}^2\bar{\theta}^2\omega + \hat{q}\omega^2}{(\lambda + \hat{V}\omega + \hat{P}\delta)^2}\right] \\ V = \mathbb{E}_\mu\left[\frac{\omega}{\lambda + \hat{V}\omega + \hat{P}\delta}\right] \\ P = \mathbb{E}_\mu\left[\zeta \frac{\hat{m}^2\bar{\theta}^2 + \hat{q}\omega^2}{(\lambda + \hat{V}\omega + \hat{P}\delta)^2}\right] \end{cases} . \tag{9}$$

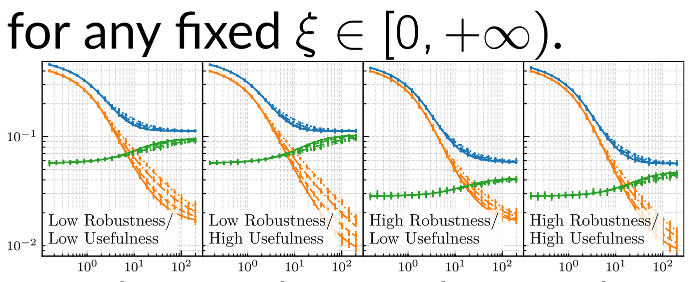
The value of A can be obtained from the solution of the same system of self consistent equations as

$$A = \mathbb{E}_\mu\left[\upsilon \frac{\hat{m}^2\bar{\theta}^2\omega + \hat{q}\omega^2}{(\lambda + \hat{V}\omega + \hat{P}\delta)^2}\right]. \tag{10}$$

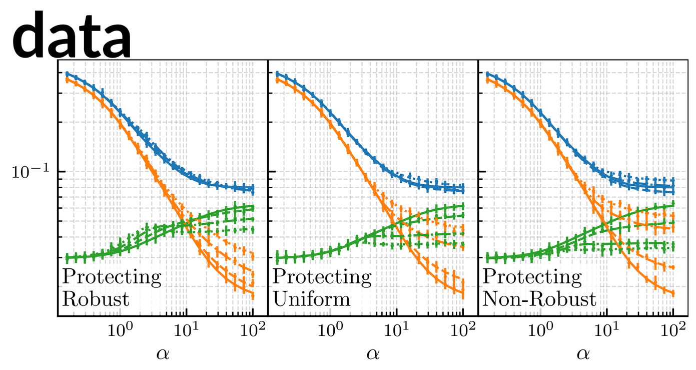
Trade-Offs

$$E_{\mathrm{adv}} = E_{\mathrm{gen}}(\vartheta, \mathcal{U}_{\boldsymbol{\theta}_0}) + \int_0^{\varepsilon_g \varkappa} f(\xi; \vartheta, \mathcal{U}_{\boldsymbol{\theta}_0}) \, \mathrm{d}\xi \,, \tag{11}$$

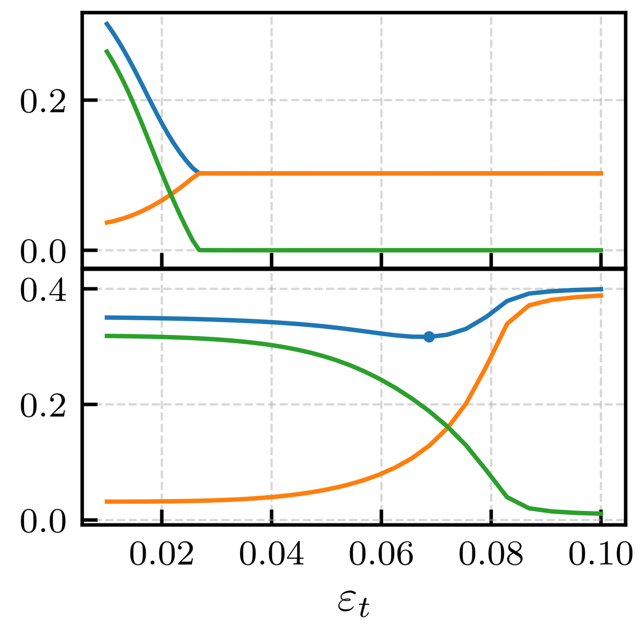
where we introduce the variable $\vartheta = m/\sqrt{\rho q}$ and $\varkappa = \sqrt{A}/\sqrt{q}$. ϑ is the cosine of the angle between the teacher weights $\boldsymbol{\theta}_0$ and the student estimate $\hat{\boldsymbol{\theta}}$ in the geometry of Σ_x and $\varkappa$ is the norm of $\hat{\boldsymbol{\theta}}$ under the attack matrix. The function $f(\xi; \vartheta)$ is positive $\forall \vartheta, \forall \xi \in [0, +\infty)$ and it is strictly increasing in ϑ for any fixed $\xi \in [0, +\infty)$.



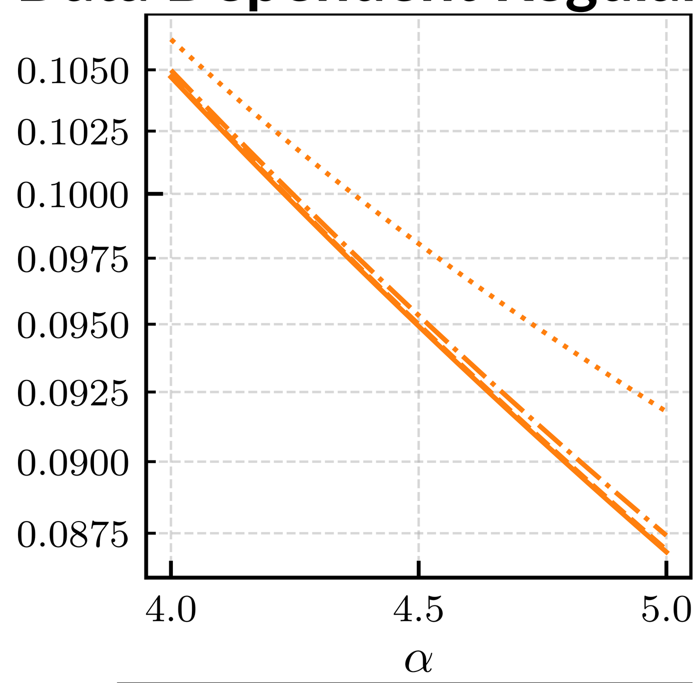
Directional Defences and structured data

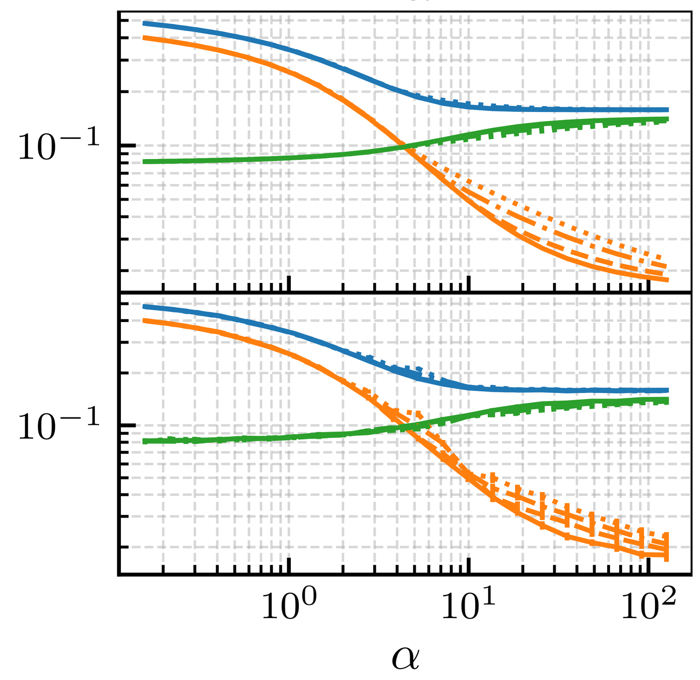


Tradeoff directions and innocuous directions



Data Dependent Regularisation





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