ADCS – VII
Attitude Determination

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Summary of last lecture

Reference frames and vectors

Rotation matrix

- Direction Cosine Matrix
- Euler angles
- Euler eigenaxis rotation
- Euler parameters / Quaternion

Angular velocity

- Transport theorem $\frac{\mathrm{d}}{\mathrm{d}t}(\ \ )_i = \frac{\mathrm{d}}{\mathrm{d}t}(\ \ )_b + \vec{\omega} \times (\ \ )$
- Velocity due to rotation → Coriolis and centripetal effect

Kinematic differential equation

Outline

Attitude determination

- Basic idea
- Static approach
- Filtering approach

Single axis determination

Three axis determination

- TRIAD method
- Q-method
- QUEST method

Attitude determination objective

<u>Objective</u>:

To determine attitude (or orientation or pointing direction) of spacecraft-fixed reference frame with respect to known (usually inertial) reference frame

Attitude determination requires two or more attitude sensors like:

- Magnetometers
- Sun sensor
- Earth sensor
- Star sensor
- Gyroscope
- GPS

Attitude determination method

What information is required to determine attitude using external sensor?

- Typically have one or more instruments mounted on satellite detecting external references (e.g. Sun, Earth, stars)

  → Measure direction to these objects in satellite coordinates

- From knowledge of directions to these objects in some known coordinate system (e.g. Earth Centered Inertial System)

  → Want to find orientation of satellite coordinates relative to this known frame

Deterministic method

- Need measurements of 2 vectors in body frame $\mathbf{u}_b$, $\mathbf{v}_b$ (e.g. Sun, Earth, …)
- Know these vectors in reference frame $\mathbf{u}_i$, $\mathbf{v}_i$ (need to know position in orbit)

  → Find rotation matrix $\mathbf{C}_{bi}$: attitude $\quad \vec{u}_b = \mathbf{C}_{bi}\vec{u}_i \ , \quad \vec{v}_b = \mathbf{C}_{bi}\vec{v}_i$

Implementation of attitude determination

Attitude described by 3 parameters

Means at least 2 vectors needed in both body- and reference frame

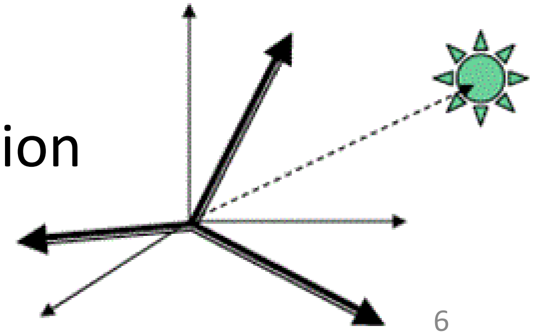
(e.g. Sun- and Earth-vector or 2 star vectors, or …)

If measure position of single star and known location (e.g. in Earth Centered Inertial System)

→ gives spacecraft orientation modulo rotation about line to star

If measure second star (not in same direction)

→ have enough information to find 3-axis orientation



Attitude determination / measurement

- Attitude determination is a process of deriving estimation of spacecraft from measured data
- Attitude determination typically requires finding three independent quantities (such as any minimal parameterization of attitude matrix)
- Exact determination is not possible (measurements have always some error)
- In order to determine spacecraft current attitude it requires use of sensors
- Possible to have absolute (sensor) and relative (gyroscope) measurements
- Real sensors measure angles with some errors (3 angular positions and 3 angular velocities)

Over-determination and under-determination

- At least 2 vectors required to determine attitude
- Remember:

  Need 3 independent parameters to determine attitude
- Unit vector has only two parameters because of unit vector constraint

  → Need 3 scalars to determine attitude

  → Requires more than 1 and less than 2 vector measurements
- Most attitude determination is **over-determinated** (if $\geq$2 vectors) or **under-determinated** (if only 1 vector)
- Over-determinated approaches use more observation information than required to compute full attitude solution

Two categories of attitude determination

Category 1:

Static determination approach which depends on measurements taken at same time (or close enough in time that satellite motion between measurements can be ignored)

Requirement is that each time is enough information available to fully compute attitude (no need of priori attitude estimation)

Category 2:

Filtering determination approach which makes explicit use of knowledge of motion of satellite to accumulate information (memory) of past measurements

Filtering approach can provide more accurate attitude estimation than static approach (Kalman filter)

We concentrate on **static attitude determination** (no time involved)

Single axis determination

Single axis attitude determination:

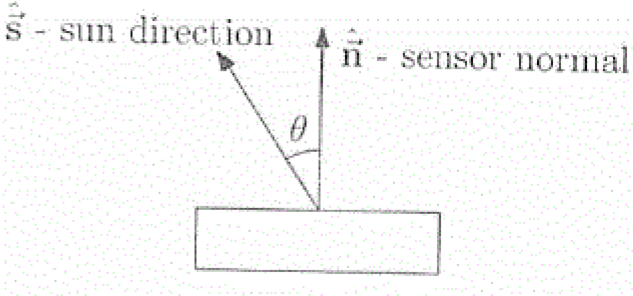
Determine orientation of single spacecraft axis in space (usually space axis)

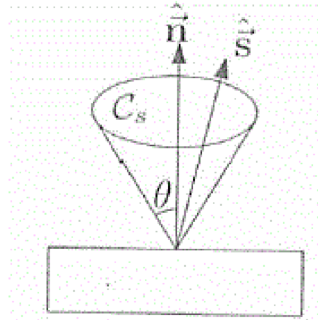
Utilize sensors that yield an arc-length measurement between sensor at spacecraft and known point (e.g. Sun, nadir)

Requisite at least two independent measurements and a scheme to chose between true and false solution

Total lack of a priori estimate (static attitude determination)

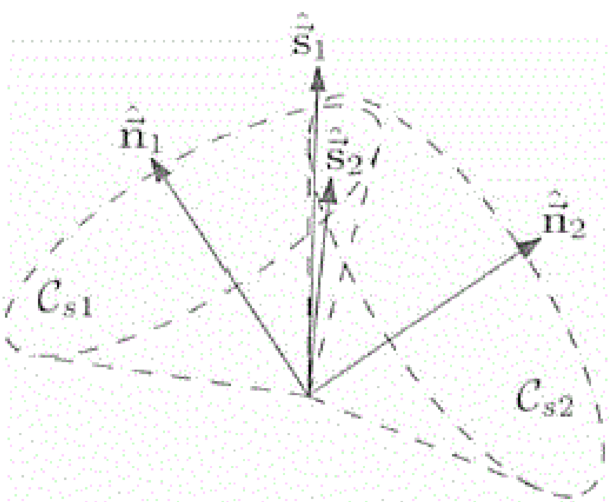
Single axis determination

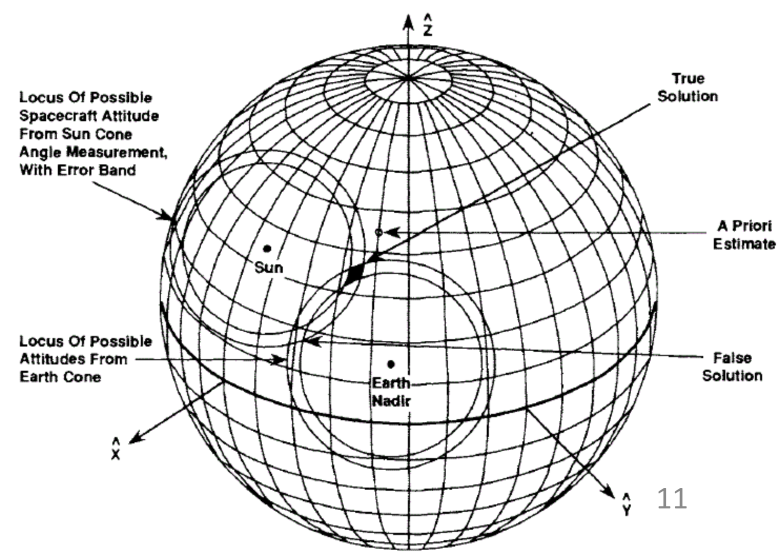




Analog Sun-sensor provides angle Θ of Sun-vector relative to sensor normal
$\Rightarrow$ Cone where Sun-vector must lie

Combinations of pair of analog sun-sensor give additional information
Pair of analog Sun-sensors with normal n_1 and n_2
$\Rightarrow$ pair of cones on which sun-vector must lie
$\Rightarrow$ pair of possible Sun-vectors s_1 and s_2
Additional measurement needed to determine true Sun-vector
Possibility add third Sun-sensor → three intersecting cones → give Sun-vector (body frame)
If know spacecraft orbit position and Earth orbit position → Sun-vector (inertial frame)





Three-axis attitude determination

Three axis attitude determination:

Determine complete orientation

Need two vectors measured in spacecraft frame and known in reference frame

Generally redundant data available, so that extend calculations to include a least-squares estimate for attitude (static attitude determination)

Triad method

Incorporates just enough observation information
to uniquely determine attitude (need of two vectors)

Triad method

- Triad method is one of earliest and simplest solutions to spacecraft attitude determination problem
- Given knowledge of two vectors in reference and body coordinates of satellite

  →Triad method obtains direction cosine matrix relating both frames
- Only works with two measured attitude observations and uses only three out of four pieces of information obtainable from this two vectors
- Requires vectors are normalized
- Requires two nonparallel vectors

  (Try what happens if vectors are parallel)
- **TRIAD** = **TRI**axial **A**ttitude **D**etermination

Triad method

- Assume: found two vectors in body coordinates $\mathbf{u}_b$ and $\mathbf{v}_b$
- Assume: know orientation of these vectors in reference coordinates $\mathbf{u}_i$ and $\mathbf{v}_i$
- Problem: Determine directional cosine matrix $\mathbf{C}_{bi}$ that rotates reference frame into body frame

$$\vec{u}_b = \mathbf{C}_{bi}\vec{u}_i \ , \quad \vec{v}_b = \mathbf{C}_{bi}\vec{v}_i$$

- Take **normalized** vector measurement

  → can form orthogonal triads in both coordinate frames

  (body and reference) from these vectors

Construct body frame triad

$$\vec{t}_{1b} = \vec{u}_b$$

$$\vec{t}_{2b} = \frac{\vec{u}_b \times \vec{v}_b}{|\vec{u}_b \times \vec{v}_b|}$$

$$\vec{t}_{3b} = \vec{t}_{1b} \times \vec{t}_{2b}$$

Construct reference frame triad in similar way

$$\vec{t}_{1i} = \vec{u}_i$$

$$\vec{t}_{2i} = \frac{\vec{u}_i \times \vec{v}_i}{|\vec{u}_i \times \vec{v}_i|}$$

$$\vec{t}_{3i} = \vec{t}_{1i} \times \vec{t}_{2i}$$

Build: Two triads of orthonormal unit vectors

Triad method

These two triads of orthonormal unit vectors are connected through unique rotation matrix $\mathbf{C}_{bi}$ which satisfies

$$\vec{t}_{jb} = \mathbf{C}_{bi}\vec{t}_{ji} \qquad j = 1, 2, 3$$

Or written out $$[\vec{t}_{1b}\ \vec{t}_{2b}\ \vec{t}_{3b}] = \mathbf{C}_{bi}[\vec{t}_{1i}\ \vec{t}_{2i}\ \vec{t}_{3i}]$$

Since triad is orthonormal matrix → Rotation matrix given by

$$\mathbf{C}_{bi} = [\vec{t}_{1b}\ \vec{t}_{2b}\ \vec{t}_{3b}][\vec{t}_{1i}\ \vec{t}_{2i}\ \vec{t}_{3i}]^T = \sum_{j=1}^{3} \vec{t}_{jb}\vec{t}_{ji}^{\,T}$$

Note: To compute rotation matrix $\mathbf{C}_{bi}$ simply arrange components of various vectors in matrix form and then multiply two matrixes

Simple algorithm, but not optimal in any statistical sense (lose information)

Triad example

Following normalized body and reference vectors are measured

$$\vec{u}_b = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \quad \vec{v}_b = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \quad \vec{u}_i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_i = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Construct body and reference frame triad and build matrixes

$$[\vec{t}_{1b}\ \vec{t}_{2b}\ \vec{t}_{3b}] = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{bmatrix} \qquad [t_{1i}\ t_{2i}\ t_{3i}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\vec{t}_{1i} = \vec{u}_i$$

$$\vec{t}_{2i} = \frac{\vec{u}_i \times \vec{v}_i}{|\vec{u}_i \times \vec{v}_i|}$$

$$\vec{t}_{3i} = \vec{t}_{1i} \times \vec{t}_{2i}$$

Calculate rotation attitude matrix $\mathbf{C}_{bi}$

$$\mathbf{C}_{bi} = [\vec{t}_{1b}\ \vec{t}_{2b}\ \vec{t}_{3b}][\vec{t}_{1i}\ \vec{t}_{2i}\ \vec{t}_{3i}]^T =$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Statistical Attitude Determination

Attitude determination using two or more measurements

Statistical attitude determination

- If more than two observations available → use all information
- Assume spacecraft equipped with *N* sensors
- Sensors provide *k*=1,….,*N* measurements in body frame $\vec{v}_{kb}$
- Given knowledge of orbital position of spacecraft, vectors assumed to be known also in inertial frame $\vec{v}_{ki}$
- For convenience all vectors are normalized
- Given all these measurement: How to get estimation of attitude of spacecraft?
- Formally how to estimate rotation matrix $\mathbf{C}_{bi}$ given *k*=1,…,*N* vectors in body and reference frame? $\vec{v}_{kb} = \mathbf{C}_{bi}\vec{v}_{ki}$
- Which additional conditions (if system is over-deterimated)?

  → Whaba´s problem

Whaba´s problem

- Given $\vec{v}_{ki}$ and $\vec{v}_{kb}$ want to find $\mathbf{C}_{bi}$ under condition $\mathbf{C}_{bi}^T\mathbf{C}_{bi} = 1$
- Obviously set of equation is over-determined if $N \geq 2$
- Want to find solution for $\mathbf{C}_{bi}$ which minimizes overall error for N vectors
- Minimize $\mathbf{r}_k$ where $\vec{r}_k = \vec{v}_{kb} - \mathbf{C}_{bi}\vec{v}_{ki}$ subject to $\mathbf{C}_{bi}^T\mathbf{C}_{bi} = 1$
- Consider following constrained optimization problem (loss function)

$$\text{Minimize} \quad J(\mathbf{C}_{bi}) = \tfrac{1}{2}\sum_{k=1}^{N} w_k \vec{r}_k^{\,T}\vec{r}$$

$$= \tfrac{1}{2}\sum_{k=1}^{N} w_k (\vec{v}_{kb} - \mathbf{C}_{bi}\vec{v}_{ki})^T(\vec{v}_{kb} - \mathbf{C}_{bi}\vec{v}_{ki})$$

such that $\mathbf{C}_{bi}^T\mathbf{C}_{bi} = 1$ where $w_k > 0$ are scalar weights

Whaba´s problem

- Loss function is sum of squared errors for each vector measurement
- **Weights** used to give more emphasis on certain measurement
- If measurements are error free $J = 0$, if any error $J > 0$
- The smaller can be made J, better approximation of $\mathbf{C}_{bi}$

Solve Whaba´s problem

Present two methods to solve Whaba´s minimization problem:

<u>Method 1:</u>

q-method

Solve for attitude matrix

<u>Method 2:</u>

Efficient approximation of q-method known as **QUEST-method** (Quaternion ESTimation)

Solve for quaternion representation of attitude matrix

q-method

Solves Whaba´s problem

q-method

Have N sensor observations

Find rotation matrix minimizing loss function

$$J(\mathbf{C}_{bi}) = \tfrac{1}{2}\sum_{k=1}^{N} w_k (\vec{v}_{kb} - \mathbf{C}_{bi}\vec{v}_{ki})^T (\vec{v}_{kb} - \mathbf{C}_{bi}\vec{v}_{ki}) \qquad \mathbf{C}_{bi}^T\mathbf{C}_{bi} = \mathbf{1}$$

where w_k weights relative importance of observation

Multiplying term gives:

$$\begin{aligned}
J(\mathbf{C}_{bi}) &= \tfrac{1}{2}\sum_{k=1}^{N} w_k (\vec{v}_{kb} - \mathbf{C}_{bi}\vec{v}_{ki})^T (\vec{v}_{kb} - \mathbf{C}_{bi}\vec{v}_{ki}) \\
&= \tfrac{1}{2}\sum_{k=1}^{N} w_k (\vec{v}_{kb}^T\vec{v}_{kb} - 2\vec{v}_{kb}^T\mathbf{C}_{bi}\vec{v}_{ki} + \vec{v}_{ki}^T\mathbf{C}_{bi}^T\mathbf{C}_{bi}\vec{v}_{ki}) \\
&= \sum_{k=1}^{N} w_k (1 - \vec{v}_{kb}^T\mathbf{C}_{bi}\vec{v}_{ki}) = \sum_{k=1}^{N} w_k - \sum_{k=1}^{N} w_k (\vec{v}_{kb}^T\mathbf{C}_{bi}\vec{v}_{ki}) \\
&= -\sum_{k=1}^{N} w_k (\vec{v}_{kb}^T\mathbf{C}_{bi}\vec{v}_{ki}) + \text{constant} \quad \text{term}
\end{aligned}$$

$\mathbf{C}_{bi}^T\mathbf{C}_{bi} = \mathbf{1}$

All vectors are normalized

q-method

Minimizing function $J(\mathbf{C}_{bi})$ is equivalent to maximizing function

$$\tilde{J}(\mathbf{C}_{bi}) = \sum_{k=1}^{N} w_k (\vec{v}_{kb}^{T} \mathbf{C}_{bi} \vec{v}_{ki})$$

Problem: $\boxed{\text{Find } \mathbf{C}_{bi} \text{ that maximizes } J(\mathbf{C}_{bi}) \text{ under constraint } \mathbf{C}_{bi}^{T}\mathbf{C}_{bi} = \mathbf{1}}$

This problem can be simplified using trace operation and quaternions

After some manipulation problem is written as follows:

$$\boxed{\tilde{J}(\mathbf{q}) = \mathbf{q}^{T}\mathbf{K}\mathbf{q} \quad \text{with constraint} \quad \mathbf{q}^{T}\mathbf{q} = 1}$$

Function $J(\mathbf{q})$ is rewritten in terms of quaternion instead of rotation matrix, where **K** is 4x4 matrix and **q** is quaternion (matrix **K** given later)

Problem: $\boxed{\text{Find } \mathbf{q} \text{ that maximizes } J(\mathbf{q}) \text{ subject to constraint } \mathbf{q}^{T}\mathbf{q} = 1}$

Note:

Reduced **matrix constraint** to **scalar constraint** by using quaternions

q-method

Problem: Find **q** that maximizes *J*(**q**) subject to constraint $\mathbf{q}^T\mathbf{q} = 1$

Use Lagrange multiplier to define new loss function, means use *J*(**q**) augmented with constraint $1 - \mathbf{q}^T\mathbf{q} = 0$

$$\widehat{J}(\mathbf{q}) = \mathbf{q}^T\mathbf{K}\mathbf{q} + \lambda(1 - \mathbf{q}^T\mathbf{q})$$

where λ is scalar Lagrange multiplier

To find optimal quaternion take partial derivative to **q** and λ and set result to zero

$$\frac{\partial \widehat{J}(\mathbf{q})}{\partial \mathbf{q}} = \mathbf{q}^T\mathbf{K} - \lambda\mathbf{q}^T = 0$$

$$\frac{\partial \widehat{J}(\mathbf{q})}{\partial \lambda} = (1 - \mathbf{q}^T\mathbf{q}) = 0 \quad \text{Quaternion constraint}$$

q-method

- Partial derivative of **q** can be written as eigenvalue problem

$$\frac{\partial \widehat{J}(\mathbf{q})}{\partial \mathbf{q}} = \mathbf{q}^T\mathbf{K} - \lambda\mathbf{q}^T = 0 \Leftrightarrow \mathbf{K}\mathbf{q} = \lambda\mathbf{q}$$

- Problem is now an eigenvalue problem with quaternion constraint for free
- **K** is 4x4matrix so have four eigenvalues and four eigenvectors
- Which of four solutions maximizes $J(\mathbf{q})$?
- Plug $\mathbf{Kq} = \lambda\mathbf{q}$ into $\widehat{J}(\mathbf{q}) = \mathbf{q}^T\mathbf{K}\mathbf{q} + \lambda(1 - \mathbf{q}^T\mathbf{q})$ leads to

$$\widehat{J}(\mathbf{q}) = \mathbf{q}^T\mathbf{K}\mathbf{q} = \mathbf{q}^T\lambda\mathbf{q} = \lambda\mathbf{q}^T\mathbf{q} = \lambda$$

- Therefore largest eigenvalue of **K** with associated eigenvector $\mathbf{q}_{larg}$ maximizes $J(\mathbf{q})$
- Finally, eigenvector $\mathbf{q}_{larg}$ is optimal quaternion together with

$$\mathbf{C}(\vec{q}, q_4) = (q_4^2 - \vec{q}^T\vec{q})\mathbf{1} + 2\vec{q}\vec{q}^T - 2q_4\mathbf{q}^\times$$

$$= \begin{bmatrix} 1-2(q_2^2+q_3^2) & 2(q_1q_2+q_3q_4) & 2(q_1q_3-q_2q_4) \\ 2(q_2q_1-q_3q_4) & 1-2(q_3^2+q_1^2) & 2(q_2q_3+q_1q_4) \\ 2(q_3q_1+q_2q_4) & 2(q_3q_2-q_1q_4) & 1-2(q_1^2+q_2^2) \end{bmatrix}$$

provides optimal estimation of $\mathbf{C}_{bi}$

q-method

K is 4 x 4 matrix given by

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \vec{\mathbf{k}}_{12} \\ \vec{\mathbf{k}}_{12}^{T} & k_{22} \end{bmatrix}$$

$$\mathbf{K} \in \mathbb{R}^{4\times 4} \quad \mathbf{K}_{11} \in \mathbb{R}^{3\times 3} \quad \vec{\mathbf{k}}_{12} \in \mathbb{R}^{3\times 1} \quad k_{22} \in \mathbb{R}$$

where

$$\mathbf{B} = \sum_{k=1}^{N} w_k (\vec{v}_{ki} \vec{v}_{kb}^{T}) \qquad (\mathbf{B} \in \mathbb{R}^{3\times 3})$$

$$k_{22} = tr\left[\mathbf{B}\right]$$

$$\mathbf{K}_{11} = \mathbf{B} + \mathbf{B}^{T} - k_{22}\mathbf{1} \qquad (\mathbf{1} \in \mathbb{R}^{3\times 3})$$

$$\mathbf{k}_{12} = \left[(B_{23} - B_{32}) \quad (B_{31} - B_{13}) \quad (B_{12} - B_{21}) \right]^{T}$$

q-method example

Following normalized body and reference vectors are measured

$$\vec{u}_b = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \quad \vec{v}_b = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \quad \vec{u}_i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_i = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Assume measurements are prefect without error → weights all equal

$$N = 2$$

$$\mathbf{B} = \sum_{k=1}^{N} w_k \left(\vec{v}_{ki} \vec{v}_{kb}^T \right) = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad k_{22} = tr\left[\mathbf{B}\right] = \sqrt{2}$$

$$\mathbf{K}_{11} = \mathbf{B} + \mathbf{B}^T - k_{22}\mathbf{1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} \end{bmatrix}$$

q-method example

K matrix is finally given by

$$\mathbf{k}_{12} = \left[(B_{23} - B_{32}) \quad (B_{31} - B_{13}) \quad (B_{12} - B_{21}) \right]^T = \left[0 \quad 0 \quad \sqrt{2} \right]$$

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{2} & \sqrt{2} \end{bmatrix}$$

Largest eigenvalue and corresponding eigenvector given by

$$\lambda_{\max} = 2$$

$$\mathbf{q} = \left[0 \quad 0 \quad 0.3827 \quad 0.9239 \right]$$

Write quaternion as 3 x 3 matrix and check if same results as before

$$\mathbf{C}(\vec{q}, q_4) = (q_4^2 - \vec{q}^T\vec{q})\mathbf{1} + 2\vec{q}\vec{q}^T - 2q_4\mathbf{q}^\times$$

$$= \begin{bmatrix} 1-2(q_2^2+q_3^2) & 2(q_1q_2+q_3q_4) & 2(q_1q_3-q_2q_4) \\ 2(q_2q_1-q_3q_4) & 1-2(q_3^2+q_1^2) & 2(q_2q_3+q_1q_4) \\ 2(q_3q_1+q_2q_4) & 2(q_3q_2-q_1q_4) & 1-2(q_1^2+q_2^2) \end{bmatrix}$$

QUEST method

Solves Whaba´s problem

QUEST-method

q-method reduces Wahba´s problem to eigenvalue problem where optimal attitude estimate corresponds to eigenvector (quaternion) associated with largest eigenvalue

Exist many methods for directly calculating eigenvalues and eigenvectors

→ q-method involves solving eigenvalue and eigenvector problem directly

Instead can use also approximation for eigenvalue problem

In 1978 Malcolm Shuster developed a solution that does not involve need to solve eigenvalue and eigenvector directly

QUEST algorithm (**QU**aternion **EST**imator)

QUEST approximates largest eigenvalue and solves for corresponding eigenvector

QUEST-method

QUEST-method used if eigenvalue problem not solved by programs like Matlab (numerically intensive)

Spacecraft flight computer might have limited memory

QUEST-method provides less intensive eigenvalue problem solution and approximates solution to eigenvalue problem

QUEST-method has two steps:

1. Find near optimal eigenvalues $\lambda_{\max}$
2. Solve eigenvector problem $\mathbf{Kq} = \lambda_{\max}\mathbf{q}$

QUEST-method

Remember minimize loss function $$J = \sum_{k=1}^{N} w_k - \sum_{k=1}^{N} w_k (\vec{v}_{kb}^T \mathbf{C}_{bi} \vec{v}_{ki})$$

Equivalent to maximize function $$\tilde{J} = \sum_{k=1}^{N} w_k (\vec{v}_{kb}^T \mathbf{C}_{bi} \vec{v}_{ki})$$

Equivalent to finding largest eigenvalue $$\tilde{J} = \lambda_{\max}$$

Follows that $$\lambda_{\max} = \sum_{k=1}^{N} w_k - J$$

Good approximation of optimal eigenvalue $$\lambda_{\max} \approx \sum_{k=1}^{N} w_k$$

(J should be small)

Step 1:

Find optimal eigenvalue $\lambda_{\max}$ using Newton-Raphson iteration starting with initial estimate value $\sum_{k=1}^{N} w_k$

QUEST-method

Step 2:

Solve eigenvector problem $\mathbf{K}\mathbf{q} = \lambda_{\max}\mathbf{q}$ for optimal quaternion $\mathbf{q}$

Define first $\vec{\mathbf{p}}$ (Gibbs vector) as $\vec{\mathbf{p}} = \frac{\vec{\mathbf{q}}}{q_4} = \vec{\mathbf{e}}\tan\frac{\phi}{2}$

For QUEST algorithm: eigenvalue problem $\mathbf{K}\mathbf{q} = \lambda_{\max}\mathbf{q}$ is rewritten as

$$\vec{\mathbf{p}} = \left[\left(\lambda_{\max} + k_{22}\right)\mathbf{1} - \mathbf{S}\right]^{-1}\vec{\mathbf{k}}_{12} \quad with \quad \mathbf{S} = \mathbf{B} + \mathbf{B}^T$$

(Note: $\mathbf{K}_{11}\vec{\mathbf{q}} + \vec{\mathbf{k}}_{12}q_4 = \lambda_{\max}\vec{\mathbf{q}} \Leftrightarrow \vec{\mathbf{k}}_{12}q_4 = [\left(\lambda_{\max} + k_{22}\right)\mathbf{1} - \mathbf{S}]\vec{\mathbf{q}} \quad and \quad divide \quad by \quad q_4$)

Instead of inverting matrix (computationally intensive operation), use Gaussian elimination or other linear system methods to solve equation

$$\left[\left(\lambda_{\max} + k_{22}\right)\mathbf{1} - \mathbf{S}\right]\vec{\mathbf{p}} = \vec{\mathbf{k}}_{12}$$

After finding $\vec{\mathbf{p}}$ ⇒ optimal quaternion is calculated by $\mathbf{q} = \frac{1}{\sqrt{1+\vec{\mathbf{p}}^T\vec{\mathbf{p}}}}\begin{bmatrix}\vec{\mathbf{p}} \\ 1\end{bmatrix}$

QUEST-method example

Following normalized body and reference vectors are measured

$$\vec{u}_b = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \quad \vec{v}_b = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \quad \vec{u}_i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_i = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Assume measurements are prefect without error → weights all equal

$$N = 2$$

$$\lambda_{\max} \approx \sum_{k=1}^{N} w_k = 2$$

$$\mathbf{B} = \sum_{k=1}^{N} w_k (\vec{v}_{ki} \vec{v}_{kb}^T) = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{S} = \mathbf{B} + \mathbf{B}^T = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

QUEST-method example

Calculate $\vec{\mathbf{p}}$

$$\vec{\mathbf{p}} = \left[\left(\lambda_{\max} + k_{22}\right)\mathbf{1} - \mathbf{S}\right]^{-1} \vec{\mathbf{k}}_{12} =$$

$$= \left[\begin{bmatrix} 2+\sqrt{2} & 0 & 0 \\ 0 & 2+\sqrt{2} & 0 \\ 0 & 0 & 2+\sqrt{2} \end{bmatrix} - \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}\right]^{-1} \begin{bmatrix} 0 \\ 0 \\ \sqrt{2} \end{bmatrix}$$

$$\vec{\mathbf{p}} = \begin{bmatrix} 2+\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & 2+\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 2+\sqrt{2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 0.3694 & 0 & 0 \\ 0 & 0.3694 & 0 \\ 0 & 0 & 0.2929 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.4142 \end{bmatrix}$$

Find quaternion $\mathbf{q}$

$$\mathbf{q} = \frac{1}{\sqrt{1+\vec{\mathbf{p}}^T\vec{\mathbf{p}}}} \begin{bmatrix} \vec{\mathbf{p}} \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0.3827 \\ 0.9239 \end{bmatrix}$$

Result same as before

More realistic examples

Goal is to compare results from TRIAD, q-method and QUEST-method

Assume have five sensor in satellite and measurements are not perfect

Given five vectors in inertial frame a (have to be normalized):

$$\mathbf{s}_{a1} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{s}_{a2} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{s}_{a3} = \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{s}_{a4} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{s}_{a5} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Our five measured vectors with noise are given by (are normalized):

$$\hat{\mathbf{s}}_{b1} = \begin{bmatrix} 0.9082 \\ 0.3185 \\ 0.2715 \end{bmatrix}, \quad \hat{\mathbf{s}}_{b2} = \begin{bmatrix} 0.5670 \\ 0.3732 \\ -0.7343 \end{bmatrix}, \quad \hat{\mathbf{s}}_{b3} = \begin{bmatrix} -0.2821 \\ 0.7163 \\ 0.6382 \end{bmatrix},$$

$$\hat{\mathbf{s}}_{b4} = \begin{bmatrix} 0.7510 \\ -0.3303 \\ 0.5718 \end{bmatrix}, \quad \hat{\mathbf{s}}_{b5} = \begin{bmatrix} 0.9261 \\ -0.2053 \\ -0.3166 \end{bmatrix},$$

More realistic examples

Note: For this numerical example five measured vectors are generated

In example assume to know attitude (rotation matrix defined by 1-2-3 Euler angle sequence with rotation of 45º in x, -30º in y and 60º in z)

True exact attitude is represented by rotation matrix

$$\mathbf{C}_{ba} = \mathbf{C}_z(60^\circ)\mathbf{C}_y(-30^\circ)\mathbf{C}_x(45^\circ) = \begin{bmatrix} 0.4330 & 0.4356 & 0.7891 \\ -0.7500 & 0.6597 & 0.0474 \\ -0.5000 & -0.6124 & 0.6124 \end{bmatrix}$$

To model that measurements are not perfect introduce some error in body frame sensor measurement with following errors

$$\sigma_1 = 0.0100, \quad \sigma_2 = 0.0325, \quad \sigma_3 = 0.0550, \quad \sigma_4 = 0.0775, \quad \sigma_5 = 0.1000.$$

Uniformly distributed random error is added to sensor measurement

So that five generated vectors with error in previous slide

are slightly different from five vectors without error

More realistic examples

Using **TRIAD method** (use only first two measurements k = 1,2) satellite attitude is estimated by

$$\bar{\mathbf{C}}_{ba} = \begin{bmatrix} 0.4156 & 0.4504 & 0.7902 \\ -0.7630 & 0.6456 & 0.0333 \\ -0.4952 & -0.6167 & 0.6119 \end{bmatrix}$$

Using **q-method** and **QUEST** provide identical results given by

$$\bar{\mathbf{C}}_{ba} = \begin{bmatrix} 0.4153 & 0.4472 & 0.7921 \\ -0.7562 & 0.6537 & 0.0274 \\ -0.5056 & -0.6104 & 0.6097 \end{bmatrix}$$

where weights are given by $w_k = 1/\sigma^2_k$

More realistic examples

Loss function of performance for each estimate of rotation matrix is given in following table

Table 25.1 Performance and error of the attitude estimate provided by the q-Method, QUEST, and the TRIAD algorithm

	TRIAD	q-Method	QUEST
$J(\bar{\mathbf{C}}_{ba})$	4.2449	4.0333	4.0333
ϕ_e	1.3622°	1.2644°	1.2644°

- Useful approach to measure values of attitude estimation is to make use of $\mathbf{C}_{ba}^{true}\mathbf{C}_{ba}^{T\ \ true} = \mathbf{1}$
- Relationship between estimated attitude $\bar{\mathbf{C}}_{ba}$ and true attitude $\mathbf{C}_{ba}^{true}$ can be given by $\mathbf{C}_{ba,e} = \bar{\mathbf{C}}_{ba}\mathbf{C}_{ba}^{T\ \ true}$ where $\mathbf{C}_{ba,e}$ is attitude error
- $\mathbf{C}_{ba,e}$ is rotation matrix from exact attitude to estimated attitude
- Rotation angle ϕ associated with $\mathbf{C}_{ba,e}$ is $\cos\phi = \frac{1}{2}[C_{11} + C_{22} + C_{33} - 1]$

Singular value decomposition (SVD)

Note:

A different way of finding directional cosine matrix that minimizes Wahba´s problem (loss function) is to use singular values decomposition of matrix **B**.

(Singular value decomposition of attitude matrix $\mathbf{B} = \mathbf{U}\,\mathbf{S}\,\mathbf{V}^T$

U and **V** are orthogonal and **S** diagonal matrix with $s_i \geq 0$ i=1,2,3.)

Summary of attitude methods

Methods	Characteristics
TRIAD-method	Geometric method Requires two nonparallel vector measurements
q-method	Closed form robust solution based on eigenvalues Can be rather slow due to required eigenvalue solution
QUEST-method	An iterated solution that avoids explicitly to solve for eigenvalues as in q-method

Summary

Static attitude determination

- Attitude determination is **over-determinated** (if $\geq$2 vectors) or **under-determinated** (if only 1 vector)
- Need of two or more sensor measurements (at given time) which defines distinct vectors known in body and reference frame
- Vectors used in algorithm to estimate attitude (represented by rotation matrix, Euler angles, quaternions, etc.)
- Simplest algorithm **Triad method** (use of only two vectors)
- More accurate methods based on Wahba´s minimization problem (allows arbitrary weights of measurements and allows to use more than two measurements)
- **Analytical solution** to minimization Whaba´s problem **q-method**
- **Approximation** to minimization Whaba´s problem **QUEST-method**

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