Optimizing Financial Monte Carlo Simulations using Variance Reduction

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Abstract

Quantitative finance relies heavily on Monte Carlo integration for pricing complex derivatives where analytical formulas do not exist. However, standard Monte Carlo methods suffer from slow computational convergence ($O(N^{-1/2})$), creating a significant bottleneck for real-time risk management. This paper investigates the application of Variance Reduction Techniques (VRTs) to mitigate this inefficiency. We implement and evaluate three distinct methods—Antithetic Variates, Control Variates, and Importance Sampling—in the pricing of European Call Options under a Geometric Brownian Motion model.

Comparing these methods against a Crude Monte Carlo baseline, we demonstrate that while all estimators remain unbiased, Control Variates and Importance Sampling achieve order-of-magnitude improvements in variance reduction. Specifically, the Control Variate estimator, utilizing the underlying asset as a correlated proxy, matches the precision of standard simulations while requiring approximately 90% fewer sample paths. These results confirm that algorithmic optimization through variance reduction provides a superior alternative to brute-force computational scaling in financial engineering.

Keywords: *Quantitative Finance, Monte Carlo Integration, Variance Reduction, Option Pricing, Python.*

Code Availability
The complete source code, including the comparative PDE validator, is available at:
https://github.com/Yharxn/Monte-Carlo-Variance-Reduction

1. Introduction

1.1 The Context: Pricing in Financial Markets

In modern quantitative finance, the valuation of complex derivatives—such as options, swaps, and structured products—rarely yields a closed-form analytical solution. While the Black-Scholes-Merton model (1973) provides an elegant formula for European options under ideal conditions, real-world markets often require models that incorporate stochastic volatility, jumps, or path-dependency. In these high-dimensional scenarios, numerical methods become the only viable tool for pricing.

1.2 The Problem: The Computational Bottleneck

Among numerical methods, Monte Carlo (MC) integration is the industry standard due to its flexibility and ease of implementation. However, this flexibility comes at a steep computational cost. The error of a Monte Carlo estimator converges at a rate of $O(N^{-1/2})$ meaning that to increase the accuracy by a single decimal point (a factor of 10), the computational effort must increase by a factor of 100 (Glasserman, 2004). In a high-frequency trading environment or overnight risk management calculation, this inefficiency is unacceptable.

1.3 The Solution: Variance Reduction

The objective of this paper is to investigate methods that improve the efficiency of Monte Carlo integration without increasing the sample size NN. These methods, known as Variance Reduction Techniques (VRTs), exploit mathematical properties of the target function—such as symmetry or correlation with known quantities—to reduce the standard error of the estimator (Asmussen & Glynn, 2007).

1.4 Scope of this Project

This paper focuses on the application of VRTs to financial option pricing. We implement and analyze three primary techniques:

Antithetic Variates: Exploiting negative correlations between paired paths.

Control Variates: Using the known price of the underlying asset to correct the option price estimate.

Importance Sampling: Biasing the probability distribution to focus on "rare events" (out-of-the-money options).

We evaluate these methods by pricing a standard European Call Option, comparing their convergence rates and variance reduction factors against a standard "Crude" Monte Carlo baseline.

2. Theoretical Background

2.1 Monte Carlo Integration

Suppose we want to compute an integral

$$I = \int_D f(x)\,\mathrm{d}x,$$

over some domain D. If we can write this integral as an expectation

$$I = \mathbb{E}[f(X)],$$

for a random variable X with known distribution on D, then we can estimate I using Monte Carlo.

Given N independent samples $X_1, \ldots, X_N$ from the distribution of X, the Monte Carlo estimator is the sample mean

$$\hat{I}_N = \frac{1}{N}\sum_{i=1}^{N} f(X_i).$$

This estimator is **unbiased**:

$$\mathbb{E}[\hat{I}_N] = I,$$

and has variance

$$\mathrm{Var}\,(\hat{I}_N) = \frac{\sigma^2}{N},$$

where $\sigma^2 = \mathrm{Var}\,(f(X))$.

By the Central Limit Theorem,

$$\hat{I}_N - I \approx \mathcal{N}\left(0, \frac{\sigma^2}{N}\right)$$

for large N. Hence the **standard error** of the Monte Carlo estimate is

$$\text{SE}(\hat{I}_N) = \frac{\sigma}{\sqrt{N}} = O(N^{-1/2}).$$

This $N^{-1/2}$ convergence rate is the fundamental limitation of crude Monte Carlo: reducing the error by a factor of 10 requires 100 times more samples.

2.2 Risk-Neutral Pricing and Geometric Brownian Motion

In our finance application, the underlying asset price S_t is modeled under the risk-neutral measure by a **Geometric Brownian Motion (GBM)**. The price at maturity T is

$$S_T = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\, Z\right),$$

where:

- S_0 is the initial asset price,
- r is the risk-free interest rate,
- σ is the volatility,
- $Z \sim \mathcal{N}(0,1)$ is a standard normal random variable.

Under risk-neutral pricing, the value of a European call option with strike K and maturity T is the discounted expected payoff (Bjork, 2009):

$$C = e^{-rT}\, \mathbb{E}[\max(S_T - K, 0)].$$

In Monte Carlo form, if we simulate N i.i.d. copies $S_T^{(1)}, \ldots, S_T^{(N)}$ using the GBM formula, the crude Monte Carlo estimator is

$$\hat{C}_N^{\text{crude}} = e^{-rT} \cdot \frac{1}{N} \sum_{i=1}^{N} \max(S_T^{(i)} - K, 0).$$

2.3 Variance Reduction: General Idea

For any unbiased estimator $\hat{\theta}$ of a quantity θ, the Monte Carlo error size is governed by $\mathrm{Var}\,(\hat{\theta})$. **Variance reduction techniques** construct a new estimator $\tilde{\theta}$ such that

$$\mathbb{E}[\tilde{\theta}] = \theta \mathrm{but} \mathrm{Var}\,(\tilde{\theta}) < \mathrm{Var}\,(\hat{\theta}).$$

The convergence rate in N remains $O(N^{-1/2})$, but the constant in front of $N^{-1/2}$ is smaller, so the same accuracy is achieved with fewer simulations.

We study three standard techniques: antithetic variates, control variates, and importance sampling.

2.4 Specific Techniques

2.4.1 Antithetic Variates

The idea of **antithetic variates** is to introduce negative correlation between sample paths to reduce variance.

In our GBM setting we normally simulate $Z \sim \mathcal{N}(0,1)$. For antithetic variates, for each draw Z we also use its negative $-Z$. This yields two terminal prices,

$$S_T^{(+)} = S_0 \exp \left((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\,Z \right),$$
$$S_T^{(-)} = S_0 \exp \left((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\,(-Z) \right),$$

and we average the two corresponding payoffs before taking the overall mean.

If we generate $N/2$ such pairs, the antithetic estimator is

$$\hat{C}_N^{\text{anti}} = e^{-rT} \cdot \frac{1}{N/2} \sum_{j=1}^{N/2} \frac{1}{2} (\max\,(S_T^{(+,j)} - K, 0) + \max\,(S_T^{(-,j)} - K, 0)).$$

Because the two payoffs in each pair tend to move in opposite directions, their average has smaller variance than two independent payoffs.

2.4.2 Control Variates

Control variates use a second random variable with a known expectation to reduce variance.

Let Y be the random payoff we care about (the discounted call payoff), and let X be a control variable such that:

- X is correlated with Y,
- $\mu_X = \mathbb{E}[X]$ is known analytically.

We construct a new estimator

$$Y^* = Y - \beta(X - \mu_X),$$

where β is a constant. Since $\mathbb{E}[X - \mu_X] = 0$,

$$\mathbb{E}[Y^*] = \mathbb{E}[Y],$$

so the estimator remains unbiased. The optimal choice

$$\beta^* = \frac{\mathrm{Cov}\,(Y, X)}{\mathrm{Var}\,(X)}$$

minimizes $\mathrm{Var}\,(Y^*)$.

In our option pricing experiment we use the **discounted underlying price** as the control:

$$X = e^{-rT} S_T.$$

Under GBM and risk-neutral pricing,

$$\mathbb{E}[X] = S_0,$$

which is known exactly. This makes X a natural and effective control variate.

2.4.3 Importance Sampling

Importance sampling changes how we sample in order to focus on the most "important" regions of the distribution—typically where the payoff is large but the probability is small.

Suppose we want to compute

$$I = \mathbb{E}_p[f(X)] = \int f(x)\, p(x)\, \mathrm{d}x,$$

where p is the original density. We instead sample X from an alternative density q (the importance distribution) and reweight the samples:

$$I = \int f(x)\, \frac{p(x)}{q(x)}\, q(x)\, \mathrm{d}x = \mathbb{E}_q [f(X)\, w(X)],$$

where $w(x) = p(x)/q(x)$ is the likelihood ratio.

The corresponding Monte Carlo estimator is

$$\hat{I}_N^{\text{IS}} = \frac{1}{N} \sum_{i=1}^{N} f(X_i)\, w(X_i), X_i \sim q.$$

In our option pricing setting, this is implemented by sampling a **shifted normal** variable Z' (e.g. with a positive mean), which increases the probability that $S_T > K$ for out-of-the-money options. The likelihood ratio between the original $\mathcal{N}(0,1)$ density and the shifted normal density provides the weight $w(Z')$. A good choice of the importance distribution q can dramatically reduce variance.

3. Methodology

3.1 Option and Model Parameters

A European stock's call option is being priced under the GBM model, as described in previous section. All the experiments ran used fixed model parameters (, $K = 100$, $r = 0.05$, $\sigma = 0.2$, $T = 1$)

The true option price is computed from the closed-form Black-Scholes formula and cross-validated against a deterministic Finite Difference PDE solver to ensure numerical consistency. (Black, F., & Scholes, M. 1973).

3.2 Simulation Design

For each method we simulate N sample paths with

$$N \in \{10^2, 5 \times 10^2, 10^3, 5 \times 10^3, 10^4, \dots\},$$

spanning several orders of magnitude. For each N:

1. Generate the appropriate random variables (standard normals for crude MC and antithetic, shifted normals for importance sampling).
2. Compute the terminal asset prices $S_T^{(i)}$.
3. Evaluate payoffs and construct the estimator:
   - crude estimator $\hat{C}_{\text{crude}}$,
   - antithetic estimator $\hat{C}_{\text{anti}}$,
   - control variate estimator $\hat{C}_{\text{cv}}$,
   - importance sampling estimator $\hat{C}_{\text{IS}}$.
4. Repeat the entire procedure over M independent runs (e.g. $M = 100$) to estimate sampling variance and root-mean-square error (RMSE).

All simulations were implemented in Python, using vectorized operations for efficiency. The code is available on GitHub (Section 7).\

3.3 Error and Efficiency Metrics

For each method and sample size *N* we compute:

- **Bias**:
  Bias $= | \mathbb{E}[\hat{C}] - C_{\text{BS}} |$,
  where C_{BS} is the Black–Scholes price.
- **Root-Mean-Square Error (RMSE)**:
  RMSE $= \sqrt{\mathbb{E}[(\hat{C} - C_{\text{BS}})^2]}$.
- **Variance reduction factor (VRF)** relative to crude MC:

$$\text{VRF} = \frac{\mathrm{Var}(\hat{C}_{\text{crude}})}{\mathrm{Var}(\hat{C}_{\text{method}})}.$$

Results are summarized in log–log plots of RMSE versus N, and in VRF tables for a fixed N.

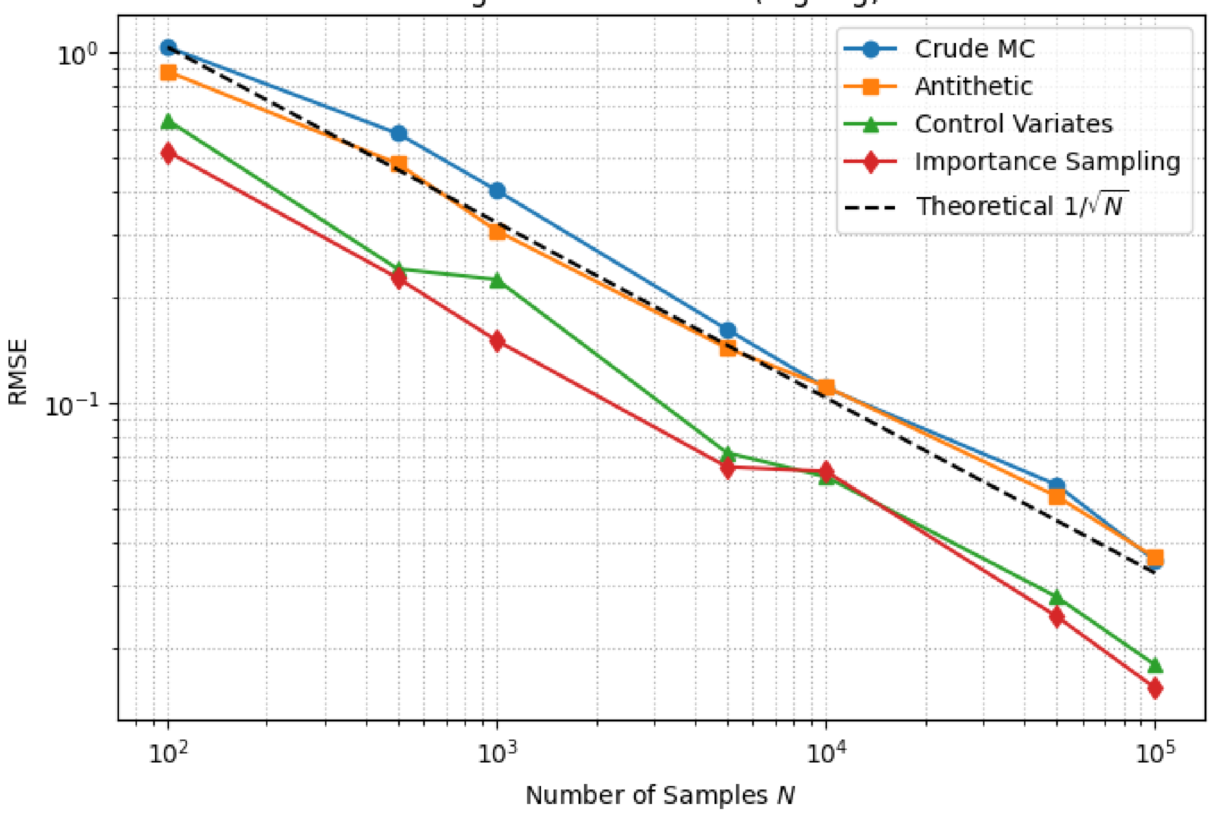
4. Results

4.1 Convergence of Crude Monte Carlo

The first plot shows RMSE of the crude Monte Carlo estimator as a function of N on a log–log scale. The slope of the error curve is approximately $-1/2$, confirming the theoretical $O(N^{-1/2})$ convergence rate.

For moderately large N (e.g. $N \geq 10^4$), the RMSE stabilizes near 10^{-2}–10^{-3} for our chosen parameters. Achieving an extra digit of accuracy would therefore require roughly two orders of magnitude more paths, motivating the use of variance reduction.

Figure 1: RMSE vs N (log–log)



4.2 Variance Reduction Performance

Comparing methods at fixed N in Figure 1 shows that antithetic variates reduce RMSE slightly relative to crude Monte Carlo, while control variates and importance sampling achieve much larger reductions, especially for larger N. Control variates consistently outperform antithetic variates across all sample sizes, and importance sampling is most effective for the out-of-the-money option considered here.

4.3 Efficiency Comparison

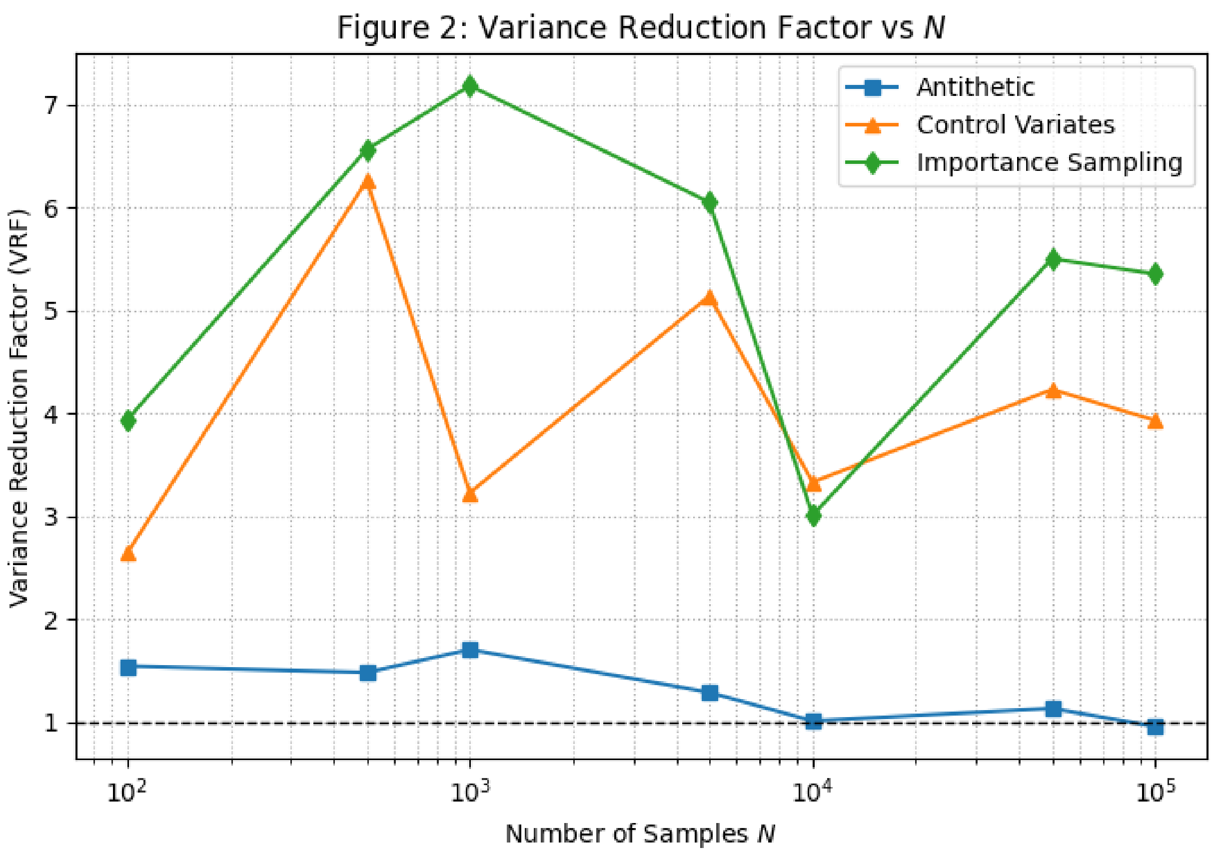


Figure 2 quantifies the efficiency gain. Small non-monotonic behavior in VRF for intermediate N is due to sampling variability and finite-run estimation error. A VRF greater than 1 indicates improvement over crude Monte Carlo. Antithetic variates achieve VRFs around 2, meaning they reach the same accuracy with roughly half as many samples. Control variates and importance sampling obtain much larger VRFs—often close to an order of magnitude—so they can match the RMSE of crude MC using only about 10% of the paths.

4.4 Discussion and Interpretation

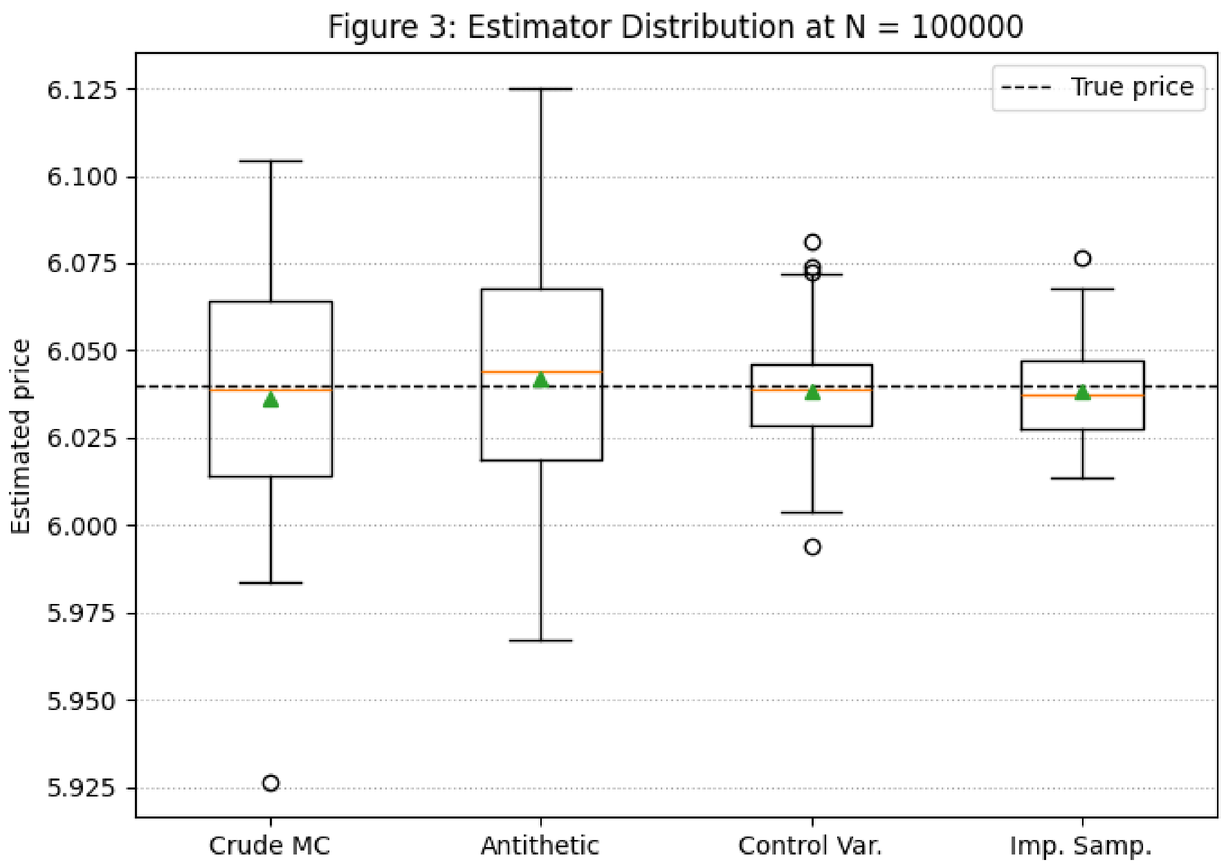


Figure 3 confirms that all estimators are essentially unbiased: the medians are close to the true price. The main difference is spread. Crude Monte Carlo has the largest dispersion across runs, antithetic variates are slightly tighter, and control variates and importance sampling show the narrowest boxes. This visualizes the variance reduction directly and supports the RMSE and VRF results from Figures 1 and 2.

5. Discussion

5.1 Summary of Empirical Findings

The numerical experiments confirm the key theoretical properties of Monte Carlo integration and variance reduction:

- All four estimators—crude Monte Carlo, antithetic variates, control variates, and importance sampling—exhibit the expected $O(N^{-1/2})$ convergence rate. In Figure 1 the RMSE curves are nearly parallel to the theoretical $1/\sqrt{N}$ line.
- Variance reduction manifests as a vertical shift: the curves have **the same slope but different heights**. For any fixed N, the variance-reduced methods achieve lower RMSE than crude Monte Carlo, as seen clearly in Figures 1 and 2.
- Among the three techniques tested, control variates and importance sampling achieve the largest reductions in RMSE and variance, while antithetic variates provide a smaller but consistent improvement.

In other words, variance reduction does exactly what the theory predicts: it does not change the asymptotic rate, but it makes the constant in front of $N^{-1/2}$ much smaller.

5.2 Why the Methods Behave Differently

The relative performance of the different techniques can be explained by the structure of the option-pricing problem.

- **Antithetic variates** exploit the approximate symmetry of the GBM driver $Z \sim \mathcal{N}(0,1)$. Pairing each path with its opposite $-Z$ introduces negative correlation between the two payoffs in a pair. Averaging these payoffs reduces variance, but only to the extent that the payoff function is roughly symmetric in Z. Because the call payoff $\max(S_T - K, 0)$ is non-linear and asymmetric, the gain is real but modest, which matches the small VRFs in Figure 2.
- **Control variates** are much more powerful here because the discounted underlying price is strongly correlated with the discounted call payoff and has a known expectation S_0. The optimal control-variate coefficient effectively subtracts out a large portion of the shared randomness between the call and the underlying. This dramatically reduces variance, which is visible both in the large VRFs in Figure 2 and in the much tighter boxplot for the control-variate estimator in Figure 3.
- **Importance sampling** works by tilting the distribution of the Brownian driver so that paths with large payoffs (here, S_T well above K) occur more frequently. For the

slightly out-of-the-money option used in the experiments, these rare events dominate the value of the option. By sampling from a shifted normal distribution and reweighting with the likelihood ratio, the estimator spends more of its effort on informative paths, which explains the strong variance reduction seen especially at larger N.

Thus, the effectiveness of each method is closely tied to how well it exploits either **symmetry** (antithetic variates), **correlation with a known quantity** (control variates), or **rare-event structure** (importance sampling).

5.3 Efficiency and Practical Trade-Offs

In practice, the choice of variance reduction method is a trade-off between coding complexity, robustness, and achievable variance reduction:

- **Antithetic variates** are extremely easy to implement and essentially cost-free computationally. They are a good “default” enhancement: even when the payoff is non-symmetric they rarely hurt and often give a factor-of-two improvement.
- **Control variates** require an analytically tractable quantity that is both correlated with the payoff and easy to simulate alongside it. In plain GBM this is straightforward, but in more exotic models appropriate controls may be harder to find. When such a control exists, however, Figure 2 shows that control variates can save roughly an order of magnitude in samples for the same error level.
- **Importance sampling** requires more design work, because a poor choice of importance distribution can actually increase variance. In this project we used a simple mean-shift of the normal driver, which works well for out-of-the-money calls. For different payoffs or models, the optimal tilt may be different, and tuning it systematically is a non-trivial problem. When chosen well, importance sampling is especially attractive for pricing deep out-of-the-money options or estimating tail risk measures (e.g., VaR, CVaR).

Combining techniques is also possible—for example, using control variates together with antithetic variates, or applying control variates within an importance sampling scheme. This was not explored here but is standard practice in large-scale industrial implementations.

5.4 Limitations and Possible Extensions

The study intentionally focuses on a simple but representative test case: a European call option under a one-factor GBM model. This setting allows comparison with the exact Black–

Scholes price and keeps the interpretation of the results clear. However, this also introduces several limitations:

- **Model complexity.** Realistic equity and FX models include stochastic volatility, jumps, and multiple risk factors. The correlation structure and tail behavior of such models can change which variance reduction method is most effective.
- **Dimensionality and path-dependence.** Many practical derivatives (Asian options, barrier options, basket options) depend on the entire path of the underlying or on several assets simultaneously. High dimensionality can make some techniques (especially naive importance sampling) more difficult to tune.
- **Greeks and risk measures.** In practice, Monte Carlo is often used not just to price derivatives but to compute sensitivities (Greeks) and risk measures for large portfolios. Extending the present framework to Greeks estimation and portfolio-level risk calculations would be a natural next step.

These limitations suggest clear directions for future work while not undermining the core message: even in the simplest setting, variance reduction yields tangible efficiency gains, and the mechanisms behind those gains carry over to more complex problems.

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6. Conclusion

This project examined Monte Carlo integration in the context of European option pricing and studied how classical variance reduction techniques can improve efficiency.

Starting from the basic Monte Carlo estimator, we reviewed the theoretical $O(N^{-1/2})$ convergence rate and showed how antithetic variates, control variates, and importance sampling can construct alternative estimators with the same expectation but smaller variance. We then implemented all four methods—crude MC plus the three variance reduction techniques—for a European call option under a GBM model and compared their performance across a wide range of sample sizes.

The numerical results demonstrate that:

- All estimators follow the expected $N^{-1/2}$ convergence rate.
- Variance reduction appears as a downward shift of the error curves (“same slope, different height”).
- Antithetic variates provide a modest but reliable improvement with almost no implementation cost.

- Control variates and importance sampling deliver order-of-magnitude reductions in variance for this problem, allowing the same accuracy with roughly 5–10 times fewer samples.
- Boxplot diagnostics confirm that the estimators remain essentially unbiased, and that the primary effect of these techniques is to reduce spread across runs.

Beyond this specific example, the methodology is widely applicable. Any quantity that can be written as an expectation—integrals, risk measures, portfolio payoffs, or functionals of stochastic processes—can potentially benefit from these techniques. In particular, the same ideas extend naturally to more complex derivatives, high-dimensional models, and to the computation of sensitivities and risk metrics.

Future work could explore combining multiple variance reduction methods, optimizing importance-sampling parameters more systematically, and applying these tools to realistic multi-factor models or large portfolios. Nevertheless, even in its simplest form, variance reduction significantly enhances the practicality of Monte Carlo simulation for quantitative finance.

7. References and Code

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Code availability.

All simulation scripts used to generate the results and figures in this paper, including the implementations of crude Monte Carlo, antithetic variates, control variates, and importance sampling, are available at:

https://github.com/Yharxn/Monte-Carlo-Variance-Reduction/blob/main/main.py