Cosmological Physics Note

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Grand Unified Theory of the Cosmology:

$$\boldsymbol{F} = m\boldsymbol{a}$$

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1 Formula Part

1.1 Robert-Walker (flat) metric

The metric form

$$g_{\mu\nu} = \mathbf{diag}\{-1,\ a^2(t),\ a^2(t),\ a^2(t)\}, \qquad g^{\mu\nu} = \mathbf{diag}\{-1,\ a^{-2}(t),\ a^{-2}(t),\ a^{-2}(t)\}. \tag{1}$$

Behaviour of the energy-momentum

$$(E,\ p^1,\ p^2,\ p^3) = \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda},\ \frac{\mathrm{d}x^1}{\mathrm{d}\lambda},\ \frac{\mathrm{d}x^2}{\mathrm{d}\lambda},\ \frac{\mathrm{d}x^3}{\mathrm{d}\lambda}\right) = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}, \tag{2}$$

$$\frac{\mathrm{d}^2 x^\beta}{\mathrm{d}\lambda^2} + \Gamma^\beta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = 0,\ \Gamma^\beta_{\mu\nu} = \frac{1}{2}g^{\beta\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}), \tag{3}$$

the nonzero terms of $\Gamma^\beta_{\mu\nu}$ are

$$\Gamma^i_{j0} = \frac{1}{2}g^{i\rho}(g_{\rho j,0} + g_{\rho 0,j} - g_{j0,\rho}) = \frac{1}{2}g^{ii}(g_{ij,0} + g_{i0,j} - g_{j0,i}) = \frac{1}{2}[a^2(t)]^{-1}\delta^i_j \cdot 2a(t)\dot{a} = \frac{\dot{a}}{a}\delta^i_j = \Gamma^i_{0j}, \tag{4}$$

$$\Gamma^0_{ij} = \frac{1}{2}g^{0\rho}(g_{\rho i,j} + g_{\rho j,i} - g_{ij,\rho}) = -\frac{1}{2}\cdot(-g_{ij,0}) = a\dot{a}\delta_{ij}. \tag{5}$$

For $\beta = 0$ component in Eq.(3),

$$\frac{\mathrm{d}E}{\mathrm{d}\lambda} + \Gamma^0_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = \frac{\mathrm{d}E}{\mathrm{d}\lambda} + \Gamma^0_{ij}p^i p^j = 0, \tag{6}$$

with $\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}}{\mathrm{d}t} = E\frac{\mathrm{d}}{\mathrm{d}t}$,

$$E\frac{\mathrm{d}E}{\mathrm{d}t} + a\dot{a}\delta_{ij}p^i p^j = 0. \tag{7}$$

For a massless particle,

$$0 = g_{\mu\nu}p^\mu p^\nu \rightarrow -E^2 + \delta_{ij}a^2 p^i p^j = 0, \tag{8}$$

with Eq.(7), there is

$$E\frac{\mathrm{d}E}{\mathrm{d}t} + \frac{\dot{a}}{a}E^2 = 0 \rightarrow \frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{\dot{a}}{a}E. \qquad (E \propto a^{-1}) \tag{9}$$

For $\beta = i$ components in Eq.(3),

$$\frac{\mathrm{d}p^i}{\mathrm{d}\lambda} + 2\Gamma^i_{0j}Ep^j = \frac{\mathrm{d}p^i}{\mathrm{d}\lambda} + 2\frac{\dot{a}}{a}\delta^i_j Ep^j = \frac{\mathrm{d}p^i}{\mathrm{d}\lambda} + 2\frac{\dot{a}}{a}Ep^i = E\frac{\mathrm{d}p^i}{\mathrm{d}t} + 2\frac{\dot{a}}{a}Ep^i = 0, \tag{10}$$

which leads to

$$\frac{\mathrm{d}p^i}{\mathrm{d}t} = -2\frac{\dot{a}}{a}p^i. \qquad (p^i \propto a^{-2}) \tag{11}$$

Note that the massless particle condition is not used for $\beta = i$ components.

Einstein equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \qquad R = g^{\mu\nu}R_{\mu\nu}. \tag{12}$$

The Ricci tensor

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\alpha,\nu} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}, \tag{13}$$

the components

$$R_{00} = -3\frac{\ddot{a}}{a}, \qquad R_{0i} = R_{i0} = 0, \qquad R_{ij} = (2\dot{a}^2 + a\ddot{a})\delta_{ij}, \tag{14}$$

Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} = 3\frac{\ddot{a}}{a} + 3\left(\frac{\ddot{a}}{a}\right) + 6\left(\frac{\dot{a}}{a}\right)^2 = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]. \tag{15}$$

Also notice that Ricci tensor can be proved to be symmetrical with
$$R_{\mu\nu} = g^{\lambda\rho} R_{\rho\mu\lambda\nu} = g^{\lambda\rho} R_{\lambda\nu\rho\mu} = g^{\rho\lambda} R_{\lambda\nu\rho\mu} = R^{\rho}_{\nu\rho\mu} = R_{\nu\mu}. \tag{16}$$

It can also be proved with Eq.(13), and the only term that is not obviously symmetrical is $\Gamma^{\alpha}_{\mu\alpha,\nu}$, expand it as

$$\Gamma^{\alpha}_{\mu\alpha,\nu} = \frac{1}{2}g^{\alpha\rho}_{,\nu}g_{\rho\mu,\alpha} + \frac{1}{2}g^{\alpha\rho}g_{\rho\mu,\alpha,\nu} + \frac{1}{2}g^{\alpha\rho}_{,\nu}g_{\rho\alpha,\mu} + \frac{1}{2}g^{\alpha\rho}g_{\rho\alpha,\mu,\nu} - \frac{1}{2}g^{\alpha\rho}_{,\nu}g_{\mu\alpha,\rho} - \frac{1}{2}g^{\alpha\rho}g_{\mu\alpha,\rho,\nu}, \tag{17}$$

notice that the fourth term is obviously symmetrical. Look at the third term,

$$\frac{1}{2}g^{\alpha\rho}_{,\nu}g_{\rho\alpha,\mu} = -\frac{1}{2}g^{\alpha\sigma}g^{\rho\tau}g_{\sigma\tau,\nu}g_{\rho\alpha,\mu} = \frac{1}{2}g^{\sigma\tau}_{,\mu}g_{\sigma\tau,\nu} = \frac{1}{2}g^{\alpha\rho}_{,\mu}g_{\rho\alpha,\nu}, \tag{18}$$

so the third term is also symmetrical. Look at the first and the fifth terms,

$$\frac{1}{2}g^{\alpha\rho}_{,\nu}g_{\rho\mu,\alpha} = \frac{1}{2}g^{\alpha\rho}_{,\nu}g_{\mu\rho,\alpha} = \frac{1}{2}g^{\rho\alpha}_{,\nu}g_{\mu\alpha,\rho} = \frac{1}{2}g^{\alpha\rho}_{,\nu}g_{\mu\alpha,\rho}, \tag{19}$$

so they can offset each other. The same for the second and the sixth terms,

$$\frac{1}{2}g^{\alpha\rho}g_{\rho\mu,\alpha,\nu} = \frac{1}{2}g^{\rho\alpha}g_{\alpha\mu,\rho,\nu} = \frac{1}{2}g^{\alpha\rho}g_{\mu\alpha,\rho,\nu}, \tag{20}$$

so they can also offset each other. $\Gamma^{\alpha}_{\mu\alpha,\nu}$ only has two symmetrical terms to be considered, so $\Gamma^{\alpha}_{\mu\alpha,\nu} = \Gamma^{\alpha}_{\nu\alpha,\mu}$.

Energy-momentum tensor
$$T^{\mu}_{\nu} = \mathbf{diag}(-\rho, \ P, \ P, \ P), \tag{21}$$
$$T_{\mu\nu} = g_{\mu\alpha}T^{\alpha}_{\nu} = \mathbf{diag}(\rho, \ a^2 P, \ a^2 P, \ a^2 P), \quad T^{\mu\nu} = g^{\mu\alpha}T^{\nu}_{\alpha} = \mathbf{diag}(\rho, \ a^{-2}P, \ a^{-2}P, \ a^{-2}P). \tag{22}$$

Consider
$$R_{00} - \frac{1}{2}g_{00}R = 8\pi G T_{00} \to -3\frac{\ddot{a}}{a} + \frac{1}{2}\times 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] = 3\left(\frac{\dot{a}}{a}\right)^2 = 8\pi G\rho \to \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho, \tag{23}$$

this is Friedmann equation. Consider
$$R_{ii} - \frac{1}{2}g_{ii}R = 8\pi G T_{ii}, \tag{24}$$

this will lead to
$$(2\dot{a}^2 + a\ddot{a})\delta_{ii} - \frac{1}{2}a^2 \times 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] = -2a\ddot{a} - \dot{a}^2 = 8\pi G a^2 P \to \frac{\dot{a}^2}{a^2} + \frac{2\ddot{a}}{a} = -8\pi G P \to \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(3P + \rho), \tag{25}$$

this is No.2 Friedmann equation.

The conservation of energy-momentum tensor
$$0 = T^{\mu}_{\nu;\mu} = \partial_{\mu}T^{\mu}_{\nu} + \Gamma^{\mu}_{\alpha\mu}T^{\alpha}_{\nu} - \Gamma^{\alpha}_{\nu\mu}T^{\mu}_{\alpha}. \tag{26}$$

For $\nu = 0$ component,
$$T^{\mu}_{0;\mu} = -\partial_t\rho - 3\frac{\dot{a}}{a}(\rho + P) = 0 \to \partial_t\rho + 3\frac{\dot{a}}{a}(\rho + P) = 0, \tag{27}$$

it can be checked with $dE + PdV = 0$, with $E = \rho V$ $(V = a^3)$, then
$$dE + PdV = \rho dV + Vd\rho + PdV = 0 = Vd\rho + (P + \rho)dV \to d\rho + (P + \rho)d(\log V) = 0, \tag{28}$$

so
$$d\rho + 3(P + \rho)d(\log a) = 0 \to \partial_t\rho + 3\frac{\dot{a}}{a}(\rho + P) = 0. \tag{29}$$

They are the same, verified.

For non-relativistic matter, $P = 0$ $(v \ll c)$,
$$\partial_t\rho = -3\frac{\dot{a}}{a}\rho \to \rho \propto a^{-3}. \tag{30}$$

For relativistic energy (radiation), $P = (1/3)\rho$,
$$\partial_t\rho = -4\frac{\dot{a}}{a}\rho \to \rho \propto a^{-4}, \tag{31}$$

Define Hubble parameter $H \equiv \dot{a}/a = \dot{R}/R$, $(a(t) = R(t)/R_0, \ \dot{a} = \dot{R}/R_0)$, with $H_0 = h \cdot 100 \ \mathrm{km} \cdot \mathrm{s}^{-1} \cdot \mathrm{Mpc}^{-1}$, with $h = 0.7$ in code.

1.2 General Robert-Walker metric

Rewrite the RW metric

$$ds^2 = -c^2 dt^2 + a^2(t) \left[dr^2 + R_0^2 S_k^2 \left(\frac{r}{R_0} \right) (d\theta^2 + \sin^2 \theta d\varphi^2) \right]. \tag{32}$$

with $a(t) = R(t)/R_0$. The function

$$S_k(\chi) = \sin \chi \ (k = 1), \ \sinh \chi \ (k = -1), \ \chi \ (k = 0, \ \text{flat}). \tag{33}$$

This will lead to

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{a^2 R_0^2}, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2} (3P + \rho c^2). \tag{34}$$

Define the critical density

$$\rho_c = \frac{3H^2}{8\pi G} \approx 1.88 \times 10^{-26} \ h^2 \cdot \text{kg} \cdot \text{m}^{-3} = 2.78 \times 10^{11} \ h^2 \ M_\odot \ \text{Mpc}^{-3}. \tag{35}$$

There is

$$\frac{H^2}{H_0^2} = \frac{8\pi G}{3H_0^2} \rho - \frac{kc^2}{a^2 R_0^2 H_0^2} = \Omega_{m0} a^{-3} + \Omega_{r0} a^{-4} + \Omega_{\Lambda 0} + \Omega_{k0} a^{-2}, \tag{36}$$

when $a = 1$, $H = H_0$, so

$$\Omega_{m0} + \Omega_{r0} + \Omega_{\Lambda 0} + \Omega_{k0} \equiv \Omega_0 + \Omega_{k0} = 1, \qquad \Omega_{m0} = \frac{\rho_{m0}}{\rho_{c0}} \ \text{(the same for other three)}, \tag{37}$$

then

$$1 = \frac{\rho(a)}{\rho_c(a)} - \frac{kc^2}{a^2 H^2 R_0^2} = \Omega(a) - \frac{kc^2}{a^2 H^2 R_0^2}, \tag{38}$$

in another way, $a(t = 0) = 1$,

$$H_0^2 = \frac{8\pi G \rho_0}{3} - \frac{kc^2}{R_0^2} \rightarrow R_0 = \frac{c}{H_0} \left(\frac{k}{\Omega_0 - 1} \right)^{\frac{1}{2}}. \tag{39}$$

$$\Omega_m(a) = \frac{\rho_m(a)}{\rho_c(a)} = \frac{\Omega_{m0} a^{-3}}{\Omega_{m0} a^{-3} + \Omega_{r0} a^{-4} + \Omega_{\Lambda 0} + (1 - \Omega_0) a^{-2}} \equiv \frac{\Omega_{m0} a^{-3}}{f(a)}, \tag{40}$$

and for radiation and dark matter terms,

$$\Omega_r(a) = \frac{\Omega_{r0} a^{-4}}{f(a)}, \qquad \Omega_\Lambda(a) = \frac{\Omega_{\Lambda 0}}{f(a)}. \tag{41}$$

And for the curvature term,

$$1 - \Omega(a) = \frac{(1 - \Omega_0) a^{-2}}{f(a)} \rightarrow 0 \ (\text{for } a \rightarrow 0), \tag{42}$$

which means the flatness in early universe.

The deceleration parameter

$$q \equiv -\frac{\ddot{R} R}{\dot{R}^2} = -\frac{\ddot{a} a}{\dot{a}^2} = -\frac{\ddot{R}}{R} \left(\frac{R}{\dot{R}} \right)^2 = \frac{4\pi G}{3c^2} (3P + \rho c^2) \cdot \frac{1}{H^2} = \frac{1}{2} \cdot \frac{8\pi G}{3H^2} \left(\rho + 3\frac{P}{c^2} \right), \tag{43}$$

for non-relativistic matter, $P = 0$, for radiation matter, $P = \rho c^2/3$, for dark matter, $P = -\rho c^2$, so

$$q = \frac{1}{2} \cdot \frac{1}{\rho_c} (\rho_m + \rho_r + \rho_\Lambda + \rho_r - 3\rho_\Lambda) = \frac{\Omega_m}{2} + \Omega_r - \Omega_\Lambda. \tag{44}$$

And if the universe is flat, $\Omega_\Lambda = 1 - \Omega_m - \Omega_r$, then $q = 3\Omega_m/2 + 2\Omega_r - 1$.

• The discussion on the equivalence of two forms of the RW metric:

The first form is as Eq. (32), the other form is

$$ds^2 = -c^2 dt^2 + a^2(t) \left[\frac{1}{1 - k(r/R_0)^2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right]. \tag{45}$$

- For $k = 1$, the two forms of the metric are ($c = 1$)

$$ds^2 = -dt^2 + a^2(t) \left[dr^2 + R_0^2 \sin^2 \left(\frac{r}{R_0} \right) d\Omega^2 \right], \tag{46}$$

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - (r/R_0)^2} + r^2 d\Omega^2 \right]. \tag{47}$$

Defining $\chi \equiv R_0 \sin(r/R_0)$, then $d\chi = \cos(r/R_0)dr$, the first form can be rewritten as

$$ds^2 = -dt^2 + a^2(t) \left[\frac{d\chi^2}{1 - (\chi/R_0)^2} + \chi^2 d\Omega^2 \right]. \qquad (\sin^2 x + \cos^2 x = 1) \tag{48}$$

So the two forms are equivalent.
- For $k = -1$, the two forms of the metric are

$$ds^2 = -dt^2 + a^2(t) \left[dr^2 + R_0^2 \sinh^2 \left(\frac{r}{R_0} \right) d\Omega^2 \right], \tag{49}$$

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 + (r/R_0)^2} + r^2 d\Omega^2 \right]. \tag{50}$$

Defining $\chi \equiv R_0 \sinh(r/R_0)$, then $d\chi = \cosh(r/R_0)dr$, the first form can be rewritten as

$$ds^2 = -dt^2 + a^2(t) \left[\frac{d\chi^2}{1 + (\chi/R_0)^2} + \chi^2 d\Omega^2 \right]. \qquad (\cosh^2 x - \sinh^2 x = 1) \tag{51}$$

So the two forms are equivalent.
- For $k = 0$, the two forms of the metric are apparently the same.
Then we can derive the Friedmann equations with spatial curvature, with the metric form of Eq. (47). The nonzero connection components are

$$\Gamma^0_{11} = \frac{a\dot{a}}{1 - k(r/R_0)^2}, \ \Gamma^0_{22} = r^2 a\dot{a}, \ \Gamma^0_{33} = r^2 \sin^2 \theta a\dot{a}, \ \Gamma^1_{01} = \frac{\dot{a}}{a} = \Gamma^1_{10}, \ \Gamma^1_{11} = -\frac{kr}{kr^2 - R_0^2},$$

$$\Gamma^1_{22} = -r + \frac{kr^3}{R_0^2}, \ \Gamma^1_{33} = -r \sin^2 \theta \left[1 - k(r/R_0)^2 \right], \ \Gamma^2_{02} = \frac{\dot{a}}{a} = \Gamma^2_{20}, \ \Gamma^2_{12} = \frac{1}{r} = \Gamma^2_{21}, \tag{52}$$

$$\Gamma^2_{33} = -\sin \theta \cos \theta, \ \Gamma^3_{03} = \frac{\dot{a}}{a} = \Gamma^3_{30}, \ \Gamma^3_{13} = \frac{1}{r} = \Gamma^3_{31}, \ \Gamma^3_{23} = \cot \theta = \Gamma^3_{32}.$$

Then

$$R_{00} = -3\frac{\ddot{a}}{a}, \ R_{11} = \frac{a\ddot{a}}{1 - k(r/R_0)^2} - \frac{2(k + R_0^2 \dot{a}^2)}{kr^2 - R_0^2},$$

$$R_{22} = r^2(2\dot{a}^2 + a\ddot{a}) + 2k \left(\frac{r}{R_0} \right)^2, \tag{53}$$

$$R_{33} = r^2 \sin^2 \theta (2\dot{a}^2 + a\ddot{a}) + 2k \left(\frac{r}{R_0} \right)^2 \sin^2 \theta.$$

And the Ricci scalar

$$R = 6 \left[\frac{k}{a^2 R_0^2} + \left(\frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} \right]. \tag{54}$$

With Einstein equation,

$$G_{00} = R_{00} - \frac{1}{2} g_{00} R = 3 \left[\left(\frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2 R_0^2} \right] = 8\pi G \rho, \tag{55}$$

which leads to the first Friedmann equation

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2 R_0^2}. \tag{56}$$

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With

$$G_{11} = R_{11} - \frac{1}{2}g_{11}R = 8\pi G \frac{a^2}{1 - k(r/R_0)^2}P, \tag{57}$$

then

$$\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\ddot{a}}{a} + \frac{k}{a^2 R_0^2} = -8\pi GP \rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P). \tag{58}$$

which is the second Friedmann equation.

1.3 Redshift

For a photon,

$$0 = ds^2 = -dt^2 + a^2(t)dx^2. \tag{59}$$

If at $x = x_0$, the source emitted two photons at t_1, t_2, and the receiver at $x = x_1$ received them at t_1', t_2', then

$$\int_{x_0}^{x_1} dx = \int_{t_1}^{t_1'} \frac{dt}{a(t)} = \int_{t_2}^{t_2'} \frac{dt}{a(t)} \rightarrow \int_{t_1}^{t_2} \frac{dt}{a(t)} = \int_{t_1'}^{t_2'} \frac{dt}{a(t)}, \tag{60}$$

then

$$\frac{\Delta t}{a(t_1)} = \frac{\Delta t'}{a(t_1')}, \tag{61}$$

if $a(t_1) < a(t_1')$, then $\Delta t < \Delta t'$. The frequency and wave length

$$\frac{\nu(t_1')}{\nu(t_1)} = \frac{\lambda(t_1)}{\lambda(t_1')} = \frac{\Delta t}{\Delta t'} = \frac{a(t_1)}{a(t_1')}, \tag{62}$$

and the redshift relation

$$\lambda_{\text{receive}} = \frac{\lambda_{\text{emit}}}{a(t_{\text{emit}})} \cdot a(t_{\text{receive}}), \tag{63}$$

$t_{\text{receive}} = t_{\text{today}}$, $a = 1$, so

$$\lambda_{\text{receive}} = \frac{\lambda_{\text{emit}}}{a(t_{\text{emit}})} = \lambda_{\text{emit}}(1 + z), \qquad a(t_{\text{emit}}) = \frac{1}{1 + z}. \tag{64}$$

• Distance-Redshift relation, for the photons

$$0 = ds^2 = -dt^2 + a^2(t)dx^2 \rightarrow dx = \frac{dt}{a(t)} \qquad (dx = |dx|) \tag{65}$$

with

$$\frac{\dot{a}}{a} = H \rightarrow dt = \frac{1}{H}d(\log a) = -\frac{1}{H}d\left[\log(1 + z)\right] = -\frac{dz}{1 + z}\frac{1}{H(z)}, \tag{66}$$

then

$$-\int_{x_0}^{0} dx = \int_{0}^{t} \frac{dt'}{a(t')} = -\int_{z_0}^{0} \frac{dz'}{1 + z'}\frac{1}{H(z')} \cdot \frac{1}{a(z')} = \int_{0}^{z_0} dz' \frac{1}{H(z')}. \tag{67}$$

If $z_0 \ll 1$,

$$x_0 = \int_{0}^{z_0} \frac{dz'}{H(z')} \approx \frac{z_0}{H_0}(c = 1). \tag{68}$$

With Eq. (36), there is

$$x_0 = \frac{1}{H_0}\int_{0}^{z_0} \frac{dz}{\sqrt{\Omega_{\Lambda 0} + \Omega_{m0}(1 + z)^3 + \Omega_{\gamma 0}(1 + z)^4 + (1 - \Omega_0)(1 + z)^2}}. \tag{69}$$

• Angular-Diameter Distance D_A, the proper transverse size of an object seen by us is

$$D_{\text{proper}} = R_0 S_k\left(\frac{r}{R_0}\right)d\theta, \tag{70}$$

and its proper size at the time of light emission is (as $a(t)$ has expanded by $(1+z)$)

$$\frac{1}{1+z}R_0 S_k\left(\frac{r}{R_0}\right)d\theta \equiv D_A d\theta, \tag{71}$$

where D_A is the angular diameter distance.

• Luminosity Distance, notice four points:

1. The photon energy will decrease by $(1+z)$;
2. The arrival rate of the photons will decrease by $(1+z)$;
3. The bandwidth of photon frequency of the photons will decrease by $(1+z)$;
4. The photon frequency will be redshifted.

The total flux

$$S_{\text{tot}} = \int_0^\infty S_0(\nu_0)d\nu_0 = \frac{1}{4\pi R_0^2 S_k^2(r/R_0)(1+z)^2}\int_0^\infty L_{\nu'}d\nu' = \frac{L_{\text{tot}}}{4\pi R_0^2 S_k^2(r/R_0)(1+z)^2} \equiv \frac{L_{\text{tot}}}{4\pi D_L^2}, \tag{72}$$

where D_L is the luminosity distance, and the $(1+z)^2$ in the denominator is from 1., 2. effects above,

$$\nu' = (1+z)\nu_0, \qquad D_L \equiv R_0 S_k\left(\frac{r}{R_0}\right)(1+z). \tag{73}$$

The redshift-drift, emission time t_1, receiving time t_2, after a small time interval, at $t_1 + \Delta t_1$ there is another emission and received at $t_2 + \Delta t_2$. As the co-moving distance is the same,

$$\int_{x_1}^{x_2} dx = \int_{t_1}^{t_2}\frac{dt}{a(t)} = \int_{t_1+\Delta t_1}^{t_2+\Delta t_2}\frac{dt}{a(t)} \rightarrow \frac{\Delta t_2}{a(t_2)} = \frac{\Delta t_1}{a(t_1)}. \tag{74}$$

There is the relation

$$z(t_2) = \frac{a(t_2)}{a(t_1)} - 1, \qquad z(t_2+\Delta t_2) = \frac{a(t_2+\Delta t_2)}{a(t_1+\Delta t_1)} - 1, \tag{75}$$

then

$$\begin{aligned}
z(t_2+\Delta t_2) - z(t_2) &\approx \dot{z}(t_2)\Delta t_2 \\
&= \frac{a(t_2+\Delta t_2)}{a(t_1+\Delta t_1)} - \frac{a(t_2)}{a(t_1)} \approx \frac{a(t_2)+\dot{a}(t_2)\Delta t_2}{a(t_1)+\dot{a}(t_1)\Delta t_1} - \frac{a(t_2)}{a(t_1)} \\
&= \frac{a(t_2)}{a(t_1)}\left[\frac{1+H(t_2)\Delta t_2}{1+H(t_1)\Delta t_1} - 1\right] \approx \frac{a(t_2)}{a(t_1)}\Delta t_2\left[H(t_2) - H(t_1)\frac{\Delta t_1}{\Delta t_2}\right].
\end{aligned} \tag{76}$$

Where the fact that $H(t_1)\Delta t_1 \ll 1$ is used. If t_2 is today, then $a(t_2) = 1$, $H(t_2) = H_0$, $a(t_1) = (1+z)^{-1}$. Then the equation above can be rewritten as

$$\dot{z}(z) = (1+z)H_0 - H(z) = H_0\left[1 + z - \sqrt{\Omega_{m0}(1+z)^3 + \Omega_{\gamma_0}(1+z)^4 + \Omega_{\Lambda 0} + \Omega_{k0}(1+z)^2}\right]. \tag{77}$$

1.4   The Metric Perturbation

Under the Newtonian gauge, the metric becomes as

$$g_{\mu\nu} = \textbf{diag}\{-[1+2\psi(t,\ \boldsymbol{x})],\ a^2[1+2\phi(t,\ \boldsymbol{x})],\ a^2[1+2\phi(t,\ \boldsymbol{x})],\ a^2[1+2\phi(t,\ \boldsymbol{x})]\}, \tag{78}$$

$$g^{\mu\nu} = \textbf{diag}\{-1+2\psi(t,\ \boldsymbol{x}),\ a^{-2}[1-2\phi(t,\ \boldsymbol{x})],\ a^{-2}[1-2\phi(t,\ \boldsymbol{x})],\ a^{-2}[1-2\phi(t,\ \boldsymbol{x})]\}. \tag{79}$$

The perturbation terms $\psi(t,\ \boldsymbol{x})$, $\phi(t,\ \boldsymbol{x})$ are treated as 1st-order small values. The nonzero connection components to the leading order are

$$\Gamma_{00}^0 = \frac{-1+2\psi}{2}g_{00,0} = \frac{-1+2\psi}{2}(-2\psi_{,0}) \approx \psi_{,0}, \tag{80}$$

$$\Gamma_{0i}^0 = \Gamma_{i0}^0 = \frac{-1+2\psi}{2}g_{00,i} = \frac{-1+2\psi}{2}(-2\psi_{,i}) \approx \psi_{,i}, \tag{81}$$

$$\Gamma_{ij}^0 = \frac{1-2\psi}{2}\frac{\partial}{\partial t}\left[a^2(t)\delta_{ij}(1+2\phi)\right] = \frac{1-2\psi}{2}\delta_{ij}\left[2a\dot{a}(1+2\phi) + 2a^2\phi_{,0}\right] \tag{82}$$

$$\Gamma^i_{00} = \frac{g^{i\nu}}{2}(-g_{00,\nu}) = -\frac{1}{2a^2}(1-2\phi)(-2\psi_{,i}) = \frac{1}{a^2}\psi_{,i}. \tag{83}$$

$$\Gamma^i_{j0} = \Gamma^i_{0j} = \frac{g^{ii}}{2}g_{ij,0} = \frac{1-2\phi}{2a^2}\frac{\partial}{\partial t}\left[a^2\delta_{ij}(1+2\phi)\right] \approx \delta_{ij}(H+\phi_{,0}). \tag{84}$$

$$\Gamma^i_{jk} = \frac{1-2\phi}{2a^2}\left\{\partial_k[\delta_{ij}a^2(1+2\phi)] + \partial_j[\delta_{ik}a^2(1+2\phi)] - \partial_i[\delta_{jk}a^2(1+2\phi)]\right\}$$
$$\approx \delta_{ik}\partial_j\phi + \delta_{ij}\partial_k\phi - \delta_{jk}\partial_i\phi. \tag{85}$$

For the Ricci tensor, the nonzero components

$$R_{00} = \frac{1}{a^2}\nabla^2\psi - 3\frac{\ddot{a}}{a} - 3\partial_t^2\phi + 3H\partial_t(\psi - 2\phi). \tag{86}$$

$$R_{i0} = R_{0i} = 2(H\partial_i\psi - \partial_i\partial_t\phi), \tag{87}$$

$$R_{ij} = a^2\delta_{ij}\left[(3H^2 + \dot{H})(1+2\phi-2\psi) + H\partial_t(6\phi-\psi) + \partial_t^2\phi\right]$$
$$- \delta_{ij}\nabla^2\phi - \partial_i\partial_j(\psi + \phi). \tag{88}$$

The Ricci scalar is made up of two parts, the first one is zero-order part, Eq. (15), and the leading order perturbed term

$$\delta R = -12\psi\left[\left(\frac{\dot{a}}{a}\right) + \frac{\ddot{a}}{a}\right] + 6\partial_t^2\phi - 6H\partial_t\psi + 24H\partial_t\phi - 2a^{-2}(2\nabla^2\phi + \nabla^2\psi), \tag{89}$$

so the Ricci scalar is

$$R = 6\left[\left(\frac{\dot{a}}{a}\right) + \frac{\ddot{a}}{a}\right](1-2\psi) + 6\partial_t^2\phi - 6H\partial_t\psi + 24H\partial_t\phi - 2a^{-2}(2\nabla^2\phi + \nabla^2\psi). \tag{90}$$

The Einstein tensor components

$$G_{00} = 3H^2 + 6H\partial_t\phi - 2a^{-2}\nabla^2\phi, \tag{91}$$

$$G_{0i} = 2(H\partial_i\psi - \partial_i\partial_t\phi), \tag{92}$$

$$G_{ij} = a^2\delta_{ij}\left[-(3H^2+2\dot{H})(1+2\phi-2\psi) - 6H\partial_t\phi + 2H\partial_i\psi - 2\partial_t^2\phi\right] + (\delta_{ij}\nabla^2 - \partial_i\partial_j)(\phi+\psi). \tag{93}$$

The energy-momentum tensor for i-type particles in the universe, with the spin-degeneracy $\mathfrak{g}_i$:

$$T^{\mu(i)}_{\nu} = \mathfrak{g}_i \int \frac{dP_1 dP_2 dP_3}{(2\pi)^3}\left[-\mathbf{det}(g_{\alpha\beta})\right]^{-\frac{1}{2}}\frac{P^\mu P_\nu}{P_0}f_i(\boldsymbol{p},\ \boldsymbol{x},\ t), \tag{94}$$

which is derived from

$$\mathfrak{g}_i \int \frac{2P^0 dP_0 dP_1 dP_2 dP_3}{(2\pi)^3}\left[-\mathbf{det}(g_{\alpha\beta})\right]^{-\frac{1}{2}}\frac{P^\mu P_\nu}{P_0}f_i(\boldsymbol{p},\ \boldsymbol{x},\ t)\delta_D\left(P^\mu P_\mu + m^2\right)$$
$$= \mathfrak{g}_i \int \frac{dP_1 dP_2 dP_3}{(2\pi)^3}\left[-\mathbf{det}(g_{\alpha\beta})\right]^{-\frac{1}{2}}\frac{P^\mu P_\nu}{P_0}f_i(\boldsymbol{p},\ \boldsymbol{x},\ t), \tag{95}$$

notice the facts that

$$\int 2P^0 dP_0 \delta_D\left(P^\mu P_\mu + m^2\right) = \int d(P^0 P_0)\delta_D\left(P^\mu P_\mu + m^2\right) = 1, \tag{96}$$

and

$$g^{\mu\nu}dx_\mu dx_\nu = g'^{\alpha\beta}dx'_\alpha dx'_\beta \rightarrow dP_0 dP_1 dP_2 dP_3\sqrt{\mathbf{det}(g^{\mu\nu})} = dP'_0 dP'_1 dP'_2 dP'_3\sqrt{\mathbf{det}(g'^{\alpha\beta})}, \tag{97}$$

this is because of the Jacobian

$$\mathbf{det}\left[\frac{\partial(x_0,\ x_1,\ x_2,\ x_3)}{\partial(x'_0,\ x'_1,\ x'_2,\ x'_3)}\right] = \sqrt{\frac{\mathbf{det}(g'^{\alpha\beta})}{\mathbf{det}(g^{\mu\nu})}}, \qquad \mathbf{det}(g^{\mu\nu})\mathbf{det}(g_{\mu\nu}) = \mathbf{det}(g'^{\alpha\beta})\mathbf{det}(g'_{\alpha\beta}) = 1. \tag{98}$$

The relation between four-momentum P^μ and physical momentum $\boldsymbol{p}$:

$$P^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \frac{\mathrm{d}x^0}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\mu}{\mathrm{d}x^0} = P^0\frac{\mathrm{d}x^\mu}{\mathrm{d}x^0} = E\frac{\mathrm{d}x^\mu}{\mathrm{d}t}, \qquad P^\mu P_\mu = -m^2, \tag{99}$$

$$|\boldsymbol{p}|^2 = g_{ij}P^i P^j = g^{ij}P_i P_j \to p^i = aP^i = \frac{1}{a}P_i. \tag{100}$$

The components of $T^\mu{}_\nu$ (for a certain type of particle, assuming $\mathfrak{g} = 1$):

$$T^0{}_0 = \int \frac{dP_1 dP_2 dP_3}{(2\pi)^3}a^{-3}P_0 f(\boldsymbol{p},\,\boldsymbol{x},\,t) = -\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}Ef(\boldsymbol{p},\,\boldsymbol{x},\,t), \tag{101}$$

$$T^i{}_j = \int \frac{dP_1 dP_2 dP_3}{(2\pi)^3}a^{-3}\frac{P^i P_j}{P^0} f(\boldsymbol{p},\,\boldsymbol{x},\,t) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{p^i p^j}{E} f(\boldsymbol{p},\,\boldsymbol{x},\,t), \tag{102}$$

$$T^0{}_j = \int \frac{dP_1 dP_2 dP_3}{(2\pi)^3}a^{-3}\frac{P^0 P_j}{P^0} f(\boldsymbol{p},\,\boldsymbol{x},\,t) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}ap^j f(\boldsymbol{p},\,\boldsymbol{x},\,t), \tag{103}$$

$$T^i{}_0 = \int \frac{dP_1 dP_2 dP_3}{(2\pi)^3}a^{-3}\frac{P^i P_0}{P^0} f(\boldsymbol{p},\,\boldsymbol{x},\,t) = -\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{p^i}{a} f(\boldsymbol{p},\,\boldsymbol{x},\,t). \tag{104}$$

With

$$T^0{}_0 = -\rho_m = -\bar\rho(1+\delta), \tag{105}$$

the relativistic Poission equation:

$$3H^2\psi - 3H\partial_t\phi + a^{-2}\nabla^2\phi = -4\pi G\bar\rho\delta, \tag{106}$$

only the third on the left side should be considered under the small scale $L \ll L_{\text{Horizon}}$.
The Fourier transformation on the Einstein tensor and energy-momentum tensor:

$$\left(\hat k_i \hat k^j - \frac{1}{3}\delta^j{}_i\right)\tilde G^i{}_j = a^{-2}\left(\hat k_i \hat k^j k^i k_j - \frac{1}{3}k^i k_i\right)\left(\tilde\phi + \tilde\psi\right) = \frac{2}{3a^2}k^2\left(\tilde\phi + \tilde\psi\right). \tag{107}$$

$$\left(\hat k_i \hat k^j - \frac{1}{3}\delta^j{}_i\right)\tilde T^i{}_j = \int \frac{d^3 p}{(2\pi)^3}\frac{p^2\left(\mu^2 - \frac{1}{3}\right)}{E}\tilde f(\boldsymbol{p},\,\boldsymbol{k},\,t), \qquad \mu \equiv \frac{\boldsymbol{p}\cdot\boldsymbol{k}}{|\boldsymbol{p}||\boldsymbol{k}|}. \tag{108}$$

For the non-relativistic matter in late universe, there is $\tilde\phi + \tilde\psi,\ \phi + \psi = 0$.

1.5 Eulerian Perturbation Theory

The conservation of energy-momentum tensor, $T^\mu{}_{\nu;\mu} = 0$. For $\nu = 0$ part,

$$
\begin{aligned}
T^\mu{}_{0;\mu} &= \partial_\mu T^\mu{}_0 + \Gamma^\mu{}_{\alpha\mu}T^\alpha{}_0 - \Gamma^\alpha{}_{\mu 0}T^\mu{}_\alpha \\
&= \partial_0 T^0{}_0 + \partial_i T^i{}_0 + \Gamma^0{}_{00}T^0{}_0 + \Gamma^i{}_{0i}T^0{}_0 + \Gamma^0{}_{i0}T^i{}_0 + \textcolor{red}{\Gamma^j{}_{ij}T^i{}_0} - \Gamma^0{}_{00}T^0{}_0 - \Gamma^0{}_{i0}T^i{}_0 - \textcolor{red}{\Gamma^i{}_{00}T^0{}_i} - \Gamma^i{}_{j0}T^j{}_i \\
&= \partial_0 T^0{}_0 + \partial_i T^i{}_0 + \Gamma^i{}_{0i}T^0{}_0 + \textcolor{red}{\Gamma^j{}_{ij}T^i{}_0} - \textcolor{red}{\Gamma^i{}_{00}T^0{}_i} - \Gamma^i{}_{j0}T^j{}_i. \\
&= -\partial_t\rho_m - \partial_i\left[\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{p^i}{a}f(\boldsymbol{p},\,\boldsymbol{x},\,t)\right] - 3(H + \partial_t\phi)\rho_m - \textcolor{red}{H\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{p^2}{E}f(\boldsymbol{p},\,\boldsymbol{x},\,t)},
\end{aligned}
\tag{109}
$$

where Eq.(80)-(85) have been used, and the red terms are of higher order perturbation/small values (ϕ, ψ, $\partial_i\phi$, $\partial_j\psi$, p^i), neglected step by step in Eq. (109). The continuity equation

$$-\partial_t\rho_m - 3H\rho_m - \frac{1}{a}\partial_i\left(\rho_m v^i\right) \approx 0, \tag{110}$$

with

$$v^i(\boldsymbol{x},\,t) = \frac{\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}p^i f(\boldsymbol{p},\,\boldsymbol{x},\,t)}{\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}mf(\boldsymbol{p},\,\boldsymbol{x},\,t)}. \tag{111}$$

With $\rho_m = \bar{\rho}(t)[1 + \delta(\boldsymbol{x}, t)]$, and Eq. (30) for non-relativistic matter, there is

$$\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}t} = -3H\bar{\rho} \to 0 = \frac{\partial\delta}{\partial t} + \frac{1}{a}\partial_i[(1+\delta)v^i]. \tag{112}$$

Defining the conformal time $d\tau \equiv dt/a$, then the continuity equation is

$$\frac{\partial\delta}{\partial\tau} = -\partial_i[(1+\delta)v^i]. \tag{113}$$

For $\nu = i$ parts,

$$\begin{aligned}
0 = T^\mu_{i;\mu} &= \partial_\mu T^\mu_i + \Gamma^\mu_{\alpha\mu}T^\alpha_i - \Gamma^\alpha_{\mu i}T^\mu_\alpha \\
&= \partial_t\left[\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}ap^i f(\boldsymbol{p}, \boldsymbol{x}, t)\right] + \partial_j\left[\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{p^i p^j}{E}f(\boldsymbol{p}, \boldsymbol{x}, t)\right] \\
&\quad + \Gamma^0_{00}T^0_i + \Gamma^j_{0j}T^0_i + \Gamma^0_{0j} + \Gamma^0_{j0}T^j_i + \Gamma^k_{jk}T^j_i - \Gamma^0_{0i}T^0_0 - \Gamma^0_{ji}T^j_0 - \Gamma^j_{0i}T^0_j - \Gamma^j_{ki}T^k_j \\
&\approx \partial_t(a\rho_m v^i) + 3Ha\rho_m v^i + \rho_m\partial_i\psi + \partial_j\left[\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{p^i p^j}{E}f(\boldsymbol{p}, \boldsymbol{x}, t)\right].
\end{aligned} \tag{114}$$

Defining

$$\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{p^i p^j}{m}f(\boldsymbol{p}, \boldsymbol{x}, t) = \rho_m v^i v^j + \sigma^{ij}(\boldsymbol{x}, t), \tag{115}$$

where σ^{ij} is the stress tensor,

$$\sigma^{ij} = -p\delta_{ij} + \eta\left(\nabla_i u_j + \nabla_j u_i - \frac{2}{3}\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{u}\right) + \zeta\delta_{ij}\boldsymbol{\nabla}\cdot\boldsymbol{u}, \tag{116}$$

notice that p is the pressure, η, ζ are viscosity coefficients. Then

$$\begin{aligned}
0 &\approx \partial_t(a\rho_m v^i) + 3Ha\rho_m v^i + \rho_m\partial_i\psi + \partial_j(\rho_m v^i v^j + \sigma^{ij}) \\
&= \partial_t\left[a\bar{\rho}(1+\delta)v^i\right] + 3Ha\bar{\rho}(1+\delta)v^i + \bar{\rho}(1+\delta)\partial_i\psi + \partial_j\left[\bar{\rho}(1+\delta)v^i v^j + \sigma^{ij}\right],
\end{aligned} \tag{117}$$

the first term combined with Eq.(112)(113),

$$\partial_t\left[a\bar{\rho}(1+\delta)v^i\right] = \dot{a}\bar{\rho}(1+\delta)v^i - 3Ha\bar{\rho}(1+\delta)v^i + a\bar{\rho}\left\{-\frac{1}{a}\partial_j\left[(1+\delta)v^j\right]\right\}v^i + a\bar{\rho}(1+\delta)\frac{\partial v^i}{\partial t}, \tag{118}$$

thus,

$$0 \approx \dot{a}\bar{\rho}(1+\delta)v^i + a\bar{\rho}(1+\delta)\partial_t v^i + \bar{\rho}(1+\delta)\partial_i\psi + \bar{\rho}(1+\delta)v^j\partial_j v^i + \partial_j\sigma^{ij}, \tag{119}$$

with both sides divided by $\bar{\rho}(1+\delta)$, the Euler equation can be derived as

$$\frac{\partial v^i}{\partial\tau} + aHv^i + \partial_i\psi + v^j\partial_j v^i + \frac{1}{\rho_m}\partial_j\sigma^{ij} = 0. \tag{120}$$

Consider the relation for non-relativistic matter $\phi + \psi = 0$, and the Poisson equation (106) under small scales,

$$\nabla^2\phi = -\nabla^2\psi = -4\pi Ga^2\bar{\rho}_m\delta_m, \tag{121}$$

notice that $\bar{\rho} \to \bar{\rho}_m$, $\delta \to \delta_m$ to denote the density and perturbation of matters, now there is

$$\nabla^2\psi = 4\pi Ga^2\bar{\rho}_m\delta_m = \frac{8\pi G}{3}\bar{\rho}_{\text{tot}}\Omega_m(a)\cdot\frac{3}{2}a^2\delta_m = \frac{3}{2}a^2 H^2\Omega_m(a)\delta_m, \tag{122}$$

this is the Poisson equation under Eulerian perturbation theory. Keeping the 1-st order, neglecting σ^{ij}, the Euler equation (120) and the continuity equation Eq. (113) become as

$$\frac{\partial\delta_m}{\partial\tau} + \partial_i v^i = 0, \qquad \frac{\partial v^i}{\partial\tau} + aHv^i = -\partial_i\psi, \tag{123}$$

with the property $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\psi = 0$, there is

$$\frac{\partial}{\partial \tau}(\boldsymbol{\nabla} \times \boldsymbol{v}) + aH(\boldsymbol{\nabla} \times \boldsymbol{v}) = 0 \to \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\nabla} \times \boldsymbol{v}) + \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v}) = 0, \tag{124}$$

then we have $\boldsymbol{\nabla} \times \boldsymbol{v} \propto a^{-1}$, the curl field of $\boldsymbol{v}$ is decayed with a.

Defining $\theta = \partial_i v^i$, with two equations in Eq. (123),

$$\frac{\partial^2 \delta_m}{\partial \tau^2} + aH\frac{\partial \delta_m}{\partial \tau} = \frac{3}{2}a^2 H^2 \Omega_m(a)\delta_m, \tag{125}$$

rewrite $\delta_m(\boldsymbol{x}, \ \tau)$ with a linear growth factor: $\delta_m(\boldsymbol{x}, \ \tau) = \delta_m(\boldsymbol{x}, \ 0)D(\tau)$, then

$$\frac{\partial^2 D}{\partial \tau^2} + aH\frac{\partial D}{\partial \tau} = \frac{3}{2}a^2 H^2 \Omega_m(a)D, \tag{126}$$

with $\frac{\mathrm{d}}{\mathrm{d}\tau} = a\frac{\mathrm{d}}{\mathrm{d}t} = a\frac{\mathrm{d}a}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}a} = a^2 H\frac{\mathrm{d}}{\mathrm{d}a}$, then

$$a^2 H^2 \frac{\mathrm{d}^2 D}{\mathrm{d}a^2} + \left(3aH^2 + a^2 H\frac{\mathrm{d}H}{\mathrm{d}a}\right)\frac{\mathrm{d}D}{\mathrm{d}a} = \frac{3}{2}H^2 \Omega_m(a)D. \tag{127}$$

Now we discuss the behaviours of $D(\tau)$ with a. Assuming that there are only Ω_Λ and Ω_m,

$$H^2 = H_0^2 \left[\Omega_m a^{-3} + \Omega_\Lambda + (1 - \Omega_m - \Omega_\Lambda)a^{-2}\right], \tag{128}$$

defining

$$M(a) = 2\Omega_m + 2(1 - \Omega_m - \Omega_\Lambda)a + 2\Omega_\Lambda a^3 = 2a^3 \frac{H^2}{H_0^2}, \tag{129}$$

$$N(a) = 3\Omega_m a^{-1} + 2(1 - \Omega_m - \Omega_\Lambda) = -a^3 \frac{1}{H_0^2}\frac{\mathrm{d}}{\mathrm{d}a}(H^2). \tag{130}$$

Eq. (127) can be simplified as

$$M\frac{\mathrm{d}^2 D}{\mathrm{d}a^2} + \left(N + \frac{\mathrm{d}M}{\mathrm{d}a}\right)\frac{\mathrm{d}D}{\mathrm{d}a} + \frac{\mathrm{d}N}{\mathrm{d}a}D = \frac{\mathrm{d}}{\mathrm{d}a}\left(M\frac{\mathrm{d}D}{\mathrm{d}a} + ND\right) = 0. \tag{131}$$

Thus,

$$M\frac{\mathrm{d}D}{\mathrm{d}a} + ND = \text{constant} \to 2a^3 H^2 \frac{\mathrm{d}D}{\mathrm{d}a} - a^3 D\frac{\mathrm{d}H^2}{\mathrm{d}a} = C, \tag{132}$$

where C is the integration constant. At last,

$$H^{-1}\frac{\mathrm{d}D}{\mathrm{d}a} - DH^{-2}\frac{\mathrm{d}H}{\mathrm{d}a} = \frac{\mathrm{d}(H^{-1}D)}{\mathrm{d}a} = \frac{C}{2a^3 H^3}, \tag{133}$$

for $C \neq 0$, the growth solution is

$$D^+(a) \propto H(a)\int_0^a \frac{\mathrm{d}a'}{a'^3 H^3(a')}, \tag{134}$$

for $C = 0$, the decaying solution is

$$D^-(a) \propto H(a). \tag{135}$$

For Einstein-Desitter universe standard cold dark matter (SCDM), $\Omega_m = 1, \ \Omega_\Lambda = 0$, then we have

$$H(a) \propto a^{-\frac{3}{2}}, \ D(a) \propto a. \tag{136}$$

Based on the perturbed metric (78), the trajectory of a particle on the geodesic with the equation of motion

$$\frac{\mathrm{d}^2 x^\alpha}{\mathrm{d}\lambda^2} = \Gamma^\alpha_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}, \tag{137}$$

there are

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} = \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 \left\{-\partial_t \psi - 2\partial_i \psi\frac{\mathrm{d}x^i}{\mathrm{d}t} - a^2[H(1 + 2\phi - 2\psi) + \partial_t \phi]\delta_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}\right\}, \tag{138}$$

11

$$\frac{d^2 x^i}{d\lambda^2} = -\Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = -a^{-2}\partial_i\psi\left(\frac{dt}{d\lambda}\right)^2 - 2(H + \partial_t\phi)\frac{dt}{d\lambda}\frac{dx^i}{d\lambda} - 2\partial_j\phi\frac{dx^j}{d\lambda}\frac{dx^i}{d\lambda} + \partial_i\phi\left(\frac{dx^j}{d\lambda}\right)^2$$

$$= \frac{d}{d\lambda}\left(\frac{dt}{d\lambda}\frac{dx^i}{dt}\right) = \frac{d^2 t}{d\lambda^2}\frac{dx^i}{dt} + \left(\frac{dt}{d\lambda}\right)^2\frac{d^2 x^i}{dt^2}. \tag{139}$$

Eq. (139) also leads to

$$\left\{-\partial_t\psi - 2\partial_i\psi\frac{dx^i}{dt} - a^2[H(1 + 2\phi - 2\psi) + \partial_t\phi]\left(\frac{dx^i}{dt}\right)^2\right\}\frac{dx^i}{dt} + \frac{d^2 x^i}{dt^2}$$

$$= -a^{-2}\partial_i\psi - 2(H + \partial_t\phi)\frac{dx^i}{dt} - 2\partial_j\phi\frac{dx^j}{dt}\frac{dx^i}{dt} + \partial_i\phi\left(\frac{dx^j}{dt}\right)^2, \tag{140}$$

for non-relativistic and sub-horizon motions, $dx^i/dt \ll 1$, ϕ, ψ, $\partial_t\phi$, $\partial_t\psi$ are also small perturbations, so

$$\frac{d^2 x^i}{dt^2} + 2H\frac{dx^i}{dt} = -a^{-2}\partial_i\psi, \tag{141}$$

with $\frac{d}{d\tau} = a\frac{d}{dt}$,

$$a\frac{d}{d\tau}\left(\frac{1}{a}\frac{dx^i}{d\tau}\right) + 2aH\frac{dx^i}{d\tau} = \frac{d^2 x^i}{d\tau^2} - \dot{a}\frac{dx^i}{d\tau} + 2aH\frac{dx^i}{d\tau} = \frac{d^2 x^i}{d\tau^2} + aH\frac{dx^i}{d\tau} = -\partial_i\psi. \tag{142}$$

Based on the framework of Lagrangian dynamics, we calculate the displacement field of the position of a particle, $\boldsymbol{x}(\tau) = \boldsymbol{q} + \boldsymbol{\phi}(\boldsymbol{q},\ \tau)$, the density field under coordinates transformation $\bar{\rho}(1 + \delta)d^3 x = \bar{\rho}d^3 q$, then the Jacobian

$$J(\boldsymbol{q},\ \tau) = \det\left(\frac{\partial x_i}{\partial q_j}\right) = \det(\delta_{ij} + \phi_{i,j}) = (1 + \delta)^{-1}, \tag{143}$$

with the Poission equation (122),

$$\partial_i\left(\frac{d^2\phi^i}{d\tau^2} + aH\frac{d\phi^i}{d\tau}\right) = -\nabla^2\psi = -\frac{3}{2}a^2 H^2\Omega_m(a)\delta = \frac{3}{2}a^2 H^2\Omega_m(a)\left[1 - J(\boldsymbol{q},\ \tau)^{-1}\right], \tag{144}$$

as $\boldsymbol{\phi}(\boldsymbol{q},\ \tau) = \boldsymbol{x} - \boldsymbol{q}$, there is $d^3\phi = d^3 x - d^3 q$, $\partial_i\phi^i = 1 - J(\boldsymbol{q},\ \tau)^{-1}$, (???) then Eq. (144) can be approximated (to the 1st-order accuracy) as

$$\frac{d^2(\partial_i\phi^i)}{d\tau^2} + aH\frac{d(\partial_i\phi^i)}{d\tau} = \frac{3}{2}a^2 H^2\Omega_m(a)\partial_i\phi^i. \tag{145}$$

Recall Eq. (125), we may express the form of $\partial_i\phi^i$ as $\partial_i\phi^i(\boldsymbol{q},\ \tau) = -D_1(\tau)\delta(\boldsymbol{q})$.

1.6   The (Extended) Press-Schechter Theory

Assuming spherical collapse, $R(\tau) = r - u(r,\ \tau) = [1 - \alpha(\tau)]r$, when $\alpha \ll 1$, there is (linear growth equation) (I do not know how to derive it...)

$$\frac{d^2\alpha}{d\tau^2} + aH\frac{d\alpha}{d\tau} = \frac{3}{2}a^2 H^2\Omega_m(a)\alpha. \tag{146}$$

tophat filter of radius

$$R(M) = \left(\frac{3M}{4\pi\bar{\rho}}\right)^{\frac{1}{3}}, \tag{147}$$

with the corresponding variance of the Gaussian random field, $\sigma^2(M)$. Critical value of the Halo formation: $\delta_c \approx 1.68$. The fraction in halos $> M$ is

$$\frac{1}{\sqrt{2\pi}\sigma(M)}\int_{\delta_c}^{\infty}d\delta\exp\left[-\frac{\delta^2}{2\sigma^2(M)}\right] = \frac{1}{2}\mathrm{erfc}\left(\frac{\nu}{\sqrt{2}}\right), \qquad \left(\text{Notice that } \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}dx e^{-x^2} = 1\right) \tag{148}$$

where

$$\nu \equiv \frac{\delta_c}{\sigma(M)}, \qquad \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp\left(-t^2\right) dt. \tag{149}$$

Differentiate in M to find fraction in range dM to get the halo number density

$$VMdn = \frac{1}{2} V \rho_m \left| \frac{\mathrm{d}}{\mathrm{d}M} \text{erfc}\left(\frac{\nu}{\sqrt{2}}\right) \right| dM, \tag{150}$$

which leads to

$$\begin{aligned}
\left| \frac{\mathrm{d}n}{\mathrm{d}M} M \right| = \left| \frac{\mathrm{d}n}{\mathrm{d}\log M} \right| &= \frac{1}{2} \frac{\rho_m}{M} \left| \frac{\mathrm{d}}{\mathrm{d}\log M} \text{erfc}\left(\frac{\nu}{\sqrt{2}}\right) \right| \\
&= \frac{1}{2} \frac{\rho_m}{M} \frac{1}{\sqrt{2}} \left| \frac{\mathrm{d}\nu}{\mathrm{d}\log M} \frac{\mathrm{d}}{\mathrm{d}(\nu/\sqrt{2})} \text{erfc}\left(\frac{\nu}{\sqrt{2}}\right) \right| \\
&= \frac{1}{2} \frac{\rho_m}{M} \sqrt{\frac{2}{\pi}} \left| -\frac{\mathrm{d}\nu}{\mathrm{d}\log M} \exp\left(-\frac{\nu^2}{2}\right) \right|,
\end{aligned} \tag{151}$$

with

$$\frac{\mathrm{d}\nu}{\mathrm{d}\log M} = \nu \frac{\mathrm{d}\log \nu}{\mathrm{d}\log M} = \nu \frac{\mathrm{d}\log \sigma^{-1}(M)}{\mathrm{d}\log M} > 0, \tag{152}$$

then

$$\left| \frac{\mathrm{d}n}{\mathrm{d}\log M} \right| = \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{\rho_m}{M} \frac{\mathrm{d}\log \sigma^{-1}}{\mathrm{d}\log M} \nu \exp\left(-\frac{\nu^2}{2}\right). \tag{153}$$

The form $P(k) \propto k^n$,

$$\sigma^2(M) \propto \int_0^{M^{-\frac{1}{3}}} dk \; k^2 P(k) \propto \int_0^{M^{-\frac{1}{3}}} dk \; k^{n+2} \propto M^{-\frac{n+3}{3}}, \tag{154}$$

$\nu = \delta_c/\sigma(M) = (M/M_*)^{\frac{n+3}{6}} \sqrt{2}$, where M_* is the critical mass for structure formation. Then

$$\begin{aligned}
\left| \frac{\mathrm{d}n}{\mathrm{d}M} \right| &= \frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{\rho_m}{M^2} \frac{n+3}{6} \left(\frac{M}{M_*}\right)^{\frac{n+3}{6}} \sqrt{2} \exp\left[-\left(\frac{M}{M_*}\right)^{\frac{n+3}{3}}\right] \\
&= \frac{1}{2} \frac{1}{\sqrt{\pi}} \frac{\rho_m}{M^2} \frac{n+3}{3} \left(\frac{M}{M_*}\right)^{\frac{n+3}{6}} \exp\left[-\left(\frac{M}{M_*}\right)^{\frac{n+3}{3}}\right].
\end{aligned} \tag{155}$$

There is a factor $1/2$ in Eq. (155) related with "cloud-in-cloud" problem. Now we define

$$S \equiv \sigma_R^2(M) = \left\langle \delta_R^2 \right\rangle, \tag{156}$$

there is the survival probability (not forming Halo)

$$P_f(\delta, \; S) d\delta = \frac{1}{\sqrt{2\pi S}} \left\{ \exp\left(-\frac{\delta^2}{2S}\right) - \exp\left[-\frac{(2\delta_c - \delta)^2}{2S}\right] \right\} d\delta, \tag{157}$$

for the first crossing (δ_c) distribution (at S), as $S - M$ are correlated, we can define the distribution function $f(S)$ and

$$MN(M)dM = M \left| \frac{\mathrm{d}n}{\mathrm{d}M} \right| dM = \bar{\rho} f(S) dS, \tag{158}$$

which depicts the total mass per volume, of the Halo at mass M, and then

$$f(S)dS = 1 - \int_{-\infty}^{\delta_c} P_f(\delta, \; S) d\delta \rightarrow f(S) = -\frac{\mathrm{d}}{\mathrm{d}S} \int_{-\infty}^{\delta_c} P_f(\delta, \; S) d\delta, \tag{159}$$

with

$$\frac{1}{\sqrt{2\pi S}} \int_{-\infty}^{\delta_c} \exp\left(-\frac{\delta^2}{2S}\right) d\delta = -\frac{1}{2} \frac{2}{\sqrt{\pi}} \int_{+\infty}^{-\frac{\delta_c}{\sqrt{2S}}} \exp\left(-\frac{\delta^2}{2S}\right) d\left(-\frac{\delta}{\sqrt{2S}}\right) = \frac{1}{2} \text{erfc}\left(-\frac{\delta_c}{\sqrt{2S}}\right), \tag{160}$$

$$\frac{1}{\sqrt{2\pi S}} \int_{-\infty}^{\delta_c} \exp\left[-\frac{(2\delta_c - \delta)^2}{2S}\right] d\delta = -\frac{1}{2}\frac{2}{\sqrt{\pi}} \int_{+\infty}^{\frac{\delta_c}{\sqrt{2S}}} \exp\left[-\frac{(2\delta_c - \delta)^2}{2S}\right] d\left(-\frac{\delta - 2\delta_c}{\sqrt{2S}}\right) = \frac{1}{2}\text{erfc}\left(\frac{\delta_c}{\sqrt{2S}}\right), \quad (161)$$

and

$$\frac{d}{dS}\left[\text{erfc}\left(-\frac{\delta_c}{\sqrt{2S}}\right)\right] = \frac{2}{\sqrt{\pi}}\frac{d}{dS}\int_{-\infty}^{\frac{\delta_c}{\sqrt{2S}}} \exp(-t^2)dt = -\frac{\delta_c}{\sqrt{2\pi S^3}}\exp\left(-\frac{\delta_c^2}{2S}\right), \quad (162)$$

$$\frac{d}{dS}\left[\text{erfc}\left(\frac{\delta_c}{\sqrt{2S}}\right)\right] = \frac{2}{\sqrt{\pi}}\frac{d}{dS}\int_{-\infty}^{-\frac{\delta_c}{\sqrt{2S}}} \exp(-t^2)dt = \frac{\delta_c}{\sqrt{2\pi S^3}}\exp\left(-\frac{\delta_c^2}{2S}\right). \quad (163)$$

Thus,

$$f(S) = -\frac{1}{2}\cdot 2\left[-\frac{\delta_c}{\sqrt{2\pi S^3}}\exp\left(-\frac{\delta_c^2}{2S}\right)\right] = \frac{\delta_c}{\sqrt{2\pi S^3}}\exp\left(-\frac{\delta_c^2}{2S}\right). \quad (164)$$

With Eq. (158), and $S \equiv \sigma^2(M)$,

$$\begin{aligned}
N(M) &= \left|\frac{dn}{dM}\right| = \frac{\bar{\rho}}{M}f(S)\left|\frac{dS}{dM}\right| \\
&= \frac{\bar{\rho}}{M}\cdot\frac{\delta_c}{\sqrt{2\pi S^3}}\exp\left(-\frac{\delta_c^2}{2S}\right)\left|\frac{dS}{dM}\right| \\
&= \sqrt{\frac{2}{\pi}}\frac{\bar{\rho}}{M^2}\frac{\delta_c}{\sigma(M)}\frac{d\log\sigma^{-1}}{d\log M}\exp\left[-\frac{\delta_c^2}{2\sigma^2(M)}\right],
\end{aligned} \quad (165)$$

compared with Eq. (153), the factor 1/2 has disappeared, which thanks to the revision (second) term in the survival probability Eq. (157). The spatial bias coefficient $b_{\rm h}$ of Halos, in the relation

$$\delta_{\rm h} = \frac{\rho_{\rm h} - \bar{\rho}_{\rm h}}{\bar{\rho}_{\rm h}} = b_{\rm h}\delta = b_{\rm h}\frac{\rho - \bar{\rho}}{\bar{\rho}}. \quad (166)$$

If the "observer" is at an overdensed position $(\delta_0,\ S_0)$, the first-crossing distribution function should be moved from the original point $(0,\ 0)$ to $(\delta_0,\ S_0)$, then

$$f(S|\delta_0,\ S_0)dS = \frac{\delta_c - \delta_0}{\sqrt{2\pi(S - S_0)^3}}\exp\left[-\frac{(\delta_c - \delta_0)^2}{2(S - S_0)}\right]dS, \quad (167)$$

$$n(M|\delta_0,\ S_0) = \frac{\bar{\rho}(1 + \delta_0)}{M}\frac{dS}{dM}f(S|\delta_0,\ S_0), \quad (168)$$

and then

$$\begin{aligned}
\delta_{\rm h} &= \frac{n(M|\delta_0,\ S_0) - n(M)}{n(M)} = \frac{d[\log n(M|\delta_0,\ S_0)]}{d\delta_0}\cdot(\delta_0 - 0) \\
&= \delta_0\left\{\frac{1}{1 + \delta_0} + \frac{d[\log f(M|\delta_0,\ S_0)]}{d\delta_0}\right\} \\
&\approx \delta_0\left\{1 + \frac{d[\log f(M|\delta_0,\ S_0)]}{d\delta_0}\right\} \\
&= \delta_0\left\{1 + \frac{d[\log(\delta_c - \delta_0)]}{d\delta_0} + \frac{\delta_c - \delta_0}{S - S_0}\right\} \\
&\approx \delta_0\left(1 - \frac{1}{\delta_c} + \frac{\delta_c}{S}\right) = \delta_0\left(1 + \frac{\nu^2 - 1}{\delta_c}\right), \qquad \left(\nu \equiv \frac{\delta_c}{\sqrt{S}}\right),
\end{aligned} \quad (169)$$

where we have used the fact that $\delta_0 \ll 1$, $S_0 \ll S$ (large scale). For $\nu = 1$, $b_{\rm h} = 1 + (\nu^2 - 1)/\delta_c = 1$.

1.7 Gravitational lensing

The fundamental lensing equation

$$\hat{\vec{\alpha}}(D_L\vec{\theta}_I) = \frac{D_S}{D_{LS}}(\vec{\theta}_I - \vec{\theta}_S). \quad (170)$$

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Lensing does not alter the surface brightness, but the image size and flux. For simple lens models with circularly symmetric lenses,

$$\alpha = \frac{4GM(<b)}{c^2 b}, \tag{171}$$

if $\theta_S = 0$, with $b = D_L \theta_I \equiv D_L \theta_E$, the Einstein ring radius can be calculated as

$$\frac{4GM}{c^2 D_L \theta_E} = \frac{D_S}{D_{LS}} \theta_E \rightarrow \theta_E = \sqrt{\frac{4GM}{c^2} \frac{D_{LS}}{D_L D_S}}. \tag{172}$$

The vector form of the deflection angle

$$\hat{\vec{\alpha}}(D_L \vec{\theta}_I) = \frac{4GM}{c^2 D_L} \frac{\vec{\theta}}{|\vec{\theta}|^2}, \tag{173}$$

Defining the lensing potential of the point-mass lens

$$\vec{\alpha}(\vec{\theta}) = \frac{D_{LS}}{D_S} \hat{\vec{\alpha}}(D_L \vec{\theta}_I) = \boldsymbol{\nabla}\hat{\Psi}(\vec{\theta}) \rightarrow \hat{\Psi}(\vec{\theta}) = \frac{4GM}{c^2} \frac{D_{LS}}{D_L D_S} \log |\vec{\theta}|. \tag{174}$$

Now, there is

$$\theta_S = \theta_I - \frac{4GM}{c^2 D_L \theta_I} \frac{D_{LS}}{D_S} = \theta_I - \frac{\theta_E^2}{\theta_I}. \tag{175}$$

Substituting with $y \equiv \theta_S/\theta_E$, $x \equiv \theta_I/\theta_E$, then Eq. (175) becomes as

$$y = x - \frac{1}{x} \rightarrow x_\pm = \frac{1}{2}(y \pm \sqrt{y^2 + 4}). \tag{176}$$

The flux increasing factor is

$$\det \left| \frac{\partial(\theta_x^I, \theta_y^I)}{\partial(\theta_x^S, \theta_y^S)} \right| = \left[\det \left| \frac{\partial(\theta_x^S, \theta_y^S)}{\partial(\theta_x^I, \theta_y^I)} \right| \right]^{-1}, \tag{177}$$

if the lens is axially-symmetric, the Jacobian determinant is (why???)

$$\det \left| \frac{\partial(\theta_x^S, \theta_y^S)}{\partial(\theta_x^I, \theta_y^I)} \right| = \frac{\theta_S}{\theta_I} \frac{\mathrm{d}\theta_S}{\mathrm{d}\theta_I} = \frac{y}{x} \frac{\mathrm{d}y}{\mathrm{d}x}, \tag{178}$$

and

$$\frac{y}{x} = 1 - \frac{1}{x^2}, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{1}{x^2}. \tag{179}$$

then the magnification is

$$\mu(x) = \left[\left(1 - \frac{1}{x^2}\right)\left(1 + \frac{1}{x^2}\right) \right]^{-1} = \left(1 - \frac{1}{x^4}\right)^{-1}. \tag{180}$$

There is also

$$\frac{x}{y} = \frac{1}{2}\left(1 \pm \frac{\sqrt{y^2+4}}{y}\right), \qquad \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{2}\left(1 \pm \frac{y}{\sqrt{y^2+4}}\right), \tag{181}$$

then

$$\mu_\pm(y) = \frac{x}{y}\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{4}\left(1 \pm \frac{\sqrt{y^2+4}}{y}\right)\left(1 \pm \frac{y}{\sqrt{y^2+4}}\right) = \frac{1}{2}\left(1 \pm \frac{y^2+2}{y\sqrt{y^2+4}}\right). \tag{182}$$

Notice that for $y > 0$, there is always $\mu_-(y) < 0$, then the total source magnification should be

$$\mu(y) = \mu_+(y) + |\mu_-(y)| = \frac{y^2+2}{y\sqrt{y^2+4}}. \tag{183}$$

For the FRW metric,

$$g_{\mu\nu} = \mathbf{diag}[-1 - 2\Psi, \ a^2(1 + 2\Phi), \ a^2(1 + 2\Phi), \ a^2(1 + 2\Phi)], \tag{184}$$

there are the equations of motion (non-zero components of $\Gamma^\alpha_{\mu\nu}$ are shown in page with Eq. (78))

$$
\begin{aligned}
0 &= \frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} + \Gamma^0_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + 2\Gamma^0_{i0}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}t}{\mathrm{d}\lambda} + \Gamma^0_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda} \\
&= \frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} + \partial_t\Psi\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + 2\partial_i\Psi\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}t}{\mathrm{d}\lambda} + a^2[H(1+2\Phi-2\Psi)+\partial_t\Phi]\left(\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\right)^2,
\end{aligned}
\tag{185}
$$

$$
\begin{aligned}
0 &= \frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} + \Gamma^i_{00}\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + 2\Gamma^i_{j0}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}t}{\mathrm{d}\lambda} + \Gamma^i_{jk}\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} \\
&= \frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} + a^{-2}\partial_i\Psi\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + 2(H+\partial_t\Phi)\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}t}{\mathrm{d}\lambda} + 2\partial_k\Phi\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} - \partial_i\Phi\left(\frac{\mathrm{d}x^j}{\mathrm{d}\lambda}\right)^2.
\end{aligned}
\tag{186}
$$

For the photons,

$$
ds^2 = 0 = -(1+2\Psi)dt^2 + a^2(1+2\Phi)dx^i dx^i \rightarrow \left(\frac{\mathrm{d}x^i}{\mathrm{d}t}\right) \approx a^{-2}(1+2\Psi-2\Phi),
\tag{187}
$$

then (treating Ψ, Φ, $\partial_\mu\Psi$, $\partial_\mu\Phi$, dx^i/dt as small values in the 1st order)

$$
\begin{aligned}
0 &= \frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} + \partial_t\Psi\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right) + 2\partial_i\Psi\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2\frac{\mathrm{d}x^i}{\mathrm{d}\lambda} \\
&\quad + a^2[H(1+2\Phi-2\Psi)+\partial_t\Phi]\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 a^{-2}(1+2\Psi-2\Phi) \\
&\approx \frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2} + \left[2\partial_i\Psi\frac{\mathrm{d}x^i}{\mathrm{d}t} + (H+\partial_t\Phi+\partial_t\Psi)\right]\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2,
\end{aligned}
\tag{188}
$$

$$
\begin{aligned}
0 &= \frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} + a^{-2}\partial_i\Psi\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + 2(H+\partial_t\Phi)\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2\frac{\mathrm{d}x^i}{\mathrm{d}t} \\
&\quad + 2\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda} - \partial_i\Phi\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 a^{-2}(1+2\Psi-2\Phi) \\
&= \frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} + a^{-2}\partial_i(\Psi-\Phi)\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 + 2H\frac{\mathrm{d}t}{\mathrm{d}\lambda}\frac{\mathrm{d}x^i}{\mathrm{d}\lambda} + 2\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda}.
\end{aligned}
\tag{189}
$$

With the relation

$$
\frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} = \frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2},
\tag{190}
$$

there is

$$
\begin{aligned}
-\frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2}\frac{\mathrm{d}x^i}{\mathrm{d}t} &\approx \left[2\partial_i\Psi\frac{\mathrm{d}x^i}{\mathrm{d}t} + (H+\partial_t\Phi+\partial_t\Psi)\right]\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2\frac{\mathrm{d}x^i}{\mathrm{d}t} \\
&\approx (H+\partial_t\Phi+\partial_t\Psi)\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2\frac{\mathrm{d}x^i}{\mathrm{d}t} \\
&= \left[\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + a^{-2}\partial_i(\Psi-\Phi) + 2H\frac{\mathrm{d}x^i}{\mathrm{d}t} + 2\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}\Phi}{\mathrm{d}t}\right]\left(\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2,
\end{aligned}
\tag{191}
$$

then

$$
\partial_t(\Phi-\Psi)\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + a^{-2}\partial_i(\Psi-\Phi) + H\frac{\mathrm{d}x^i}{\mathrm{d}t} + 2\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}(\Phi-\Psi)}{\mathrm{d}t} \approx 0.
\tag{192}
$$

Assuming that the light propagates along the $z-$ direction, and in the weal lensing limit, then ($z \rightarrow l$)

$$
\frac{\mathrm{d}^2 l}{\mathrm{d}t^2} + H\frac{\mathrm{d}l}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + a^{-2}\partial_x(\Psi-\Phi) + H\frac{\mathrm{d}x}{\mathrm{d}t} \approx 0, \qquad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + a^{-2}\partial_y(\Psi-\Phi) + H\frac{\mathrm{d}y}{\mathrm{d}t} \approx 0,
\tag{193}
$$

the first equation leads to

$$
\frac{\dot{a}}{a} = -\frac{\ddot{l}}{\dot{l}} \rightarrow a\frac{\mathrm{d}l}{\mathrm{d}t} = constant = 1 \rightarrow adl = dt,
\tag{194}
$$

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then the second and third equations becomes as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a\frac{\mathrm{d}x}{\mathrm{d}t}\right)+a^{-1}\partial_x(\Psi-\Phi)=0\rightarrow\frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{\mathrm{d}x}{\mathrm{d}l}\right)+\partial_x(\Psi-\Phi)=0\rightarrow\frac{\mathrm{d}x}{\mathrm{d}l}=\left.\frac{\mathrm{d}x}{\mathrm{d}l}\right|_0+\int_0^l\partial_x(\Phi-\Psi)(l')dl',\quad(195)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(a\frac{\mathrm{d}y}{\mathrm{d}t}\right)+a^{-1}\partial_y(\Psi-\Phi)=0\rightarrow\frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{\mathrm{d}y}{\mathrm{d}l}\right)+\partial_y(\Psi-\Phi)=0\rightarrow\frac{\mathrm{d}y}{\mathrm{d}l}=\left.\frac{\mathrm{d}y}{\mathrm{d}l}\right|_0+\int_0^l\partial_y(\Phi-\Psi)(l')dl'.\quad(196)$$

If the emitter is at $l=\chi$, and the observer is at $l=0$ (Weak lensing approximation???),

$$0=\int_0^\chi\frac{\mathrm{d}x}{\mathrm{d}l}dl=\int_0^\chi\left[\left.\frac{\mathrm{d}x}{\mathrm{d}l}\right|_0+\int_0^l\partial_x(\Phi-\Psi)(l')dl'\right]dl=\chi\left.\frac{\mathrm{d}x}{\mathrm{d}l}\right|_0+\int_0^\chi dl\int_0^l\partial_x(\Phi-\Psi)(l')dl',\quad(197)$$

by changing the integration order,

$$\chi\left.\frac{\mathrm{d}x}{\mathrm{d}l}\right|_0=-\int_0^\chi dl'\ \partial_x(\Phi-\Psi)(l')\int_{l'}^\chi dl=-\int_0^\chi dl\ \partial_x(\Phi-\Psi)\cdot(\chi-l),\quad(198)$$

which leads to

$$\left.\frac{\mathrm{d}x}{\mathrm{d}l}\right|_0=\int_0^\chi dl\ \partial_x(\Psi-\Phi)\frac{\chi-l}{\chi}\rightarrow\theta_{\text{source}}^x=\theta_{\text{image}}^x+\int_0^\chi dl\ \partial_x(\Phi-\Psi)\left(1-\frac{l}{\chi}\right),\quad(199)$$

where θ^i is the angular position. The same for the $y-$direction.

Using $ds^2=0$, and $dt=ad\tau$, there is also

$$ds^2=-(1+2\Psi)dt^2+a^2(1+2\Phi)\left(dx^2+dy^2+dz^2\right)=a^2(1+2\Phi)\left\{-(1+2\Psi-2\Phi)d\tau^2+dx^2+dy^2+dz^2\right\}=0,\quad(200)$$

thus, the "velocity"

$$V=\sqrt{\frac{dx^2+dy^2+dz^2}{d\tau^2}}=1+\Psi-\Phi,\quad(201)$$

the time interval between emission and reception is

$$T=\int_{\text{emission}}^{\text{reception}}d\tau=\int_0^\chi\frac{dz}{1+\Psi-\Phi}\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2}$$
$$\approx\int_0^\chi\left[1+\Phi-\Psi+\frac{1}{2}\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2+\frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}z}\right)^2\right]dz,\quad(202)$$

using Fermat's principle, and notice that (the same for y part)

$$\frac{1}{2}\left[\frac{\mathrm{d}(x+\delta x)}{\mathrm{d}z}\right]^2-\frac{1}{2}\left[\frac{\mathrm{d}x}{\mathrm{d}z}\right]^2=\frac{1}{2}\frac{\partial}{\partial x}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)^2\right]\delta x=\frac{\mathrm{d}x}{\mathrm{d}z}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}x}{\mathrm{d}z}\right)\delta x=\frac{\mathrm{d}x}{\mathrm{d}z}\frac{\mathrm{d}\delta x}{\mathrm{d}z},\quad(203)$$

then

$$\delta T=\int_0^\chi\left[\partial_x(\Phi-\Psi)\delta x+\partial_y(\Phi-\Psi)\delta y+\frac{\mathrm{d}x}{\mathrm{d}z}\frac{\mathrm{d}\delta x}{\mathrm{d}z}+\frac{\mathrm{d}y}{\mathrm{d}z}\frac{\mathrm{d}\delta y}{\mathrm{d}z}\right]$$
$$=\int_0^\chi\left\{\left[\partial_x(\Phi-\Psi)-\frac{\mathrm{d}^2x}{\mathrm{d}z^2}\right]\delta x+\left[\partial_y(\Phi-\Psi)-\frac{\mathrm{d}^2y}{\mathrm{d}z^2}\right]\delta y+\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}x}{\mathrm{d}z}\delta x+\frac{\mathrm{d}y}{\mathrm{d}z}\delta y\right)\right\}dz\quad(204)$$
$$=0,$$

the last term is surely to be zero, and

$$\partial_x(\Phi-\Psi)=\frac{\mathrm{d}^2x}{\mathrm{d}z^2},\qquad\partial_y(\Phi-\Psi)=\frac{\mathrm{d}^2y}{\mathrm{d}z^2}.\quad(205)$$

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The lensing potential (copied from the lecture PPT 15), with $\Phi = -\Psi$ is

$$\vec{\theta}_I - \vec{\theta}_S = 2 \int_0^\chi dl \ \boldsymbol{\nabla}\Psi \cdot \left(\frac{\chi - l}{\chi}\right) = 2\boldsymbol{\nabla}_\theta \int_0^\chi dl \ \Psi \cdot \left(\frac{\chi - l}{\chi l}\right) = \boldsymbol{\nabla}_\theta \varphi(\vec{\theta}_I), \tag{206}$$

then

$$\varphi = 2 \int_0^\chi dl \ \Psi \cdot \left(\frac{\chi - l}{\chi l}\right), \tag{207}$$

and

$$\nabla_\theta^2 \varphi = 2 \int_0^\chi dl \ l^2 \partial_\perp^2 \Psi \cdot \frac{\chi - l}{\chi l} \approx 2 \int_0^\chi dl \ \nabla^2\Psi \cdot \frac{l(\chi - l)}{\chi}, \tag{208}$$

with the Poission equation

$$\nabla^2 \Psi = 4\pi G a^2 \rho, \tag{209}$$

then

$$\nabla_\theta^2 \varphi \approx 8\pi G \int_0^\chi dl \ \rho a^2 \cdot \frac{l(\chi - l)}{\chi}, \tag{210}$$

and for thin lens,

$$\nabla_\theta^2 \varphi \approx \frac{a_L l_L (l_S - l_L)}{l_S} \cdot 8\pi G \int a dl \cdot \rho \equiv \frac{a_L l_L (l_S - l_L)}{l_S} \cdot 8\pi G \Sigma, \tag{211}$$

with L, S respectively meaning lens and source, Σ is the surface density, the above formula can be rewritten as

$$\nabla_\theta^2 \varphi = \frac{D_L^A D_{LS}^A}{D_S^A} \frac{8\pi G}{c^2} \Sigma \equiv 2\frac{\Sigma}{\Sigma_c}, \tag{212}$$

with the surface density and the critical surface density

$$\Sigma = \int a dl \cdot \rho, \qquad \Sigma_c = \frac{c^2}{4\pi G} \frac{D_S^A}{D_L^A D_{LS}^A}. \tag{213}$$

1.8 Dynamics

• Dynamical friction: a particle of mass M and velocity V passing through another particle of mass m and $v = 0$, $M \gg m$, impact parameter b. The final velocity of m

$$v_\perp = \int_{-\infty}^\infty \frac{GM}{r^2} \frac{b}{r} dt = \int_{-\infty}^\infty \frac{GMb}{(x^2 + b^2)^{\frac{3}{2}}} \frac{dx}{V} = \frac{GM}{bV} \int_{-\infty}^\infty \frac{dy}{(y^2 + 1)^{\frac{3}{2}}} = \frac{2GM}{bV}, \qquad (y \equiv x/b) \tag{214}$$

$$v_{x\ \text{Final}} = \sqrt{V^2 - V_\perp^2} = \sqrt{V^2 - \left(\frac{2GM}{bV}\right)^2} \approx V\left[1 - 2\left(\frac{GM}{bV^2}\right)^2\right]. \tag{215}$$

The momentum loss every unit time is

$$2V\left(\frac{GM}{bV^2}\right)^2 dm = 4\pi V^2 \left(\frac{GM}{bV^2}\right)^2 \rho b \ dt \ db = \frac{G^2 M^2}{V^2} \frac{db}{b} 4\pi \rho dt, \tag{216}$$

the equivalent dynamical friction is

$$f_{\text{equal}} = 4\pi \rho \frac{G^2 M^2}{V^2} \int \frac{db}{b}. \tag{217}$$

The scattering angle $\theta \approx V_\perp/V \approx 2GM/(bV^2)$. For a collision to happen, $b \sim Gm/V^2$, the number of collisions a star experiences is

$$N_c \sim nb^2 V \Delta t \sim n\left(\frac{Gm}{V^2}\right)^2 V \Delta t \sim \frac{G^2 m \rho}{V^3} \Delta t, \qquad (n = \rho/m) \tag{218}$$

the velocity of the star with the virial theorem

$$V^2 \sim \frac{GM}{R} \sim \frac{GM}{(M/\rho)^{\frac{1}{3}}} \sim GM^{\frac{2}{3}} \rho^{\frac{1}{3}}, \tag{219}$$

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then

$$N_c \sim \frac{G^2 m \rho}{G^{\frac{2}{3}} M \rho^{\frac{1}{2}}} \Delta t \sim \frac{\sqrt{G\rho}}{N_{\text{star}}} \Delta t = \frac{1}{N_{\text{star}}} \frac{\Delta t}{T_{\text{dyn}}}, \tag{220}$$

with M the total mass of the whole system, $N_{\text{star}} \equiv M/m$, $T_{\text{dyn}} \equiv 1/\sqrt{G\rho}$.

The Boltzmann equation with the Liouville's theorem

$$\frac{\mathrm{d}f(\boldsymbol{x}, \boldsymbol{v}, t)}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} f + \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \boldsymbol{\nabla}_v f = \frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} f - \boldsymbol{\nabla} \Psi \cdot \boldsymbol{\nabla}_v f = 0. \tag{221}$$

If the system is steady, then $\partial f / \partial t = 0$, then

$$\boldsymbol{v} \cdot \boldsymbol{\nabla} f + \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \cdot \boldsymbol{\nabla}_v f = \left(\dot{r} \frac{\partial f}{\partial r} + \dot{\theta} \frac{\partial f}{\partial \theta} + \dot{\phi} \frac{\partial f}{\partial \phi} \right) + \left(\dot{v}_r \frac{\partial f}{\partial v_r} + \dot{v}_\theta \frac{\partial f}{\partial v_\theta} + \dot{v}_\phi \frac{\partial f}{\partial v_\phi} \right) = 0. \tag{222}$$

With the relation

$$\ddot{\boldsymbol{r}} = -\frac{\partial \Psi}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial \Psi}{\partial \theta} - \frac{1}{r \sin\theta} \frac{\partial \Psi}{\partial \phi} \hat{\phi}, \tag{223}$$

and the expressions of the time derivatives of the vectors, there is

$$\dot{v}_r = \frac{1}{r} \left(v_\theta^2 + v_\phi^2 \right) - \frac{\partial \Psi}{\partial r}, \quad \dot{v}_\theta = \frac{1}{r} \left(v_\phi^2 \cot\theta - v_r v_\theta - \frac{\partial \Psi}{\partial \theta} \right), \quad \dot{v}_\phi = -\frac{1}{r} \left(v_r v_\phi + \cot\theta v_\theta v_\phi + \frac{1}{\sin\theta} \frac{\partial \Psi}{\partial \phi} \right). \tag{224}$$

with $df/dt = 0$, there is

$$\frac{1}{4\pi} \int \sin\theta d\theta d\phi \int dv_r \int dv_\theta \int dv_\phi \left(v_r \cdot \frac{\mathrm{d}f}{\mathrm{d}t} \right), \tag{225}$$

which will lead to the Jeans equation

$$-\frac{\partial \Psi}{\partial r} = \frac{1}{\rho} \frac{\partial (\rho \sigma_r^2)}{\partial r} + \frac{2\sigma_r^2 - (\sigma_\theta^2 + \sigma_\phi^2)}{r}, \tag{226}$$

with

$$\rho(r) = \frac{1}{4\pi} \int \sin\theta d\theta d\phi \int dv_r \int v_\theta \int dv_\phi f, \qquad \rho \sigma_i^2 = \frac{1}{4\pi} \int \sin\theta d\theta d\phi \int dv_r dv_\theta dv_\phi (v_i^2 f). \tag{227}$$

Assuming the isothermal density profile, $\sigma_r = \sigma_\theta = \sigma_\phi = \sigma = \text{const}$, then

$$-\frac{\mathrm{d}\Psi}{\mathrm{d}r} = \frac{1}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}r} \sigma^2 = -\frac{GM(<r)}{r^2}. \tag{228}$$

When r is small, $\rho \propto r^\alpha$, $M(<r) \propto \rho r^3 \propto r^{\alpha+3}$, $(1/\rho)(d\rho/dr) \propto r^{-1}$, $M(<r)/r^2 \propto r^{\alpha+1}$, then $\alpha = -2$, $\rho \propto r^{-2}$. The hydrostatic equilibrium

$$-\frac{1}{r^2} \frac{\mathrm{d}[r^2 p_{\text{gas}}(r)]}{\mathrm{d}r} - \rho_{\text{gas}}(r) \frac{\mathrm{d}\Psi}{\mathrm{d}r} + \frac{2}{r} p_{\text{gas}}(r) = 0, \tag{229}$$

for galaxies,

$$-\frac{\partial \Psi}{\partial r} = \frac{1}{\rho} \frac{\partial (\rho \sigma_r^2)}{\partial r} + \frac{2\sigma_r^2 - (\sigma_\theta^2 + \sigma_\phi^2)}{r}, \tag{230}$$

thus,

$$-\frac{GM(<r)}{\sigma_r^2 r} = \frac{\mathrm{d}\log\rho}{\mathrm{d}\log r} + \frac{\mathrm{d}\log\sigma_r^2}{\mathrm{d}\log r} + 2 - \frac{(\sigma_\theta^2 + \sigma_\phi^2)}{\sigma_r^2}, \tag{231}$$

the mass

$$M(<r) = -\frac{\sigma_r^2 r}{G} \left\{ \frac{\mathrm{d}\log\rho}{\mathrm{d}\log r} + \frac{\mathrm{d}\log\sigma_r^2}{\mathrm{d}\log r} + 2 \left[1 - \frac{(\sigma_\theta^2 + \sigma_\phi^2)}{2\sigma_r^2} \right] \right\}. \tag{232}$$

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1.9 Redshift Space Distortion

Consider a galaxy with redshift z, and peculiar velocity $v = dx/dt$ (> 0, to the observer, otherwise, < 0), then its observed redshift is calculated as

$$\int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} \frac{dt}{a(t)}, \qquad \int_{x_1+v\Delta t_1}^{x_2} dx = \int_{t_1+\Delta t_1}^{t_2+\Delta t_2} \frac{dt}{a(t)}, \tag{233}$$

then

$$-v\Delta t_1 = \frac{\Delta t_2}{a(t_2)} - \frac{\Delta t_1}{a(t_1)} \rightarrow \Delta t_1 \left(\frac{1}{a(t_1)} - v\right) = \frac{\Delta t_2}{a(t_2)} \rightarrow 1 + z_{\text{obs}} = \frac{\lambda_2}{\lambda_1} = \frac{1}{a(z)} - v = \frac{1}{a_{\text{obs}}} = 1 + z - v, \tag{234}$$

so $\Delta z = -v/c$ ($c = 1$). With $\Delta x = c\,\Delta z/H$, the redshift space distance s for a galaxy with radial peculiar velocity v_r, and the real distance r is

$$s = r + \frac{v_r}{H}. \tag{235}$$

2 Concepts part

2.1 Sunyaev-Zeldovich (SZ) effect

• Thermal SZ effect: Photons passing through the cluster will undergo (inverse) Compton scattering by collisions with electrons in the ionised IGM, causing distortions in the spectrum of the emergent radiation. CMB can be used to detect such distortions. The intensity of the low-frequency photons will decrease, while the intensity of the high-frequency photons will increase (low-energy photons$\rightarrow$ high-energy photons).
• Kinetic SZ effect: Caused when a galaxy cluster is moving relative to the Hubble flow, a second-order effect where the CMB photons interact with electrons that have high energies due to their bulk motion. The effect is relatively weak for most galaxy clusters.

2.2 Sachs-Wolfe (SW) effect

CMB photons are gravitationally redshifted, causing the CMB spectrum to appear uneven.
• Non-integrated SW effect: CMB photons gravitationally redshifted at the surface of last scattering. Photons lose energy when getting out of the gravitational "wells". Occuring at the end part of the radiation dominating stage, early part of the matter domination stage.
• Integrated SW effect: CMB photons gravitationally redshifted between the surface of last scattering and the Earth (not part of the primordial CMB). Occuring at the stages that the universe is dominated in its energy density by something other than matter.

2.3 Formation of Compact Objects

• Kelvin-Helmholtz time: $t \sim \frac{GM^2}{RL}$, where L is the stellar luminosity.
• According to the virial theorem, $T \propto M/R$, when $n^{-1/3}p \sim \hbar$, $n^{-1/3}p \propto (M/R^3)^{-1/3}\sqrt{T} \sim \hbar$, then $M^{2/3} \sim T^{1/2}$, then $T_{\text{max}} \approx 6 \times 10^8 (M/M_{\text{sun}})^{4/3}$ K.
• The Jeans mass $M_J = \rho\lambda_J^3 \sim \left(\frac{k_{\text{B}}T}{G\mu m_{\text{H}}}\right)^{3/2} \rho^{-1/2}$. The Jeans length $\lambda_J = c_s\sqrt{\pi/(G\rho)}$.

• The cooling luminosity to be sufficient to account for the rate of increasing binding energy is $L \sim \left(\frac{GM^2}{R}\right)(G\rho)^{1/2}$. With $L \leq 4\pi R^2 \sigma T^4$ (black body radiation) and the virial theorem $k_{\text{B}}T \sim G\mu m_{\text{p}}/(3R)$, there is the mass of fragmentation limit

$$M \geq 6\mu^{-2}\left(\frac{k_{\text{B}}T}{\mu m_{\text{p}}c^2}\right)^{1/4}\left(\frac{\hbar c}{Gm_{\text{p}}^2}\right)^{3/2} m_{\text{p}}. \tag{236}$$