Stellar Structure and Evolution Note

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Grand Unified Theory of the Stellar Physics:

$$k_{\mathrm{B}}T_c \approx \frac{GM_* m_{\mathrm{p}}}{R_*}$$

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Contents

1 Formula Part

1.1 Basic Structure Related Equations

$$\frac{\partial P}{\partial r} = -\rho g = -\frac{Gm}{r^2}\rho, \qquad \frac{\partial m}{\partial r} = 4\pi r^2 \rho, \to \ \frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4}. \tag{1}$$

Shell acceleration equation

$$\frac{\mathrm{d}m}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} = f_P + f_g \to \ \frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} = -\frac{\partial P}{\partial m} - \frac{Gm}{4\pi r^4}. \tag{2}$$

Free-fall time scale (estimated with the Sun)

$$\tau_{\mathrm{ff}}(\tau_{\mathrm{hydr}}) \sim \left(\frac{R_*^3}{GM}\right)^{1/2} \sim 30\text{ min} \sim 10^3 \text{ sec}. \tag{3}$$

The Virial theorem

$$\int_0^M 4\pi r^3 \frac{\partial P}{\partial m} dm = \left[4\pi r^3 P\right]|_0^M - \int_0^M 12\pi r^2 \frac{\partial r}{\partial m} P dm, \tag{4}$$

with (1),

$$\int_0^M \frac{Gm}{r} dm = 3\int_0^M \frac{P}{\rho} dm = 3\int_0^{V_*} PdV. \tag{5}$$

Derivation of GUT (the Virial theorem aspect)

$$3\int_0^{V_*} PdV = -E_g \sim \frac{GM^2}{R} \to nk_{\mathrm{B}}TR^3 \sim \frac{GM^2}{R} \to \frac{\rho}{m_p}k_{\mathrm{B}}TR^3 \sim \frac{M}{R^3 m_p}k_{\mathrm{B}}TR^3 \sim \frac{GM^2}{R} \to k_{\mathrm{B}}T_c \sim \frac{GMm_p}{R}. \tag{6}$$

Derivation of GUT (the structure equation (1) aspect)

$$\frac{P-P_c}{M} \sim -\frac{G\cdot\frac{M}{2}}{4\pi\left(\frac{R}{2}\right)^4} \sim -\frac{GM}{R^4} \to P_c \approx \rho_c \frac{k_{\mathrm{B}}}{\mu m_{\mathrm{u}}}T_c \sim \frac{M}{R^3}\frac{k_{\mathrm{B}}}{\mu m_{\mathrm{u}}}T_c \sim -\frac{GM}{R^4} \to k_{\mathrm{B}}T_c \sim \frac{GMm_{\mathrm{u}}}{R} \sim \frac{GMm_p}{R}. \tag{7}$$

1.2 Chemistry Basis

Classical gas and radiation gas

$$P_{\mathrm{class}} = nk_{\mathrm{B}}T = \frac{\rho}{\mu m_{\mathrm{u}}}k_{\mathrm{B}}T = \rho\frac{\mathscr{R}}{\mu}T, \qquad P_{\mathrm{rad}} = \frac{1}{3}aT^4. \tag{8}$$

The mean molecular weight

$$P = nk_{\mathrm{B}}T = \mathscr{R}\sum_i \frac{X_i(1+Z_i)}{\mu_i}\rho T = \rho\frac{\mathscr{R}}{\mu}T \to \frac{1}{\mu} = \sum_i \frac{X_i(1+Z_i)}{\mu_i}, \tag{9}$$

for electrons,

$$\frac{1}{\mu_e} = \sum_i \frac{X_i Z_i}{\mu_i}, \ n_e = \frac{\rho}{\mu_e m_{\mathrm{u}}}. \tag{10}$$

Defining $X \equiv X_{\mathrm{H}},\ Y \equiv X_{\mathrm{He}}, 1-X-Y = X_{\mathrm{others}}$, assuming the elements after He satisfy $\mu_i/Z_i \approx 2$,

$$\mu_e \approx \left[X + \frac{1}{2}Y + \frac{1}{2}(1-X-Y)\right]^{-1} = \frac{2}{1+X}, \tag{11}$$

electron opacity ($n_e\sigma_{\mathrm{T}} = \kappa_e\rho = \kappa_e n_e \mu_e m_{\mathrm{u}}$)

$$\kappa_e = \frac{\sigma_{\mathrm{T}}}{\mu_e m_{\mathrm{H}}} = \frac{0.4\left(\mathrm{cm}^2\cdot\mathrm{g}^{-1}\right)}{\mu_e} \approx 0.2(1+X)\left(\mathrm{cm}^2\cdot\mathrm{g}^{-1}\right). \tag{12}$$

For ions, $Y \approx 1-X$, and

$$\mu_I \approx \left[X + \frac{1}{4}(1-X)\right]^{-1} = \frac{4}{1+3X} \to \mu = \left(\frac{1}{\mu_I} + \frac{1}{\mu_e}\right)^{-1} \approx \frac{4}{3+5X}. \tag{13}$$

1.3 Thermodynamics

The first law of thermodynamics is $dq = du + Pdv$, specific heats

$$c_p \equiv \left(\frac{\mathrm{d}q}{\mathrm{d}T}\right)_P = \left(\frac{\partial u}{\partial T}\right)_P + P\left(\frac{\partial v}{\partial T}\right)_P, c_v = \left(\frac{\mathrm{d}q}{\mathrm{d}T}\right)_v = \left(\frac{\partial u}{\partial T}\right)_v. \tag{14}$$

Below contents, more details in textbook *SSE* Pages 25-27. Defining

$$\alpha \equiv \left(\frac{\partial \log \rho}{\partial \log P}\right)_T, \ \delta \equiv -\left(\frac{\partial \log \rho}{\partial \log T}\right)_P, \ \varphi \equiv \left(\frac{\partial \log \rho}{\partial \log \mu}\right), \tag{15}$$

it can be derived that

$$c_p - c_v = \frac{P\delta^2}{\rho T \alpha}, \ dq = \left(c_v + \frac{P\delta^2}{\rho T\alpha}\right)dT - \frac{\delta}{\rho}dP = c_p dT - \frac{\delta}{\rho}dP. \tag{16}$$

defining adiabatic temperature gradient

$$\nabla_{\mathrm{ad}} \equiv \left(\frac{\mathrm{d}\log T}{\mathrm{d}\log P}\right)_s = \frac{P\delta}{T\rho c_p}. \tag{17}$$

Apparently, for ideal classical gas $\alpha = \delta = 1$, and $c_p - c_v = \mathscr{R}/\mu$, and $c_v = \frac{3}{2}\frac{\mathscr{R}}{\mu}$, then $\nabla_{\mathrm{ad}} = \frac{2}{5}$. More rigorously (complete derivation process in Chapter 13 of textbook *SSE*, pages 123-125),

$$P = P_{\mathrm{gas}} + P_{\mathrm{rad}} = \frac{\mathscr{R}}{\mu}\rho T + \frac{a}{3}T^4 \rightarrow \rho = \frac{\mu}{\mathscr{R}T}\left(P - \frac{1}{3}aT^4\right), \tag{18}$$

the gas pressure ratio

$$\beta \equiv \frac{P_{\mathrm{gas}}}{P}, \quad 1 - \beta = \frac{P_{\mathrm{rad}}}{P}, \tag{19}$$

with definitions (15),

$$\alpha = \frac{1}{\beta}, \quad \delta = \frac{4-3\beta}{\beta}, \quad \varphi = 1, \tag{20}$$

and

$$\nabla_{\mathrm{ad}} = \frac{P\delta}{T\rho c_p} = \frac{P_{\mathrm{gas}}\delta}{\beta T\rho c_p} = \frac{\mathscr{R}\delta}{\beta\mu c_p} = \left(1 + \frac{(1-\beta)(4+\beta)}{\beta^2}\right) / \left(\frac{5}{2} + \frac{4(1-\beta)(4+\beta)}{\beta^2}\right). \tag{21}$$

```
Wolfram Language 14.1.0 Engine for Mac OS X ARM (64-bit)
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In[1]:= f[x_]:=(1+(1-x)*(4+x)/x^2)/(5/2+4*(1-x)*(4+x)/x^2);

In[2]:= Limit[f[x],x->0]

         1
Out[2]= -
         4

In[3]:= Limit[f[x],x->1]

         2
Out[3]= -
         5
```

- For radiation gas ($\beta \to 0$), $\nabla_{\mathrm{ad}} = 1/4$.
- For perfect monatomic gas ($\beta \to 1$), $\nabla_{\mathrm{ad}} = 2/5$.

Defining adiabatic index

$$\gamma_{\mathrm{ad}} = \left(\frac{\mathrm{d}\log P}{\mathrm{d}\log \rho}\right)_{\mathrm{ad}}, \tag{22}$$

with

$$\left(\frac{\mathrm{d}\log \rho}{\mathrm{d}\log P}\right)_{\mathrm{ad}} = \left(\frac{\partial \log \rho}{\partial \log P}\right)_{\mathrm{ad}} + \left(\frac{\partial \log \rho}{\partial \log T}\right)\left(\frac{\partial \log T}{\partial \log P}\right)_{\mathrm{ad}} = \alpha - \delta\nabla_{\mathrm{ad}}, \tag{23}$$

then

$$\gamma_{\rm ad} = \frac{1}{\alpha - \delta \nabla_{\rm ad}}. \tag{24}$$

```
In[1]:= f[x_]:=(1+(1-x)*(4+x)/x^2)/(5/2+4*(1-x)*(4+x)/x^2);

In[2]:= alpha[x_]:=1/x;

In[3]:= delta[x_]:=(4-3*x)/x;

In[4]:= g[x_]:=1/(alpha[x]-delta[x]*f[x]);

In[5]:= Limit[g[x],x->1]

        5
Out[5]= -
        3

In[6]:= Limit[g[x],x->0]

        4
Out[6]= -
        3
```

- For perfect monatomic gas ($\beta \to 1$), $\gamma_{\rm ad} = 5/3$.
- For radiation gas ($\beta \to 0$), $\gamma_{\rm ad} = 4/3$.

Opposite path in textbook *SI* pages 174-175, $P \propto \rho^{\chi_\rho} T^{\chi_T}$ (equivalent with α and δ expressions), for adiabatic case, and with energy-pressure relation $u = \phi(P/\rho)$ (internal energy per mass),

$$du = -Pd\left(\frac{1}{\rho}\right) = \frac{P}{\rho^2} d\rho = \phi \frac{dP}{\rho} - \phi \frac{P}{\rho^2} d\rho \to \frac{dP}{P} = \frac{\phi + 1}{\phi} \frac{d\rho}{\rho}, \tag{25}$$

with the definition of (22), $\gamma_{\rm ad} = (\phi + 1)/\phi = 5/3$ for classical gas with $\phi = 3/2$, and $\gamma_{\rm ad} = 4/3$ for radiation gas with $\phi = 3$. Based on the power relation there is

$$\frac{dT}{T} = \frac{1}{\chi_T} \frac{dP}{P} - \frac{\chi_\rho}{\chi_T} \frac{d\rho}{\rho} \to \nabla_{\rm ad} = \left(\frac{\partial \log T}{\partial \log P}\right)_{\rm ad} - \left(\frac{\partial \log T}{\partial \log \rho}\right) \left(\frac{\partial \log \rho}{\partial \log P}\right)_{\rm ad} = \frac{1}{\chi_T} - \frac{\chi_\rho}{\chi_T} \frac{1}{\gamma_{\rm ad}} = \frac{\gamma_{\rm ad} - \chi_\rho}{\gamma_{\rm ad} \cdot \chi_T}. \tag{26}$$

- For ideal gas, $P \propto \rho T$, $\gamma_{\rm ad} = 5/3$, then $\nabla_{\rm ad} = 2/5$.
- For radiation gas, $P \propto T^4$, $\gamma_{\rm ad} = 4/3$, then $\nabla_{\rm ad} = 1/4$.

In *SSE*, there is $\nabla_{\rm ad} \to \gamma_{\rm ad}$, and in *SI*, there is $\gamma_{\rm ad} \to \nabla_{\rm ad}$.
Now we have, for a perfect gas,

$$\frac{P}{\rho} = \frac{\mathscr{R}}{\mu} T = (c_P - c_v) T = (\gamma - 1) c_v T = (\gamma - 1) u, \tag{27}$$

defining

$$\zeta u \equiv 3(\gamma - 1) u = 3 \frac{P}{\rho}, \tag{28}$$

with the virial theorem

$$E_{\rm i} \equiv \int_0^M u dm, \;\; E_{\rm g} \equiv -\int_0^M \frac{Gm}{r} dm \to \;\; \zeta E_{\rm i} + E_{\rm g} = 0. \tag{29}$$

Then the total energy

$$W = E_{\rm i} + E_{\rm g} = (1 - \zeta) E_{\rm i} = \frac{\zeta - 1}{\zeta} E_{\rm g}, \tag{30}$$

the luminosity

$$L = -\frac{{\rm d}W}{{\rm d}t} = (\zeta - 1) \frac{{\rm d}E_{\rm i}}{{\rm d}t} = -\frac{\zeta - 1}{\zeta} \frac{{\rm d}E_{\rm g}}{{\rm d}t}. \tag{31}$$

Notice that for $\gamma = 4/3$, $\zeta = 1$, then $W = 0$, while for $\gamma = 5/3$, $\zeta = 2$, then $L = -\dot{E}_{\rm g}/2 = \dot{E}_{\rm i} > 0$.

1.4 Quantum Statistics

Generally, particle number, internal energy, pressure

$$n=\int_0^{\infty} n(p)dp,\quad u=\int_0^{\infty} n(p)\varepsilon_k dp,\quad P=\frac{1}{3}\int_0^{\infty} pvn(p)dp=\frac{1}{3}n\left\langle pv\right\rangle. \tag{32}$$

Kinetic energy, velocity-momentum relation ($c=1$)

$$\varepsilon_k=\sqrt{p^2+m^2}-m,\ v=\frac{\partial\varepsilon}{\partial p}=\frac{p}{\sqrt{p^2+m^2}}. \tag{33}$$

Non-relativistic case,

$$u\approx\int_0^{\infty} n(p)\frac{p^2}{2m}dp,\quad P\approx\frac{1}{3}\int_0^{\infty} n(p)\frac{p^2}{m}dp\rightarrow\ P=\frac{2}{3}u. \tag{34}$$

Relativistic case (radiation gas),

$$u\approx\int_0^{\infty} n(p)pdp,\quad P\approx\frac{1}{3}\int_0^{\infty} n(p)pdp\rightarrow P=\frac{1}{3}u. \tag{35}$$

Example of degenerate electron gas (degeneracy number $\mathfrak{g}=2$)

$$n_e=\mathfrak{g}\times\int_0^{p_{\rm F}}\frac{4\pi p^2}{h^3}dp=\frac{8\pi}{3h^3}p_{\rm F}^3\rightarrow p_{\rm F}\sim n_e^{\frac{1}{3}}, \tag{36}$$

$$P_e=2\times\frac{1}{4\pi}\int_0^{2\pi}{\rm d}\varphi\int_0^{\frac{\pi}{2}}{\rm d}\theta\int_0^{p_{\rm F}}{\rm d}p\ \mathfrak{g}\times\frac{4\pi p^2}{h^3}v(p)p\cos^2\theta\sin\theta=\frac{8\pi}{3h^3}\int_0^{p_{\rm F}}{\rm d}p\ p^3v(p). \tag{37}$$

With $\varepsilon_k=\sqrt{m_e^2c^4+p^2c^2}-m_ec^2$, $v(p)=\partial\varepsilon_k/\partial p$, it can be obtained that

$$P_e=\frac{\pi m_e^4c^5}{3h^3}\left[x(2x^2-3)(x^2+1)^{\frac{1}{2}}+3{\rm arcsinh}(x)\right]\equiv\frac{\pi m_e^4c^5}{3h^3}f(x).\ \left(x_{\rm F}\equiv\frac{p_{\rm F}}{m_ec}\right) \tag{38}$$

Internal energy

$$U_e=\mathfrak{g}\times\int_0^{p_{\rm F}}\varepsilon_k\frac{4\pi p^2}{h^3}dp=\frac{\pi m_e^4c^5}{3h^3}\left\{8x^3\left[(x^2+1)^{\frac{1}{2}}-1\right]-f(x)\right\}\equiv\frac{\pi m_e^4c^5}{3h^3}g(x). \tag{39}$$

Non-relativistic case, $x\ll 1$, expand P_e and U_e at $x=0$, and reserve the lowest-power term,

$$P_e\approx\frac{8\pi m_e^4c^5}{15h^3}x^5,\ U_e\approx\frac{4\pi m_e^4c^5}{5h^3}x^5\rightarrow P_e=\frac{2}{3}U_e. \tag{40}$$

Relativistic case, $x\gg 1$, expand P_e and U_e at $x\rightarrow\infty$, and reserve the highest-power term,

$$P_e\approx\frac{2\pi m_e^4c^5}{3h^3}x^4,\ U_e\approx\frac{2\pi m_e^4c^5}{h^3}x^4\rightarrow P_e=\frac{1}{3}U_e. \tag{41}$$

Numerical forms, with relation (36),

$$P_e=K_{\rm non-rel}\left(\frac{\rho}{\mu_e}\right)^{\frac{5}{3}}=\left[1.0036\times10^{13}\ ({\rm erg\ g^{-\frac{5}{3}}\ cm^2})\right]\left(\frac{\rho}{\mu_e}\right)^{\frac{5}{3}},\qquad (x\ll 1) \tag{42}$$

$$P_e=K_{\rm rel}\left(\frac{\rho}{\mu_e}\right)^{\frac{4}{3}}=\left[1.2435\times10^{15}\ ({\rm erg\ g^{-\frac{4}{3}}\ cm})\right]\left(\frac{\rho}{\mu_e}\right)^{\frac{4}{3}},\qquad (x\gg 1) \tag{43}$$

The boundary between non-relativistic and relativistic cases is $x=1$, with $\rho=\mu_em_{\rm u}n_e(x=1)\approx10^6\mu_e\ {\rm g\ cm^{-3}}$. Similarly, for the pure neutron gas, just make changes with $\mu_e\rightarrow\mu_{\rm n}\sim1$, $n_e\rightarrow n_{\rm n}$, m_e and $m_{\rm u}\rightarrow m_{\rm n}$, and the boundary density is $\rho=m_{\rm n}n_{\rm n}(x=1)\approx6.1\times10^{15}\ {\rm g\ cm^{-3}}$.

• One important mind exercise here!!! For extreme-relativistic electron gas, $P_e\propto\rho^{\frac{4}{3}}$, while for extreme-relativistic pure neutron gas, $P_{\rm n}\propto\rho$, why?

• The key point is that the electron gas is not pure, and $m_{\rm u}\gg m_e$, so the momentum of electrons will not affect the masses of heavy particles like protons, and as $\rho=\mu_en_em_{\rm u}$, so $P_e\propto p_{\rm F}^4\propto n_e^{\frac{4}{3}}\propto\rho^{\frac{4}{3}}$.

• But for pure neutron gas, $\rho=n_{\rm n}m_{\rm n}$, and when $p_{\rm F}\gg m_{\rm n}c$, there is $E_{\rm tot}=m_{\rm n}c^2+E_k\approx E_k$ for a single neutron. As $m'_{\rm n}=E_{\rm tot}/c^2\sim p_{\rm F}$, and $n_{\rm n}\sim p_{\rm F}^3$, with $P_{\rm n}\sim p_{\rm F}^4$, there is $\rho'=n_{\rm n}m'_{\rm n}\sim p_{\rm F}^4\sim P_{\rm n}$.

• For degenerate electron gas, $P_e\propto\rho^{\frac{5}{3}}$ (non-relativistic), $P_e\propto\rho^{\frac{4}{3}}$ (relativistic).

• For pure neutron gas, $P_{\rm n}\propto\rho^{\frac{5}{3}}$ (non-relativistic), $P_{\rm n}\propto\rho$ (relativistic).

1.5 Hydrodynamics

Energy flux ($l_{\rm free}$ is the free path)

$$\mathbf{F} = -\frac{1}{3} c l_{\rm free} \boldsymbol{\nabla} U = -\frac{c}{3\kappa\rho} \boldsymbol{\nabla}\left(aT^4\right) = -\frac{4acT^3}{3\kappa\rho} \boldsymbol{\nabla} T \equiv -k \boldsymbol{\nabla} T. \tag{44}$$

Consider the spherical-symmetric case (l is luminosity),

$$F_r = -\frac{4acT^3}{3\kappa\rho}\frac{\partial T}{\partial r} = \frac{l}{4\pi r^2} \to \frac{\partial T}{\partial r} = -\frac{3}{16\pi ac}\frac{\kappa\rho l}{r^2 T^3}. \tag{45}$$

With $dm = 4\pi r^2 \rho dr$, and Eq.(1), radiation gradient

$$\nabla_{\rm rad} \equiv \left(\frac{\mathrm{d}\log T}{\mathrm{d}\log P}\right)_{\rm rad} = \frac{P}{T}\frac{(\partial T/\partial m)}{(\partial P/\partial m)} = \frac{3}{16\pi acG}\frac{\kappa l P}{mT^4}. \tag{46}$$

Discussion on dynamical stability. Consider fluctuated elements in the stellar surrounding environment, being raised from r to $r + \Delta r$. The difference of density

$$D\rho = \left[\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\rm e} - \left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\rm s}\right]\Delta r, \tag{47}$$

the net force of gravity and buoyancy is $K_r = -gD\rho$, for stability, $K_r < 0$ to make the element come back. As $\Delta r > 0$, there should be

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\rm e} - \left(\frac{\mathrm{d}\rho}{\mathrm{d}r}\right)_{\rm s} > 0. \tag{48}$$

With the definition of (15), and the pressure balance condition that $DP = 0$, the gradients should satisfy such a relation to achieve stability

$$\left(\frac{\mathrm{d}\log T}{\mathrm{d}\log P}\right)_{\rm s} < \left(\frac{\mathrm{d}\log T}{\mathrm{d}\log P}\right)_{\rm e} + \frac{\varphi}{\delta}\left(\frac{\mathrm{d}\log \mu}{\mathrm{d}\log P}\right)_{\rm s} \to \nabla < \nabla_{\rm e} + \frac{\varphi}{\delta}\nabla_\mu. \tag{49}$$

With the approximation that $\nabla \to \nabla_{\rm rad}$, $\nabla_{\rm e} \to \nabla_{\rm ad}$, the Ledoux criterion can be derived as

$$\nabla_{\rm rad} < \nabla_{\rm ad} + \frac{\varphi}{\delta}\nabla_\mu, \tag{50}$$

the Schwarzschild criterion can be derived under the homogeneous chemical composition condition as

$$\nabla_{\rm rad} < \nabla_{\rm ad}. \tag{51}$$

Details of the above contents can be found in textbook *SSE* pages 48-51. Defining scale height of pressure

$$H_P \equiv -\frac{\mathrm{d}r}{\mathrm{d}\log P} = -P\frac{\mathrm{d}r}{\mathrm{d}P}. \tag{52}$$

The mix length

$$l_m \sim H_P \approx \frac{P}{\rho g}, \tag{53}$$

the temperature difference

$$DT = \left[\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\rm e} - \left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\rm s}\right]\Delta r \approx \left[\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\rm e} - \left(\frac{\mathrm{d}T}{\mathrm{d}r}\right)_{\rm s}\right] l_m = \Delta\left(\frac{\mathrm{d}T}{\mathrm{d}r}\right) l_m, \tag{54}$$

and

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{T}{r}\left(\frac{\mathrm{d}\log T}{\mathrm{d}r}\right) = T\left(\frac{\mathrm{d}\log T}{\mathrm{d}\log P}\right)\left(\frac{\mathrm{d}\log P}{\mathrm{d}r}\right) = -\frac{T}{H_P}\nabla. \tag{55}$$

Combined with (54),

$$DT = T\frac{l_m}{H_P}\left(\nabla - \nabla_{\rm rad}\right). \tag{56}$$

Convection energy flux (with the pressure balance assumption)

$$F_{\rm conv} \approx v_c \rho \Delta u \approx v_c \rho c_p DT, \tag{57}$$

estimating typical mixing speed v_c

$$a = -g\frac{D\rho}{\rho} \approx g\frac{DT}{T},\ l_m = \frac{1}{2}at^2 \rightarrow\ v_c \approx \frac{l_m}{t} = \sqrt{\frac{1}{2}al_m} = \sqrt{\frac{l_m^2}{2H_P}g(\nabla - \nabla_{\rm ad})}, \tag{58}$$

then

$$F_{\rm conv} = \rho c_p T \left(\frac{l_m}{H_P}\right)^2 \sqrt{\frac{1}{2}gH_P}(\nabla - \nabla_{\rm ad})^{\frac{3}{2}}. \tag{59}$$

In radiation dominated regions (no convection), $\nabla \approx \nabla_{\rm rad} < \nabla_{\rm ad}$ (the first case). While for convection dominated cases, $\nabla_{\rm ad} < \nabla_{\rm rad}$, the difference between ∇ and $\nabla_{\rm ad}$ is called as "super-adiabatic". With (59), assuming

$$F_{\rm conv} \approx \frac{L_*}{4\pi R_*^2}, \tag{60}$$

with

$$\rho \sim \frac{M}{R_*^3}, \quad T \approx \frac{GMm_{\rm p}}{k_{\rm B}R_*} \approx \frac{GM}{\mathscr{R}R_*}, \quad c_p \approx \frac{5}{2}\frac{\mathscr{R}}{\mu} \sim \frac{5}{2}\mathscr{R}, \tag{61}$$

so

$$c_p T \sim \frac{GM}{R_*}, \quad \sqrt{gH_P} = \sqrt{\frac{P}{\rho}} \sim \sqrt{\mathscr{R}T} \sim \sqrt{\frac{GM}{R_*}}, \tag{62}$$

at last,

$$F_{\rm conv} \sim \frac{M}{R_*^3}\left(\frac{GM}{R_*^3}\right)^{\frac{3}{2}}(\nabla - \nabla_{\rm ad})^{\frac{3}{2}}, \rightarrow \nabla - \nabla_{\rm ad} \sim \left(\frac{L_* R_*}{M}\right)^{\frac{2}{3}}\frac{R_*}{GM} \sim 10^{-8}, \tag{63}$$

which shows that $\nabla \approx \nabla_{\rm ad} < \nabla_{\rm rad}$ (the second case). In stellar convective interior regions, the thermal equilibrium equation is

$$\frac{{\rm d}T}{{\rm d}m} = \frac{{\rm d}T}{{\rm d}P}\frac{{\rm d}P}{{\rm d}r}\frac{{\rm d}r}{{\rm d}m} = \frac{T}{P}\cdot\left(-\frac{Gm}{r^2}\rho\right)\cdot\left(\frac{1}{4\pi r^2\rho}\right)\cdot\nabla = -\frac{Gm}{4\pi r^4}\frac{T}{P}\nabla. \tag{64}$$

Based on Eq.(59), if $\rho\downarrow$, $T\downarrow$, then $F_{\rm conv}\downarrow$, when $F_{\rm conv} \ll F_{\rm rad}$, radiation will dominate energy transport, $\nabla \approx \nabla_{\rm rad} > \nabla_{\rm ad}$ (the third case), convection will happen, and this is stellar convective envelope.
Calculation of mixing speed and timescale:

$$v_c \approx \sqrt{\frac{1}{2}al_m} \approx \sqrt{\frac{l_m^2}{H_P}g(\nabla - \nabla_{\rm ad})} \approx \sqrt{gH_P(\nabla - \nabla_{\rm ad})} \qquad (l_m \approx H_P), \tag{65}$$

sound speed $v_s \approx \sqrt{P/\rho} \approx \sqrt{gH_P}$, then

$$v_c \approx v_s\sqrt{\nabla - \nabla_{\rm ad}} \approx \sqrt{\frac{GM}{R_*}}\sqrt{\nabla - \nabla_{\rm ad}} \ll v_s, \tag{66}$$

which means subsonic convection. The timescale (estimated with the Sun)

$$\tau_{\rm mix} \approx \frac{R_*}{v_c} \approx \frac{R_*}{\sqrt{\frac{GM}{R_*}}\left(\frac{L_*R_*}{M}\right)^{\frac{1}{3}}\sqrt{\frac{R_*}{GM}}} = \frac{R_*}{\left(\frac{L_*R_*}{M}\right)^{\frac{1}{3}}} \sim 10^7 \text{ sec.} \tag{67}$$

• The first comparison of timescales: $\tau_{\rm ff} \ll \tau_{\rm mix}$.
At the boundary of convection, $\nabla_{\rm rad} = \nabla_{\rm ad}$,

$$a = -g\frac{D\rho}{\rho} \approx g\frac{DT}{T} \approx g\frac{l_m}{H_P}(\nabla - \nabla_{\rm ad}), \tag{68}$$

this acceleration will lead to overshooting distance $d_{\rm ov} \equiv \alpha_{\rm ov}H_P$, with $\alpha_{\rm ov}$ the overshooting coefficient.

1.6 Advanced Equations of Stellar Structure and Evolution

1.6.1 The full set of stellar equations

The full set of stellar equations (details can be found in textbook *SSE* Chapter 10, pages 89):

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}, \tag{69}$$

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} - \frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2}, \qquad \left(\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4}\right) \tag{70}$$

$$\frac{\partial l}{\partial m} = \varepsilon_{\rm n} - \varepsilon_\nu - c_P\frac{\partial T}{\partial t} + \frac{\delta}{\rho}\frac{\partial P}{\partial t}, \qquad \left(\frac{\partial l}{\partial m} = \varepsilon_{\rm n} - \varepsilon_\nu\right) \tag{71}$$

$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla, \tag{72}$$

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\rho}\left(\sum_i r_{ji} - \sum_k r_{ik}\right). \tag{73}$$

The nuclear and Kelvin-Helmholtz timescales (estimated with the Sun):
- $\tau_{\rm n} \equiv E_{\rm n}/L \sim 10^{11}$ yr (*SSE* page 35),
- $\tau_{\rm KH} \equiv E_g/L \approx GM^2/(2R_* L) \sim 10^7$ yr (*SSE* page 22).
- The second comparison of timescales: $\tau_{\rm ff} \ll \tau_{\rm mix} \ll \tau_{\rm KH} \ll \tau_{\rm n}$.

The forms in the brackets in Eq. (70) and Eq. (71) are with the consideration of evolution under hydrostatic equilibrium and nuclear timescale, respectively.

1.6.2 The Eddington luminosity

The Eddington luminosity (*SSE* pages 261-262), the radiation pressure

$$P_{\rm rad} = \frac{1}{3}U = \frac{a}{3}T^4 \rightarrow \frac{{\rm d}P_{\rm rad}}{{\rm d}r} = \frac{4}{3}aT^3\frac{{\rm d}T}{{\rm d}r}, \tag{74}$$

with Eq.(44), $dP_{\rm rad}/dr = -(\kappa\rho/c)F_{\rm rad}$, and in balance state, $\rho g = -dP_{\rm rad}/dr$, then

$$g = \frac{GM}{r^2} = \frac{\kappa F_{\rm rad}}{c} = \frac{\kappa L_r}{4\pi r^2 c} \rightarrow L_{\rm Edd} = \frac{4\pi c GM}{\kappa}. \tag{75}$$

1.6.3 Lane-Emden equation

Polytropic relations $P = K\rho^\gamma \equiv K\rho^{1+\frac{1}{n}}$, with hydrostatic equilibrium, and Poisson's equation under the spherical symmetry

$$\frac{{\rm d}P}{{\rm d}r} = -\frac{{\rm d}\Phi}{{\rm d}r}\rho, \qquad \frac{1}{r^2}\frac{\rm d}{{\rm d}r}\left(r^2\frac{{\rm d}\Phi}{{\rm d}r}\right) = 4\pi G\rho, \tag{76}$$

the Lane-Emden equation can be derived as

$$\frac{1}{z^2}\frac{\rm d}{{\rm d}z}\left(z^2\frac{{\rm d}w}{{\rm d}z}\right) + w^n = 0, \tag{77}$$

with

$$z = Ar, \qquad w = \frac{\Phi}{\Phi_c} = \left(\frac{\rho}{\rho_c}\right)^{\frac{1}{n}}, \tag{78}$$

and

$$A^2 = \frac{4\pi G}{(n+1)^n K^n}(-\Phi_c)^{n-1} = \frac{4\pi G}{(n+1)K}\rho_c^{\frac{n-1}{n}}. \tag{79}$$

Mass distribution

$$m(r) = \int_0^r 4\pi r^2\rho{\rm d}r = 4\pi\rho_c\int_0^r w^n r^2{\rm d}r = 4\pi\rho_c\frac{r^3}{z^3}\int_0^z w^n z^2{\rm d}z = 4\pi\rho_c r^3\left(-\frac{1}{z}\frac{{\rm d}w}{{\rm d}z}\right), \tag{80}$$

then $M = m(r = R,\ z = z_n)$. With the definition,

$$\left(\frac{r}{z}\right)^2 = \frac{1}{A^2} = \frac{1}{4\pi G}(n+1)K\rho_c^{\frac{1-n}{n}} \to R \sim \rho_c^{\frac{1-n}{2n}}, \tag{81}$$

with $M \sim \rho_c R^3$, then $M \sim \rho_c^{\frac{3-n}{2n}}$, if $n \neq 3$, $R \sim M^{\frac{1-n}{3-n}}$. For $n = 3$, $P = K\rho^{\frac{4}{3}}$, with an extreme assumption of extreme-relativistic electron gas, $K = K_{\rm rel}$ in Eq. (43), and the Chandrasekhar mass (expressed with the mass of the Sun $M_\odot$)

$$M_{\rm ch} = 4\pi \left(-\frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z}\right)_{z_3} z_3^3 \left(\frac{K_{\rm rel}}{\pi G}\right)^{\frac{3}{2}} = \frac{5.836}{\mu_e^2} M_\odot. \tag{82}$$

Polytropic spheres $P_c = C_1 G M^{\frac{2}{3}} \rho_c^{\frac{4}{3}}$ $(n = 3)$, $P_c = C_2 G M^{\frac{1}{3}} R \rho_c^{\frac{5}{3}}$ $(n = 3/2)$, with $C_1,\ C_2$ both dimensionless constants. As

$$\left(\frac{r}{z}\right)^2 = \left(\frac{R}{z_n}\right)^2 = \frac{1}{A^2} = \frac{1}{4\pi G}(n+1)K\rho_c^{\frac{1-n}{n}} \to \rho_c R^3 = z_n^3 \left(\frac{n+1}{4\pi G}\right)^{\frac{3}{2}} K^{\frac{3}{2}} \rho_c^{\frac{3-n}{2n}}, \tag{83}$$

then

$$\begin{aligned} M &= 4\pi\rho_c R^3 \left(-\frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z}\right)_{z=z_n} \\ &= 4\pi \left(-\frac{1}{z}\frac{\mathrm{d}w}{\mathrm{d}z}\right)_{z=z_n} z_n^3 \left(\frac{n+1}{4\pi G}\right)^{\frac{3}{2}} K^{\frac{3}{2}} \rho_c^{\frac{3-n}{2n}} \\ &= 4\pi \left(-z^2\frac{\mathrm{d}w}{\mathrm{d}z}\right)_{z=z_n} \left(\frac{n+1}{4\pi G}\right)^{\frac{3}{2}} K^{\frac{3}{2}} \rho_c^{\frac{3-n}{2n}}, \end{aligned} \tag{84}$$

defining $h(n) \equiv 4\pi \left(-z^2 \frac{\mathrm{d}w}{\mathrm{d}z}\right)_{z=z_n}$, then

$$K = [h(n)]^{-\frac{2}{3}} \left(\frac{4\pi G}{n+1}\right) M^{\frac{2}{3}} \rho_c^{\frac{n-3}{3n}}. \tag{85}$$

For $n = 3$, $K \approx 0.364 G M^{\frac{2}{3}}$, for $n = 3/2$, $K \approx 0.424 G M^{\frac{1}{3}} R$. Thus, $C_1 \approx 0.364$, $C_2 \approx 0.424$.

1.6.4 Homology

Homology condition of two stars (Details in *SSE* Chapter 20):

$$\frac{r(\xi)}{r'(\xi)} = \frac{R}{R'}, \qquad \xi \equiv \frac{m}{M} = \frac{m'}{M'}. \tag{86}$$

Defining the ratio parameters

$$x = \frac{M}{M'},\quad y = \frac{\mu}{\mu'},\quad \frac{r}{r'} = z = \frac{R}{R'},\quad \frac{P}{P'} = p = \frac{P_{\rm c}}{P'_{\rm c}},\quad \frac{T}{T'} = t = \frac{T_{\rm c}}{T'_c},\quad \frac{l}{l'} = s = \frac{L}{L'}, \tag{87}$$

and also

$$\frac{\rho}{\rho'} = d,\quad \frac{\varepsilon}{\varepsilon'} = e,\quad \frac{\kappa}{\kappa'} = k, \qquad (\varepsilon = \varepsilon_{\rm n} - \varepsilon_\nu). \tag{88}$$

With equation set (69)(70)(71)(72),

$$\frac{x}{z^3 d} = 1,\quad \frac{x^2}{z^4 p} = 1,\quad \frac{ex}{s} = 1,\quad \frac{ksx}{z^4 t^4} = 1. \tag{89}$$

Power laws

$$\rho \sim P^\alpha T^{-\delta} \mu^\varphi,\ \varepsilon \sim \rho^\lambda T^\nu,\ \kappa \sim P^a T^b, \tag{90}$$

then

$$d = p^\alpha t^{-\delta} y^\varphi,\ e = p^{\lambda\alpha} t^{\nu - \lambda\delta} y^{\lambda\varphi},\ k = p^a t^b. \tag{91}$$

Describe $r,\ P,\ T, l$ with $M,\ \mu$

$$z = x^{z_1}y^{z_2},\ p = x^{p_1}y^{p_2},\ t = x^{t_1}y^{t_2},\ s = x^{s_1}y^{s_2}. \tag{92}$$

Take $x/(z^3 d) = 1$ as an example,

$$\frac{x}{z^3 d} = \frac{x}{x^{3z_1}y^{3z_2}p^{\alpha}t^{-\delta}y^{\varphi}} = x^{1-3z_1-\alpha p_1+\delta t_1}y^{-3t_2-\varphi-\alpha+\delta t_2} = 1, \tag{93}$$

there should be

$$-3z_1 - \alpha p_1 + \delta t_1 = -1, \qquad -3t_2 - \alpha + \delta t_2 = \varphi, \tag{94}$$

the same way for another three relations in Eq. (89). So Eq. (20.13) and Eq. (20.14) in *SSE* page 236 can be derived. For ideal gas with constant opacity, $\alpha = \delta = \varphi = 1,\ a = b = 0$, then

$$z_1 = \frac{\nu+\lambda-2}{\nu+3\lambda},\ z_2 = \frac{\nu-4}{\nu+3\lambda},\ p_1 = 2-4z_1,\ p_2 = -4z_2,\ t_1 = 1-z_1,\ t_2 = 1-z_2,\ s_1 = 3,\ s_2 = 4, \tag{95}$$

assuming $\varepsilon \sim \rho T^{\nu}$ ($\lambda = 1$), then

$$R \propto \mu^{\frac{\nu-4}{\nu+3}}M^{\frac{\nu-1}{\nu+3}},\ P_{\rm c} \propto \mu^{-\frac{4(\nu-4)}{\nu+3}}M^{\frac{-2\nu+10}{\nu+3}},\ T_{\rm c} \propto \mu^{\frac{7}{\nu+3}}M^{\frac{4}{\nu+3}},\ L \propto \mu^4 M^3, \tag{96}$$

and also

$$\rho_c \sim P_{\rm c}T_{\rm c}^{-1}\mu \sim \mu^{\frac{-3\nu+12}{\nu+3}}M^{\frac{-2\nu+6}{\nu+3}}. \tag{97}$$

As for the $L-M$ relation, with

$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla = -\frac{GmT}{4\pi r^4 P}\cdot\frac{3\kappa}{16\pi acG}\cdot\frac{lP}{mT^4} \sim \frac{\kappa l}{r^4 T^3}, \tag{98}$$

so (recall GUT)

$$l \sim \frac{T}{M}\frac{R^4T^3}{\kappa} = \frac{T^4R^4}{\kappa M} \sim \left(\frac{\mu M}{R}\right)^4 R^4\frac{1}{\kappa M} = \frac{\mu^4 M^3}{\kappa}. \tag{99}$$

With Kramers' dominant opacity $\kappa = \kappa_0\rho T^{-3.5}$, and $\rho \sim MR^{-3}$, there is also

$$l \sim \frac{\mu^4 M^3 T^{3.5}}{\rho} \sim \mu^{\frac{15}{2}}M^{\frac{16}{3}}\rho^{\frac{1}{6}} \sim \frac{\mu^{7.5}M^{5.5}}{R^{0.5}}. \tag{100}$$

For homologous contraction, two models separated by a time interval Δt, with $r' = r + \dot{r}\Delta t$, requiring

$$\frac{r'}{r} = \frac{R'}{R} = \text{constant}, \tag{101}$$

throughout the stellar interior. Thus,

$$\frac{\dot{r}}{r} = \frac{\dot{R}}{R} = \text{constant} \to \frac{\partial}{\partial m}\left(\frac{\dot{r}}{r}\right) = \frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial r}{\partial m}\right) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi r^3\rho}\right) = \frac{1}{4\pi r^3\rho}\left(-3\frac{\dot{r}}{r} - \frac{\dot{\rho}}{\rho}\right) = 0, \tag{102}$$

with Eq. (70), there is also

$$\dot{P} = -\int_m^M \frac{\partial}{\partial t}\left(\frac{1}{r^4}\right)\frac{Gm}{4\pi}dm = 4\frac{\dot{r}}{r}\int_m^M \frac{Gm}{4\pi r^4}dm, \tag{103}$$

above calculations give

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{r}}{r}, \qquad \frac{\dot{P}}{P} = -4\frac{\dot{r}}{r}, \tag{104}$$

with $\rho \sim P^{\alpha}T^{-\delta}$, the $T-\rho$ relation is derived as

$$\frac{\dot{T}}{T} = -\frac{4\alpha-3}{\delta}\frac{\dot{r}}{r}, \tag{105}$$

if $\alpha = \delta = 1$, there is $(\dot{T}/T) \sim \frac{1}{3}(\dot{\rho}/\rho)$, an important physical picture can be depicted with $T_{\rm c} \propto \rho_{\rm c}^{\frac{1}{3}}$.

1.6.5 White Dwarf

The Chandrasekhar's differential equation for the structure of WD

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}\zeta^2}+\frac{2}{\zeta}\frac{\mathrm{d}\varphi}{\mathrm{d}\zeta}+\left(\varphi^2-\frac{1}{z_{\mathrm{c}}^2}\right)^{\frac{3}{2}}=0, \tag{106}$$

with

$$\zeta\equiv\frac{r}{\alpha},\ \alpha=\sqrt{\frac{2C_1}{\pi G}}\frac{1}{C_2 z_{\mathrm{c}}},\ \varphi\equiv\frac{z}{z_{\mathrm{c}}},\ z^2\equiv x^2+1, \tag{107}$$

with $C_1,\ C_2,\ x$ defined from electron gas part

$$P_e=\frac{\pi m_e^4c^5}{3h^3}f(x)\equiv C_1f(x),\qquad \rho=\frac{8\pi\mu_e m_{\mathrm{u}}m_e^3c^3}{3h^3}x^3\equiv C_2x^3,\qquad x\equiv\frac{p_{\mathrm{F}}}{m_ec}. \tag{108}$$

Details found in *SSE* Chapter 37.1, pages 475-477, note that Eq. (37.3) in *SSE* page 476 is derived with

$$\begin{aligned}\frac{1}{x^3}\frac{\mathrm{d}f(x)}{\mathrm{d}x}&=\frac{1}{x^3}\frac{\mathrm{d}x}{\mathrm{d}r}\frac{\mathrm{d}f(x)}{\mathrm{d}x}=\frac{1}{x^3}\frac{\mathrm{d}x}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}x}\left[8\int_0^x\frac{\xi^4}{(1+\xi^2)^{\frac{1}{2}}}d\xi\right]\\&=\frac{1}{x^3}\cdot\frac{\mathrm{d}x}{\mathrm{d}r}\cdot\frac{8x^4}{(1+x^2)^{\frac{1}{2}}}=\frac{8x}{(1+x^2)^{\frac{1}{2}}}\frac{\mathrm{d}x}{\mathrm{d}r}=8\frac{\mathrm{d}}{\mathrm{d}r}\left[(x^2+1)^{\frac{1}{2}}\right].\end{aligned} \tag{109}$$

1.6.6 Neutron Star

This chapter will show the derivation of Tolman-Oppenheimer-Volkoff (TOV) equation. Simplify the interior of the Neutron Star (NS) as static and spherical symmetric, the metric form is

$$\mathrm{d}s^2=-\mathrm{e}^{-a(r)}c^2\mathrm{d}t^2+\mathrm{e}^{b(r)}\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2), \tag{110}$$

the Christoffel symbol

$$\Gamma^\rho_{\mu\nu}=\frac{1}{2}g^{\rho\lambda}(g_{\nu\lambda,\mu}+g_{\lambda\mu,\nu}-g_{\mu\nu,\lambda}), \tag{111}$$

with the non-zero components

$$\begin{aligned}&\Gamma^0_{01}=a'/2,\ \Gamma^1_{00}=a'\mathrm{e}^{a-b}/2,\ \Gamma^1_{11}=b'/2,\\&\Gamma^1_{22}=-r\mathrm{e}^{-b},\ \Gamma^1_{33}=-r\sin^2\theta\mathrm{e}^{-b},\ \Gamma^2_{12}=1/r,\\&\Gamma^2_{33}=-\sin\theta\cos\theta,\ \Gamma^3_{13}=1/r,\Gamma^3_{23}=\cot\theta.\end{aligned} \tag{112}$$

The curvature tensor

$$R^\sigma{}_{\rho\mu\nu}=\Gamma^\sigma_{\nu\rho,\mu}-\Gamma^\sigma_{\mu\rho,\nu}+\Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\nu\rho}-\Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\mu\rho}, \tag{113}$$

and curvature scalar

$$\begin{aligned}R&=g^{\mu\nu}R_{\mu\nu}\\&=g^{00}R_{00}+g^{11}R_{11}+g^{22}R_{22}+g^{33}R_{33}\\&=-\frac{1}{2}\mathrm{e}^{-b}(2a''-a'b'+(a')^2)-2r^{-1}\mathrm{e}^{-b}(a'-b')-2r^{-2}(\mathrm{e}^{-b}-1),\end{aligned} \tag{114}$$

with

$$\begin{aligned}R_{00}&=\frac{1}{4}\mathrm{e}^{a-b}(2a''-a'b'+(a')^2+4r^{-1}a')=0,\\R_{11}&=-\frac{1}{4}(2a''-a'b'+(a')^2-4r^{-1}b')=0,\\R_{22}&=-\frac{1}{2}\mathrm{e}^{-b}(ra'-rb'+2-2\mathrm{e}^b)=0,\\R_{33}&=R_{22}\sin^2\theta=0.\end{aligned} \tag{115}$$

The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, with the components

$$\begin{aligned} G_{00} &= r^{-2} \mathrm{e}^{a-b}(rb' + \mathrm{e}^b - 1), \\ G_{11} &= r^{-2}(ra' - \mathrm{e}^b + 1) \\ G_{22} &= \frac{1}{4} r \mathrm{e}^{-b}(2a' - 2b' - ra'b' + 2ra'' + r(a')^2) \\ G_{33} &= \sin^2\theta G_{22}. \end{aligned} \tag{116}$$

The energy-momentum tensor and four-velocity

$$T_{\mu\nu} = \left[\rho(r) + \frac{P(r)}{c^2}\right] u^\mu u^\nu + P g^{\mu\nu}, \tag{117}$$

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\sqrt{-g_{00}}\mathrm{d}t} = \mathbf{diag}\left[c\mathrm{e}^{-a(r)/2}, 0, 0, 0\right]. \tag{118}$$

With the Einstein equation $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$, there are three indepedent equations

$$\begin{aligned} r^{-2}\mathrm{e}^{-b}(rb' + \mathrm{e}^b - 1) &= \frac{8\pi G}{c^4}\rho, \\ r^{-2}\mathrm{e}^{-b}(ra' - \mathrm{e}^b + 1) &= \frac{8\pi G}{c^4}P, \\ \frac{1}{4}r^{-1}\mathrm{e}^{-b}(2a' - 2b' - ra'b' + 2a''r + r(a')^2) &= \frac{8\pi G}{c^4}P. \end{aligned} \tag{119}$$

Under energy-momentum conservation, $T_{\mu\nu}$ satisfies

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \left(\rho + \frac{P}{c^2}\right) u^\nu \nabla_\mu u^\mu + \left(\rho + \frac{P}{c^2}\right) u^\mu \nabla_\mu u^\nu \\ &\quad + u^\mu u^\nu \nabla_\mu \left(\rho + \frac{P}{c^2}\right) + \nabla_\mu P g^{\mu\nu} + P \nabla_\mu g^{\mu\nu} \\ &= \left(\rho + \frac{P}{c^2}\right) u^\mu \nabla_\mu u^\nu + g^{\mu\nu} \nabla_\mu P = 0, \end{aligned} \tag{120}$$

where $\nabla_\rho g_{\mu\nu} = 0$, and the simplified form of u^μ have been used. This will lead to another equation

$$\frac{\mathrm{d}P}{\mathrm{d}r}\mathrm{e}^{-b(r)} + \left(\rho + \frac{P}{c^2}\right) c\mathrm{e}^{-\frac{a(r)}{2}} \Gamma^1_{00} u^1 = \frac{\mathrm{d}P}{\mathrm{d}r}\mathrm{e}^{-b(r)} + \left(\rho + \frac{P}{c^2}\right) c\mathrm{e}^{-b(r)} \frac{a'}{2} = 0 \to \frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{1}{2} a'(P + \rho c^2). \tag{121}$$

Eq.(119) together with Eq.(121), there will be TOV equation

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{1}{2}(P + \rho c^2)\left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1} \left[\frac{8\pi G P}{c^4} r + \frac{2Gm(r)}{r^2 c^2}\right]. \tag{122}$$

Notice that there are four functions $a(r)$, $b(r)$, $\rho(r)$, $P(r)$ to be solved, and there are four equations, but Bianchi identity will "consume" one freedom, so another one equation is in need, that is the Equation of State (EOS) $P = P(\rho)$ of the NS interior.

1.7 Nuclear Interaction

The tunnelling probability

$$P_0 = p_0 E^{-\frac{1}{2}} e^{-2\pi\eta},\ \eta = \left(\frac{m}{2}\right)^{\frac{1}{2}} \frac{Z_1 Z_2 e^2}{\hbar E^{\frac{1}{2}}}. \tag{123}$$

For the interaction between j, k particles, the energy

$$E = \frac{1}{2} m v^2 = \frac{1}{2}\left(\frac{m_j m_k}{m_j + m_k}\right) v^2, \tag{124}$$

Maxwell-Boltzmann distribution

$$f(E)dE = \frac{2}{\sqrt{\pi}} \frac{E^{\frac{1}{2}}}{(k_\mathrm{B}T)^{\frac{3}{2}}} e^{-\frac{E}{k_\mathrm{B}T}} dE. \tag{125}$$

With the corss section $\sigma(E) = S(E)E^{-1}e^{-2\pi\eta} \approx S_0 E^{-1}e^{-2\pi\eta}$ (where $S(E)$ is approximately considered as constant), the averaged probability

$$\langle\sigma v\rangle = \int_0^\infty \sigma(E) v f(E) dE = \frac{2^{\frac{3}{2}}}{(m\pi)^{\frac{1}{2}}} \frac{1}{(k_\mathrm{B}T)^{\frac{3}{2}}} \int_0^\infty S_0 \exp\left(-\frac{E}{k_\mathrm{B}T} - \frac{\bar{\eta}}{E^{\frac{1}{2}}}\right) dE, \quad \left(\bar{\eta} = 2\pi\eta E^{\frac{1}{2}} = \pi(2m)^{\frac{1}{2}} \frac{Z_j Z_k e^2}{\hbar}\right) \tag{126}$$

consider the function in **exp**,

$$f(E) = -\frac{E}{k_\mathrm{B}T} - \frac{\bar{\eta}}{E^{\frac{1}{2}}},\ f'(E_0) = 0 \to E_0 = \left(\frac{1}{2}\bar{\eta} k_\mathrm{B}T\right)^{\frac{2}{3}} = \left[\left(\frac{m}{2}\right)^{\frac{1}{2}} \pi \frac{Z_i Z_k e^2 kT}{\hbar}\right]^{\frac{2}{3}}, \tag{127}$$

where m is still the reduced mass. Expand $f(E)$ to the second order approximately at E_0 (Gamov peak), with $f'(E_0) = 0$, then

$$f(E) \approx -\tau - \frac{1}{4}\tau\left(\frac{E}{E_0} - 1\right)^2, \qquad \left(\tau = \frac{3E_0}{k_\mathrm{B}T} = 3\left[\pi\left(\frac{m}{2k_\mathrm{B}T}\right)^{\frac{1}{2}} \frac{Z_j Z_k e^2}{\hbar}\right]^{\frac{2}{3}}\right) \tag{128}$$

defining $\xi \equiv \left(\frac{E}{E_0} - 1\right)\frac{\sqrt{\tau}}{2}$, then

$$\begin{aligned} \int_0^\infty \exp\left[f(E)\right] dE &\approx \int_0^\infty \exp\left[-\tau - \frac{\tau}{4}\left(\frac{E}{E_0} - 1\right)^2\right] = \frac{2}{3} k_\mathrm{B}T \tau^{\frac{1}{2}} e^{-\tau} \int_{-\frac{\sqrt{\tau}}{2}}^\infty e^{-\xi^2} d\xi \\ &\approx \frac{2}{3} k_\mathrm{B}T \tau^{\frac{1}{2}} e^{-\tau} \int_0^\infty e^{-\xi^2} d\xi = \frac{2}{3} k_\mathrm{B}T \pi^{\frac{1}{2}} \tau^{\frac{1}{2}} e^{-\tau}, \end{aligned} \tag{129}$$

so

$$\langle\sigma v\rangle \approx \frac{4}{3}\left(\frac{2}{m}\right)^{\frac{1}{2}} \frac{1}{(k_\mathrm{B}T)^{\frac{1}{2}}} S_0 \tau^{\frac{1}{2}} e^{-\tau}. \tag{130}$$

Assuming

$$\langle\sigma v\rangle = \langle\sigma v\rangle_0 \left(\frac{T}{T_0}\right)^\nu, \tag{131}$$

with Eq. (128) and Eq. (130), $\langle\sigma v\rangle \sim T^{-\frac{1}{2}}\tau^{\frac{1}{2}}e^{-\tau} \sim T^{-\frac{2}{3}}e^{-\tau}$, and $\tau \sim T^{-\frac{1}{3}}$, then

$$\nu = \frac{\partial \log\langle\sigma v\rangle}{\partial \log T} = -\frac{2}{3} - \frac{\partial\tau}{\partial \log T} = -\frac{2}{3} - \tau\frac{\partial \log\tau}{\partial \log T} = \frac{\tau}{3} - \frac{2}{3}. \tag{132}$$

Note that the real value of ν is calculated with the function form of $\langle\sigma v\rangle(T)$, and

$$\nu = \frac{\partial \log\langle\sigma v\rangle}{\partial \log T} = \frac{T}{\langle\sigma v\rangle}\frac{\partial\langle\sigma v\rangle}{\partial T}. \tag{133}$$

Two energy levels, $\varepsilon_1,\ \varepsilon_2$, with the number densities $n_1,\ n_2$, degeneracy numbers $g_1,\ g_2$, at the temperature T, satisfies

$$\frac{n_1}{n_2} = \frac{g_1}{g_2}\exp\left(-\frac{\varepsilon_1 - \varepsilon_2}{k_\mathrm{B}T}\right). \tag{134}$$

The example of the Saha equation (textbook *SI* Chapter 3.4, page 155) for $\mathrm{H}^+ + \mathrm{e}^- \rightleftharpoons \mathrm{H}^0 + \chi_\mathrm{H}$, with $\chi_\mathrm{H} = 13.6$ eV the ionization potential from the ground state of hydrogen. The number density of e^-, H^+, H^0

$$n_e = \frac{2(2\pi m_e k_\mathrm{B}T)^{\frac{3}{2}}}{h^3} e^{\frac{\mu^-}{k_\mathrm{B}T}}, \qquad n^+ = \frac{(2\pi m_p k_\mathrm{B}T)^{\frac{3}{2}}}{h^3} e^{\frac{\mu^+}{k_\mathrm{B}T}}, \qquad n^0 = \frac{2\left[2\pi(m_e + m_p)k_\mathrm{B}T\right]^{\frac{3}{2}}}{h^3} e^{\frac{\mu^0 + \chi_\mathrm{H}}{k_\mathrm{B}T}}. \tag{135}$$

The equilibrium condition requires that $\mu^+ + \mu^- - \mu^0 = 0$, then

$$\frac{n^+ n_e}{n^0} = \left(\frac{2\pi k_\mathrm{B} T}{h^2} \cdot \frac{m_e m_p}{m_e + m_p}\right)^{\frac{3}{2}} e^{-\frac{\chi_\mathrm{H}}{k_\mathrm{B} T}} \approx \left(\frac{2\pi m_e k_\mathrm{B} T}{h^2}\right)^{\frac{3}{2}} e^{-\frac{\chi_\mathrm{H}}{k_\mathrm{B} T}}. \tag{136}$$

The example of the nuclear version of Saha equation (*SI* page 308) for $\alpha + \alpha \rightleftharpoons {}^8\mathrm{Be}$, the mass of α is $4.0026\ m_\mathrm{u}$, while the mass of ^{8}Be is $8.0053\ \mathrm{m_u}$, and from left to right is endothermic, $\chi = -|Q| = -91.84$ keV, so with $m_{^8\mathrm{Be}} \approx 2m_\alpha$,

$$\frac{n_\alpha^2}{n(^8\mathrm{Be})} = \left(\frac{2\pi k_\mathrm{B} T}{h^2} \cdot \frac{m_\alpha^2}{m_{^8\mathrm{Be}}}\right)^{\frac{3}{2}} e^{-\frac{\chi}{k_\mathrm{B} T}} \approx \left(\frac{\pi m_\alpha k_\mathrm{B} T}{h^2}\right)^{\frac{3}{2}} e^{\frac{|Q|}{k_\mathrm{B} T}}. \tag{137}$$

For the part of neutrino oscillation, you may refer to Griffiths' textbook on particle physics.

1.8 Asteroseismology

1.8.1 Spherical Oscillations

The perturbed oscillation of pressure, radius, density

$$P(m,t) = P_0(m)\left[1 + p(m)e^{i\omega t}\right], \ r(m,t) = r_0(m)\left[1 + x(m)e^{i\omega t}\right], \ \rho(m,t) = \rho_0(m)\left[1 + d(m)e^{i\omega t}\right]. \tag{138}$$

There are $p(m),\ x(m),\ d(m) \ll 1$ and $P_0(m),\ r_0(m),\ \rho_0(m)$ satisfying spherically symmetric equations. With Eq. (70), the linearization process (assuming all terms with $x^n, n \geq 2$ vanishes)

$$\begin{aligned}
\frac{\partial P}{\partial m} &= \frac{\partial}{\partial m}\left(P_0 + P_0 p e^{i\omega t}\right) \\
&= \frac{\partial P_0}{\partial m} + \frac{\partial P_0}{\partial m} p e^{i\omega t} + P_0 \frac{\partial p}{\partial m} e^{i\omega t} \\
&= -\frac{Gm}{4\pi r^4} - \frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} \\
&= -\frac{Gm}{4\pi r_0^4 \left(1 + x e^{i\omega t}\right)^4} - \frac{1}{4\pi r_0^2 \left(1 + x e^{i\omega t}\right)^2}(-\omega^2) r_0 x e^{i\omega t} \\
&\approx -\frac{Gm}{4\pi r_0^4}\left(1 - 4x e^{i\omega t}\right) + \frac{\omega^2}{4\pi r_0} x e^{i\omega t},
\end{aligned} \tag{139}$$

with

$$\frac{\partial P_0}{\partial m} = -\frac{Gm}{4\pi r_0^4}, \qquad g_0 = \frac{Gm}{r_0^2}, \qquad \frac{\partial P_0}{\partial r_0} = -\rho_0 g_0, \tag{140}$$

then

$$\frac{\partial (P_0 p)}{\partial m} = \left(4 g_0 + r_0 \omega^2\right) \frac{x}{4\pi r_0^2}. \tag{141}$$

$$\frac{\partial}{\partial m} = \frac{1}{4\pi r_0^2 \rho_0}\frac{\partial}{\partial r_0} \to \frac{\partial (P_0 p)}{\partial m} = \frac{1}{4\pi r_0^2 \rho_0}\left(p\frac{\partial P_0}{\partial r_0} + P_0 \frac{\partial p}{\partial r_0}\right), \tag{142}$$

and then

$$\frac{P_0}{\rho_0}\frac{\partial p}{\partial r_0} = g_0(p + 4x) + \omega^2 r_0 x, \tag{143}$$

The mass-radius equation

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \tag{144}$$

leads to

$$\begin{aligned}
\frac{\partial r}{\partial m} &= \frac{1}{4\pi r_0^2 \rho_0}\frac{\partial}{\partial r_0}\left[r_0 + r_0 x e^{i\omega t}\right] \\
&= \frac{1}{4\pi r_0^2 \rho_0}\left[1 + x e^{i\omega t} + r_0 \frac{\partial x}{\partial r_0} e^{i\omega t}\right] \\
&= \frac{1}{4\pi r^2 \rho} \approx \frac{1}{4\pi r_0^2 \rho_0}\left(1 - 2x e^{i\omega t} - d e^{i\omega t}\right)
\end{aligned} \tag{145}$$

and then

$$r_0 \frac{\partial x}{\partial r_0} = -3x - d. \tag{146}$$

Based on *SSE* pages 55-56, chapter 6.4, the thermal adjustment time $\tau_{\rm adj}$ is equivalent to $\tau_{\rm adj}$, and $\tau_{\rm hydr} \ll \tau_{\rm adj}$, which means adiabatic oscillations, and $p/d = \gamma_{\rm ad}$ (first order expansion, assuming $\gamma_{\rm ad}$ is a constant), then

$$\begin{aligned} \frac{\partial x}{\partial r_0} + r_0 \frac{\partial^2 x}{\partial r_0^2} &= -3\frac{\partial x}{\partial r_0} - \frac{1}{\gamma_{\rm ad}}\frac{\partial p}{\partial r_0} \\ &= -3\frac{\partial x}{\partial r_0} - \frac{1}{\gamma_{\rm ad}}\frac{\rho_0}{P_0}\left[\omega^2 r_0 x + g_0(p+4x)\right], \end{aligned} \tag{147}$$

rearranged to

$$\frac{\partial^2 x}{\partial r_0^2} + \frac{4}{r_0}\frac{\partial x}{\partial r_0} + \frac{\rho_0}{\gamma_{\rm ad} P_0}\left(\omega^2 x + \frac{4g_0}{r_0}x + \frac{g_0}{r_0}p\right) = 0. \tag{148}$$

With $p/d = \gamma_{\rm ad}$ and Eq. (146), there is

$$\frac{g_0 p}{r_0}\frac{\rho_0}{\gamma_{\rm ad} P_0} = \frac{g_0 \rho_0 d}{P_0 r_0} = \frac{\rho_0 g_0}{P_0 r_0}\left(-3x - r_0\frac{\partial x}{\partial r_0}\right), \tag{149}$$

at last (prime corresponding to the derivative over r_0),

$$x'' + \left(\frac{4}{r_0} - \frac{\rho_0 g_0}{P_0}\right)x' + \frac{\rho_0}{\gamma_{\rm ad} P_0}\left[\omega^2 + (4-3\gamma_{\rm ad})\frac{g_0}{r_0}\right]x = 0. \tag{150}$$

Eq. (150) can be transformed into $Strum - Liouville$ equation and then solved, details found in *SSE* chapter 40.1, pages 520-523.

1.8.2 Non-radial Oscillations

Three components $v_r,\ v_\theta,\ v_\varphi(r,\ \theta,\ \varphi)$, defining the displacement parameter $\vec{\xi}$.
The Lagrangian form (displacement)

$$P = P_0 + DP, \qquad \rho = \rho_0 + D\rho, \qquad \Phi = \Phi_0 + D\Phi, \qquad \vec{v} = \frac{{\rm d}\vec{\xi}}{{\rm d}t}. \tag{151}$$

The Eulerian perturbations (local)

$$P = P_0 + \delta P, \qquad \rho = \rho_0 + \delta\rho, \qquad \Phi = \Phi_0 + \delta\Phi, \qquad \vec{v} = \frac{\partial\vec{\xi}}{\partial t}. \tag{152}$$

there is

$$Dq = \delta q + \vec{\xi}\cdot\boldsymbol{\nabla} q_0 = \delta q + \xi_r \frac{\partial q_0}{\partial r}. \tag{153}$$

The gravity acceleration and Poisson's equation

$$\vec{g} = -\boldsymbol{\nabla}\Phi, \qquad \nabla^2\delta\Phi = 4\pi G\delta\rho. \tag{154}$$

The equation of motion

$$\rho\frac{{\rm d}\vec{v}}{{\rm d}t} = \rho\frac{{\rm d}^2\vec{\xi}}{{\rm d}t^2} = \rho\vec{g} - \boldsymbol{\nabla}P = (\rho_0 + D\rho)(\vec{g}_0 + D\vec{g}) - \boldsymbol{\nabla}(P_0 + DP) \approx \vec{g}_0 D\rho + \rho_0 D\vec{g} - \boldsymbol{\nabla}(DP). \tag{155}$$

As

$$\vec{g}_0 D\rho = -\delta\rho\boldsymbol{\nabla}\Phi_0 - (\vec{\xi}\cdot\boldsymbol{\nabla}\rho_0)\boldsymbol{\nabla}\Phi_0 = -\delta\rho\boldsymbol{\nabla}\Phi_0 + (\vec{\xi}\cdot\boldsymbol{\nabla}\rho_0)\vec{g}_0, \tag{156}$$

$$\rho_0 D\vec{g} = \rho_0\left[-\boldsymbol{\nabla}(D\Phi)\right] = -\rho_0\boldsymbol{\nabla}\delta\Phi - \rho_0\boldsymbol{\nabla}\left(\vec{\xi}\cdot\boldsymbol{\nabla}\Phi_0\right) = -\rho_0\boldsymbol{\nabla}\delta\Phi + \rho_0\boldsymbol{\nabla}\left(\vec{\xi}\cdot\vec{g}_0\right), \tag{157}$$

$$-\boldsymbol{\nabla}(DP) = -\boldsymbol{\nabla}\delta P - \boldsymbol{\nabla}\left(\vec{\xi}\cdot\boldsymbol{\nabla}P_0\right), \tag{158}$$

with

$$-\nabla\left(\vec{\xi}\cdot\nabla P_0\right) = -\nabla\left[\vec{\xi}\cdot(\rho_0\vec{g}_0)\right] = -\rho_0\nabla\left(\vec{\xi}\cdot\vec{g}_0\right) - \left(\vec{\xi}\cdot\vec{g}_0\right)\nabla\rho_0, \tag{159}$$

and $\nabla\rho_0$ is parallel with $\vec{g}_0$, there is

$$\rho_0\frac{\partial^2\vec{\xi}}{\partial t^2} = -\rho_0\nabla\delta\Phi - \delta\rho\nabla\Phi_0 - \nabla\delta P. \tag{160}$$

The equation of continuity (local, Eulerian form, *SSE* Eq. (42.9) is wrong)

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = \frac{\partial(\rho_0+\delta\rho)}{\partial t} + \nabla\cdot\left[(\rho_0+\delta\rho)\frac{\partial\vec{\xi}}{\partial t}\right] = 0, \tag{161}$$

with $\partial\rho_0/\partial t = 0$, and the process of linearization,

$$\delta\rho + \nabla\cdot(\rho_0\vec{\xi}) = \delta\rho + \rho_0\nabla\cdot\vec{\xi} + \nabla\rho_0\cdot\vec{\xi} = 0. \tag{162}$$

For adiabatic perturbation (displacement, Lagrangian form),

$$\frac{DP}{P_0} = \gamma_{\rm ad}\frac{D\rho}{\rho_0}, \tag{163}$$

equivalent to

$$\delta P + \vec{\xi}\cdot\nabla P_0 = \frac{P_0}{\rho_0}\gamma_{\rm ad}\left(\delta\rho + \vec{\xi}\cdot\nabla\rho_0\right). \tag{164}$$

At the surface, $D\nabla\Phi = 0$, $DP = 0$, equivalent forms (typo in the first equation in Eq. (42.13) of *SSE*)

$$\left(\delta\Phi + \vec{\xi}\cdot\nabla\Phi_0\right)_{\rm in} = \left(\delta\Phi + \vec{\xi}\cdot\nabla\Phi_0\right)_{\rm out},\ \ \delta P + \vec{\xi}\cdot\nabla P_0 = 0. \tag{165}$$

At the center, $\delta P = 0$, $\delta\Phi = 0$. These are four boundary conditions to solve the oscillation modes, and also four ODEs shown in *SSE* chapter 42.2, pages 545-548. With Cowling approximation, the perturbation of the gravitational potential can be neglected, only two ODEs and two parameters $\tilde{\eta}_1$, $\tilde{\eta}_2$ are calculated. Now discuss on the perturbation parameters and dimensionless frequency

$$\eta_j = \tilde{\eta}_j(r)Y_l^m(\theta,\ \varphi)e^{i\omega t}, \qquad \sigma^2 = \omega^2\frac{R^3}{GM}. \tag{166}$$

• Large σ^2,

$$r\frac{\partial\tilde{\eta}_1}{\partial r} = \left(3 - \frac{V}{\gamma_{\rm ad}}\right)\tilde{\eta}_1 + \frac{V}{\gamma_{\rm ad}}\tilde{\eta}_2, \tag{167}$$

$$r\frac{\partial\tilde{\eta}_2}{\partial r} = (W + C\sigma^2)\tilde{\eta}_1 + (1 - U - W)\tilde{\eta}_2, \tag{168}$$

local analysis $\tilde{\eta}_{1,2} \sim \exp(ik_r r)$ $(k_r r \gg 1)$, there is the solution

$$k_r^2 = \frac{l(l+1)}{r^2}\frac{1}{\sigma^2 S_l^2}(\sigma^2 - N^2)(\sigma^2 - S_l^2), \tag{169}$$

and dispersion relations

$$S_l^2 = \frac{l(l+1)}{r^2}\frac{\gamma_{\rm ad}P_0}{\rho_0}, \qquad N^2 = -g\left[\frac{\mathrm{d}\log\rho_0}{\mathrm{d}\log r} - \frac{1}{\gamma_{\rm ad}}\frac{\mathrm{d}\log P_0}{\mathrm{d}\log r}\right], \tag{170}$$

$$\sigma^2 = \left[k_r^2 + \frac{l(l+1)}{r^2}\right]\frac{\gamma_{\rm ad}P_0}{\rho_0} = \left[k_r^2 + \frac{l(l+1)}{r^2}\right]c_s^2. \tag{171}$$

These oscillation wave modes driven by the pressure are called as $p-$modes, at stellar surface.
• Small $\sigma^2 \ll N^2$, S_l^2, there is

$$\sigma^2 \approx \frac{l(l+1)}{k_r^2 r^2 + l(l+1)}N^2. \tag{172}$$

These oscillation wave modes driven by gravitational forces are called as $g-$modes, at stellar interior. The energy integral

$$W = \int_0^M dm \oint \frac{\mathrm{d}q}{\mathrm{d}t} dt, \tag{173}$$

• $W > 0$ corresponds to amplitudes increasing in time (excitation),
• $W < 0$ corresponds to the damped oscillation.
For the specific entropy s there is $ds = dq/T$, $T(t) = T_0 + \delta T(t)$, and

$$0 = \oint \frac{\mathrm{d}s}{\mathrm{d}t} dt = \oint \frac{1}{T}\frac{\mathrm{d}q}{\mathrm{d}t} dt \approx \oint \frac{1}{T_0}\left(1 - \frac{\delta T}{T_0}\right)\frac{\mathrm{d}q}{\mathrm{d}t} dt \to \oint \frac{\mathrm{d}q}{\mathrm{d}t} dt = \oint \frac{\mathrm{d}q}{\mathrm{d}t}\frac{\delta T(t)}{T_0} dt. \tag{174}$$

For details you may read *SSE* Chapter 41.3 and listen carefully in the class !!!

1.9 Binary

The binary system with $(m_1,\ r_1)$, $(m_2,\ r_2)$, angular velocity

$$\omega = \sqrt{\frac{GM}{a^3}}. \qquad (M = m_1 + m_2,\ a = r_1 + r_2) \tag{175}$$

The total energy of the system

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{Gm_1 m_2}{a} = -\frac{Gm_1 m_2}{2a}, \tag{176}$$

The total angular momentum

$$J = m_1 r_1^2 \omega + m_2 r_2^2 \omega = \frac{m_1 m_2}{M} a^2 \omega = m_1 m_2 \sqrt{\frac{Ga}{M}}. \tag{177}$$

The evolution rate of J

$$\frac{\mathrm{d}J}{\mathrm{d}t} = (\dot{m}_1 m_2 + m_1 \dot{m}_2)\sqrt{\frac{Ga}{M}} + m_1 m_2 \sqrt{\frac{G}{M}}\frac{\dot{a}}{2\sqrt{a}}. \tag{178}$$

$$\frac{\dot{J}}{J} = \frac{\dot{m}_1}{m_1} + \frac{\dot{m}_2}{m_2} + \frac{\dot{a}}{2a}, \tag{179}$$

which leads to

$$\frac{\dot{a}}{a} = \frac{2\dot{J}}{J} - \frac{2\dot{m}_1}{m_1} - \frac{2\dot{m}_2}{m_2}. \tag{180}$$

If $\dot{M} = \dot{m}_1 + \dot{m}_2 = 0$,

$$\frac{\dot{a}}{a} = 2\frac{\dot{J}}{J} - \frac{2\dot{m}_1}{m_1}\left(1 - \frac{m_1}{m_2}\right) \to \frac{\dot{a}}{a} = -\frac{2\dot{m}_1}{m_1}\left(1 - \frac{m_1}{m_2}\right). \quad (\text{if } \dot{J} = 0) \tag{181}$$

Assuming $\dot{m}_1 < 0$,
• $m_1 < m_2 \to \dot{a}/a > 0$, orbital expansion.
• $m_1 > m_2 \to \dot{a}/a < 0$, orbit shrinks, Roche lobe shrinks, more mass transferred.
• The paper (Hills 1983) discussed the sudden mass loss effects on binary orbit, with the ratio of the amount of mass lost in the supernova explosion to the pre-explosion total mass of the system, and r/a_0, the ratio of the separation of the binary components to the semi-major axis of the binary at the time of the explosion. The new semi-major axis a satisfies

$$\frac{a}{a_0} = \frac{1}{2}\left[\frac{(M_0 - \Delta M)}{\frac{1}{2}M_0 - (a_0/r)\Delta M}\right], \tag{182}$$

and the binary dissociates if the mass loss occurs where

$$\frac{r}{2a_0} < \frac{r_c}{2a_0} = \frac{\Delta M}{M_0}. \tag{183}$$

There is

$$\frac{r}{2a_0} \in \left(\frac{1-e}{2},\ \frac{1+e}{2}\right), \tag{184}$$

with the conservation of the angular momentum,

$$\frac{1-e^2}{1-e_0^2} = \frac{M_0}{M_0 - \Delta M} \cdot \frac{a_0}{a} = \frac{1-(2a_0/r)(\Delta M/M_0)}{\left[1-(\Delta M/M_0)\right]^2}. \tag{185}$$

For asymmetric explosion (kick velocity), before supernova (SN),

$$E_0 = -\frac{GM_1^0 M_2}{2a_0} = -\frac{GM_1^0 M_2}{r} + \frac{1}{2}\mu_0 V_0^2, \qquad \mu_0 = \frac{M_1^0 M_2}{M_1^0 + M_2}, \tag{186}$$

after SN,

$$E = -\frac{GM_1 M_2}{2a} = -\frac{GM_1 M_2}{r} + \frac{1}{2}\mu V^2, \qquad M_1 = M_1^0 - \Delta M, \quad \mu = \frac{M_1 M_2}{M_1 + M_2}. \tag{187}$$

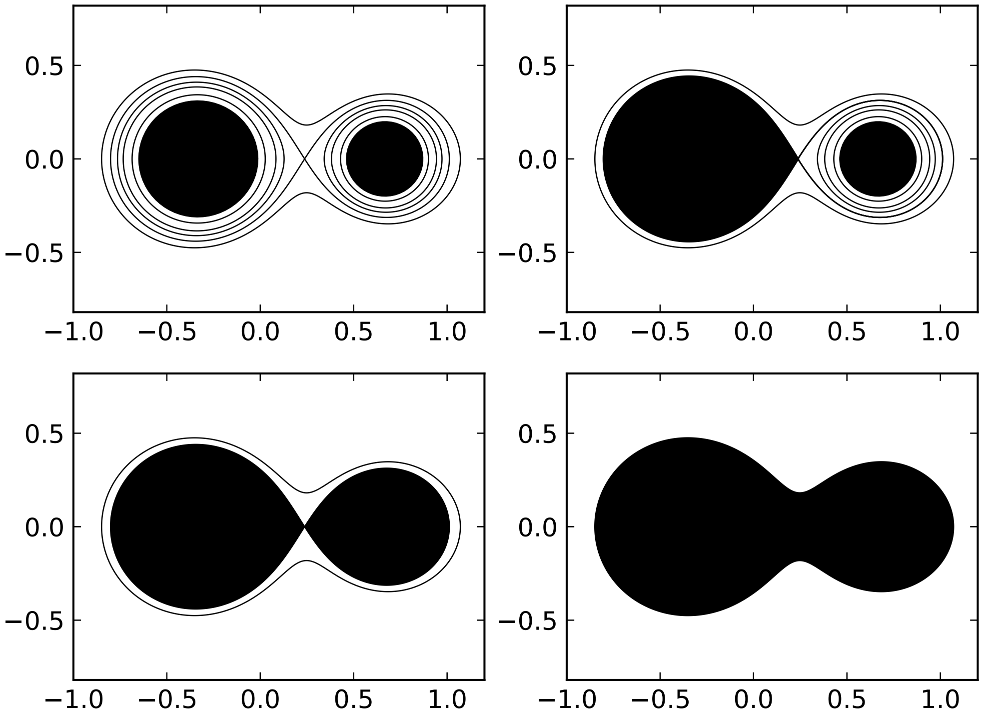


Figure 1: The accretion-overflow process of a binary system, you may know the concept of the Roche lobes.

2 Physical Picture Part

2.1 Hydrodynamics

• In radiative equilibrium (no convection), $\nabla \approx \nabla_{\rm rad} < \nabla_{\rm ad}$.
• Convection dominated energy transport, $\nabla \approx \nabla_{\rm ad} < \nabla_{\rm rad}$, in stellar convective interior, $\nabla - \nabla_{\rm ad} \ll 1$, nearly adiabatic temperature stratifation, independent of MLT (mix length theory) details.
• In stellar convective envelope (convection exists), $\nabla \approx \nabla_{\rm rad} > \nabla_{\rm ad}$, radiation dominates energy transport, MLT details make a difference ($l_m = \alpha_m H_{\rm P}$). See 3 different regions in Fig 22.8 of *SSE*.
• $\tau_{\rm dyn} \ll \tau_{\rm mix} \rightarrow$ efficient mixing (chemically homogeneous) even in envelope.
• Convective core $\rightarrow$ more nuclear burning fuel.
• Convective envelope $\rightarrow$ dredge-up burning products to the surface.
• Convective overshooting happens at the boundary of convection (Schwarzschild), $\nabla_{\rm rad} = \nabla_{\rm ad}$. For acceleration a, and velocity v, $a > 0$ in convection zone, where $v > 0$, then $a < 0$ in non-convection zone, $v > 0$, then a vanishes, and also v.

2.2 Nuclear Interaction

• Two interactions transforming ^{1}H into ^{4}He: proton-proton (pp) chain (*SSE* page 193), CNO cycle (*SSE* page 195). Recall that $\langle \sigma v \rangle \sim T^\nu$, for pp chain, when $T_6 = 5$, $\nu \approx 6$, when $T_6 \approx 20$, $\nu \approx 3.5$. While for CNO cycle, $T_6 = 10...50$, $\nu \approx 23...13$. Thus, pp chain dominates at lower temperature ($\sim \mathcal{O}(10^6)$ K), CNO cycle dominates at relatively higher temerature. (*SSE* Fig. 18.8, page 196)
• Notice that for $pp1$ and $pp3$, the net result is

$$4{}^1\text{H} \rightarrow^4 \text{He} + 2\text{e}^+ + 2\gamma + 2\nu_e, \tag{188}$$

while for $pp2$, the net result is

$$4{}^1\text{H} + \text{e}^- \rightarrow^4 \text{He} + \text{e}^+ + 2\gamma + 2\nu_e, \tag{189}$$

they are not totally equivalent, as e$^+$ will interact with electrons in the surroundings, $\text{e}^+ + \text{e}^- \rightarrow 2\gamma$, for $pp1$ and $pp3$, there are two e$^+$ for each interaction chains, while for $pp2$, there is only one (and actually even "consumes" an electron from the surroundings). Note that the energy of photons from e$^+$, e$^-$ interaction is also considered as part of the nuclear energy released.
• Neutrinos will carry away part of energy released, which should be removed when calculating on different interactions. (*SSE* page 194, for different types of neutrino energy in $pp1$, $pp2$, $pp3$)
• Helium burning happens at $T_8 \geq 1$, triple α (3α) interaction 3 ${}^4\text{He} \rightarrow^{12}$ C (*SSE* page 197), and ^{12}C, ^{16}O will also interact with ^{4}He.
• Higher temperature, $T_8 = 5...10$, ^{12}C burning happens. $T_9 \geq 1$, ^{16}O burning happens.
• $T_9 > 1.5$, ^{20}Ne disintegration happens. $T_9 \approx 3$, ^{28}Si photodisintegration happens (^{28}Si burning into ^{56}Fe). $T_9 \geq 5$, ^{56}Fe photodisintegration also happens. (Details in *SSE* page 201)

2.3 Homology Contraction (part)

• $T_c - \rho_c$ evolution diagram, for an example of $\mu \approx \mu_\odot \approx 0.62$, $\mu_e \approx \mu_{e,\odot} \approx 1.176$. For radiation pressure dominated condition, $P_{\rm rad} = aT^4/3$, with $a = 7.57 \times 10^{-15}$ erg cm^{-3} K^{-4}. For classical gas pressure dominated condition, $P_{\rm gas} = \rho \mathscr{R} T/\mu$. For relativistic degenerate electrons regimes, there is the relation that

$$P_e = \left(\frac{3}{\pi}\right)^{\frac{1}{3}} \frac{hc}{8m_u^{\frac{4}{3}}} \left(\frac{\rho}{\mu_e}\right)^{\frac{4}{3}}, P_c = C_0\ GM^{\frac{2}{3}} \rho_c^{\frac{4}{3}}, \quad C_0 = 0.364. \tag{190}$$

For $\mu_e = \mu_{e,\odot}$, $M = 5.836\ M_\odot/\mu_e^2 \approx 4.22\ M_\odot$. For non-relativistic degenerate electrons regimes,

$$P_e = \frac{1}{20} \left(\frac{3}{\pi}\right)^{\frac{2}{3}} \frac{h^2}{m_e m_u^{\frac{5}{3}}} \left(\frac{\rho}{\mu_e}\right)^{\frac{5}{3}} \equiv k_{\rm NR} \left(\frac{\rho}{\mu_e}\right)^{\frac{5}{3}}. \tag{191}$$

In order to calculate in cgs unit, we obtain that $\mathscr{R} \equiv k_{\rm B}/m_u = 8.31 \times 10^7$ erg g^{-1} K^{-1}. And we define dimensionless parameters $\bar{T} \equiv T_c/(10^7\ \text{K})$, $\bar{\rho} \equiv \rho_c/(10^2\ \text{g cm}^{-3})$. Thus, the first boundary, which is between

$P_{\rm gas}$ regime and $P_{\rm rad}$ regime, should follow the relation

$$T_c = \left(\frac{3}{a}\frac{\mathscr{R}}{\mu}\right)^{\frac{1}{3}} \rho_c^{\frac{1}{3}} \rightarrow \bar{T} \approx 17.448\ \bar{\rho}^{\frac{1}{3}}. \tag{192}$$

The second boundary, which is between $P_{\rm gas}$ regime and $P_{\rm R}$ regime, should follow the relation

$$T_c = \frac{\mu}{\mathscr{R}} C_0 G M^{\frac{2}{3}} \rho_c^{\frac{1}{3}} \rightarrow \bar{T} \approx 3.487\ \bar{\rho}^{\frac{1}{3}}. \tag{193}$$

The third boundary, which is between $P_{\rm gas}$ regime and $P_{\rm NR}$ regime, should follow the relation

$$T_c = k_{\rm NR} \frac{\mu}{\mathscr{R}\mu_e^{\frac{5}{3}}} \rho^{\frac{2}{3}} \rightarrow \bar{T} \approx 0.123\ \bar{\rho}^{\frac{2}{3}}. \tag{194}$$

where $k_{\rm NR} = 1.0036 \times 10^{13}$ erg g$^{-\frac{5}{3}}$ cm^2. The fourth boundary, which is between $P_{\rm NR}$ regime and $P_{\rm R}$ regime, should follow the relation

$$\rho_c = \left(\frac{C_0 G}{k_{\rm NR}}\right)^3 \mu_e^5 M^2, \rightarrow \bar{\rho} \approx 2.27 \times 10^4. \tag{195}$$

The four regimes in $T_c - \rho_c$ diagram and evolution tracks (not real simulation) are shown in Fig. 2.

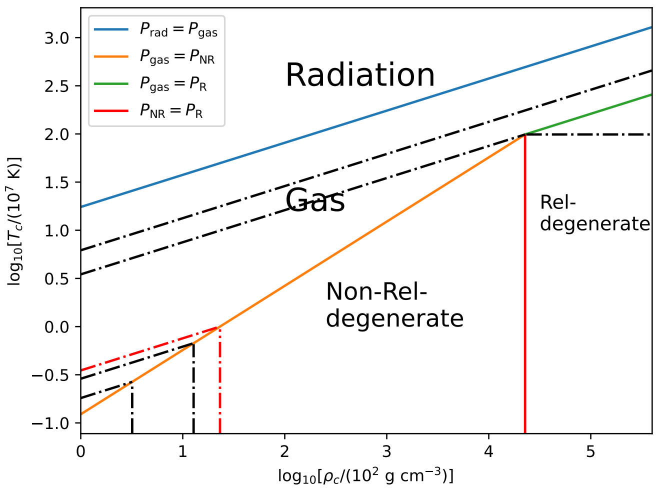


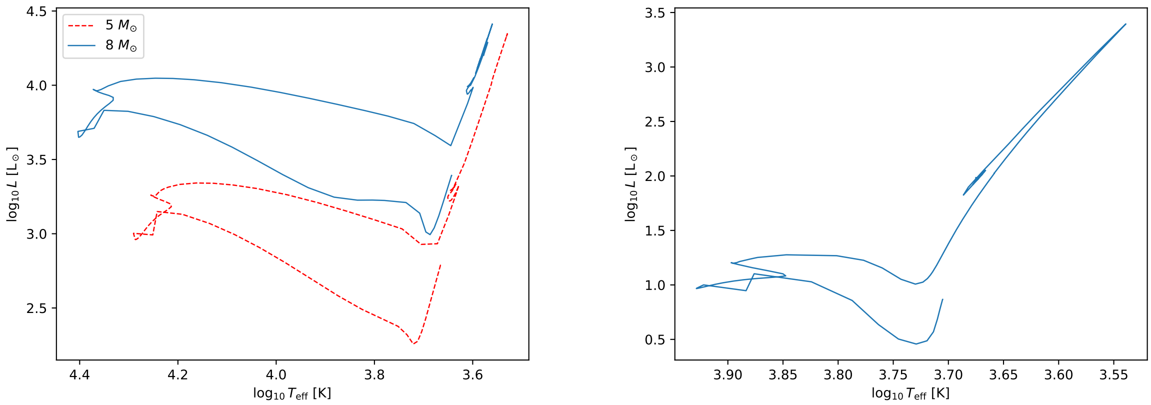
Figure 2: The black dot-dashed lines are four evolution tracks for 0.05 $M_\odot$, 0.1 $M_\odot$, $M_{\rm ch}$, 10 $M_\odot$ (From lower to upper). Only in the case of 10 $M_\odot$, T_c rises up in the whole evolution. The red dot-dashed line corresponds to the minimum mass that can reach 10^7 K during the evolution.

• From the red dot-dashed line in Fig. 2, which passes through the point of $T_c = 10^7$ K on $P_{\rm gas} - P_{\rm NR}$ boundary, it can be obtained that when $\rho_c = 10^2$ g cm^{-3}, $T_c \approx 3.51 \times 10^6$ K. Thus, $m_{\rm min} \approx 0.135\ M_\odot$ for being able to reach $T_c = 10^7$ K during the evolution, which also fits well with 0.1 $M_\odot$ for H-burning mentioned in the class.
• Stars with masses lower than $m_{\rm min}$ (which evolve downwards in NR regime in Fig. 2) correspond to brown dwarfs (no H-burning). $m_{\rm min}$ correspond to the minimum mass for Zero-Age-Main-Sequence (ZAMS) stage of evolution.

2.4 Stellar Evolution

• Pre-MS: Homologous contraction, under the timescale of $\tau_{\rm KH}$, $\rho_c \uparrow$, $T_c \uparrow$, with $T_c \propto \rho_c^{\frac{1}{3}}$, until $\sim 10^7$ K, or brown dwarfs.
• MS: H-core burning, under the timescale of $\tau_{\rm nuc}$, ρ_c, T_c slowly rises up (almost constant values).
• There are several stages and cases after MS. The symbol of the end of MS is the exhaust of H-core.

• SGB: isothermal He-core, while H-shell burning. Core contraction, then $\rho_c \uparrow \rightarrow T_c \uparrow$, H-shell burning $\uparrow$, the He-core mass $\uparrow$, again the He-core contraction $\uparrow$.
• RGB: H-shell burning, He-core mass $\uparrow$, luminosity $\uparrow$, until the tip of RGB (TRGB).
• Stellar mass $m_* <\sim 0.4 M_\odot$, density increase wins, $\rho_c \approx 10^6$ g cm^{-3}, $T_c < 10^8$ K, no He-burning, the star evolves as a He-white dwarf (WD).
• Stellar mass $m_* \in (\sim 0.4, \sim 2.6) M_\odot$, $\rho_c \approx 10^6$ g cm^{-3}, $T_c \approx 10^8$ K, He-cores ignites (He-core mass $\approx 0.4 - 0.5\ M_\odot$), He-flash happens. Inside He-core, degenerate pressure dominates, $T_c \uparrow$, P_c increase very mildly, He-core burning $\uparrow$ until degeneracy breaks down, then $P_c \uparrow \rightarrow$ He-core expands. He-flash (TRGB of this case) can be used as standard candles.
• Stellar mass $m_* >\sim 2.6\ M_\odot$, temperature wins, $T_c \approx 10^8$ K, $\rho_c \ll 10^6$ g cm^{-3}, He-core ignites and burns peacefully.
• Horizontal branch (HB): He-core burning, C-O core forms and mass $\uparrow$ (blueward HB). He-core exhausts, He-shell burning, C-O core mass $\uparrow$ (redward HB). There is also H-shell burning at HB stage.
• AGB: C-O core burning and double shell (H, He) burning.

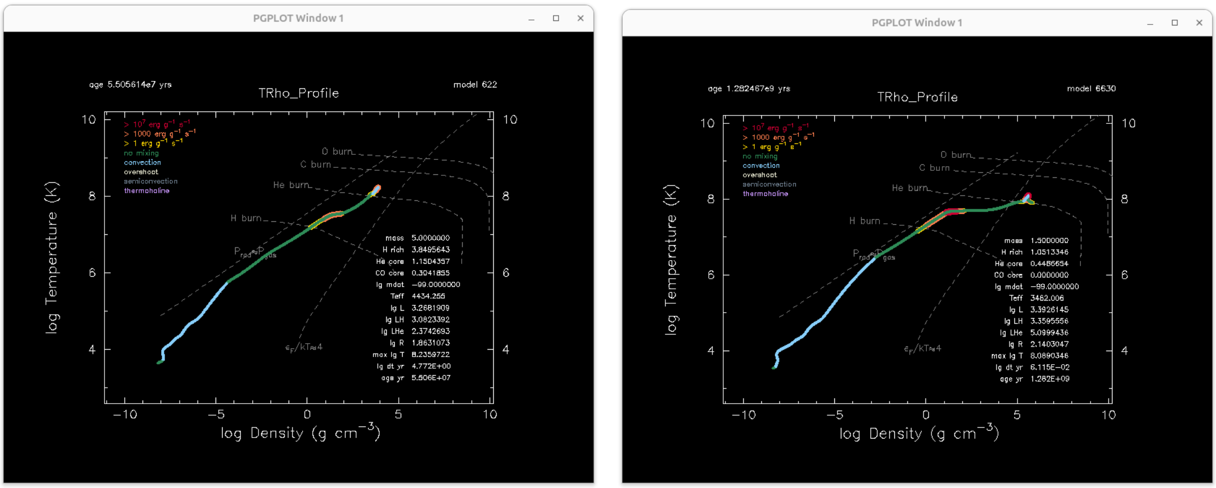


(a) H-R diagram contrast of 5 $M_\odot$ and 8 $M_\odot$ stars

(b) H-R diagram of 1.5 $M_\odot$ star

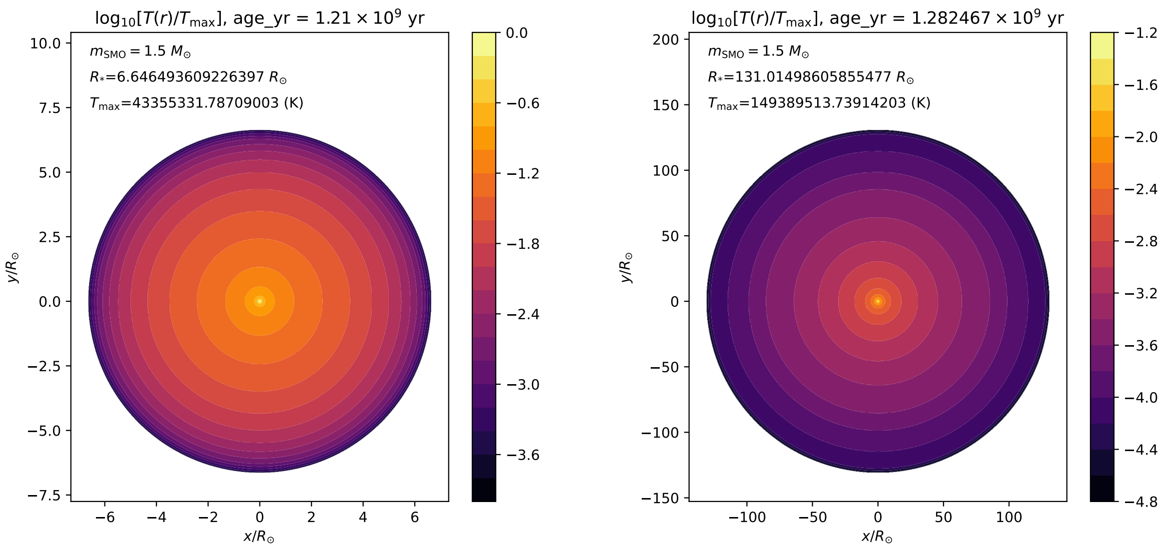
Figure 3: The H-R diagrams, from gas contraction to a little time after He-core burning. In figure (a), high mass stars have no He-flash, HB can be seen after TRGB. In figure (b), the huge tip marks the He-flash. Higher mass star evolves more rapidly, the ages of these three evolution tracks are 5.8381×10^7 years (5 $M_\odot$), 2.1267×10^7 years (8 $M_\odot$), 1.3075×10^9 years (1.5 $M_\odot$).

• Simulation with `MESA` code is shown in Fig. 3, Fig. 4 and Fig. 5.
• After AGB, there are Post-AGB, planet nebula (PN), White Dwarf stages (final, high effective temperature, low luminosity).
• For solar type stars, the evolution time of each stages: 10 Gyr (MS), 2 Gyr (SGB), $\sim$ 1 Gyr (RGB, 10%), HB, AGB (1%).
• Massive stars $m_* \in (\sim 8, \sim 10)\ M_\odot$, evolution track is MS $\rightarrow$ supergiant stage, no SGB, RGB, AGB. C-O core contracts $\rightarrow T_c \approx 5 \times 10^8$ K, core mass $\gtrsim 1.1\ M_\odot$, carbon-flash happens.
• The final type of WDs: $m_* <\sim 0.8\ M_\odot$, He-WD ($<\sim 0.5\ M_\odot$). $m_* \in (\sim 0.8, 8)\ M_\odot$, C-O-WD ($0.5 - 1.1\ M_\odot$). $m_* \in (\sim 8, \sim 10)\ M_\odot$, O-Ne-Mg-WD ($1.1 - 1.3\ M_\odot$).
• Hayashi line (*SSE* Chapter 24), fully convective star ($\nabla_{\rm ad} \approx 0.4,\ P \propto \rho T \rightarrow P = K \rho^{\frac{5}{3}}$)+radiative atmoshpere. Fully convective star luminosity is determined by the structure of the outermost layers (*SSE* Chapter 24.1). Scaling analysis: $T_{\rm eff} \propto L^A M^B$, for higher mass, Hayashi line shifts left.
• In the H-R diagram, Hayashi track leftside: $\bar{\nabla} < \nabla_{\rm ad}$, partially convective/radiative. On the Hayashi track: $\bar{\nabla} = \nabla_{\rm ad}$. Hayashi track rightside: $\bar{\nabla} > \nabla_{\rm ad}$, energy transport is so efficient that interior cools down quickly, recovers $\bar{\nabla} = \nabla_{\rm ad}$, which makes this regime a forbidden region (pre-MS case).



(a) 5 $M_\odot$ star $T-\rho$ diagram (b) 1.5 $M_\odot$ star $T-\rho$ diagram

Figure 4: $T-\rho$ diagrams after He-core ignited. Notice that 5 $M_\odot$ reached the He-burning line in classical regime, while 1.5 $M_\odot$ was in degeneracy regime. (marked with $\epsilon_{\mathrm{F}}/k_{\mathrm{B}}T \approx 4$ boundary)



(a) $T(r)$ distribution of RGB 1.5 $M_\odot$ star (b) $T(r)$ distribution of He-flashing 1.5 $M_\odot$ star

Figure 5: Fig.(b), in contrast of Fig.(a), shows the dramatic expansion of the radius and also extremely concentrated temperature distribution at He-flash stage.

2.5 Supernova (SN)

• Categories: thermal-nuclear SN, core collapse SN. Type Ia, Ib, Ic, Type II − b, −L, −P, −n.
• Type Ia SN, no H-lines, with Si-lines, thermal-nuclear, the WD in the binary system accretes the materials from the companion star → WD mass rises up to Chandrasekhar limit $M_{\rm ch}$, degeneracy pressure dominates, nuclear interactions happen furiously → thermal-nuclear explosion with stable luminosity → standard candles.
• Type Ib has no H-lines, but has He-lines, Type Ic has neither H lines nor He-lines, (both) core collapse, massive star losing H-envelope.
• Type II SN is dominated by H-lines, core collapse, Type II-P (Plateau), -L (Linear), -b (spectral signatures between II and Ia), -n (narrow line).
•Type II-P: massive H-envelope, H-ion$-$e$^-$ recombination plays the role of energy source of lightcurve plateau, Type II-L has no massive H-envelope.
• $m_* \in (\sim 8,\ \sim 10)M_\odot$ → O-Ne-Mg WD ($< M_{\rm Ch}$). $m_* \in (\sim 10,\ \sim 12)M_\odot$ → O-Ne-Mg core ($< M_{\rm ch}$), O-Ne-Mg core+shell burning ($> M_{\rm ch}$).
• electron capture

$$\mathrm{e}^- + {}^{24}\mathrm{Mg} \to {}^{24}\mathrm{Na} + \nu_e,\ \mathrm{e}^- + {}^{24}\mathrm{Na} \to {}^{24}\mathrm{Ne} + \nu_e,\ \mathrm{e}^- + {}^{20}\mathrm{Ne} \to {}^{20}\mathrm{F} + \nu_e,\ \mathrm{e}^- + {}^{20}\mathrm{F} \to {}^{20}\mathrm{O} + \nu_e, \tag{196}$$

$M_{\rm ch} \approx (5.8/\mu_e^2)M_\odot \downarrow$, then pressure ↓, core collapse → O-burning (not sufficient for stopping collapse)→ electron capture SN, $E_{\rm exp} \sim 10^{50}$ erg.
• $m_* > 12\ M_\odot$, ^{56}Fe core, (1) electron capture → decreasing P_e → decreasing $M_{\rm ch}$, (2) $\gamma-$ disintegration with $T \geq 10^{10}$ K, $P_e \sim P_\gamma$,

$$^{56}\mathrm{Fe} + \gamma \to 13\alpha + 4\mathrm{n}, \tag{197}$$

$\gamma_{\rm ad} \downarrow \to \gamma_{\rm ad} < 4/3$, (Virial) $E_{\rm tot} > 0$, dynamically unstable, endothermic, $P_\gamma \downarrow$, collapse with the timescale of $\tau_{\rm dyn}$, $\rho_c(\sim 10^{10}$ g cm$^{-3}) \uparrow$, $T_c \uparrow$ until $\rho_{\rm nuc} \sim 10^{14}$ g cm^{-3}, then

$$^{4}\mathrm{He} + \gamma \to 2\mathrm{p} + 2\mathrm{n},\ \mathrm{p} + \mathrm{e}^- \to \mathrm{n} + \nu_e, \tag{198}$$

neutron-rich → superheavy nucleus.
• Energy of core collapse, initial and final core radius: $R_{c,i}$, $R_{c,f}$, gravity energy release

$$E_{\rm gr} = GM_c^2\left(\frac{1}{R_{c,f}} - \frac{1}{R_{c,i}}\right) \approx \frac{GM_c^2}{R_{c,f}} \approx 3\times 10^{53}\ \mathrm{erg}, \qquad (M_c = 1.4\ M_\odot,\ R_f = 20\ \mathrm{km}) \tag{199}$$

$$E_{\rm env} = \int_{M_c}^{M} \frac{Gm}{r}dm \ll \frac{GM^2}{R_{c,i}} \approx 10^{53}\ \mathrm{erg},\ (M = 10\ M_\odot) \tag{200}$$

realistic mass profile $E_{\rm env} \sim 10^{50}$ erg.
• Typical Type II, $M_{\rm env} \approx 10\ M_\odot$ is blown away, $v \approx 10^4$ km s^{-1}, $E_{\rm kin} \sim 10^{51}$ erg, $L \approx 2\times 10^8\ L_\odot \approx 10^{42}$ erg s^{-1}, several months → $E_{\rm rad} \sim 10^{49}$ erg $\sim 1\% E_{\rm kin} \sim 10^{-4} E_{\rm gr}$.
• $E_{\rm exp}$: core collapse SN $\sim 10^{51}$ erg, electron capture $\sim 10^{50}$ erg.
• Neutrino trapping, $\langle l_\nu \rangle > l_{\nu,c} \approx 10^9$ cm > 3000 km, $\rho_c > 3\times 10^{11}$ g cm^{-3}, then neutrinos are trapped→ ν_e-degeneracy → neutron-rich process ($\mathrm{p} + \mathrm{e}^- \to \mathrm{n} + \nu_e$) stops.
• Bounce, shock formation, $E_{\rm shock,ini}$ sufficient to blow away burning shells. Shock propagation, post-shock density $\rho \leq 10^{11}$ g cm^{-3}, $\mathrm{p} + \mathrm{e}^- \to \mathrm{n} + \nu_e$, ν_e burst, $L_{\nu_e}^{\rm max} \approx 4\times 10^{53}$ erg s^{-1} for ~ 5 ms.
• Shock installing, ν heating, proto-NS (PNS), core $\rho_c \geq 10^{14}$ g cm^{-3}, $R \approx (10,\ 20)$ km. Delayed ν-heating mechanism, heated layers expand, successful expansion. Neutrinos diffuse out of R_ν (neutrino sphere), taking away $\sim 90\%\ E_{\rm gr}$.
• Structure of a pre-SN supergiant (*SSE* Fig. (36.4)): onion structure, Fe ← Si/S ← Ne/Mg/O ← C/O ← He ← He, H, on-going burning direction.
• $m_* = 20\ M_\odot$, lifetime of each stage: MS $\sim 10^7$ yr, He-burning (3α) $\sim 10^6$ yr, Carbon ~ 300 yr, Oxygen ~ 1 yr, Silicon ~ 2 days.

2.6 WD Component

• WD: $R \sim 10^3$ km, $\rho \sim 10^6$ g cm^{-3}, $v_{\rm E} \sim \mathcal{O}(10^{-2})c$. NS: $R \sim 10$ km, $\rho \sim 10^{14}$ g cm^{-3}, $v_{\rm E} \approx c/3$.
• Cooling of WDs, model: degenerate core + non-degenerate envelope. The luminosity of WDs comes from

cooling, degenerate electrons cannot cool, ions can cool. Assuming that the thin photoshpere of WDs follows Kramers' opacity $\kappa = \kappa_0 \rho T^{-3.5}(\mathscr{R}/\mu) = \kappa_0 P T^{-4.5}$ (*SSE* page 487), then

$$\frac{\mathrm{d}T}{\mathrm{d}P} = \frac{3\kappa_0}{16\pi acG}\frac{L}{M}PT^{-7.5} \equiv APT^{-7.5}, \tag{201}$$

with the outer boundary conditions $P \to 0,\ T \to 0$,

$$T^{8.5} = BP^2\ (B \equiv 4.25A) \to \rho = B^{-0.5}\frac{\mu}{\mathscr{R}}T^{3.25}.\ (P = \rho\mathscr{R}T/\mu)\,. \tag{202}$$

With

$$\rho_{\mathrm{tr}} = C_1^{-\frac{3}{2}}T_{\mathrm{tr}}^{\frac{3}{2}}, \quad C_1 = 1.21\times 10^5\frac{\mu}{\mu_e^{5/3}}, \tag{203}$$

"tr" means the transition of $P = K\rho^{\frac{5}{3}} = \rho(\mathscr{R}/\mu)T$, then

$$T_{\mathrm{tr}}^{3.5} = \frac{B}{C_1^3}\left(\frac{\mathscr{R}}{\mu}\right)^2 = \vartheta\frac{L/L_\odot}{M/M_\odot} \to T_{\mathrm{tr}} = \vartheta^{\frac{2}{7}}\left(\frac{L/L_\odot}{M/M_\odot}\right)^{\frac{2}{7}} \approx 5.9\times 10^7\ \mathrm{K}\left(\frac{L/L_\odot}{M/M_\odot}\right)^{\frac{2}{7}}. \tag{204}$$

The thermal energy of the ions $E_{\mathrm{th}} = c_{\mathrm{v}}MT,\ c_{\mathrm{v}} = 3k/2m_{\mathrm{ion}}$. Luminosity of the WD due to ions cooling

$$L = -\frac{\mathrm{d}E_{\mathrm{th}}}{\mathrm{d}t} = -c_{\mathrm{v}}M\frac{\mathrm{d}T}{\mathrm{d}t} = -c_{\mathrm{v}}M\frac{\mathrm{d}T_{\mathrm{tr}}}{\mathrm{d}t}. \tag{205}$$

With Eq. (204), $L = C_2MT_{\mathrm{tr}}^{3.5}$, where C_2 is a constant. Then

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\left(\frac{C_2}{c_{\mathrm{v}}}\right)T^{3.5} \to T^{-2.5} = T_0^{-2.5} + \left(\frac{5C_2}{2c_{\mathrm{v}}}\right)t, \tag{206}$$

after the initial fast cooling, $T \ll T_0$, $T \sim t^{-0.4}$ and

$$\frac{L}{L_\odot} \approx 4.3\times 10^{-2}\frac{M}{M_\odot}\mu_{\mathrm{ion}}^{-7/5}\left(\frac{t}{\mathrm{Gyrs}}\right)^{-\frac{7}{5}}, \tag{207}$$

which depicts the luminosity evolution with time of a cooling WD. The cooling time

$$\begin{aligned}\tau &= \int_{T_{\mathrm{ini}}}^{T}\frac{dT}{\dot{T}} \approx \frac{2}{5}\frac{M_\odot}{L_\odot}c_{\mathrm{v}}\vartheta T^{-5/2} \\ &= \frac{2}{5}c_{\mathrm{v}}\frac{MT}{L} \approx \frac{4.7\times 10^7\ \mathrm{yr}}{A}\left(\frac{M/M_\odot}{L/L_\odot}\right)^{\frac{5}{7}}. \qquad (A \text{ is atom number})\end{aligned} \tag{208}$$

- The Virial theorem in WDs: $1 < \zeta < 2$ for electrons + ions interior,

$$L = -\dot{E}_{\mathrm{tot}} = -\frac{\zeta - 1}{\zeta}\dot{E}_g = (\zeta - 1)\dot{E}_{\mathrm{int}}, \tag{209}$$

for normal stars, $E_e \sim E_{\mathrm{ion}} \sim T$, $L > 0 \to \dot{E}_{\mathrm{int}} > 0 \to T > 0$, heating. For WDs, assuming non-relativistic, $\zeta = 2$, $L = -\dot{E}_g/2$, as $-E_g \sim R^{-1} \sim \rho^{1/3}$,

$$\frac{\dot{E}_g}{E_g} = \frac{1}{3}\frac{\dot{\rho}}{\rho},\ E_e \approx E_{\mathrm{F}} \sim p_{\mathrm{F}}^2 \sim \rho^{\frac{2}{3}} \to \frac{\dot{E}_e}{E_e} = \frac{2}{3}\frac{\dot{\rho}}{\rho}, \tag{210}$$

then

$$\dot{E}_e = 2\frac{E_e}{E_g}\dot{E}_g = -\frac{E_e}{E_{\mathrm{int}}}\dot{E}_g. \qquad (E_g = -2E_{\mathrm{int}}) \tag{211}$$

For cool WDs, $E_{\mathrm{ion}} \ll E_e,\ E_{\mathrm{int}} \approx E_e$, then $\dot{E}_e = -\dot{E}_g = 2L$. $L = -\dot{E}_{\mathrm{ion}} - \dot{E}_e - \dot{E}_g \approx -\dot{E}_{\mathrm{ion}} \sim -\dot{T}$. At last, $\dot{E}_{\mathrm{int}} = \dot{E}_e + \dot{E}_{\mathrm{ion}} \approx 2L + (-L) = L$, $L > 0$, $\dot{E}_{\mathrm{int}} > 0$, $\dot{T} < 0 \to$ minus specific heat capacity. Cold black dwarf, all internal energy are in E_{F}.

2.7 NS Component

Just learn this part carefully in the class and read lecture notes.

2.8 Numerical Structure Calculation

• Lane-Emden equation, for $w(z_n) = 0$, z_n corresponds to the stellar surface. The Lane-Emden curves with zero-point solutions are shown in Fig. 6.

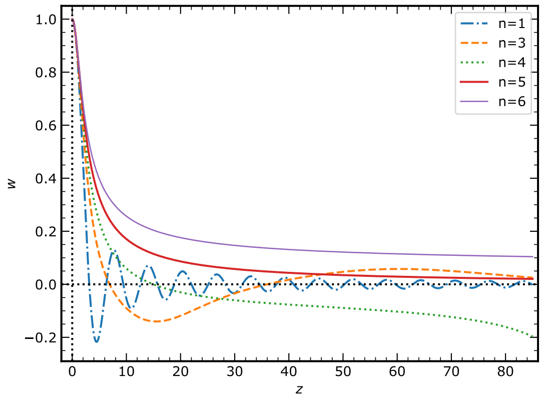
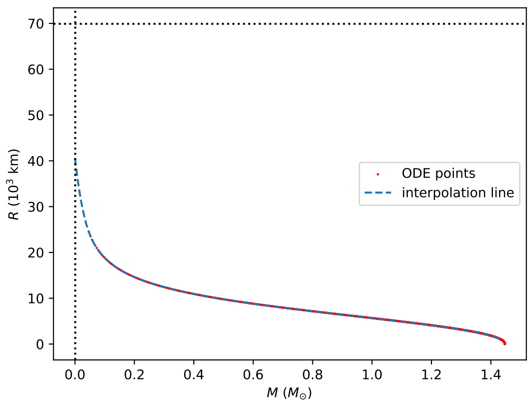
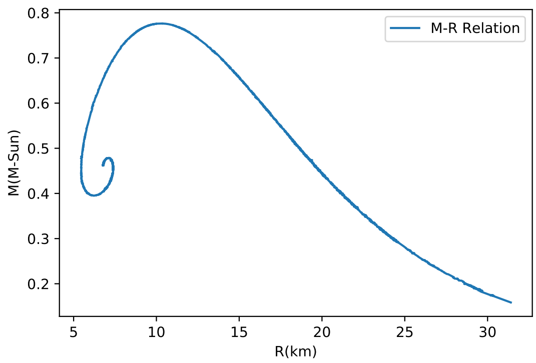


Figure 6: Lane-Emden curves

• WD structure equations, $z_c = x_c^2 + 1$ corresponds to the central $p_{\rm F}$, and also central pressure P_c. Different central pressure (z_c) values gives the distribution of mass and radius, the integration is truncated at $z = 1$, corresponding to $p_{\rm F} = 0$, $P = 0$, at star surface.
• NS structure equations (TOV equations), the M-R relation in the textbooks use a form of $\rho(P) = C_1 P + C_2 P^{3/5}$, in combination of both relativistic and non-relativistic cases for different depths of NS interior. The calculation method is then the same as WDs.
• The M-R relations of WDs and NSs are shown in Fig. 7(a)(b).



(a) WDs



(b) NSs

Figure 7: M-R relations, simplified version of model, not realistic.

2.9 Non-adiabatic Spherical Pulsations

• Category of Cepheids: (a) Classical, helium burning, inter-mass ($4 - 20\ M_\odot$), sinusoidal light curves. (b) Anomalous, helium burning, low-mass ($0.8 - 1.8\ M_\odot$), anomalous light curve. (c) Type-II, double-shell (H+He)

burning, AGB/post-AGB, low-mass ($< 0.8\ M_\odot$), light curves with some features.
• Some elements absorb heat, drive the pulsation. Others lose heat, damp the pulsation.
• ε-mechanism, energy production, $T\uparrow$, $\epsilon_{\rm nuc}\uparrow$, $dQ > 0$,, obvious effect for very massive stars ($\geq 10^2\ M_\odot$).
• κ-mechanism, opacity, during compression, the layer becomes opaque, energy flow becomes trapped, $T\uparrow$, $P\uparrow$, pushing the layer outward/expansion, transport out, then $T\downarrow$, $P\downarrow$.
• Condition (discussion in *SSE* page 534)

$$\left(\frac{\mathrm{d}\log\kappa}{\mathrm{d}\log P}\right)_{\rm ad} = \left(\frac{\partial\log\kappa}{\partial\log P}\right)_T + \left(\frac{\partial\log\kappa}{\partial\log T}\right)_P \left(\frac{\mathrm{d}\log T}{\mathrm{d}\log P}\right)_{\rm ad} = \kappa_P + \kappa_T\cdot\nabla_{\rm ad} > 0. \tag{212}$$

Kramers' law $\kappa \propto \rho T^{-3.5}$, $\kappa_P \approx 1$, $\kappa_T \approx -4.5$. ionized ideal gas $\nabla_{\rm ad} = 0.4$, then $\kappa_P + \kappa_T\cdot\nabla_{\rm ad} < 0$, the condition not working.
If $\kappa_T > 0$, H^- opacity dominates at $T < 10^4$ K, very cool stars (Mira variables), not for Cepheids.
In Kramers' law, if $\nabla_{\rm ad}$ is small ($\nabla_{\rm ad} \leq 0.222$), partially ionized gas.
At $T \approx 1.5\times 10^4$ K, two ionization zones: $\mathrm{H} \rightleftharpoons \mathrm{p} + \mathrm{e}^-$, $\mathrm{He} \rightleftharpoons \mathrm{He}^+ + \mathrm{e}^-$.
At $T \approx 4\times 10^4$ K, $\mathrm{He}^+ \rightleftharpoons \mathrm{He}^{2+} + \mathrm{e}^-$.
For $T_{\rm eff} \geq 7500$ K (blue edge), Both ionization zones $\to$ surface, weak effect, low density.
For $T_{\rm eff} \leq 5500$ K (red edge), ionization zones $\to$ convective, adiabatic.

3 Appendix

For this class you may refer to three textbooks:
• Stellar Structure and Evolution,
• Stellar Interiors,
• Lamers Levesque 2017 Understanding Stellar Evolution (Superme !!!).
• WARNING: If you find the descriptions of certain concepts in these notes to be abstract, you should be especially cautious and pay close attention during class, as this indicates that the author has not fully grasped them, even to this day.
• If you notice any obvious errors, have better suggestions, or would like to further contribute as a co-editor, please try in any way to find and contact the author, this is much easier than learning this class.