Adversarial training protects the **non-robust** features. A **trade-off** emerges if those features are **useful**.

A High Dimensional Statistical Model for Adversarial Training: Geometry and Trade-Offs

Empirical Misk Minimization

$$\sum_{i=1}^{n} g\left(y_i \frac{\boldsymbol{\theta}^\top \boldsymbol{x}_i}{\sqrt{d}} - \varepsilon_t \frac{\sqrt{\boldsymbol{\theta}^\top \Sigma_{\boldsymbol{\delta}} \boldsymbol{\theta}}}{\sqrt{d}}\right) + r(\boldsymbol{\theta}) \,. \tag{1}$$

Block Features

$$\begin{aligned} \Sigma_{\boldsymbol{x}} &= \mathrm{blockdiag}\left(\psi_1 \mathbb{1}_{d_1}, \dots, \psi_k \mathbb{1}_{d_k}\right), \\ \Sigma_{\boldsymbol{\delta}} &= \mathrm{blockdiag}\left(\Delta_1 \mathbb{1}_{d_1}, \dots, \Delta_k \mathbb{1}_{d_k}\right), \\ \Sigma_{\boldsymbol{\upsilon}} &= \mathrm{blockdiag}\left(\Upsilon_1 \mathbb{1}_{d_1}, \dots, \Upsilon_k \mathbb{1}_{d_k}\right), \\ \Sigma_{\boldsymbol{\theta}} &= \mathrm{blockdiag}\left(t_1 \mathbb{1}_{d_1}, \dots, t_k \mathbb{1}_{d_k}\right), \end{aligned} \tag{2}$$

Usefulness and robustness

$$\mathcal{U}_{\boldsymbol{\theta}_0} = \frac{1}{\sqrt{d}} \mathbb{E}_{\boldsymbol{x},y}[y \boldsymbol{\theta}_0^\top \boldsymbol{x}] \,, \tag{3}$$

$$\mathcal{R}_{\boldsymbol{\theta}_0} = \frac{1}{\sqrt{d}} \mathbb{E}_{\boldsymbol{x},y}\left[\inf_{\|\boldsymbol{\delta}\|_{\Sigma_{\boldsymbol{\upsilon}}^{-1}} \le \varepsilon_g} y \boldsymbol{\theta}_0^\top (\boldsymbol{x} + \boldsymbol{\delta})\right]. \tag{4}$$

Main Theorem

For the ERM estimator of the risk function with ℓ_2 regularisation $r(\boldsymbol{\theta}) = \frac{\lambda}{2}\|\boldsymbol{\theta}\|_2^2$ and $\lambda \ge 0$, under the data model defined in ?? and in the high dimensional proportional limit, the generalisation error E_{gen} and the boundary error E_{bnd} concentrate to

$$E_{\text{gen}} = \frac{1}{\pi} \arccos\left(m / \sqrt{(\rho + \tau^2) q}\right), \tag{5}$$

$$E_{\text{bnd}} = \int_0^{\varepsilon_g \frac{\sqrt{A}}{\sqrt{q}}} \mathrm{erfc}\left(\frac{-\frac{m}{\sqrt{q}} \nu}{\sqrt{2(\rho + \tau^2 - m^2/q)}}\right) \frac{e^{-\frac{\nu^2}{2}}}{\sqrt{2\pi}} \,\mathrm{d}\nu \,, \tag{6}$$

and the adversarial generalisation error concentrates to $E_{\text{adv}} = E_{\text{gen}} + E_{\text{bnd}}$.

The values of m and q are the solutions of a system of eight self-consistent equations for the unknowns $(m, q, V, P, \hat{m}, \hat{q}, \hat{V}, \hat{P})$. The first four equations are dependant on the loss function g and the adversarial training strength ε_t and read

$$\begin{cases} \hat{m} = \alpha \mathbb{E}_\xi\left[\int_{\mathbb{R}} \mathrm{d}y \, \partial_\omega \mathcal{Z}_0 f_g(y, \sqrt{q}\xi, P)\right] \\ \hat{q} = \alpha \mathbb{E}_\xi\left[\int_{\mathbb{R}} \mathrm{d}y \, \mathcal{Z}_0 f_g^2(y, \sqrt{q}\xi, P)\right] \\ \hat{V} = -\alpha \mathbb{E}_\xi\left[\int_{\mathbb{R}} \mathrm{d}y \, \mathcal{Z}_0 \partial_\omega f_g(y, \sqrt{q}\xi, P)\right] \\ \hat{P} = -\frac{\varepsilon_t}{2\sqrt{P}} \alpha \mathbb{E}_\xi\left[\int_{\mathbb{R}} \mathrm{d}y \, y \mathcal{Z}_0 f_g(y, \sqrt{q}\xi, P)\right] \end{cases}, \tag{7}$$

where $\xi \sim \mathcal{N}(0,1)$ and $\mathcal{Z}_0 = 1/2\, \mathrm{erfc}(-y\omega / \sqrt{2(V + \tau^2)})$ and $f_g(y, \omega, V, P) = (\mathcal{P}(\omega) - \omega)/V$, where $\mathcal{P}$ is the following proximal operator

$$\mathcal{P}(\omega) = \min_x \left[\frac{(x - \omega)^2}{2V} + g(yx - \varepsilon_t \sqrt{P})\right]. \tag{8}$$

The second set of equation depend on the spectral distribution of the matrices Σ_x, $\Sigma_{\boldsymbol{\delta}}$ and on the limiting distribution of the elements of $\bar{\boldsymbol{\theta}}$. The equations read

$$\begin{cases} m = \mathbb{E}_\mu\left[\frac{\hat{m}\bar{\theta}^2}{\lambda + \hat{V}\omega + \hat{P}\delta}\right] \\ q = \mathbb{E}_\mu\left[\frac{\hat{m}^2 \bar{\theta}^2 \omega + \hat{q}\omega^2}{(\lambda + \hat{V}\omega + \hat{P}\delta)^2}\right] \\ V = \mathbb{E}_\mu\left[\frac{\omega}{\lambda + \hat{V}\omega + \hat{P}\delta}\right] \\ P = \mathbb{E}_\mu\left[\zeta \frac{\hat{m}^2 \bar{\theta}^2 + \hat{q}\omega^2}{(\lambda + \hat{V}\omega + \hat{P}\delta)^2}\right] \end{cases}. \tag{9}$$

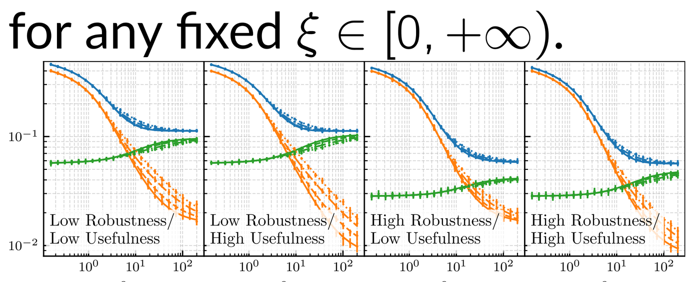
The value of A can be obtained from the solution of the same system of self consistent equations as

$$A = \mathbb{E}_\mu\left[\upsilon \frac{\hat{m}^2 \bar{\theta}^2 \omega + \hat{q}\omega^2}{(\lambda + \hat{V}\omega + \hat{P}\delta)^2}\right]. \tag{10}$$

Trade-Offs

$$E_{\text{adv}} = E_{\text{gen}}(\vartheta, \mathcal{U}_{\boldsymbol{\theta}_0}) + \int_0^{\varepsilon_g \varkappa} f(\xi; \vartheta, \mathcal{U}_{\boldsymbol{\theta}_0}) \,\mathrm{d}\xi \,, \tag{11}$$

where we introduce the variable $\vartheta = m / \sqrt{\rho q}$ and $\varkappa = \sqrt{A} / \sqrt{q}$. ϑ is the cosine of the angle between the teacher weights $\boldsymbol{\theta}_0$ and the student estimate $\hat{\boldsymbol{\theta}}$ in the geometry of Σ_x and $\varkappa$ is the norm of $\hat{\boldsymbol{\theta}}$ under the attack matrix. The function $f(\xi; \vartheta)$ is positive $\forall \vartheta, \forall \xi \in [0, +\infty)$ and it is strictly increasing in ϑ for any fixed $\xi \in [0, +\infty)$.



We notice that the values for E_{gen} and E_{bnd} change by varying the usefulness and robustness of the features for fixed types of attacks. Intuitively, we have that the more usefulness one has the less generalisation error one makes, indeed we can write a lower bound for the generalisation error

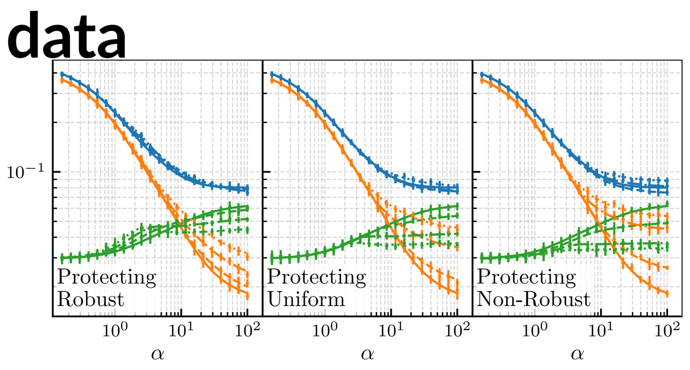
$$E_{\text{gen}} \ge \frac{1}{\pi} \arccos\left(\sqrt{\frac{\pi}{2\rho}} \mathcal{U}_{\boldsymbol{\theta}_0}\right). \tag{12}$$

We note that ρ and $\mathcal{U}_{\boldsymbol{\theta}_0}$ only depend on Σ_x and $\boldsymbol{\theta}_0$. Robustness only affects the boundary error. High robustness implies less sensibility to adversarial attacks: robust features have less samples within an attack range of the student decision boundary. The highest value that the boundary error can achieve is limited by both the robustness and the usefulness as

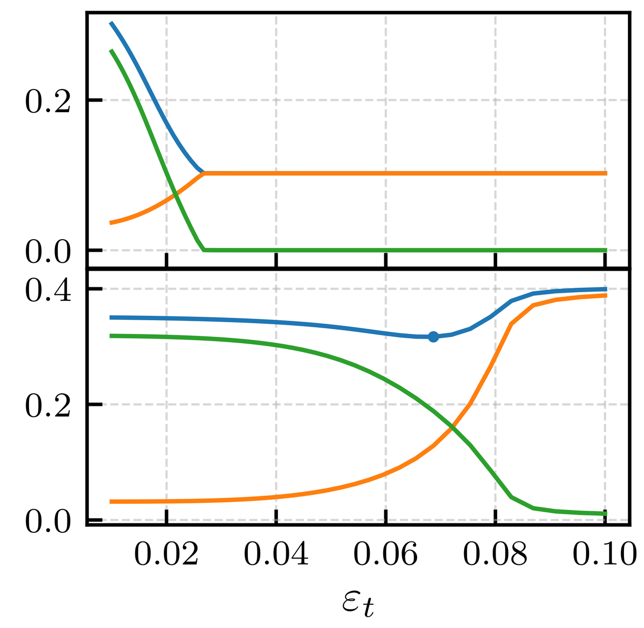
$$E_{\text{bnd}} \le 2\mathrm{T}\left(\varepsilon_g \mathcal{A}\mathcal{B}, \mathcal{A}^{-1}\right) - \frac{1}{\pi} \arctan\left(\mathcal{A}^{-1}\right) - \frac{1}{\pi} \mathrm{erf}\left(\frac{\varepsilon_g \mathcal{B}}{\sqrt{2}}\right) \mathrm{erfc}\left(\frac{\varepsilon_g \mathcal{A}\mathcal{B}}{\sqrt{2}}\right), \tag{13}$$

where $\mathcal{B} = \max_i \sqrt{(\Sigma_{\boldsymbol{\upsilon}})_{ii} / (\Sigma_{\boldsymbol{x}})_{ii}}$, $\mathcal{A} = \sqrt{\pi} \mathcal{U}_{\boldsymbol{\theta}_0} / \sqrt{2\rho}$ and T is the Owen function. This previous bound is a decreasing function of the robustness.

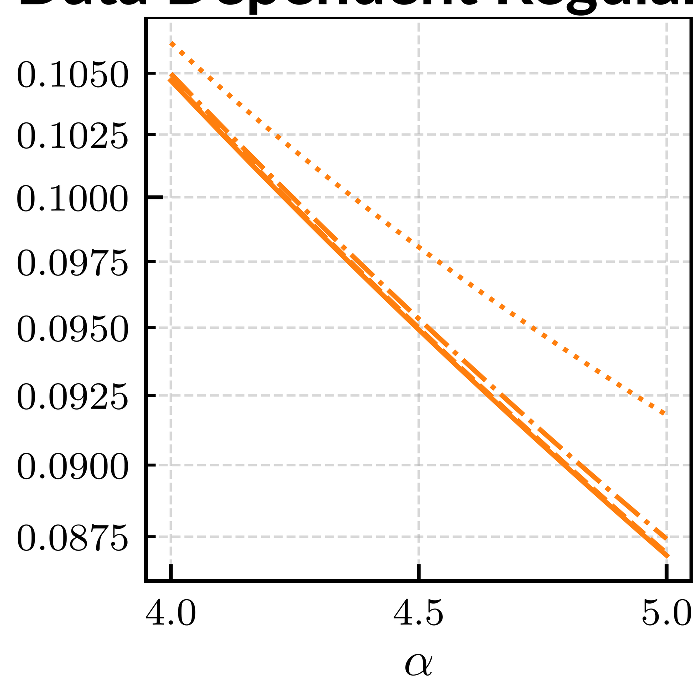
Directional Defences and structured data

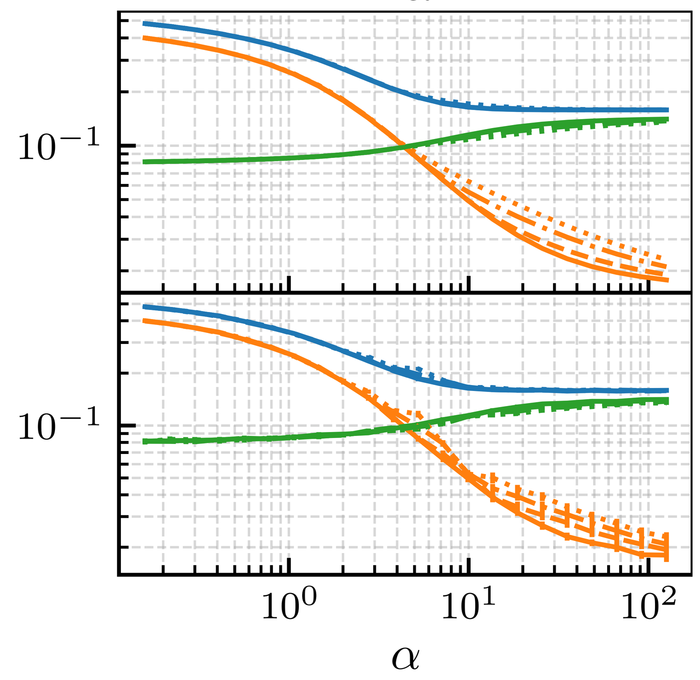


Tradeoff directions and innocuous directions



Data Dependent Regularisation





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