# Adversarial training protects the non-robust features. A trade-off emerges if those features are useful.

# A High Dimensional Statistical Model for Adversarial Training: Geometry and Trade-Offs

## Empirical Misk Minimization

$$\sum_{i=1}^{n} g\left(y_i \frac{\boldsymbol{\theta}^\top \boldsymbol{x}_i}{\sqrt{d}} - \varepsilon_t \frac{\sqrt{\boldsymbol{\theta}^\top \Sigma_{\boldsymbol{\delta}} \boldsymbol{\theta}}}{\sqrt{d}}\right) + r(\boldsymbol{\theta}) \,. \tag{1}$$

### Block Features

$$\begin{aligned}
\Sigma_{\boldsymbol{x}} &= \text{blockdiag}\left(\psi_1 \mathbb{1}_{d_1}, \ldots, \psi_k \mathbb{1}_{d_k}\right), \\
\Sigma_{\boldsymbol{\delta}} &= \text{blockdiag}\left(\Delta_1 \mathbb{1}_{d_1}, \ldots, \Delta_k \mathbb{1}_{d_k}\right), \\
\Sigma_{\boldsymbol{\upsilon}} &= \text{blockdiag}\left(\Upsilon_1 \mathbb{1}_{d_1}, \ldots, \Upsilon_k \mathbb{1}_{d_k}\right), \\
\Sigma_{\boldsymbol{\theta}} &= \text{blockdiag}\left(t_1 \mathbb{1}_{d_1}, \ldots, t_k \mathbb{1}_{d_k}\right),
\end{aligned} \tag{2}$$

### Usefulness and robustness

$$\mathcal{U}_{\boldsymbol{\theta}_0} = \frac{1}{\sqrt{d}} \mathbb{E}_{\boldsymbol{x},y}[y \boldsymbol{\theta}_0^\top \boldsymbol{x}] \,, \tag{3}$$

$$\mathcal{R}_{\boldsymbol{\theta}_0} = \frac{1}{\sqrt{d}} \mathbb{E}_{\boldsymbol{x},y}\left[\inf_{\|\boldsymbol{\delta}\|_{\Sigma_{\boldsymbol{\upsilon}}^{-1}} \leq \varepsilon_g} y \boldsymbol{\theta}_0^\top (\boldsymbol{x} + \boldsymbol{\delta})\right]. \tag{4}$$

## Main Theorem

For the ERM estimator of the risk function with $\ell_2$ regularisation $r(\boldsymbol{\theta}) = \frac{\lambda}{2}\|\boldsymbol{\theta}\|_2^2$ and $\lambda \geq 0$, under the data model defined in ?? and in the high dimensional proportional limit, the generalisation error $E_{\text{gen}}$ and the boundary error $E_{\text{bnd}}$ concentrate to

$$E_{\text{gen}} = \frac{1}{\pi} \arccos\left(m / \sqrt{(\rho + \tau^2) q}\right), \tag{5}$$

$$E_{\text{bnd}} = \int_0^{\varepsilon_g \frac{\sqrt{A}}{\sqrt{q}}} \text{erfc}\left(\frac{-\frac{m}{\sqrt{q}} \nu}{\sqrt{2(\rho + \tau^2 - m^2/q)}}\right) \frac{e^{-\frac{\nu^2}{2}}}{\sqrt{2\pi}} \, \mathrm{d}\nu \,, \tag{6}$$

and the adversarial generalisation error concentrates to $E_{\text{adv}} = E_{\text{gen}} + E_{\text{bnd}}$.

The values of $m$ and $q$ are the solutions of a system of eight self-consistent equations for the unknowns $(m, q, V, P, \hat{m}, \hat{q}, \hat{V}, \hat{P})$. The first four equations are dependant on the loss function $g$ and the adversarial training strength $\varepsilon_t$ and read

$$\begin{cases}
\hat{m} = \alpha \mathbb{E}_{\xi}\left[\int_{\mathbb{R}} \mathrm{d}y \, \partial_\omega \mathcal{Z}_0 f_g(y, \sqrt{q}\xi, P)\right] \\
\hat{q} = \alpha \mathbb{E}_{\xi}\left[\int_{\mathbb{R}} \mathrm{d}y \, \mathcal{Z}_0 f_g^2(y, \sqrt{q}\xi, P)\right] \\
\hat{V} = -\alpha \mathbb{E}_{\xi}\left[\int_{\mathbb{R}} \mathrm{d}y \, \mathcal{Z}_0 \partial_\omega f_g(y, \sqrt{q}\xi, P)\right] \\
\hat{P} = -\frac{\varepsilon_t}{2\sqrt{P}} \alpha \mathbb{E}_{\xi}\left[\int_{\mathbb{R}} \mathrm{d}y \, y \mathcal{Z}_0 f_g(y, \sqrt{q}\xi, P)\right]
\end{cases}, \tag{7}$$

where $\xi \sim \mathcal{N}(0,1)$ and $\mathcal{Z}_0 = {}^1\!/_2 \, \text{erfc}\left({}^{-y\omega}\!/\!\sqrt{2(V+\tau^2)}\right)$ and $f_g(y, \omega, V, P) = (\mathcal{P}(\omega) - \omega)/V$, where $\mathcal{P}$ is the following proximal operator

$$\mathcal{P}(\omega) = \min_x \left[\frac{(x - \omega)^2}{2V} + g(yx - \varepsilon_t \sqrt{P})\right]. \tag{8}$$

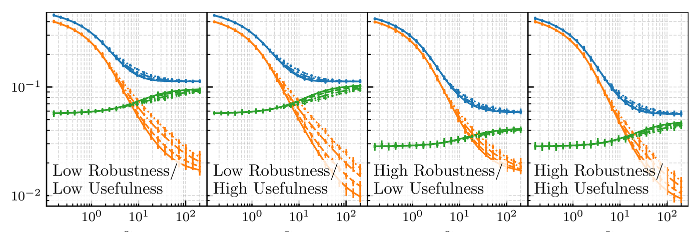
The second set of equation depend on the spectral distribution of the matrices $\Sigma_x$, $\Sigma_\delta$ and on the limiting distribution of the elements of $\bar{\theta}$. The equations read

$$\begin{cases}
m = \mathbb{E}_\mu\left[\frac{\hat{m}\bar{\theta}^2}{\lambda + \hat{V}\omega + \hat{P}\delta}\right] \\
q = \mathbb{E}_\mu\left[\frac{\hat{m}^2\bar{\theta}^2\omega + \hat{q}\omega^2}{(\lambda + \hat{V}\omega + \hat{P}\delta)^2}\right] \\
V = \mathbb{E}_\mu\left[\frac{\omega}{\lambda + \hat{V}\omega + \hat{P}\delta}\right] \\
P = \mathbb{E}_\mu\left[\zeta \frac{\hat{m}^2\bar{\theta}^2 + \hat{q}\omega^2}{(\lambda + \hat{V}\omega + \hat{P}\delta)^2}\right]
\end{cases} . \tag{9}$$

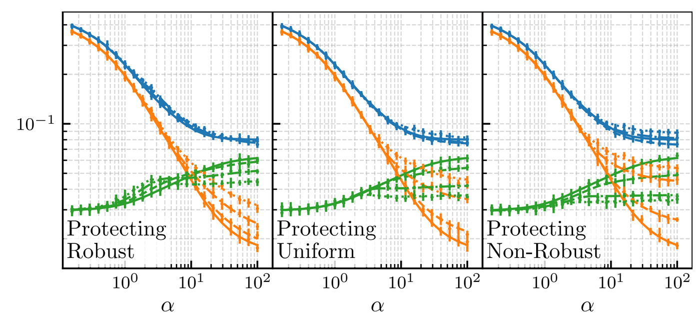
The value of $A$ can be obtained from the solution of the same system of self consistent equations as

$$A = \mathbb{E}_\mu\left[\upsilon \frac{\hat{m}^2\bar{\theta}^2\omega + \hat{q}\omega^2}{(\lambda + \hat{V}\omega + \hat{P}\delta)^2}\right]. \tag{10}$$

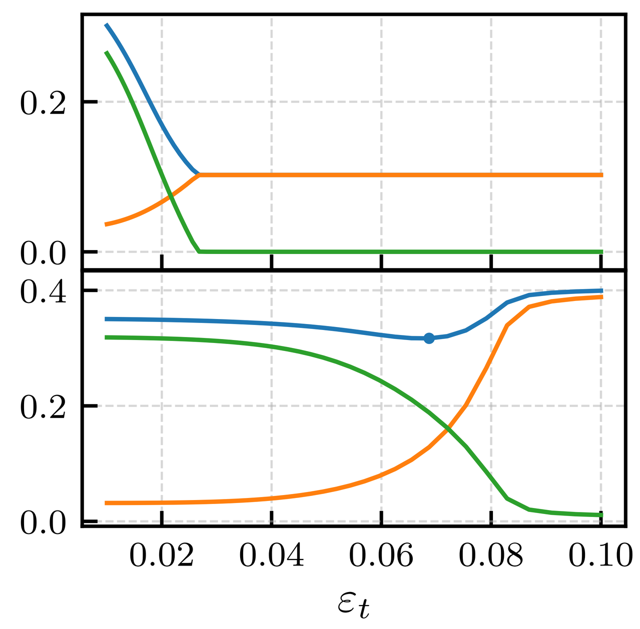
## Trade-Offs



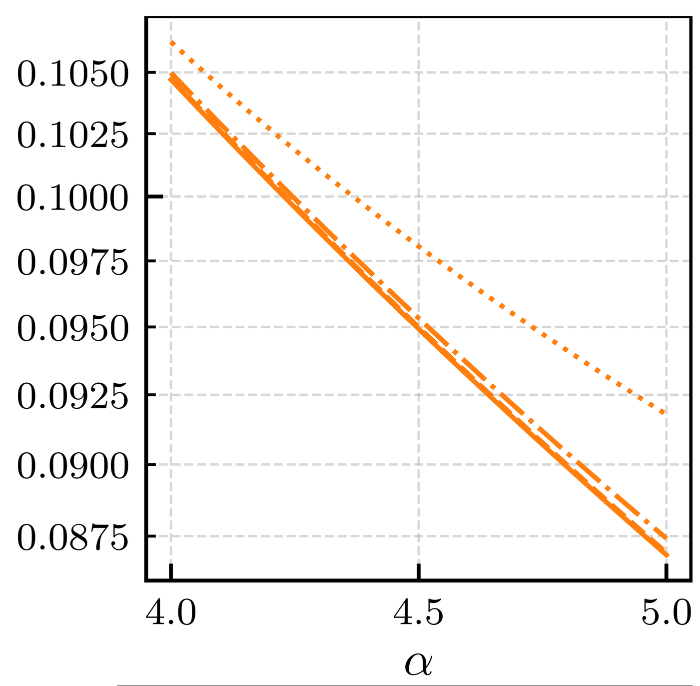

### Directional Defences and structured data

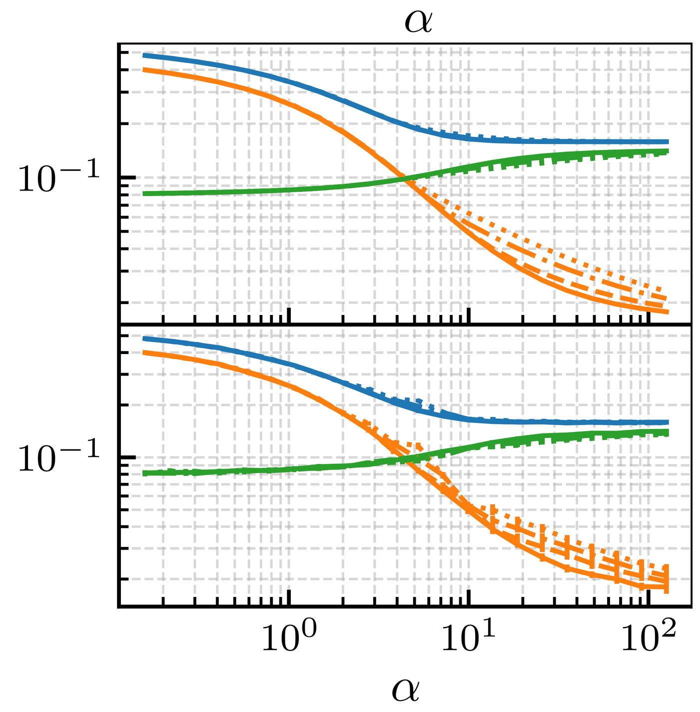



### Tradeoff directions and innocuous directions




### Data Dependent Regularisation

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