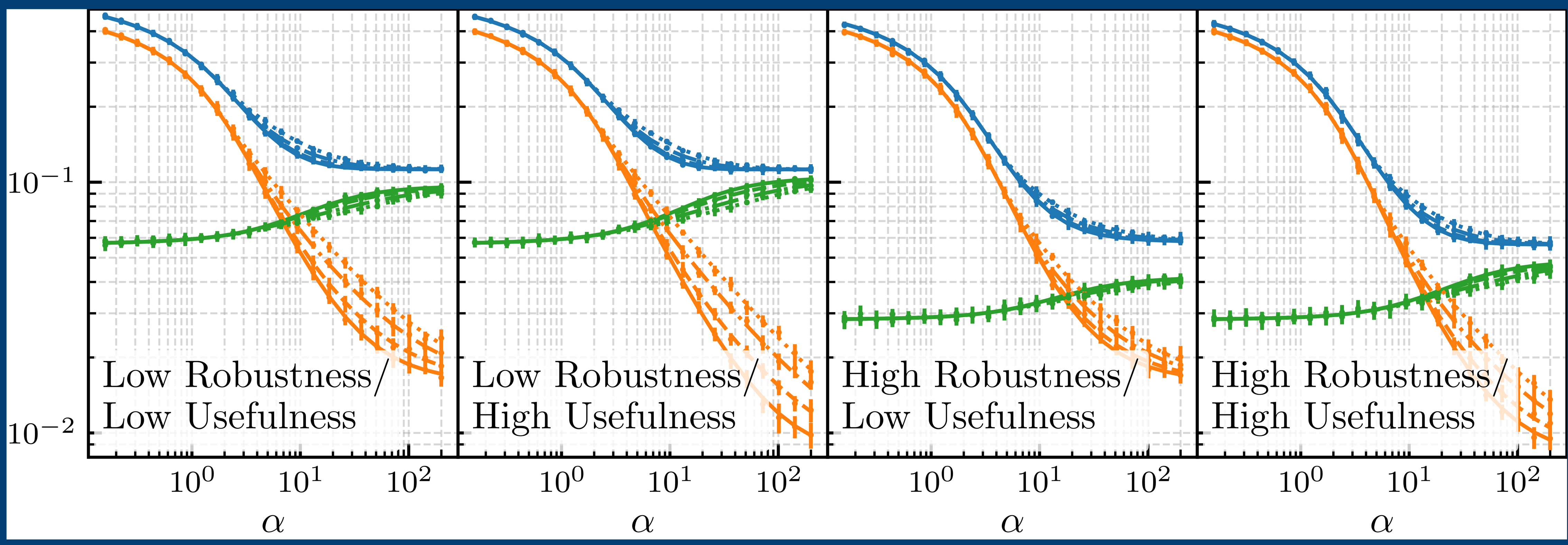
# A rigorous, closed-form characterisation of adversarial generalisation errors.




# A High Dimensional Statistical Model for Adversarial Training: Geometry and Trade-Offs

## Problem Setup

**Binary Classification Setting:**

- Training data $\mathcal{D} = \{(x_i, y_i)\}_{i=1}^n \in \mathbb{R}^d \times \{-1, +1\}$
- Probit model with noise parameter $\tau > 0$
- High-dimensional limit: $d, n \to \infty$ with fixed $\alpha = n/d$
- Structured data with block features: covariance matrices $\Sigma_x, \Sigma_\delta, \Sigma_\upsilon, \Sigma_\theta$ are block diagonal with $k$ blocks of sizes $d_1, \ldots, d_k$

**Metrics of Interest:**

- Generalisation Error:

$$E_{\text{gen}} = \mathbb{E}_{y,x}\left[\mathbb{1}(y \neq \hat{y}(\hat{\theta}, x))\right] \quad (1)$$

- Adversarial Generalisation Error:

$$E_{\text{adv}} = \mathbb{E}_{y,x}\left[\max_{\|\delta\|_{\Sigma_\upsilon^{-1}} \leq \varepsilon_g} \mathbb{1}(y \neq \hat{y}(\hat{\theta}, x + \delta))\right] \quad (2)$$

- Boundary Error:

$$E_{\text{adv}} = E_{\text{gen}} + E_{\text{bnd}} \quad (3)$$

where $E_{\text{bnd}}$ are the attackable samples.

- Usefulness and Robustness:

$$\mathcal{U}_{\theta_0} = \frac{1}{\sqrt{d}} \mathbb{E}_{x,y}[y \theta_0^\top x] \quad (4)$$

$$\mathcal{R}_{\theta_0} = \frac{1}{\sqrt{d}} \mathbb{E}_{x,y}\left[\inf_{\|\delta\|_{\Sigma_\upsilon^{-1}} \leq \varepsilon_g} y \theta_0^\top (x + \delta)\right] \quad (5)$$

**Adversarial ERM:**

$$\sum_{i=1}^n g\left(y_i \frac{\theta^\top x_i}{\sqrt{d}} - \varepsilon_t \frac{\sqrt{\theta^\top \Sigma_\delta \theta}}{\sqrt{d}}\right) + r(\theta) \quad (6)$$

## Main Result

**Theorem:** Adversarial generalization errors are *provably* characterized by a system of 8 order parameters $(m, q, V, P, \hat{m}, \hat{q}, \hat{V}, \hat{P})$ and an additional parameter $A$ through:

$$E_{\text{gen}} = \frac{1}{\pi} \arccos\left(m / \sqrt{(\rho + \tau^2) q}\right) \quad (7)$$

$$E_{\text{bnd}} = \int_0^{\varepsilon_g \frac{\sqrt{A}}{\sqrt{q}}} \operatorname{erfc}\left(\frac{-\frac{m}{\sqrt{q}}\nu}{\sqrt{2(\rho + \tau^2 - m^2/q)}}\right) \frac{e^{-\frac{\nu^2}{2}}}{\sqrt{2\pi}} \, d\nu \quad (8)$$

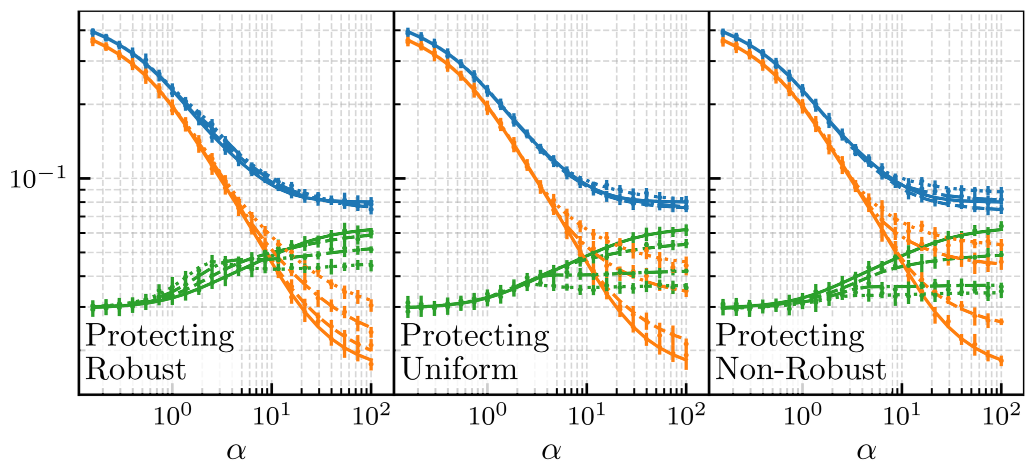
## Implications

### Trade-off between Usefulness and Robustness :

- Usefulness relates to generalisation error
- Robustness relates to boundary error
- Trade-off emerges when protecting useful but non-robust features

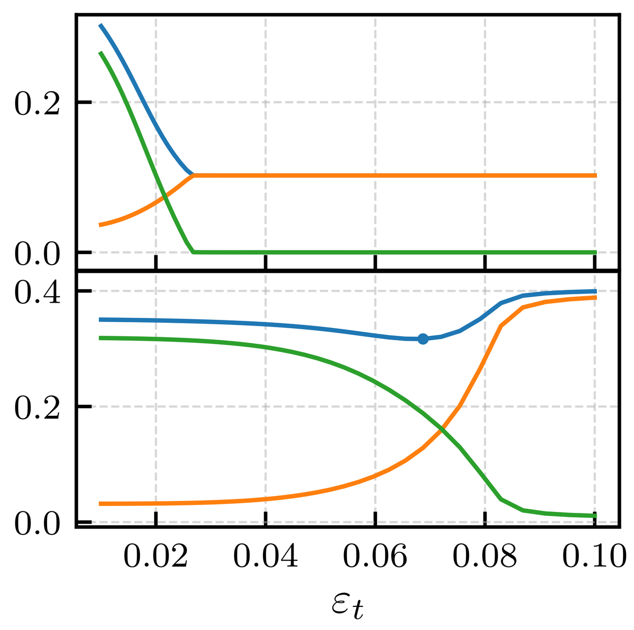
**Key Bounds:**

$$E_{\text{gen}} \geq \frac{1}{\pi} \arccos\left(\sqrt{\frac{\pi}{2\rho}} \mathcal{U}_{\theta_0}\right) \quad (9)$$




**Impact of different defense strategies on generalization ($E_{\text{gen}}$) and boundary ($E_{\text{bnd}}$) errors**

- **Defending robust features**: Low $E_{\text{gen}}$ but high $E_{\text{bnd}}$
- **Uniform defense**: Better balance, improves overall $E_{\text{adv}}$
- **Defending non-robust features**: Increases $E_{\text{gen}}$ while decreasing $E_{\text{bnd}}$



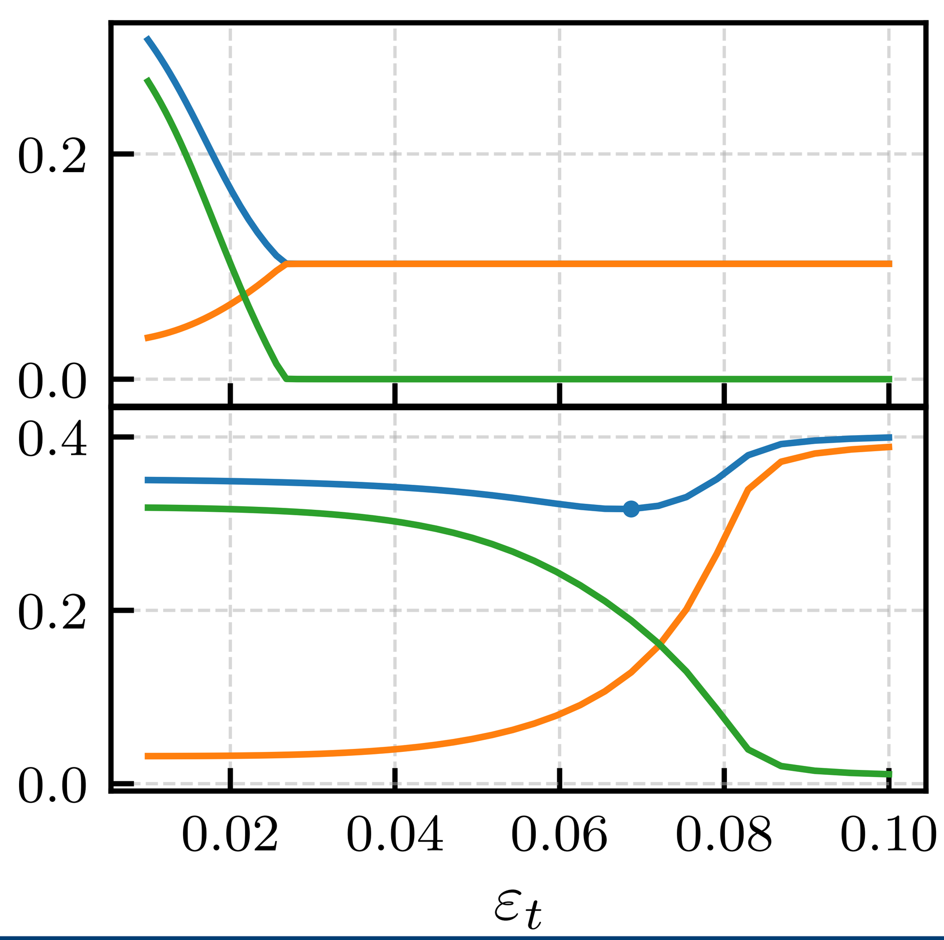

**Optimal defense strategy depends on feature geometry**

**Analytical Result:** For structured data with two feature blocks, we prove that protecting non-robust features:

- Always increases $E_{\text{gen}}$ and decreases $E_{\text{bnd}}$
- Can improve $E_{\text{adv}}$ when attack size is small enough

### Tradeoff directions and innocuous directions

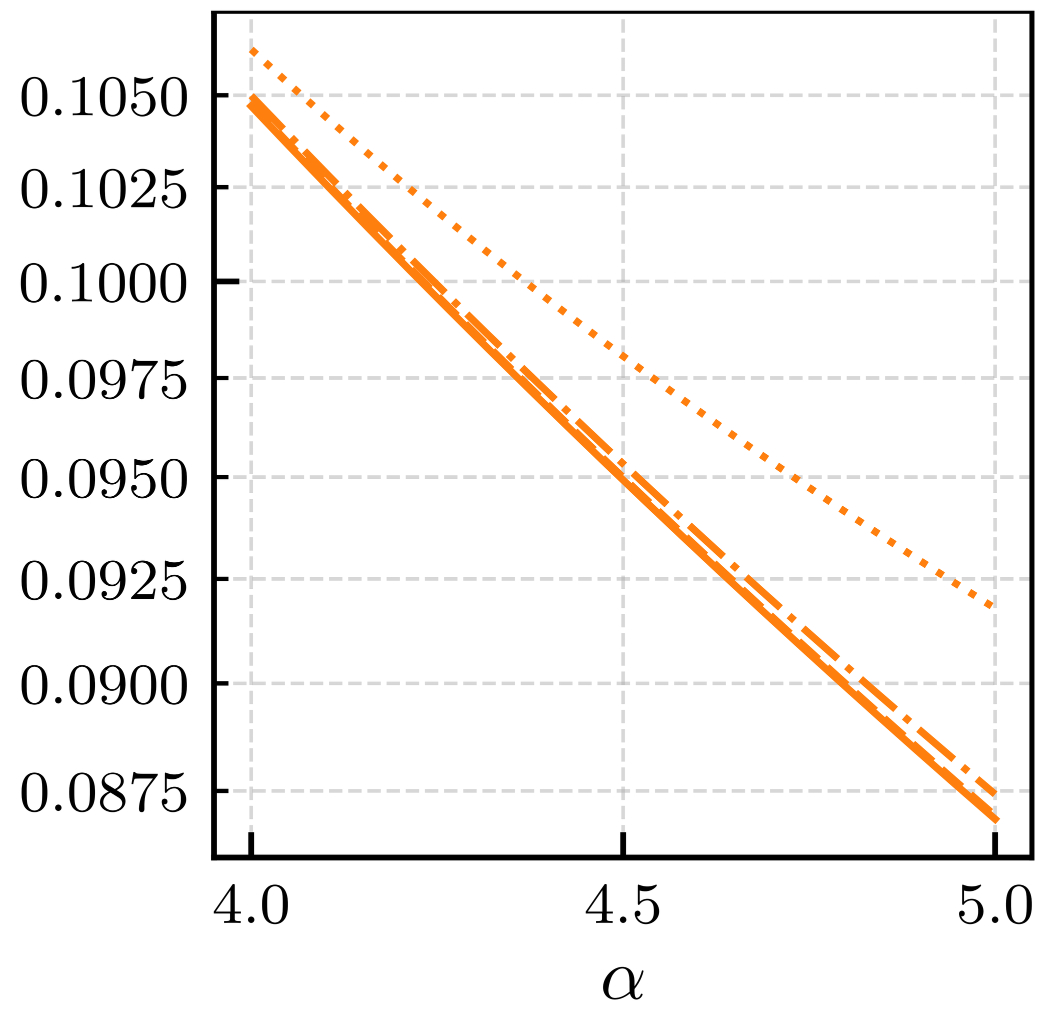
**Key Insight:** The geometry of features determines whether adversarial training leads to a trade-off:

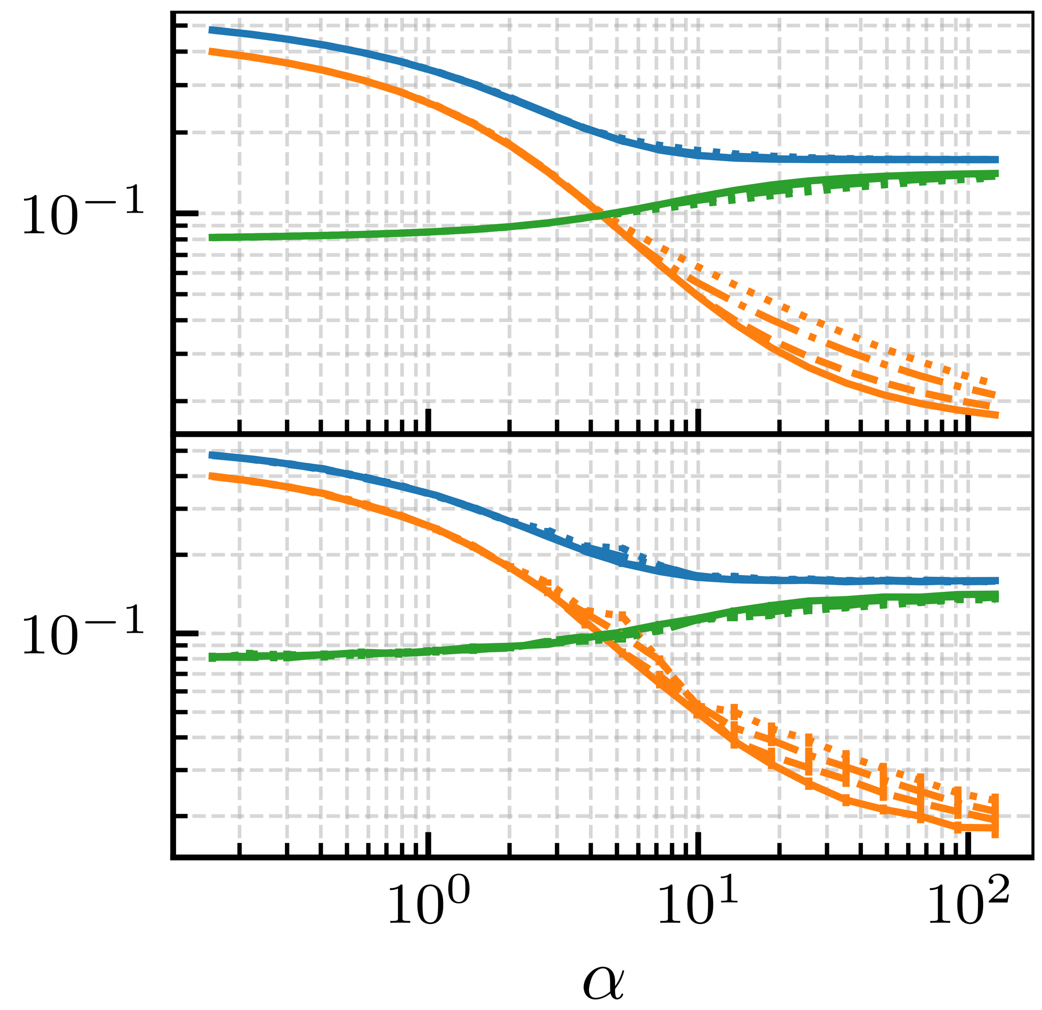



- **Trade-off Features** (aligned with teacher):
  - Fundamental trade-off between $E_{\text{gen}}$ and $E_{\text{bnd}}$
  - Optimal performance at specific $\varepsilon_t$
  - Requires careful hyperparameter tuning

## Data Dependent Regularisation

**Key Finding:** Adversarial training can be approximated as a data-dependent regularisation:




**Learning curves for adversarial training (top) and its regularisation approximation (bottom)**

**Approximate Loss:**

$$\sum_{i=1}^n g\left(y_i \frac{\theta^\top x_i}{\sqrt{d}}\right) + \tilde{\lambda}_1 \sqrt{\theta^\top \Sigma_\delta \theta} + \tilde{\lambda}_2 \theta^\top \Sigma_\delta \theta \quad (11)$$

**Key Properties:**

- **Not just $\ell_2$:** Performance depends on $\varepsilon_t$ even with optimal $\lambda$
- **Effective Regularisation:** is a directional $\sqrt{\ell_2} + \ell_2$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