Cosmological Physics Note

Author: Zan-KingJun

Grand Unified Theory of the Cosmology:

$\boldsymbol{F} = m\boldsymbol{a}$

Latest Revision Date: April 4, 2025

ZASA Editorial Department

Contents

1 Formula Part

1.1 Robert-Walker (flat) metric

The metric form

$$g_{\mu\nu} = \mathbf{diag}\{-1,\ a^2(t),\ a^2(t),\ a^2(t)\}, \qquad g^{\mu\nu} = \mathbf{diag}\{-1,\ a^{-2}(t),\ a^{-2}(t),\ a^{-2}(t)\}. \tag{1}$$

Behaviour of the energy-momentum

$$(E,\ p^1,\ p^2,\ p^3) = \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda},\ \frac{\mathrm{d}x^1}{\mathrm{d}\lambda},\ \frac{\mathrm{d}x^2}{\mathrm{d}\lambda},\ \frac{\mathrm{d}x^3}{\mathrm{d}\lambda}\right) = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}, \tag{2}$$

$$\frac{\mathrm{d}^2 x^\beta}{\mathrm{d}\lambda^2} + \Gamma^\beta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0,\ \Gamma^\beta_{\mu\nu} = \frac{1}{2}g^{\beta\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}), \tag{3}$$

the nonzero terms of $\Gamma^\beta_{\mu\nu}$ are

$$\Gamma^i_{j0} = \frac{1}{2}g^{i\rho}(g_{\rho j,0} + g_{\rho 0,j} - g_{j0,\rho}) = \frac{1}{2}g^{ii}(g_{ij,0} + g_{i0,j} - g_{j0,i}) = \frac{1}{2}[a^2(t)]^{-1}\delta^i_j \cdot 2a(t)\dot{a} = \frac{\dot{a}}{a}\delta^i_j = \Gamma^i_{0j}, \tag{4}$$

$$\Gamma^0_{ij} = \frac{1}{2}g^{0\rho}(g_{\rho i,j} + g_{\rho j,i} - g_{ij,\rho}) = -\frac{1}{2}\cdot(-g_{ij,0}) = a\dot{a}\delta_{ij}. \tag{5}$$

For $\beta = 0$ component in Eq.(3),

$$\frac{\mathrm{d}E}{\mathrm{d}\lambda} + \Gamma^0_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = \frac{\mathrm{d}E}{\mathrm{d}\lambda} + \Gamma^0_{ij}p^i p^j = 0, \tag{6}$$

with $\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}t} = E\frac{\mathrm{d}}{\mathrm{d}t}$,

$$E\frac{\mathrm{d}E}{\mathrm{d}t} + a\dot{a}\delta_{ij}p^i p^j = 0. \tag{7}$$

For a massless particle,

$$0 = g_{\mu\nu}p^\mu p^\nu \to -E^2 + \delta_{ij}a^2 p^i p^j = 0, \tag{8}$$

with Eq.(7), there is

$$E\frac{\mathrm{d}E}{\mathrm{d}t} + \frac{\dot{a}}{a}E^2 = 0 \to \frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{\dot{a}}{a}E. \qquad (E \propto a^{-1}) \tag{9}$$

For $\beta = i$ components in Eq.(3),

$$\frac{\mathrm{d}p^i}{\mathrm{d}\lambda} + 2\Gamma^i_{0j}Ep^j = \frac{\mathrm{d}p^i}{\mathrm{d}\lambda} + 2\frac{\dot{a}}{a}\delta^i_j Ep^j = \frac{\mathrm{d}p^i}{\mathrm{d}\lambda} + 2\frac{\dot{a}}{a}Ep^i = E\frac{\mathrm{d}p^i}{\mathrm{d}t} + 2\frac{\dot{a}}{a}Ep^i = 0, \tag{10}$$

which leads to

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = -2\frac{\dot{a}}{a}p^i. \qquad (p^i \propto a^{-2}) \tag{11}$$

Note that the massless particle condition is not used for $\beta = i$ components.

Einstein equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \qquad R = g^{\mu\nu}R_{\mu\nu}. \tag{12}$$

The Ricci tensor

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}, \tag{13}$$

the components

$$R_{00} = -3\frac{\ddot{a}}{a}, \qquad R_{0i} = R_{i0} = 0, \qquad R_{ij} = (2\dot{a}^2 + a\ddot{a})\delta_{ij}, \tag{14}$$

Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} = 3\frac{\ddot{a}}{a} + 3\left(\frac{\ddot{a}}{a}\right) + 6\left(\frac{\dot{a}}{a}\right)^2 = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]. \tag{15}$$

Also notice that Ricci tensor can be proved to be symmetrical with

$$R_{\mu\nu} = g^{\lambda\rho} R_{\rho\mu\lambda\nu} = g^{\lambda\rho} R_{\lambda\nu\rho\mu} = g^{\rho\lambda} R_{\lambda\nu\rho\mu} = R^{\rho}_{\nu\rho\mu} = R_{\nu\mu}. \tag{16}$$

It can also be proved with Eq.(13), and the only term that is not obviously symmetrical is $\Gamma^{\alpha}_{\mu\alpha,\nu}$, expand it as

$$\Gamma^{\alpha}_{\mu\alpha,\nu} = \frac{1}{2} g^{\alpha\rho}{}_{,\nu} g_{\rho\mu,\alpha} + \frac{1}{2} g^{\alpha\rho} g_{\rho\mu,\alpha,\nu} + \frac{1}{2} g^{\alpha\rho}{}_{,\nu} g_{\rho\alpha,\mu} + \frac{1}{2} g^{\alpha\rho} g_{\rho\alpha,\mu,\nu} - \frac{1}{2} g^{\alpha\rho}{}_{,\nu} g_{\mu\alpha,\rho} - \frac{1}{2} g^{\alpha\rho} g_{\mu\alpha,\rho,\nu}, \tag{17}$$

notice that the fourth term is obviously symmetrical. Look at the third term,

$$\frac{1}{2} g^{\alpha\rho}{}_{,\nu} g_{\rho\alpha,\mu} = -\frac{1}{2} g^{\alpha\sigma} g^{\rho\tau} g_{\sigma\tau,\nu} g_{\rho\alpha,\mu} = \frac{1}{2} g^{\sigma\tau}{}_{,\mu} g_{\sigma\tau,\nu} = \frac{1}{2} g^{\alpha\rho}{}_{,\mu} g_{\rho\alpha,\nu}, \tag{18}$$

so the third term is also symmetrical. Look at the first and the fifth terms,

$$\frac{1}{2} g^{\alpha\rho}{}_{,\nu} g_{\rho\mu,\alpha} = \frac{1}{2} g^{\alpha\rho}{}_{,\nu} g_{\mu\rho,\alpha} = \frac{1}{2} g^{\rho\alpha}{}_{,\nu} g_{\mu\alpha,\rho} = \frac{1}{2} g^{\alpha\rho}{}_{,\nu} g_{\mu\alpha,\rho}, \tag{19}$$

so they can offset each other. The same for the second and the sixth terms,

$$\frac{1}{2} g^{\alpha\rho} g_{\rho\mu,\alpha,\nu} = \frac{1}{2} g^{\rho\alpha} g_{\alpha\mu,\rho,\nu} = \frac{1}{2} g^{\alpha\rho} g_{\mu\alpha,\rho,\nu}, \tag{20}$$

so they can also offset each other. $\Gamma^{\alpha}_{\mu\alpha,\nu}$ only has two symmetrical terms to be considered, so $\Gamma^{\alpha}_{\mu\alpha,\nu} = \Gamma^{\alpha}_{\nu\alpha,\mu}$.

Energy-momentum tensor

$$T^{\mu}_{\ \nu} = \mathbf{diag}(-\rho,\ P,\ P,\ P), \tag{21}$$

$$T_{\mu\nu} = g_{\mu\alpha} T^{\alpha}_{\ \nu} = \mathbf{diag}(\rho,\ a^2 P,\ a^2 P,\ a^2 P), \quad T^{\mu\nu} = g^{\mu\alpha} T^{\nu}_{\ \alpha} = \mathbf{diag}(\rho,\ a^{-2} P,\ a^{-2} P,\ a^{-2} P). \tag{22}$$

Consider

$$R_{00} - \frac{1}{2} g_{00} R = 8\pi G T_{00} \to -3\frac{\ddot{a}}{a} + \frac{1}{2} \times 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right] = 3 \left(\frac{\dot{a}}{a} \right)^2 = 8\pi G \rho \to \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho, \tag{23}$$

this is Friedmann equation. Consider

$$R_{ii} - \frac{1}{2} g_{ii} R = 8\pi G T_{ii}, \tag{24}$$

this will lead to

$$(2\dot{a}^2 + a\ddot{a})\delta_{ii} - \frac{1}{2} a^2 \times 6 \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right] = -2a\ddot{a} - \dot{a}^2 = 8\pi G a^2 P \to \frac{\dot{a}^2}{a^2} + \frac{2\ddot{a}}{a} = -8\pi G P \to \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3P + \rho), \tag{25}$$

this is No.2 Friedmann equation.

The conservation of energy-momentum tensor

$$0 = T^{\mu}_{\ \nu;\mu} = \partial_\mu T^{\mu}_{\ \nu} + \Gamma^{\mu}_{\alpha\mu} T^{\alpha}_{\ \nu} - \Gamma^{\alpha}_{\nu\mu} T^{\mu}_{\ \alpha}. \tag{26}$$

For $\nu = 0$ component,

$$T^{\mu}_{\ 0;\mu} = -\partial_t \rho - 3\frac{\dot{a}}{a}(\rho + P) = 0 \to \partial_t \rho + 3\frac{\dot{a}}{a}(\rho + P) = 0, \tag{27}$$

it can be checked with $dE + PdV = 0$, with $E = \rho V$ $(V = a^3)$, then

$$dE + PdV = \rho dV + V d\rho + PdV = 0 = V d\rho + (P + \rho) dV \to d\rho + (P + \rho) d(\log V) = 0, \tag{28}$$

so

$$d\rho + 3(P + \rho) d(\log a) = 0 \to \partial_t \rho + 3\frac{\dot{a}}{a}(\rho + P) = 0. \tag{29}$$

They are the same, verified.

For non-relativistic matter, $P = 0$ $(v \ll c)$,

$$\partial_t \rho = -3\frac{\dot{a}}{a}\rho \to \rho \propto a^{-3}. \tag{30}$$

For relativistic energy (radiation), $P = (1/3)\rho$,

$$\partial_t \rho = -4\frac{\dot{a}}{a}\rho \to \rho \propto a^{-4}, \tag{31}$$

Define Hubble parameter $H \equiv \dot{a}/a = \dot{R}/R$, $(a(t) = R(t)/R_0,\ \dot{a} = \dot{R}/R_0)$, with $H_0 = h \cdot 100\ \mathrm{km \cdot s^{-1} \cdot Mpc^{-1}}$, with $h = 0.7$ in code.

1.2 General Robert-Walker metric

Rewrite the RW metric

$$ds^2 = -c^2dt^2 + a^2(t)\left[dr^2 + R_0^2 S_k^2\left(\frac{r}{R_0}\right)(d\theta^2 + \sin^2\theta d\varphi^2)\right]. \tag{32}$$

with $a(t) = R(t)/R_0$. The function

$$S_k(\chi) = \sin\chi\ (k=1),\ \sinh\chi\ (k=-1),\ \chi\ (k=0,\ \text{flat}). \tag{33}$$

This will lead to

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2R_0^2}, \qquad \frac{\ddot a}{a} = -\frac{4\pi G}{3c^2}(3P+\rho c^2). \tag{34}$$

Define the critical density

$$\rho_c = \frac{3H^2}{8\pi G} \approx 1.88\times 10^{-26}\ h^2\cdot \text{kg}\cdot\text{m}^{-3} = 2.78\times 10^{11}\ h^2\ M_\odot\ \text{Mpc}^{-3}. \tag{35}$$

There is

$$\frac{H^2}{H_0^2} = \frac{8\pi G}{3H_0^2}\rho - \frac{kc^2}{a^2R_0^2H_0^2} = \Omega_{m0}a^{-3} + \Omega_{r0}a^{-4} + \Omega_{\Lambda 0} + \Omega_{k0}a^{-2}, \tag{36}$$

when $a=1$, $H=H_0$, so

$$\Omega_{m0} + \Omega_{r0} + \Omega_{\Lambda 0} + \Omega_{k0} \equiv \Omega_0 + \Omega_{k0} = 1, \qquad \Omega_{m0} = \frac{\rho_{m0}}{\rho_{c0}}\ \text{(the same for other three)}, \tag{37}$$

then

$$1 = \frac{\rho(a)}{\rho_c(a)} - \frac{kc^2}{a^2H^2R_0^2} = \Omega(a) - \frac{kc^2}{a^2H^2R_0^2}, \tag{38}$$

in another way, $a(t=0)=1$,

$$H_0^2 = \frac{8\pi G\rho_0}{3} - \frac{kc^2}{R_0^2} \rightarrow R_0 = \frac{c}{H_0}\left(\frac{k}{\Omega_0 - 1}\right)^{\frac{1}{2}}. \tag{39}$$

$$\Omega_m(a) = \frac{\rho_m(a)}{\rho_c(a)} = \frac{\Omega_{m0}a^{-3}}{\Omega_{m0}a^{-3} + \Omega_{r0}a^{-4} + \Omega_{\Lambda 0} + (1-\Omega_0)a^{-2}} \equiv \frac{\Omega_{m0}a^{-3}}{f(a)}, \tag{40}$$

and for radiation and dark matter terms,

$$\Omega_r(a) = \frac{\Omega_{r0}a^{-4}}{f(a)}, \qquad \Omega_\Lambda(a) = \frac{\Omega_{\Lambda 0}}{f(a)}. \tag{41}$$

And for the curvature term,

$$1 - \Omega(a) = \frac{(1-\Omega_0)a^{-2}}{f(a)} \rightarrow 0\ (\text{for } a\rightarrow 0), \tag{42}$$

which means the flatness in early universe.

The deceleration parameter

$$q \equiv -\frac{\ddot R R}{\dot R^2} = -\frac{\ddot a a}{\dot a^2} = -\frac{\ddot R}{R}\left(\frac{R}{\dot R}\right)^2 = \frac{4\pi G}{3}(3P+\rho c^2)\cdot\frac{1}{H^2} = \frac{1}{2}\cdot\frac{8\pi G}{3H^2}\left(\rho + 3\frac{P}{c^2}\right), \tag{43}$$

for non-relativistic matter, $P=0$, for radiation matter, $P=\rho c^2/3$, for dark matter, $P=-\rho$, so

$$q = \frac{1}{2}\cdot\frac{1}{\rho_c}(\rho_m + \rho_r + \rho_\Lambda + \rho_r - \rho_\Lambda) = \frac{\Omega_m}{2} + \Omega_r - \Omega_\Lambda. \tag{44}$$

And if the universe is flat, $\Omega_\Lambda = 1 - \Omega_m - \Omega_r$, then $q = 3\Omega_m/2 + 2\Omega_r - 1$.

• The discussion on the equivalence of two forms of the RW metric:

The first form is as Eq. (32), the other form is

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{1}{1-k(r/R_0)^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]. \tag{45}$$

• For $k=1$, the two forms of the metric are ($c=1$)

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + R_0^2 \sin^2\left(\frac{r}{R_0}\right) d\Omega^2\right], \tag{46}$$

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-(r/R_0)^2} + r^2 d\Omega^2\right]. \tag{47}$$

Defining $\chi \equiv R_0 \sin(r/R_0)$, then $d\chi = \cos(r/R_0)dr$, the first form can be rewritten as

$$ds^2 = -dt^2 + a^2(t)\left[\frac{d\chi^2}{1-(\chi/R_0)^2} + \chi^2 d\Omega^2\right]. \qquad (\sin^2 x + \cos^2 x = 1) \tag{48}$$

So the two forms are equivalent.
• For $k=-1$, the two forms of the metric are

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + R_0^2 \sinh^2\left(\frac{r}{R_0}\right) d\Omega^2\right], \tag{49}$$

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1+(r/R_0)^2} + r^2 d\Omega^2\right]. \tag{50}$$

Defining $\chi \equiv R_0 \sinh(r/R_0)$, then $d\chi = \cosh(r/R_0)dr$, the first form can be rewritten as

$$ds^2 = -dt^2 + a^2(t)\left[\frac{d\chi^2}{1+(\chi/R_0)^2} + \chi^2 d\Omega^2\right]. \qquad (\cosh^2 x - \sinh^2 x = 1) \tag{51}$$

So the two forms are equivalent.
• For $k=0$, the two forms of the metric are apparently the same.
Then we can derive the Friedmann equations with spatial curvature, with the metric form of Eq. (47). The nonzero connection components are

$$\begin{aligned}
&\Gamma^0_{11} = \frac{a\dot{a}}{1-k(r/R_0)^2},\ \Gamma^0_{22} = r^2 a\dot{a},\ \Gamma^0_{33} = r^2\sin^2\theta a\dot{a},\ \Gamma^1_{01} = \frac{\dot{a}}{a} = \Gamma^1_{10},\ \Gamma^1_{11} = -\frac{kr}{kr^2 - R_0^2},\\
&\Gamma^1_{22} = -r + \frac{kr^3}{R_0^2},\ \Gamma^1_{33} = -r\sin^2\theta\left[1-k(r/R_0)^2\right],\ \Gamma^2_{02} = \frac{\dot{a}}{a} = \Gamma^2_{20},\ \Gamma^2_{12} = \frac{1}{r} = \Gamma^2_{21},\\
&\Gamma^2_{33} = -\sin\theta\cos\theta,\ \Gamma^3_{03} = \frac{\dot{a}}{a} = \Gamma^3_{30},\ \Gamma^3_{13} = \frac{1}{r} = \Gamma^3_{31},\ \Gamma^3_{23} = \cot\theta = \Gamma^3_{32}.
\end{aligned} \tag{52}$$

Then

$$\begin{aligned}
R_{11} &= -3\frac{\ddot{a}}{a},\ R_{22} = \frac{a\ddot{a}}{1-k(r/R_0)^2} - \frac{2(k+R_0^2\dot{a}^2)}{kr^2 - R_0^2},\\
R_{33} &= r^2(2\dot{a}^2 + a\ddot{a}) + 2k\left(\frac{r}{R_0}\right)^2,\\
R_{44} &= r^2\sin^2\theta(2\dot{a}^2 + a\ddot{a}) + 2k\left(\frac{r}{R_0}\right)^2\sin^2\theta.
\end{aligned} \tag{53}$$

And the Ricci scalar

$$R = 6\left[\frac{k}{a^2R_0^2} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right]. \tag{54}$$

With Einstein equation,

$$G_{00} = R_{00} - \frac{1}{2}g_{00}R = 3\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2R_0^2}\right] = 8\pi G\rho, \tag{55}$$

which leads to the first Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2R_0^2}. \tag{56}$$

With

$$G_{11} = R_{11} - \frac{1}{2} g_{11} R = 8\pi G \frac{a^2}{1 - k(r/R_0)^2} P, \tag{57}$$

then

$$\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} = -8\pi G P \rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P). \tag{58}$$

which is the second Friedmann equation.

1.3 Redshift

For a photon,

$$0 = ds^2 = -dt^2 + a^2(t) d\boldsymbol{x}^2. \tag{59}$$

If at $x = x_0$, the source emitted two photons at $t1,\ t_2$, and the receiver at $x = x_1$ received them at $t_1',\ t_2'$, then

$$\int_{x_0}^{x_1} dx = \int_{t_1}^{t_1'} \frac{dt}{a(t)} = \int_{t_2}^{t_2'} \frac{dt}{a(t)} \rightarrow \int_{t_1}^{t_2} \frac{dt}{a(t)} = \int_{t_1'}^{t_2'} \frac{dt}{a(t)}, \tag{60}$$

then

$$\frac{\Delta t}{a(t_1)} = \frac{\Delta t'}{a(t_1')}, \tag{61}$$

if $a(t_1) < a(t_1')$, then $\Delta t < \Delta t'$. The frequency and wave length

$$\frac{\nu(t_1')}{\nu(t_1)} = \frac{\lambda(t_1)}{\lambda(t_1')} = \frac{\Delta t}{\Delta t'} = \frac{a(t_1)}{a(t_1')}, \tag{62}$$

and the redshift relation

$$\lambda_{\text{receive}} = \frac{\lambda_{\text{emit}}}{a(t_{\text{emit}})} \cdot a(t_{\text{receive}}), \tag{63}$$

$t_{\text{receive}} = t_{\text{today}}$, $a = 1$, so

$$\lambda_{\text{receive}} = \frac{\lambda_{\text{emit}}}{a(t_{\text{emit}})} = \lambda_{\text{emit}}(1+z), \qquad a(t_{\text{emit}}) = \frac{1}{1+z}. \tag{64}$$

• Distance-Redshift relation, for the photons

$$0 = ds^2 = -dt^2 + a^2(t) d\boldsymbol{x}^2 \rightarrow dx = \frac{dt}{a(t)} \qquad (dx = |d\boldsymbol{x}|) \tag{65}$$

with

$$\frac{\dot{a}}{a} = H \rightarrow dt = \frac{1}{H} d(\log a) = -\frac{1}{H} d\left[\log(1+z)\right] = -\frac{dz}{1+z}\frac{1}{H(z)}, \tag{66}$$

then

$$-\int_{x_0}^{0} dx = \int_0^t \frac{dt'}{a(t')} = -\int_{z_0}^{0} \frac{dz'}{1+z'}\frac{1}{H(z')} \cdot \frac{1}{a(z')} = \int_0^{z_0} dz' \frac{1}{H(z')}. \tag{67}$$

If $z_0 \ll 1$,

$$x_0 = \int_0^{z_0} \frac{dz'}{H(z')} \approx \frac{z_0}{H_0} (c = 1). \tag{68}$$

With Eq. (36), there is

$$x_0 = \frac{1}{H_0} \int_0^{z_0} \frac{dz}{\sqrt{\Omega_{\Lambda 0} + \Omega_{m0}(1+z)^3 + \Omega_{\gamma 0}(1+z)^4 + (1-\Omega_0)(1+z)^2}}. \tag{69}$$

• Angular-Diameter Distance D_A, the proper transverse size of an object seen by us is

$$D_{\text{proper}} = R_0 S_k \left(\frac{r}{R_0}\right) d\theta, \tag{70}$$

and its proper size at the time of light emission is

$$D_A = \frac{1}{1+z} R_0 S_k \left(\frac{r}{R_0} \right) d\theta. \tag{71}$$

• Luminosity Distance, notice four points:
1. The photon energy will decrease by $(1+z)$;
2. The arrival rate of the photons will decrease by $(1+z)$;
3. The bandwidth of photon frequency of the photons will decrease by $(1+z)$;
4. The photon frequency will be redshifted.
The total flux

$$S_{\text{tot}} = \int_0^\infty S_0(\nu_0) d\nu_0 = \frac{1}{4\pi R_0^2 S_k^2(r/R_0)(1+z)^2} \int_0^\infty L_{\nu'} d\nu' = \frac{L_{\text{tot}}}{4\pi R_0^2 S_k^2(r/R_0)(1+z)^2} \equiv \frac{L_{\text{tot}}}{4\pi D_L^2}, \tag{72}$$

where D_L is the luminosity distance, and the $(1+z)^2$ in the denominator is from 1., 2. effects above,

$$\nu' = (1+z)\nu_0, \qquad D_L \equiv R_0 S_k \left(\frac{r}{R_0} \right) (1+z). \tag{73}$$

The redshift-drift, emission time t_1, receiving time t_2, after a small time interval, at $t_1 + \Delta t_1$ there is another emission and received at $t_2 + \Delta t_2$. As the co-moving distance is the same,

$$\int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} \frac{dt}{a(t)} = \int_{t_1+\Delta t_1}^{t_2+\Delta t_2} \frac{dt}{a(t)} \rightarrow \frac{\Delta t_2}{a(t_2)} = \frac{\Delta t_1}{a(t_1)}. \tag{74}$$

There is the relation

$$z(t_2) = \frac{a(t_2)}{a(t_1)} - 1, \qquad z(t_2 + \Delta t_2) = \frac{a(t_2 + \Delta t_2)}{a(t_1 + \Delta t_1)} - 1, \tag{75}$$

then

$$\begin{aligned} & z(t_2 + \Delta t_2) - z(t_2) \approx \dot{z}(t_2)\Delta t_2 \\ &= \frac{a(t_2 + \Delta t_2)}{a(t_1 + \Delta t_1)} - \frac{a(t_2)}{a(t_1)} \approx \frac{a(t_2) + \dot{a}(t_2)\Delta t_2}{a(t_1) + \dot{a}(t_1)\Delta t_1} - \frac{a(t_2)}{a(t_1)} \\ &= \frac{a(t_2)}{a(t_1)} \left[\frac{1 + H(t_2)\Delta t_2}{1 + H(t_1)\Delta t_1} - 1 \right] \approx \frac{a(t_2)}{a(t_1)} \Delta t_2 \left[H(t_2) - H(t_1) \frac{\Delta t_1}{\Delta t_2} \right]. \end{aligned} \tag{76}$$

Where the fact that $H(t_1)\Delta t_1 \ll 1$ is used. If t_2 is today, then $a(t_2) = 1$, $H(t_2) = H_0$, $a(t_1) = (1+z)^{-1}$. Then the equation above can be rewritten as

$$\dot{z}(z) = (1+z)H_0 - H(z) = H_0 \left[1 + z - \sqrt{\Omega_{m0}(1+z)^3 + \Omega_{\gamma_0}(1+z)^4 + \Omega_{\Lambda 0} + \Omega_{k0}(1+z)^2} \right]. \tag{77}$$

1.4 The Metric Perturbation

Under the Newtonian gauge, the metric becomes as

$$g_{\mu\nu} = \textbf{diag}\{-[1 + 2\psi(t,\ \boldsymbol{x})],\ a^2[1 + 2\phi(t,\ \boldsymbol{x})],\ a^2[1 + 2\phi(t,\ \boldsymbol{x})],\ a^2[1 + 2\phi(t,\ \boldsymbol{x})]\}, \tag{78}$$

$$g^{\mu\nu} = \textbf{diag}\{-1 + 2\psi(t,\ \boldsymbol{x}),\ a^{-2}[1 - 2\phi(t,\ \boldsymbol{x})],\ a^{-2}[1 - 2\phi(t,\ \boldsymbol{x})],\ a^{-2}[1 - 2\phi(t,\ \boldsymbol{x})]\}. \tag{79}$$

The perturbation terms $\psi(t,\ \boldsymbol{x})$, $\phi(t,\ \boldsymbol{x})$ are treated as 1st-order small values. The nonzero connection components to the leading order are

$$\Gamma^0_{00} = \frac{-1 + 2\psi}{2} g_{00,0} = \frac{-1 + 2\psi}{2}(-2\psi_{,0}) \approx \psi_{,0}, \tag{80}$$

$$\Gamma^0_{0i} = \Gamma^0_{i0} = \frac{-1 + 2\psi}{2} g_{00,i} = \frac{-1 + 2\psi}{2}(-2\psi_{,i}) \approx \psi_{,i}, \tag{81}$$

$$\Gamma^0_{ij} = \frac{1 - 2\psi}{2} \frac{\partial}{\partial t} \left[a^2(t)\delta_{ij}(1 + 2\phi) \right] = \frac{1 - 2\psi}{2} \delta_{ij} \left[2a\dot{a}(1 + 2\phi) + 2a^2\phi_{,0} \right] \tag{82}$$

$$\Gamma^i_{00} = \frac{g^{i\nu}}{2}(-g_{00,\nu}) = -\frac{1}{2a^2}(1-2\phi)(-2\psi_{,i}) = \frac{1}{a^2}\psi_{,i}. \tag{83}$$

$$\Gamma^i_{j0} = \Gamma^i_{0j} = \frac{g^{ii}}{2}g_{ij,0} = \frac{1-2\phi}{2a^2}\frac{\partial}{\partial t}\left[a^2\delta_{ij}(1+2\phi)\right] \approx \delta_{ij}(H+\phi_{,0}). \tag{84}$$

$$\begin{aligned}\Gamma^i_{jk} &= \frac{1-2\phi}{2a^2}\left\{\partial_k[\delta_{ij}a^2(1+2\phi)] + \partial_j[\delta_{ik}a^2(1+2\phi)] - \partial_i[\delta_{jk}a^2(1+2\phi)]\right\} \\ &\approx \delta_{ik}\partial_j\phi + \delta_{ij}\partial_k\phi - \delta_{jk}\partial_i\phi.\end{aligned} \tag{85}$$

For the Ricci tensor, the nonzero components

$$R_{00} = \frac{1}{a^2}\nabla^2\psi - 3\frac{\ddot{a}}{a} - 3\partial_t^2\phi + 3H\partial_t(\psi - 2\phi). \tag{86}$$

$$R_{i0} = R_{0i} = 2(H\partial_i\psi - \partial_i\partial_t\phi), \tag{87}$$

$$\begin{aligned}R_{ij} &= a^2\delta_{ij}\left[(3H^2+\dot{H})(1+2\phi-2\psi) + H\partial_t(6\phi-\psi) + \partial_t^2\phi\right] \\ &- \delta_{ij}\nabla^2\phi - \partial_i\partial_j(\psi+\phi).\end{aligned} \tag{88}$$

The Ricci scalar is made up of two parts, the first one is zero-order part, Eq. (15), and the leading order perturbed term

$$\delta R = -12\psi\left[\left(\frac{\dot{a}}{a}\right) + \frac{\ddot{a}}{a}\right] + 6\partial_t^2\phi - 6H\partial_t\psi + 24H\partial_t\phi - 2a^{-2}(2\nabla^2\phi + \nabla^2\psi), \tag{89}$$

so the Ricci scalar is

$$R = 6\left[\left(\frac{\dot{a}}{a}\right) + \frac{\ddot{a}}{a}\right](1-2\psi) + 6\partial_t^2\phi - 6H\partial_t\psi + 24H\partial_t\phi - 2a^{-2}(2\nabla^2\phi + \nabla^2\psi). \tag{90}$$

The Einstein tensor components

$$G_{00} = 3H^2 + 6H\partial_t\phi - 2a^{-2}\nabla^2\phi, \tag{91}$$

$$G_{0i} = 2(H\partial_i\psi - \partial_i\partial_t\phi), \tag{92}$$

$$G_{ij} = a^2\delta_{ij}\left[-(3H^2+2\dot{H})(1+2\phi-2\psi) - 6H\partial_t\phi + 2H\partial_i\psi - 2\partial_t^2\phi\right] + (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\phi+\psi). \tag{93}$$

The energy-momentum tensor for i-type particles in the universe, with the spin-degeneracy $\mathfrak{g}_i$:

$$T^{\mu(i)}_{\nu} = \mathfrak{g}_i\int\frac{dP_1dP_2dP_3}{(2\pi)^3}\left[-\mathbf{det}(g_{\alpha\beta})\right]^{-\frac{1}{2}}\frac{P^\mu P_\nu}{P_0}f_i(\boldsymbol{p},\ \boldsymbol{x},\ t), \tag{94}$$

which is derived from

$$\begin{aligned}&\mathfrak{g}_i\int\frac{2P^0dP_0dP_1dP_2dP_3}{(2\pi)^3}\left[-\mathbf{det}(g_{\alpha\beta})\right]^{-\frac{1}{2}}\frac{P^\mu P_\nu}{P_0}f_i(\boldsymbol{p},\ \boldsymbol{x},\ t)\delta_D\left(P^\mu P_\mu + m^2\right) \\ &= \mathfrak{g}_i\int\frac{dP_1dP_2dP_3}{(2\pi)^3}\left[-\mathbf{det}(g_{\alpha\beta})\right]^{-\frac{1}{2}}\frac{P^\mu P_\nu}{P_0}f_i(\boldsymbol{p},\ \boldsymbol{x},\ t),\end{aligned} \tag{95}$$

notice the facts that

$$\int 2P^0dP_0\delta_D\left(P^\mu P_\mu + m^2\right) = \int d(P^0P_0)\delta_D\left(P^\mu P_\mu + m^2\right) = 1, \tag{96}$$

and

$$g^{\mu\nu}dx_\mu dx_\nu = g'^{\alpha\beta}dx'_\alpha dx'_\beta \to dP_0dP_1dP_2dP_3\sqrt{\mathbf{det}(g^{\mu\nu})} = dP'_0dP'_1dP'_2dP'_3\sqrt{\mathbf{det}(g'^{\alpha\beta})}, \tag{97}$$

this is because of the Jacobian

$$\mathbf{det}\left[\frac{\partial(x_0,\ x_1,\ x_2,\ x_3)}{\partial(x'_0,\ x'_1,\ x'_2,\ x'_3)}\right] = \sqrt{\frac{\mathbf{det}(g'^{\alpha\beta})}{\mathbf{det}(g^{\mu\nu})}}, \qquad \mathbf{det}(g^{\mu\nu})\mathbf{det}(g_{\mu\nu}) = \mathbf{det}(g'^{\alpha\beta})\mathbf{det}(g'_{\alpha\beta}) = 1. \tag{98}$$

The relation between four-momentum P^μ and physical momentum $\boldsymbol{p}$:

$$P^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^0}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}x^0} = P^0\frac{\mathrm{d}x^\mu}{\mathrm{d}x^0} = E\frac{\mathrm{d}x^\mu}{\mathrm{d}t}, \qquad P^\mu P_\mu = -m^2, \tag{99}$$

$$|\boldsymbol{p}|^2 = g_{ij}P^iP^j = g^{ij}P_iP_j \rightarrow p^i = aP^i = \frac{1}{a}P_i. \tag{100}$$

The components of $T^\mu_{\ \nu}$ (for a certain type of particle, assuming $\mathfrak{g}=1$):

$$T^0_{\ 0} = \int \frac{dP_1dP_2dP_3}{(2\pi)^3}a^{-3}P_0 f(\boldsymbol{p},\ \boldsymbol{x},\ t) = -\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}Ef(\boldsymbol{p},\ \boldsymbol{x},\ t), \tag{101}$$

$$T^i_{\ j} = \int \frac{dP_1dP_2dP_3}{(2\pi)^3}a^{-3}\frac{P^iP_j}{P^0} f(\boldsymbol{p},\ \boldsymbol{x},\ t) = \int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{p^ip^j}{E}f(\boldsymbol{p},\ \boldsymbol{x},\ t), \tag{102}$$

$$T^0_{\ j} = \int \frac{dP_1dP_2dP_3}{(2\pi)^3}a^{-3}\frac{P^0P_j}{P^0} f(\boldsymbol{p},\ \boldsymbol{x},\ t) = \int\frac{d^3\boldsymbol{p}}{(2\pi)^3}ap^jf(\boldsymbol{p},\ \boldsymbol{x},\ t), \tag{103}$$

$$T^i_{\ 0} = \int \frac{dP_1dP_2dP_3}{(2\pi)^3}a^{-3}\frac{P^iP_0}{P^0} f(\boldsymbol{p},\ \boldsymbol{x},\ t) = -\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{p^i}{a}f(\boldsymbol{p},\ \boldsymbol{x},\ t). \tag{104}$$

With

$$T^0_{\ 0} = -\rho_m = -\bar{\rho}(1+\delta), \tag{105}$$

the relativistic Poission equation:

$$3H^2\psi - 3H\partial_t\phi + a^{-2}\nabla^2\phi = -4\pi G\bar{\rho}\delta, \tag{106}$$

only the third on the left side should be considered under the small scale $L \ll L_{\mathrm{Horizon}}$.
The Fourier transformation on the Einstein tensor and energy-momentum tensor:

$$\left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta^j_{\ i}\right)\tilde{G}^i_{\ j} = a^{-2}\left(\hat{k}_i\hat{k}^jk^ik_j - \frac{1}{3}k^ik_i\right)\left(\tilde{\phi}+\tilde{\psi}\right) = \frac{2}{3a^2}k^2\left(\tilde{\phi}+\tilde{\psi}\right). \tag{107}$$

$$\left(\hat{k}_i\hat{k}^j - \frac{1}{3}\delta^j_{\ i}\right)\tilde{T}^i_{\ j} = \int\frac{d^3p}{(2\pi)^3}\frac{p^2\left(\mu^2-\frac{1}{3}\right)}{E}\tilde{f}(\boldsymbol{p},\ \boldsymbol{k},\ t), \qquad \mu\equiv\frac{\boldsymbol{p}\cdot\boldsymbol{k}}{|\boldsymbol{p}||\boldsymbol{k}|}. \tag{108}$$

For the non-relativistic matter in late universe, there is $\tilde{\phi}+\tilde{\psi},\ \phi+\psi=0$.

1.5 Eulerian Perturbation Theory

The conservation of energy-momentum tensor, $T^\mu_{\ \nu;\mu}=0$. For $\nu=0$ part,

$$\begin{aligned}
T^\mu_{\ 0;\mu} &= \partial_\mu T^\mu_{\ 0} + \Gamma^\mu_{\alpha\mu}T^\alpha_{\ 0} - \Gamma^\alpha_{\mu 0}T^\mu_{\ \alpha}\\
&= \partial_0T^0_{\ 0} + \partial_iT^i_{\ 0} + \Gamma^0_{00}T^0_{\ 0} + \Gamma^i_{0i}T^0_{\ 0} + \Gamma^0_{i0}T^i_{\ 0} + \Gamma^j_{ij}T^i_{\ 0} - \Gamma^0_{00}T^0_{\ 0} - \Gamma^0_{i0}T^i_{\ 0} - \Gamma^i_{00}T^0_{\ i} - \Gamma^i_{j0}T^j_{\ i}\\
&= \partial_0T^0_{\ 0} + \partial_iT^i_{\ 0} + \Gamma^i_{0i}T^0_{\ 0} + \Gamma^j_{ij}T^i_{\ 0} - \Gamma^i_{00}T^0_{\ i} - \Gamma^i_{j0}T^j_{\ i}.\\
&= -\partial_t\rho_m - \partial_i\left[\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{p^i}{a}f(\boldsymbol{p},\ \boldsymbol{x},\ t)\right] - 3(H+\partial_t\phi)\rho_m - H\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{p^2}{E}f(\boldsymbol{p},\ \boldsymbol{x},\ t),
\end{aligned} \tag{109}$$

where Eq.(80)-(85) have been used, and the red terms are of higher order perturbation/small values ($\phi,\ \psi,\ \partial_i\phi,\ \partial_j\psi,\ p^i$), neglected step by step in Eq. (109). The continuity equation

$$\partial_t\rho_m - 3H\rho_m - \frac{1}{a}\partial_i\left(\rho_m v^i\right) \approx 0, \tag{110}$$

with

$$v^i(\boldsymbol{x},\ t) = \frac{\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}p^if(\boldsymbol{p},\ \boldsymbol{x},\ t)}{\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}mf(\boldsymbol{p},\ \boldsymbol{x},\ t)}. \tag{111}$$

With $\rho_m = \bar{\rho}(t)[1+\delta(\boldsymbol{x},\ t)]$, and Eq. (30) for non-relativistic matter, there is

$$\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}t} = -3H\bar{\rho} \to 0 = \frac{\partial \delta}{\partial t} + \frac{1}{a}\partial_i[(1+\delta)v^i]. \tag{112}$$

Defining the conformal time $d\tau \equiv dt/a$, then the continuity equation is

$$\frac{\partial \delta}{\partial \tau} = -\partial_i[(1+\delta)v^i]. \tag{113}$$

For $\nu = i$ parts,

$$\begin{aligned}
0 = T^{\mu}_{\ i;\mu} &= \partial_\mu T^\mu_i + \Gamma^\mu_{\alpha\mu}T^\alpha_i - \Gamma^\alpha_{\mu i}T^\mu_\alpha \\
&= \partial_t\left[\int \frac{d^3\boldsymbol{p}}{(2\pi)^3} a p^i f(\boldsymbol{p},\ \boldsymbol{x},\ t)\right] + \partial_j\left[\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{p^ip^j}{E} f(\boldsymbol{p},\ \boldsymbol{x},\ t)\right] \\
&+ \Gamma^0_{00}T^0_i + \Gamma^j_{0j}T^0_i + \Gamma^0_{0j} + \Gamma^0_{j0}T^j_i + \Gamma^k_{jk}T^j_i - \Gamma^0_{0i}T^0_0 - \Gamma^0_{ji}T^j_0 - \Gamma^j_{0i}T^0_j - \Gamma^j_{ki}T^k_j \\
&\approx \partial_t(a\rho_m v^i) + 3Ha\rho_m v^i + \rho_m\partial_i\psi + \partial_j\left[\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{p^ip^j}{E} f(\boldsymbol{p},\ \boldsymbol{x},\ t)\right].
\end{aligned} \tag{114}$$

Defining

$$\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{p^ip^j}{m} f(\boldsymbol{p},\ \boldsymbol{x},\ t) = \rho_m v^iv^j + \sigma^{ij}(\boldsymbol{x},\ t), \tag{115}$$

where σ^{ij} is the stress tensor,

$$\sigma^{ij} = -p\delta_{ij} + \eta\left(\nabla_i u_j + \nabla_j u_i - \frac{2}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{u}\right) + \zeta\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{u}, \tag{116}$$

notice that p is the pressure, $\eta,\ \zeta$ are viscosity coefficients. Then

$$\begin{aligned}
0 &\approx \partial_t(a\rho_m v^i) + 3Ha\rho_m v^i + \rho_m\partial_i\psi + \partial_j(\rho_m v^iv^j + \sigma^{ij}) \\
&= \partial_t\left[a\bar{\rho}(1+\delta)v^i\right] + 3Ha\bar{\rho}(1+\delta)v^i + \bar{\rho}(1+\delta)\partial_i\psi + \partial_j\left[\bar{\rho}(1+\delta)v^iv^j + \sigma^{ij}\right],
\end{aligned} \tag{117}$$

the first term combined with Eq.(112)(113),

$$\partial_t\left[a\bar{\rho}(1+\delta)v^i\right] = \dot{a}\bar{\rho}(1+\delta)v^i - 3Ha\bar{\rho}(1+\delta)v^i + a\bar{\rho}\left\{-\frac{1}{a}\partial_j\left[(1+\delta)v^j\right]\right\}v^i + a\bar{\rho}(1+\delta)\frac{\partial v^i}{\partial t}, \tag{118}$$

thus,

$$0 \approx \dot{a}\bar{\rho}(1+\delta)v^i + a\bar{\rho}(1+\delta)\partial_t v^i + \bar{\rho}(1+\delta)\partial_i\psi + \bar{\rho}(1+\delta)v^j\partial_j v^i + \partial_j\sigma^{ij}, \tag{119}$$

with both sides divided by $\bar{\rho}(1+\delta)$, the Euler equation can be derived as

$$\frac{\partial v^i}{\partial \tau} + aHv^i + \partial_i\psi + v^j\partial_j v^i + \frac{1}{\rho_m}\partial_j\sigma^{ij} = 0. \tag{120}$$

Consider the relation for non-relativistic matter $\phi + \psi = 0$, and the Poisson equation (106) under small scales,

$$\nabla^2\phi = -\nabla^2\psi = -4\pi Ga^2\bar{\rho}_m\delta_m, \tag{121}$$

notice that $\bar{\rho} \to \bar{\rho}_m,\ \delta \to \delta_m$ to denote the density and perturbation of matters, now there is

$$\nabla^2\psi = 4\pi Ga^2\bar{\rho}_m\delta_m = \frac{8\pi G}{3}\bar{\rho}_{\mathrm{tot}}\Omega_m(a)\cdot\frac{3}{2}a^2\delta = \frac{3}{2}a^2H^2\Omega_m(a)\delta_m, \tag{122}$$

this is the Poisson equation under Eulerian perturbation theory. Keeping the 1-st order, neglecting σ^{ij}, the Euler equation (120) and the continuity equation Eq.(113) become as

$$\frac{\partial \delta_m}{\partial \tau} + \partial_i v^i = 0, \qquad \frac{\partial v^i}{\partial \tau} + aHv^i = -\partial_i\psi, \tag{123}$$

with the property $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi = 0$, there is

$$\frac{\partial}{\partial \tau}(\boldsymbol{\nabla} \times \boldsymbol{v}) + aH\boldsymbol{v} = 0 \rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\nabla} \times \boldsymbol{v}) + \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} = 0, \tag{124}$$

then we have $\boldsymbol{\nabla} \times \boldsymbol{v} \propto a^{-1}$, the curl field of $\boldsymbol{v}$ is decayed with a.
Defining $\theta = \partial_i v^i$, with two equations in Eq. (123),

$$\frac{\partial^2 \delta_m}{\partial \tau^2} + aH\frac{\partial \delta_m}{\partial \tau} = \frac{3}{2}a^2H^2\Omega_m(a)\delta_m, \tag{125}$$

rewrite $\delta_m(\boldsymbol{x},\ \tau)$ with a linear growth factor: $\delta(\boldsymbol{x},\ \tau) = \delta(\boldsymbol{x},\ 0)D(\tau)$, then

$$\frac{\partial^2 D}{\partial \tau^2} + aH\frac{\partial D}{\partial \tau} = \frac{3}{2}a^2H^2\Omega_m(a)D, \tag{126}$$

with $\frac{\mathrm{d}}{\mathrm{d}\tau} = a\frac{\mathrm{d}}{\mathrm{d}t} = a\frac{\mathrm{d}a}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}a} = a^2H\frac{\mathrm{d}}{\mathrm{d}a}$, then

$$a^2H^2\frac{\mathrm{d}^2 D}{\mathrm{d}a^2} + \left(3aH^2 + a^2H\frac{\mathrm{d}H}{\mathrm{d}a}\right)\frac{\mathrm{d}D}{\mathrm{d}a} = \frac{3}{2}H^2\Omega_m(a)D. \tag{127}$$

Now we discuss the behaviours of $D(\tau)$ with a. Assuming that there are only Ω_Λ and Ω_m,

$$H^2 = H_0^2\left[\Omega_m a^{-3} + \Omega_\Lambda + (1 - \Omega_m - \Omega_\Lambda)a^{-2}\right], \tag{128}$$

defining

$$M(a) = 2\Omega_m + 2(1 - \Omega_m - \Omega_\Lambda)a + 2\Omega_\Lambda a^3 = 2a^3\frac{H^2}{H_0^2}, \tag{129}$$

$$N(a) = 3\Omega_m a^{-1} + 2(1 - \Omega_m - \Omega_\Lambda) = -a^3\frac{1}{H_0^2}\frac{\mathrm{d}}{\mathrm{d}a}(H^2). \tag{130}$$

Eq. (127) can be simplified as

$$M\frac{\mathrm{d}^2 D}{\mathrm{d}a^2} + \left(N + \frac{\mathrm{d}M}{\mathrm{d}a}\right)\frac{\mathrm{d}D}{\mathrm{d}a} + \frac{\mathrm{d}N}{\mathrm{d}a}D = \frac{\mathrm{d}}{\mathrm{d}a}\left(M\frac{\mathrm{d}D}{\mathrm{d}a} + ND\right) = 0. \tag{131}$$

Thus,

$$M\frac{\mathrm{d}D}{\mathrm{d}a} + ND = C = 2a^3H^2\frac{\mathrm{d}D}{\mathrm{d}a} - a^3D\frac{\mathrm{d}H^2}{\mathrm{d}a}, \tag{132}$$

where C is the integration constant. At last,

$$H^{-1}\frac{\mathrm{d}D}{\mathrm{d}a} + DH^{-2}\frac{\mathrm{d}H}{\mathrm{d}a} = \frac{\mathrm{d}H^{-1}D}{\mathrm{d}a} = \frac{C}{2a^3H^3}, \tag{133}$$

for $C \neq 0$, the growth solution is

$$D^+(a) \propto H(a)\int_0^a \frac{a'}{a'^3H'(a')}, \tag{134}$$

for $C = 0$, the decaying solution is

$$D^-(a) \propto H(a). \tag{135}$$

In Einstein-Desitter universe standard cold dark matter (SCDM), $\Omega_m = 1,\ \Omega_\Lambda = 0$, then we have

$$H(a) \propto a^{-\frac{3}{2}},\ D(a) \propto a. \tag{136}$$

Based on the perturbed metric (78), the trajectory of a particle on the geodesic with the equation of motion

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} = \Gamma^\alpha_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}, \tag{137}$$

there are

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} = \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2\left\{-\partial_t\psi - 2\partial_i\psi\frac{\mathrm{d}x^i}{\mathrm{d}t} - a^2[H(1 + 2\phi - 2\psi) + \partial_t\phi]\delta_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}\right\}, \tag{138}$$

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}\lambda^2} = -\Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} = -a^{-2}\partial_i\psi\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 - 2(H+\partial_t\phi)\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda} - 2\partial_j\phi\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda} + \partial_i\phi\left(\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right)^2$$
$$= \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}x^i}{\mathrm{d}t}\right) = \frac{\mathrm{d}^2t}{\mathrm{d}\lambda^2}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}. \tag{139}$$

Eq. (139) also leads to

$$\left\{-\partial_t\psi - 2\partial_i\psi\frac{\mathrm{d}x^i}{\mathrm{d}t} - a^2[H(1+2\phi-2\psi)+\partial_t\phi]\left(\frac{\mathrm{d}x^i}{\mathrm{d}t}\right)^2\right\}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\mathrm{d}^2x^i}{\mathrm{d}t^2}$$
$$= -a^{-2}\partial_i\psi - 2(H+\partial_t\phi)\frac{\mathrm{d}x^i}{\mathrm{d}t} - 2\partial_j\phi\frac{\mathrm{d}x^j}{\mathrm{d}t}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \partial_i\phi\left(\frac{\mathrm{d}x^j}{\mathrm{d}t}\right)^2, \tag{140}$$

for non-relativistic and sub-horizon motions, $dx^i/dt \ll 1$, ϕ, ψ, $\partial_t\phi$, $\partial_t\psi$ are also small perturbations, so

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} + 2H\frac{\mathrm{d}x^i}{\mathrm{d}t} = -a^{-2}\partial_i\psi, \tag{141}$$

with $\frac{\mathrm{d}}{\mathrm{d}\tau} = a\frac{\mathrm{d}}{\mathrm{d}t}$,

$$a\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{a}\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\right) + 2aH\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} - \dot{a}\frac{\mathrm{d}x^i}{\mathrm{d}\tau} + 2aH\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{\mathrm{d}^2x^i}{\mathrm{d}\tau^2} + aH\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = -\partial_i\psi. \tag{142}$$

Based on the framework of Lagrangian dynamics, we calculate the displacement field of the position of a particle, $\boldsymbol{x}(\tau) = \boldsymbol{q} + \boldsymbol{\phi}(\boldsymbol{q},\ \tau)$, the density field under coordinates transformation $\bar{\rho}(1+\delta)d^3x = \bar{\rho}d^3q$, then the Jacobian

$$J(\boldsymbol{q},\ \tau) = \mathbf{det}\left(\frac{\partial x_i}{\partial q_j}\right) = \mathbf{det}(\delta_{ij} + \phi_{i,j}) = (1+\delta)^{-1}, \tag{143}$$

with the Poission equation (122),

$$\partial_i\left(\frac{\mathrm{d}^2\phi^i}{\mathrm{d}\tau^2} + aH\frac{\mathrm{d}\phi^i}{\mathrm{d}\tau}\right) = -\nabla^2\psi = -\frac{3}{2}a^2H^2\Omega_m(a)\delta = \frac{3}{2}a^2H^2\Omega_m(a)\left[1 - J(\boldsymbol{q},\ \tau)^{-1}\right], \tag{144}$$

as $\boldsymbol{\phi}(\boldsymbol{q},\ \tau) = \boldsymbol{x} - \boldsymbol{q}$, there is $d^3\boldsymbol{\phi} = d^3x - d^3q$, $\partial_i\phi^i = 1 - J(\boldsymbol{q},\ \tau)^{-1}$, (???) then Eq. (144) can be approximated (to the 1st-order accuracy) as

$$\frac{\mathrm{d}^2(\partial_i\phi^i)}{\mathrm{d}\tau^2} + aH\frac{\mathrm{d}(\partial_i\phi^i)}{\mathrm{d}\tau} = \frac{3}{2}a^2H^2\Omega_m(a)\partial_i\phi^i. \tag{145}$$

Recall Eq. (125), we may express the form of $\partial_i\phi^i$ as $\partial_i\phi^i(\boldsymbol{q},\ \tau) = -D_1(\tau)\delta(\boldsymbol{q})$.

1.6 The (Extended) Press-Schechter Theory

Assuming spherical collapse, $R(\tau) = r - u(r,\ \tau) = [1-\alpha(\tau)]r$, when $\alpha \ll 1$, there is (linear growth equation) (I do not know how to derive it...)

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}\tau^2} + aH\frac{\mathrm{d}\alpha}{\mathrm{d}\tau} = \frac{3}{2}a^2H^2\Omega_m(a)\alpha. \tag{146}$$

tophat filter of radius

$$R(M) = \left(\frac{3M}{4\pi\bar{\rho}}\right)^{\frac{1}{3}}, \tag{147}$$

with the corresponding variance of the Gaussian random field, $\sigma^2(M)$. Critical value of the Halo formation: $\delta_c \approx 1.68$. The fraction in halos $> M$ is

$$\frac{1}{\sqrt{2\pi}\sigma(M)}\int_{\delta_c}^{\infty} d\delta \exp\left[-\frac{\delta^2}{2\sigma^2(M)}\right] = \frac{1}{2}\mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right), \qquad \left(\text{Notice that } \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} dxe^{-x^2} = 1\right) \tag{148}$$

where

$$\nu \equiv \frac{\delta_c}{\sigma(M)}, \qquad \mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-t^2) dt. \tag{149}$$

Differentiate in M to find fraction in range dM to get the halo number density

$$VMdn = \frac{1}{2} V \rho_m \left| \frac{\mathrm{d}}{\mathrm{d}M} \mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right) \right| dM, \tag{150}$$

which leads to

$$\begin{aligned} \left| \frac{\mathrm{d}n}{\mathrm{d}M} M \right| = \left| \frac{\mathrm{d}n}{\mathrm{d}\log M} \right| &= \frac{1}{2} \frac{\rho_m}{M} \left| \frac{\mathrm{d}}{\mathrm{d}\log M} \mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right) \right| \\ &= \frac{1}{2} \frac{\rho_m}{M} \frac{1}{\sqrt{2}} \left| \frac{\mathrm{d}\nu}{\mathrm{d}\log M} \frac{\mathrm{d}}{\mathrm{d}(\nu/\sqrt{2})} \mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right) \right| \\ &= \frac{1}{2} \frac{\rho_m}{M} \sqrt{\frac{2}{\pi}} \left| -\frac{\mathrm{d}\nu}{\mathrm{d}\log M} \exp\left(-\frac{\nu^2}{2}\right) \right|, \end{aligned} \tag{151}$$

with

$$\frac{\mathrm{d}\nu}{\mathrm{d}\log M} = \nu \frac{\mathrm{d}\log\nu}{\mathrm{d}\log M} = \nu \frac{\mathrm{d}\log\sigma^{-1}(M)}{\mathrm{d}\log M} > 0, \tag{152}$$

then

$$\left| \frac{\mathrm{d}n}{\mathrm{d}\log M} \right| = \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{\rho_m}{M} \frac{\mathrm{d}\log\sigma^{-1}}{\mathrm{d}\log M} \nu \exp\left(-\frac{\nu^2}{2}\right). \tag{153}$$

The form $P(k) \propto k^n$,

$$\sigma^2(M) \propto \int_0^{M^{-\frac{1}{3}}} dk\ k^2 P(k) \propto \int_0^{M^{-\frac{1}{3}}} dk\ k^{n+2} \propto M^{-\frac{n+3}{3}}, \tag{154}$$

$\nu = \delta_c/\sigma(M) = (M/M_*)^{\frac{n+3}{6}}\sqrt{2}$, where M_* is the critical mass for structure formation. Then

$$\begin{aligned} \left| \frac{\mathrm{d}n}{\mathrm{d}M} \right| &= \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{\rho_m}{M^2} \frac{n+3}{6} \left(\frac{M}{M_*}\right)^{\frac{n+3}{6}} \sqrt{2} \exp\left[-\left(\frac{M}{M_*}\right)^{\frac{n+3}{3}}\right] \\ &= \frac{1}{2} \frac{1}{\sqrt{\pi}} \frac{\rho_m}{M^2} \frac{n+3}{3} \left(\frac{M}{M_*}\right)^{\frac{n+3}{6}} \exp\left[-\left(\frac{M}{M_*}\right)^{\frac{n+3}{3}}\right]. \end{aligned} \tag{155}$$

There is a factor 1/2 in Eq. (155) related with "cloud-in-cloud" problem. Now we define

$$S \equiv \sigma_R^2(M) = \left\langle \delta_R^2 \right\rangle, \tag{156}$$

there is the survival probability (not forming Halo)

$$P_f(\delta,\ S) d\delta = \frac{1}{\sqrt{2\pi S}} \left\{ \exp\left(-\frac{\delta^2}{2S}\right) - \exp\left[-\frac{(2\delta_c - \delta)^2}{2S}\right] \right\} d\delta, \tag{157}$$

for the first crossing (δ_c) distribution (at S), as $S - M$ are correlated, we can define the distribution function $f(S)$ and

$$MN(M)dM = M \left| \frac{\mathrm{d}n}{\mathrm{d}M} \right| dM = \bar{\rho} f(S) dS, \tag{158}$$

which depicts the total mass per volume, of the Halo at mass M, and then

$$f(S)dS = 1 - \int_{-\infty}^{\delta_c} P_f(\delta,\ S) d\delta \rightarrow f(S) = -\frac{\mathrm{d}}{\mathrm{d}S} \int_{-\infty}^{\delta_c} P_f(\delta,\ S) d\delta, \tag{159}$$

with

$$\frac{1}{\sqrt{2\pi S}} \int_{-\infty}^{\delta_c} \exp\left(-\frac{\delta^2}{2S}\right) d\delta = -\frac{1}{2} \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\delta_c} \exp\left(-\frac{\delta^2}{2S}\right) d\left(-\frac{\delta}{\sqrt{2S}}\right) = \frac{1}{2} \mathrm{erfc}\left(-\frac{\delta_c}{\sqrt{2S}}\right), \tag{160}$$

$$\frac{1}{\sqrt{2\pi S}}\int_{-\infty}^{\delta_c}\exp\left[-\frac{(2\delta_c-\delta)^2}{2S}\right]d\delta=-\frac{1}{2}\frac{2}{\sqrt{\pi}}\int_{-\infty}^{\delta_c}\exp\left[-\frac{(2\delta_c-\delta)^2}{2S}\right]d\left(-\frac{\delta-2\delta_c}{\sqrt{2S}}\right)=\frac{1}{2}\mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2S}}\right),\tag{161}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}S}\left[\mathrm{erfc}\left(-\frac{\delta_c}{\sqrt{2S}}\right)\right]=\frac{2}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}S}\int_{-\infty}^{\frac{\delta_c}{\sqrt{2S}}}\exp(-t^2)dt=-\frac{\delta_c}{\sqrt{2\pi S^3}}\exp\left(-\frac{\delta_c^2}{2S}\right),\tag{162}$$

$$\frac{\mathrm{d}}{\mathrm{d}S}\left[\mathrm{erfc}\left(\frac{\delta_c}{\sqrt{2S}}\right)\right]=\frac{2}{\sqrt{\pi}}\frac{\mathrm{d}}{\mathrm{d}S}\int_{-\infty}^{-\frac{\delta_c}{\sqrt{2S}}}\exp(-t^2)dt=-\frac{\delta_c}{\sqrt{2\pi S^3}}\exp\left(-\frac{\delta_c^2}{2S}\right).\tag{163}$$

Thus,

$$f(S)=-\frac{1}{2}\cdot 2\left[-\frac{\delta_c}{\sqrt{2\pi S^3}}\exp\left(-\frac{\delta_c^2}{2S}\right)\right]=\frac{\delta_c}{\sqrt{2\pi S^3}}\exp\left(-\frac{\delta_c^2}{2S}\right).\tag{164}$$

With Eq. (158), and $S\equiv\sigma^2(M)$,

$$\begin{aligned}N(M)=\left|\frac{\mathrm{d}n}{\mathrm{d}M}\right|&=\frac{\bar{\rho}}{M}f(S)\left|\frac{\mathrm{d}S}{\mathrm{d}M}\right|\\&=\frac{\bar{\rho}}{M}\cdot\frac{\delta_c}{\sqrt{2\pi S^3}}\exp\left(-\frac{\delta_c^2}{2S}\right)\left|\frac{\mathrm{d}S}{\mathrm{d}M}\right|\\&=\sqrt{\frac{2}{\pi}}\frac{\bar{\rho}}{M^2}\frac{\delta_c}{\sigma(M)}\frac{\mathrm{d}\log\sigma^{-1}}{\mathrm{d}\log M}\exp\left[-\frac{\delta_c^2}{2\sigma^2(M)}\right],\end{aligned}\tag{165}$$

compared with Eq. (153), the factor 1/2 has disappeared, which thanks to the revision (second) term in the survival probability Eq. (157). The spatial bias coefficient b_{h} of Halos, in the relation

$$\delta_{\mathrm{h}}=\frac{\rho_{\mathrm{h}}-\bar{\rho}_{\mathrm{h}}}{\bar{\rho}_{\mathrm{h}}}=b_{\mathrm{h}}\delta=b_{\mathrm{h}}\frac{\rho-\bar{\rho}}{\rho}.\tag{166}$$

If the "observer" is at an overdensed position $(\delta_0,\ S_0)$, the first-crossing distribution function should be moved from the original point $(0,\ 0)$ to $(\delta_0,\ S_0)$, then

$$f(S|\delta_0,\ S_0)dS=\frac{\delta_c-\delta_0}{\sqrt{2\pi(S-S_0)^3}}\exp\left[-\frac{(\delta_c-\delta_0)^2}{2(S-S_0)}\right]dS,\tag{167}$$

$$n(M|\delta_0,\ S_0)dS=\frac{\bar{\rho}(1+\delta_0)}{M}\frac{\mathrm{d}S}{\mathrm{d}M}f(S|\delta_0,\ S_0),\tag{168}$$

and then

$$\begin{aligned}\delta_{\mathrm{h}}=\frac{n(M|\delta_0,\ S_0)-n(M)}{n(M)}&=\frac{\mathrm{d}[\log n(M|\delta_0,\ S_0)]}{\mathrm{d}\delta_0}\\&=\delta_0\left\{\frac{1}{1+\delta_0}+\frac{\mathrm{d}[\log f(M|\delta_0,\ S_0)]}{\mathrm{d}\delta_0}\right\}\\&\approx\delta_0\left\{1+\frac{\mathrm{d}[\log f(M|\delta_0,\ S_0)]}{\mathrm{d}\delta_0}\right\}\\&=\delta_0\left\{1+\frac{\mathrm{d}[\log(\delta_c-\delta_0)]}{\mathrm{d}\delta_0}+\frac{\delta_c-\delta_0}{S-S_0}\right\}\\&\approx\delta_0\left(1-\frac{1}{\delta_c}+\frac{\delta_c}{S}\right)=\delta_0\left(1+\frac{\nu^2-1}{\delta_c}\right),\qquad\left(\nu\equiv\frac{\delta_c}{\sqrt{S}}\right),\end{aligned}\tag{169}$$

where we have used the fact that $\delta_0\ll 1,\ S_0\ll S$ (large scale). For $\nu=1$, $b_{\mathrm{h}}=1+(\nu^2-1)/\delta_c=1$.