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A novel Romberg integration method for neutrosophic valued functions

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A B S T R A C T

This study proposes a numerical integration technique to determine the approximate integral of the neutrosophic valued function. Newton Cot's method with a positive coefficient has been used for neutrosophic integration, and then the suggested technique is used to find the approximate value of the integral. This method is called the Romberg integration method. In this method, the composite trapezoidal rule is applied initially, and Richardson's extrapolation technique is used next to improve the solution. Richardson's extrapolation is the basis of Romberg's integration. Formulas with high-order truncation errors are created using Richardson extrapolation, which employs an averaging method. Richardson's extrapolation helps the proposed method to achieve high computational efficiency. The convergence of the numerical approximation is provided in terms of theorems. A comparative analysis of the proposed method with other existing methods is demonstrated to show the efficiency and reliability of the proposed method.

1. Introduction

In the field of uncertain mathematics, Smarandache [1] Introduced the idea of the neutrosophic set. After that, many researchers have devoted their interest to work on the application of the neutrosophic set. Broumi et al. [2] introduced the idea of bipolar and complex neutrosophic sets for multi-criteria decision-making problems. After that Nguyen et al. [3] used neutrosophic sets in different fields of medical diagnoses. They have shown that all the incomplete and inconsistent information can be managed by a neutrosophic set for modeling patients' medical situations. Mallick et al. [4] developed the idea of a penta-partitioned neutrosophic set, the modified version of single-valued neutrosophic sets. This type of neutrosphic set can be applied to solve multi-criteria decision-making problems. Dat et al. [5] proposed a single-valued linguistic complex neutrosophic set and an interval linguistic complex neutrosophic set. Also, they have given the idea of some set-theoretic operations for single-valued linguistic complex neutrosophic set and interval linguistic complex neutrosophic set. In the same article, Dat et al. [5] presented a new decision-making problem for a single-valued and interval linguistic complex neutrosophic set. In [6] Pratihar et al. studied a transportation problem in a neutrosophic environment. They used a modified northwest corner method to find the basic feasible solution to a neutrosophic transportation problem. Lu et al. [7] proposed a new emergency transport model on a single-valued neutrosophic set. They have transformed the emergency transshipment problem into a multi-attribute decision-making problem in a neutrosophic environment. Then Deli [8] developed a linear optimization method of a single-valued neutrosophic set. Also, the author has described the sensitivity analysis of attribute weights. Hamido et al. [9] proposed a new concept of neutrosophic topological space. Also, they have proved that neutrosophic topological space is not the classical topological space. In [10], Ihsan et al. described the idea of a single-valued neutrosophic hypersoft expert set. Also, they have proposed a new type of algorithm to solve decision-making problems. After that Rahman et al. [11] introduced the idea of a neutrosophic parameterized Hypersoft set. In the same research work, they proposed an algorithm to study neutrosophic decision-making problems. Bhaumik et al. [12] developed matrix game theory to solve multi-objective linguistic problems in a neutrosophic environment. They have shown that their research work can be applied to different tourism management problems. In [13], Saeed et al. studied neutrosophic hypersoft set and neutrosophic hypersoft set mapping. Also, they have discussed the diagnosis of hepatitis with their related problems in a neutrosophic environment. Broumi et al. [14] proposed the idea of an n-valued interval neutrosophic set. In the same research work, they have proposed an efficient algorithm for solving the n-valued neutrosophic multi-criteria decision-making problem. Ali et al. [15] developed a single valued neutrosophic multi-criteria decision-making problem. They have developed the idea of Hausdroff distance between single-valued neutrosophic sets. Luo et al. [16] described a novel distance between single-valued neutrosophic sets and they have applied this distance to studying pattern recognition and medical diagnoses. Antonysamy et al. [17] used neutrosophic set to determine the effect

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of corona virus on a patient. Recently, many researchers have been working on the application of neutrosophic set [18–22].

Neutrosophic calculus plays an important role in different scientific fields such as mathematical science, physical science, statistical science and engineering science. So, many authors have dedicated their interests to work in neutrosophic calculus. In [23], Smarandache defined the neutrosophic integral with the help of neutrosophic measure. After that, Smarandache [24] proposed the notion of neutrosophic derivatives, limits and integrals. Smarandache et al. [25] provided the concept of neutrosophic definite and indefinite integral. The definition of the definite integral is exactly same as the definition of the classical integral. The idea of neutrosophic thick function and its integration was first presented by Alaswad [26]. In [27], Son et al. presented the notion of granular fractional neutrosophic integration. They originally proposed the concept of granular fractional Riemann–Liouville and Caputo derivatives of neutrosophic-valued functions in the same publication. Also, they have proposed the definition of neutrosophic granular partial integrals. Then, Alhasan [28] studied neutrosophic integration by substitution method. In the same article, Alhasan [28] defined neutrosophic trigonometric identities. Also, Alhasan [29] discussed the neutrosophic integrals by parts method. In addition, he introduced a tabular method to compute the neutrosophic integrals and he has shown that the tabular method is easier to implement than by by-part method. Biswas et al. [30] defined neutrosophic integration with the help of Riemann integration approach and they also proposed some significant characteristics of neutrosophic numbers and neutrosophic functions. In [31], Alhasan used partial fraction in neutrosophic integration. In the same work. Alhasan discussed neutrosophic proper rational function. Also, they introduced the concept of the integral of neutrosophic improper rational functions. Recently, Biswas et al. [32] used Gaussian quadrature methods to compute the numerical integration of neutrosophic functions. Also, they have introduced a distance function for neutrosophic numbers. In the same work, they have defined the continuity of neutrosophic functions on closed-bounded intervals.

Neutrosophic differential and integral equation play an important role in the development of neutrosophic differential and integral calculus. In recent years, many works [33–35] on neutrosophic differential and integral equation [36] are being published for further development of this topic.

1.1. Motivation

It has been noted from the literature review that not much study has been done on the numerical integration of a neutrosophic function. Some authors have already used the Romberg integration method for crisp and fuzzy valued functions [37,38]. Newton Cot's methods have been used to integrate fuzzy functions by Tofigh Allahviranloo [39]. So, there is a huge opportunity to develop and modify this type of work in a neutrosophic environment. In recent years, researchers have been working on the numerical solution of neutrosophic differential, integral, and integro-differential equations. So, it is necessary to work on different types of numerical integration of neutrosophic valued functions for the development of neutrosophic differential, integral, and integro-differential equations. So, from there, we get the motivation to develop a numerical integration method for neutrosophic valued function. In addition, the concept of Richardson's extrapolation method needs to be developed in a neutrosophic environment for further development of this topic. In the literature review, it has been observed that no work has been studied on Richardson's extrapolation method in a neutrosophic environment. So, from there, we got the motivation to explore this research work and write this article on the Romberg integration method for neutrosophic valued function.

1.2. Novelty

The objective of this article has been given below.

- To propose a numerical integration technique to determine the approximate integral of the neutrosophic valued function.
- To use Newton Cot's formula in the neutrosophic integral.
- To introduce the Romberg integration method for neutrosophic function.
- To use composite Trapezoidal rule for finding the approximation of the given function.
- To use the Richardson extrapolation method for finding the better approximation of the given function.

1.3. Arrangement of the article

This work is organized as follows: The preliminary idea of this work is given in Section 2. In Section 3, Newton Cot's formula is discussed for neutrosophic function. Romberg integration for a neutrosophic function is provided in Section 4. Some numerical examples are examined in Section 5. Finally, a brief concluding remark is given in Section 6.

2. Preliminary ideas

Definition 2.1 (*[40]*). Let $\mathcal{N} = \{\langle n, T_{\mathcal{N}}(n), I_{\mathcal{N}}(n), F_{\mathcal{N}}(n)\rangle : n \in X\}$. Then $\mathcal{N}$ is called single valued neutrosophic set over the universal set X, if $T_{\mathcal{N}} : X \to [0,1]$, $I_{\mathcal{N}} : X \to [0,1]$, $F_{\mathcal{N}} : X \to [0,1]$ such that $0 \le T_{\mathcal{N}}(n) + I_{\mathcal{N}}(n) + F_{\mathcal{N}}(n) \le 3, \forall n \in X$. Here $T_{\mathcal{N}}(n), I_{\mathcal{N}}(n)$ and $F_{\mathcal{N}}(n)$ are denoted as truth, indeterminacy, and falsity membership function respectively.

Definition 2.2 (*[41]*). Let $\tilde{M}, \tilde{N} \in \mathcal{N}$ on $Y = \{y_1, y_2, \dots, y_n\}$. Then the Hausdorff distance $\mathcal{D}$ between $\tilde{M}$ and $\tilde{N}$ on X defined as follows

$$\mathcal{D}(\tilde{M}, \tilde{N}) = \frac{1}{n}\sum_{i=1}^{n} \max\{|T_{\tilde{M}}(y_i) - T_{\tilde{N}}(y_i)|, |I_{\tilde{M}}(y_i) - I_{\tilde{N}}(y_i)|, |F_{\tilde{M}}(y_i) - F_{\tilde{N}}(y_i)|\}$$

Proposition 2.1 (*[34]*). *Consider two neutrosophic numbers $\tilde{m}$ and $\tilde{n}$ such that*

1. $(\tilde{m}\odot\tilde{n})_{(\alpha,\beta,\gamma)} = \tilde{m}_{(\alpha,\beta,\gamma)}\odot\tilde{n}_{(\alpha,\beta,\gamma)}$, *where $\odot$ denotes any binary operation $'+', '-'$ and $'\times'$.*
2. $(\eta\tilde{m})_{(\alpha,\beta,\gamma)} = \eta\tilde{m}_{(\alpha,\beta,\gamma)}$, *here $0 \ne \eta \in \mathbb{R}$.*

Definition 2.3 (*[25]*). Let $g : [c,d] \to \mathcal{N}$ be a neutrosophic continuous function on [c, d]. Then neutrosophic integral of g on [c, d] is defined as

$$\int_c^d g(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} g(P_i)\frac{d-c}{n}$$

here $P_i \in [x_{i-1}, x_i]$, for $i \in \{1, 2, \dots, n\}$ and $c = x_0 < x_1 < \cdots < x_{n-1} < x_n = d$ are the partitions of the interval $[c,d]$. Here $g(P_i)$ may be a real set or may have some indeterminacy.

3. Newton Cot's formula

Consider g to be a neutrosophic function. Then for any $n \in \mathbb{N}$, where $\mathbb{N}$ be the set of all natural numbers, Newton Cot's formula for a neutrosophic valued function is given as

$$\int_a^b g(x) = h\sum_{i=1}^{n} g_i w_i + E, \qquad g_i = g(a+ih), \quad h = \frac{b-a}{n} \tag{1}$$

These provides the approximate solutions of the integral $\int_a^b g(x)dx$.

Now, the (α, β, γ)−cuts of the Eq. (1) are as follows :

$$\int_a^b g_{T_j}(x;\alpha)dx = h\sum_{i=1}^{n} w_i g_{T_j}(x;\alpha) + E(g_{T_j};\alpha) \tag{2}$$

$$\int_a^b g_{I_j}(x;\beta)dx = h\sum_{i=1}^{n} w_i g_{I_j}(x;\beta) + E(g_{T_j};\beta) \tag{3}$$

$$\int_a^b g_{F_j}(x;\gamma)dx = h\sum_{i=1}^{n} w_i g_{F_j}(x;\gamma) + E(g_{T_j};\gamma) \tag{4}$$

where $j = 1,2$ denotes the left and right branches respectively. Here $w_i, i = 1,2,\dots,n$ be the weights and rational numbers with the property that $\sum_{i=1}^{n} w_i = n$. This will hold when we take $g_{T_j}(x;\alpha) = g_{I_j}(x;\beta) = g_{F_j}(x;\gamma) = 1$.

Now, the approximation error can be expressed as follows:

$$E(g_{T_j};\alpha) = h^{p+1}.k.g_{T_j}^{(p)}(\delta_{T_j};\alpha), \quad \delta_{T_j} \in (a,b) \tag{5}$$

$$E(g_{I_j};\beta) = h^{p+1}.k.g_{I_j}^{(p)}(\delta_{I_j};\beta), \quad \delta_{I_j} \in (a,b) \tag{6}$$

$$E(g_{F_j};\gamma) = h^{p+1}.k.g_{F_j}^{(p)}(\delta_{F_j};\gamma), \quad \delta_{F_j} \in (a,b) \tag{7}$$

Here the values of p and k depended on the value of n but not on the integrand g. Sometimes the value of w_i becomes negative for a large value of n and the given formulas are not suitable for numerical approximation as many terms are canceled to compute the sum.

Let

$$J(g_{T_j};\alpha) = h\sum_{i=1}^{n} w_i g_{T_j}(x_i;\alpha) \tag{8}$$

$$J(g_{I_j};\beta) = h\sum_{i=1}^{n} w_i g_{I_j}(x_i;\beta) \tag{9}$$

$$J(g_{F_j};\gamma) = h\sum_{i=1}^{n} w_i g_{F_j}(x_i;\gamma) \tag{10}$$

Then from the Eqs. (8), (9) and (10), we have

$$L(g_{T_j};\alpha) = \int_a^b g_{T_j}(x;\alpha)dx = J(g_{T_j};\alpha) + E(g_{T_j};\alpha) \tag{11}$$

$$L(g_{I_j};\beta) = \int_a^b g_{I_j}(x;\beta)dx = J(g_{I_j};\beta) + E(g_{I_j};\beta) \tag{12}$$

$$L(g_{F_j};\gamma) = \int_a^b g_{F_j}(x;\gamma)dx = J(g_{F_j};\gamma) + E(g_{F_j};\gamma) \tag{13}$$

The following theorem shows that $J(g_{T_j};\alpha)$, $J(g_{I_j};\beta)$, $J(g_{F_j};\gamma)$ convergent to $L(g_{T_j};\alpha)$, $L(g_{I_j};\beta)$ and $L(g_{F_j};\gamma)$ respectively, where $j = 1, 2$ denotes the left and right branch respectively.

Theorem 3.1. *Let $g(t)$ be a neutrosophic valued continuous function in the metric D. Then $J(g_{T_j};\alpha)$, $J(g_{I_j};\beta)$, $J(g_{F_j};\gamma)$ converge uniformly to $L(g_{T_j};\alpha)$, $L(g_{I_j};\beta)$ and $L(g_{F_j};\gamma)$ respectively.*

Proof. Since A converges to the integral $\int_a^b g(t)dt$ in metric D if

$$\lim_{n\to\infty}\left[\max_{1\le i\le n}(t_i - t_{i-1})\right] = 0$$

Now, the Eqs. (8), (9) and (10) can be represented in the form of $A_{T_1}(\alpha), A_{T_2}(\alpha), A_{I_1}(\beta), A_{I_2}(\beta), A_{F_1}(\gamma)$, and $A_{F_2}(\gamma)$.

Then we have

$$D(A, L(g)) = \sup_{\alpha,\beta,\gamma}\Big\{\max\{|A_{T_1}(\alpha) - L(g_{T_1};\alpha)|, |A_{T_2}(\alpha) - L(g_{T_2};\alpha)|\} + \max\{|A_{I_1}(\beta) - L(g_{I_1};\beta)|, |A_{I_2}(\beta) - L(g_{I_2};\beta)|\} + \max\{|A_{F_1}(\gamma) - L(g_{F_1};\gamma)|, |A_{F_2}(\gamma) - L(g_{F_2};\gamma)|\}\Big\}$$

Since $\lim_{n\to\infty} D(A, L(g)) = 0$, $\max_{1\le i\le n}(t_i - t_{i-1}) \to 0$ as $n \to \infty$.

So, $A_{T_1}(\alpha), A_{T_2}(\alpha), A_{I_1}(\beta), A_{I_2}(\beta), A_{F_1}(\gamma)$, and $A_{F_2}(\gamma)$ converges uniformly to $L(g_{T_1};\alpha), L(g_{T_2};\alpha)$, $L(g_{I_1};\beta)$, $L(g_{I_2};\beta), L(g_{F_1};\gamma)$ and $L(g_{F_2};\gamma)$ respectively.

Here $J(g_{T_1};\alpha), J(g_{T_2};\alpha), J(g_{I_1};\beta), J(g_{I_2};\beta), J(g_{F_1};\gamma), J(g_{F_2};\gamma)$ are the particular case of $A_{T_1}(\alpha), A_{T_2}(\alpha)$, $A_{I_1}(\beta), A_{I_2}(\beta), A_{F_1}(\gamma)$, and $A_{F_2}(\gamma)$. Since $(t_i - t_{i-1}) := hw_i, i = 1,\dots,n$ This completes the proof. □

3.1. Peano's error representation

From the Eqs. (11), (12) and (13) we have,

$$E(g_{T_j};\alpha) = L(g_{T_j};\alpha) - J(g_{T_j};\alpha) \tag{14}$$

$$E(g_{I_j};\beta) = L(g_{I_j};\beta) - J(g_{I_j};\beta) \tag{15}$$

$$E(g_{F_j};\gamma) = L(g_{F_j};\gamma) - J(g_{F_j};\gamma) \tag{16}$$

The error function $E(g)$ is linear operator so,

$$E(\lambda f_{T_j}(x;\alpha) + g_{T_j}(x;\alpha)) = \lambda E(f_{T_j};\alpha) + E(g_{T_j};\alpha) \tag{17}$$

$$E(\lambda f_{I_j}(x;\beta) + g_{I_j}(x;\beta)) = \lambda E(f_{I_j};\beta) + E(g_{I_j};\beta) \tag{18}$$

$$E(\lambda f_{F_j}(x;\gamma) + g_{F_j}(x;\gamma)) = \lambda E(f_{F_j};\gamma) + E(g_{F_j};\gamma) \tag{19}$$

for $f, g \in \mathcal{N}$ and $\lambda \in \mathbb{R}$ where $\mathcal{N}$ is the linear neutrosophic function space.

The following theorem shows that the integral representation of the error $E(g)$ is a classical result due to Peano.

Theorem 3.2. *Let us consider E(p)=0 holds for all $p \in \mathbb{P}_n$, where $\mathbb{P}_n$ be the set of all polynomials whose degree is less than or equal to n. Then for all neutrosophic functions in (α,β,γ)-cut form $g_{T_1}(x;\alpha), g_{T_2}(x;\alpha), g_{I_1}(x;\beta)$, $g_{I_2}(x;\beta), g_{F_1}(x;\gamma), g_{F_2}(x;\gamma) \in C^{n+1}[a,b]$,*

$$E(g_{T_j};\alpha) = \int_a^b g_{T_j}^{n+1}(t;\alpha)K(t)dt \tag{20}$$

$$E(g_{I_j};\beta) = \int_a^b g_{T_j}^{n+1}(t;\beta)K(t)dt \tag{21}$$

$$E(g_{F_j};\gamma) = \int_a^b g_{F_j}^{n+1}(t;\gamma)K(t)dt \tag{22}$$

where

$$K(t) = \frac{1}{n!}E_x[(x-t)_+^n], \qquad (x-t)_+^n = \begin{cases}(x-t)^n, & x \ge t\\ 0, & x < t\end{cases}$$

Here $E_x[(x-t)_+^n]$ denotes the error $(x-t)_+^n$.

Proof. Consider the Taylor expansions of $g_{T_j}(x;\alpha)$, $g_{I_j}(x;\beta)$ and $g_{F_j}(x;\gamma), \in [a,b]$ at the point c are

$$g_{T_j}(x;\alpha) = g_{T_j}(c;\alpha) + g'_{T_j}(c;\alpha)(x-c) + \cdots + \frac{g^{(n)}(c;\alpha)}{n!}(x-c)^n + R_{T_jn}(x;\alpha) \tag{23}$$

$$g_{I_j}(x;\beta) = g_{I_j}(c;\beta) + g'_{I_j}(c;\beta)(x-c) + \cdots + \frac{g^{(n)}(c;\beta)}{n!}(x-c)^n + R_{I_jn}(x;\beta) \tag{24}$$

$$g_{F_j}(x;\gamma) = g_{F_j}(c;\gamma) + g'_{F_j}(c;\gamma)(x-c) + \cdots + \frac{g^{(n)}(c;\gamma)}{n!}(x-c)^n + R_{F_jn}(x;\gamma) \tag{25}$$

So, the Remainder terms can be written as follows

$$R_{T_jn}(x;\alpha) = \frac{1}{n!}\int_a^x g_{T_j}^{n+1}(t;\alpha)(x-t)^n dt = \frac{1}{n!}\int_a^b g_{T_j}^{n+1}(t;\alpha)(x-t)_+^n dt \tag{26}$$

$$R_{I_jn}(x;\beta) = \frac{1}{n!}\int_a^x g_{I_j}^{n+1}(t;\beta)(x-t)^n dt = \frac{1}{n!}\int_a^b g_{I_j}^{n+1}(t;\beta)(x-t)_+^n dt \tag{27}$$

$$R_{F_jn}(x;\gamma) = \frac{1}{n!}\int_a^x g_{F_j}^{n+1}(t;\gamma)(x-t)^n dt = \frac{1}{n!}\int_a^b g_{F_j}^{n+1}(t;\gamma)(x-t)_+^n dt \tag{28}$$

Applying the linear operator E to the Eqs. (23), (24) and (25) we have

$$E(g_{T_j};\alpha) = E(R_{T_jn};\alpha) = \frac{1}{n!}E_x\left(\int_a^b g_{T_j}^{n+1}(t;\alpha)(x-t)_+^n dt\right) \tag{29}$$

$$E(g_{I_j};\beta) = E(R_{I_jn};\beta) = \frac{1}{n!}E_x\left(\int_a^b g_{I_j}^{n+1}(t;\beta)(x-t)_+^n dt\right) \tag{30}$$

$$E(g_{F_j};\gamma) = E(R_{F_jn};\gamma) = \frac{1}{n!}E_x\left(\int_a^b g_{F_j}^{n+1}(t;\gamma)(x-t)_+^n dt\right) \tag{31}$$

Since $E(p) = 0$ for $p \in \mathbb{P}_n$. Now, we have to interchange the E_x operator with the integration. Then from [42], it has been shown that E_x commutes with integration. This Completes the proof. □

4. Romberg integration

In Romberg integration, the composite Trapezoidal rule has been used to find the approximation of the given function. Then the Richardson extrapolation method has been used to find a better approximation of the given function.

The Richardson extrapolation [43] can be used in the following manner :

$$U - V(h) = C_1 h + C_2 h^2 + \cdots + C_n h^n \tag{32}$$

where $C_1, C_2, \dots, C_n$ are constants and $V(h)$ is an approximation to the unknown value U. The truncation in Eq. (32) is dominated by $C_1 h$, where h is small. In Richardson's extrapolation formula, the averaging technique has been used to find higher-order truncation errors. Also, the extrapolation was used to find the approximation of the definite neutrosophic integral. Now, the composite Trapezoidal rule is used to approximate the definite neutrosophic integral on the interval $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, where $[a, b]$ divided into n sub-intervals. Then,

$$L(g_{T_j};\alpha) = \frac{h}{2}\left[g_{T_j}(x_0;\alpha) + 2\sum_{i=1}^{n-1} g_{T_j}(x_i;\alpha) + g_{T_j}(x_n;\alpha)\right] + E(g_{T_j};\alpha) \tag{33}$$

$$L(g_{I_j};\beta) = \frac{h}{2}\left[g_{I_j}(x_0;\beta) + 2\sum_{i=1}^{n-1} g_{I_j}(x_i;\beta) + g_{I_j}(x_n;\beta)\right] + E(g_{I_j};\beta) \tag{34}$$

$$L(g_{F_j};\gamma) = \frac{h}{2}\left[g_{F_j}(x_0;\gamma) + 2\sum_{i=1}^{n-1} g_{F_j}(x_i;\gamma) + g_{F_j}(x_n;\gamma)\right] + E(g_{F_j};\gamma) \tag{35}$$

where,

$$E(g_{T_j};\alpha) = -\frac{h^2}{12}(b-a)g^{(2)}_{T_j}(\delta_{T_j};\alpha) \tag{36}$$

$$E(g_{I_j};\beta) = -\frac{h^2}{12}(b-a)g^{(2)}_{I_j}(\delta_{I_j};\beta) \tag{37}$$

$$E(g_{F_j};\gamma) = -\frac{h^2}{12}(b-a)g^{(2)}_{F_j}(\delta_{F_j};\gamma) \tag{38}$$

At first, find the composite trapezoidal rule for finding the approximation with $n_1 = 1, n_2 = 2, n_3 = 4, \dots, n_m = 2^{m-1}$, where $m \in \mathbb{N}$. Let h_k be the step size to n_k, then $h_k = \frac{b-a}{n_k} = \frac{b-a}{2^{k-1}}$.

Then, from the Trapezoidal rule,

$$L(g_{T_j};\alpha) = \frac{h_k}{2}\left[g_{T_j}(x_0;\alpha) + 2\sum_{i=1}^{2^{k-1}-1} g_{T_j}(x_i;\alpha) + g_{T_j}(x_n;\alpha)\right] + E(g_{T_j};\alpha) \tag{39}$$

$$L(g_{I_j};\beta) = \frac{h_k}{2}\left[g_{I_j}(x_0;\beta) + 2\sum_{i=1}^{2^{k-1}-1} g_{I_j}(x_i;\beta) + g_{I_j}(x_n;\beta)\right] + E(g_{I_j};\beta) \tag{40}$$

$$L(g_{F_j};\gamma) = \frac{h_k}{2}\left[g_{F_j}(x_0;\gamma) + 2\sum_{i=1}^{2^{k-1}-1} g_{F_j}(x_i;\gamma) + g_{F_j}(x_n;\gamma)\right] + E(g_{F_j};\gamma) \tag{41}$$

where,

$$E(g_{T_j};\alpha) = -\frac{h_k^2}{12}(b-a)g^{(2)}_{T_j}(\delta_{T_j};\alpha) \tag{42}$$

$$E(g_{I_j};\beta) = -\frac{h_k^2}{12}(b-a)g^{(2)}_{I_j}(\delta_{I_j};\beta) \tag{43}$$

$$E(g_{F_j};\gamma) = -\frac{h_k^2}{12}(b-a)g^{(2)}_{F_j}(\delta_{F_j};\gamma) \tag{44}$$

Now, $J_{k,1}(g_{T_j};\alpha)$, $J_{k,1}(g_{I_j};\beta)$ and $J_{k,1}(g_{F_j};\gamma)$ have been used to denote the portion of the Eqs. (39), (40), (41), (42), (43) and (44) for Trapezoidal approximation. Then,

$$J_{1,1}(g_{T_j};\alpha) = \frac{h_1}{2}\left[g_{T_j}(x_0;\alpha) + g_{T_j}(x_n;\alpha)\right] \tag{45}$$

$$J_{1,1}(g_{I_j};\beta) = \frac{h_1}{2}\left[g_{I_j}(x_0;\beta) + g_{I_j}(x_n;\beta)\right] \tag{46}$$

$$J_{1,1}(g_{F_j};\gamma) = \frac{h_1}{2}\left[g_{F_j}(x_0;\gamma) + g_{F_j}(x_n;\gamma)\right] \tag{47}$$

Then,

$$\begin{aligned} J_{2,1}(g_{T_j};\alpha) &= \frac{h_2}{2}\left[g_{T_j}(x_0;\alpha) + 2g_{T_j}(x_0+h_2;\alpha) + g_{T_j}(x_n;\alpha)\right] \\ &= \frac{x_n - x_0}{4}\left[g_{T_j}(x_0;\alpha) + g_{T_j}(x_n;\alpha) + 2g_{T_j}\left(x_0 + \frac{x_n - x_0}{2};\alpha\right)\right] \\ &= \frac{1}{2}\left[J_{1,1}(g_{T_j};\alpha) + h_1 g_{T_j}(x_0+h_2;\alpha)\right] \end{aligned} \tag{48}$$

Similarly,

$$J_{2,1}(g_{I_j};\beta) = \frac{1}{2}\left[J_{1,1}(g_{I_j};\beta) + h_1 g_{I_j}(x_0+h_2;\beta)\right] \tag{49}$$

$$J_{2,1}(g_{F_j};\gamma) = \frac{1}{2}\left[J_{1,1}(g_{F_j};\gamma) + h_1 g_{F_j}(x_0+h_2;\gamma)\right] \tag{50}$$

$$J_{3,1}(g_{T_j};\alpha) = \frac{1}{2}\left\{J_{2,1}(g_{T_j};\alpha) + h_2\left[g_{T_j}(x_0+h_3;\alpha) + g_{T_j}(x_0+3h_3;\alpha)\right]\right\} \tag{51}$$

$$J_{3,1}(g_{I_j};\beta) = \frac{1}{2}\left\{J_{2,1}(g_{I_j};\beta) + h_2\left[g_{I_j}(x_0+h_3;\beta) + g_{I_j}(x_0+3h_3;\beta)\right]\right\} \tag{52}$$

$$J_{3,1}(g_{F_j};\gamma) = \frac{1}{2}\left\{J_{2,1}(g_{F_j};\gamma) + h_2\left[g_{F_j}(x_0+h_3;\gamma) + g_{F_j}(x_0+3h_3;\gamma)\right]\right\} \tag{53}$$

In general

$$J_{k,1}(g_{T_j};\alpha) = \frac{1}{2}\left\{J_{k-1,1}(g_{T_j};\alpha) + h_{k-1}\sum_{j=1}^{2^{k-2}} g_{T_j}(x_0 + (2j-1)h_k;\alpha)\right\} \tag{54}$$

$$J_{k,1}(g_{I_j};\beta) = \frac{1}{2}\left\{J_{k-1,1}(g_{I_j};\beta) + h_{k-1}\sum_{j=1}^{2^{k-2}} g_{I_j}(x_0 + (2j-1)h_k;\beta)\right\} \tag{55}$$

$$J_{k,1}(g_{F_j};\gamma) = \frac{1}{2}\left\{J_{k-1,1}(g_{F_j};\gamma) + h_{k-1}\sum_{j=1}^{2^{k-2}} g_{F_j}(x_0 + (2j-1)h_k;\gamma)\right\} \tag{56}$$

where $k = 2, 3, \dots, n$ and $j = 1, 2$ denotes the left and right branch respectively.

Here the Richardson extrapolation has been used to show the speed of the Convergence of the method. If $g_{T_j}, g_{I_j}, g_{F_j} \in C^{\infty}[a, b]$ then the composite Trapezoidal rule can be written as

$$L(g_{T_j};\alpha) - J_{k,1}(g_{T_j};\alpha) = \sum_{i=1}^{\infty} C_{ij} h_k^{2i} = C_{11}h_k^2 + \sum_{i=2}^{\infty} C_{ij} h_k^{2i} \tag{57}$$

$$L(g_{I_j};\beta) - J_{k,1}(g_{I_j};\beta) = \sum_{i=1}^{\infty} C'_{ij} h_k^{2i} = C'_{11}h_k^2 + \sum_{i=2}^{\infty} C'_{ij} h_k^{2i} \tag{58}$$

$$L(g_{F_j};\gamma) - J_{k,1}(g_{F_j};\gamma) = \sum_{i=1}^{\infty} C''_{ij} h_k^{2i} = C''_{11}h_k^2 + \sum_{i=2}^{\infty} C''_{ij} h_k^{2i} \tag{59}$$

where $C_{ij}, C'_{ij}, C''_{ij}$ are all independent of the value of h_k and the values of $C_{ij}, C'_{ij}, C''_{ij}$ depends only on $g^{2i-1}_{T_j}(a;\alpha)$, $g^{2i-1}_{I_j}(a;\beta)$, $g^{2i-1}_{F_j}(a;\gamma)$, $g^{2i-1}_{T_j}(b;\alpha)$, $g^{2i-1}_{I_j}(b;\beta)$ and $g^{2i-1}_{F_j}(b;\gamma)$. So, in the form of the Composite Trapezoidal rule, the term involving h_k^2 can eliminate and replace h_k by $h_{k+1} = \frac{h_k}{2}$. Then,

$$L(g_{T_j};\alpha) - J_{k,1}(g_{T_j};\alpha) = \sum_{i=1}^{\infty} C_{ij} h_{k+1}^{2i} = \sum_{i=1}^{\infty} \frac{C_{ij}h_k^{2i}}{2^{2i}} = \frac{C_{11}h_K^2}{4} + \sum_{i=2}^{\infty} \frac{C_{ij}h_k^{2i}}{4^i} \tag{60}$$

$$L(g_{I_j};\beta) - J_{k,1}(g_{I_j};\beta) = \sum_{i=1}^{\infty} C'_{ij} h_{k+1}^{2i} = \sum_{i=1}^{\infty} \frac{C'_{ij} h_k^{2i}}{2^{2i}} = \frac{C'_{11} h_K^2}{4} + \sum_{i=2}^{\infty} \frac{C'_{ij} h_k^{2i}}{4^i} \tag{61}$$

$$L(g_{F_j};\gamma) - J_{k,1}(g_{F_j};\gamma) = \sum_{i=1}^{\infty} C''_{ij} h_{k+1}^{2i} = \sum_{i=1}^{\infty} \frac{C''_{ij} h_k^{2i}}{2^{2i}} = \frac{C''_{11} h_K^2}{4} + \sum_{i=2}^{\infty} \frac{C''_{ij} h_k^{2i}}{4^i} \tag{62}$$

Now, multiply the Eq. (60) by 4 and subtracting the Eq. (57) from this we have

$$\begin{aligned} L(g_{T_j};\alpha) - \left[J_{k+1,1}(g_{T_j};\alpha) + \frac{J_{k+1,1}(g_{T_j};\alpha) - J_{k,1}(g_{T_j};\alpha)}{3} \right] \\ = \sum_{i=2}^{\infty} \frac{C_{ij}}{3} \left(\frac{h_k^{2i}}{4^{i-1}} - h_k^{2i} \right) \\ = \sum_{i=2}^{\infty} \frac{C_i}{3} \left(\frac{1-4^{i-1}}{4^{i-1}} \right) h_k^{2i} \end{aligned} \tag{63}$$

$$\begin{aligned} L(g_{I_j};\beta) - \left[J_{k+1,1}(g_{I_j};\beta) + \frac{J_{k+1,1}(g_{I_j};\beta) - J_{k,1}(g_{I_j};\beta)}{3} \right] \\ = \sum_{i=2}^{\infty} \frac{C'_{ij}}{3} \left(\frac{h_k^{2i}}{4^{i-1}} - h_k^{2i} \right) \\ = \sum_{i=2}^{\infty} \frac{C'_i}{3} \left(\frac{1-4^{i-1}}{4^{i-1}} \right) h_k^{2i} \end{aligned} \tag{64}$$

$$\begin{aligned} L(g_{F_j};\gamma) - \left[J_{k+1,1}(g_{F_j};\gamma) + \frac{J_{k+1,1}(g_{F_j};\gamma) - J_{k,1}(g_{F_j};\gamma)}{3} \right] \\ = \sum_{i=2}^{\infty} \frac{C''_{ij}}{3} \left(\frac{h_k^{2i}}{4^{i-1}} - h_k^{2i} \right) \\ = \sum_{i=2}^{\infty} \frac{C''_i}{3} \left(\frac{1-4^{i-1}}{4^{i-1}} \right) h_k^{2i} \end{aligned} \tag{65}$$

Now, simplify the notion then

$$J_{k,2}(g_{T_j};\alpha) = J_{k,1}(g_{T_j};\alpha) + \frac{J_{k,1}(g_{T_j};\alpha) - J_{k-1,1}(g_{T_j};\alpha)}{3} \tag{66}$$

$$J_{k,2}(g_{I_j};\beta) = J_{k,1}(g_{I_j};\beta) + \frac{J_{k,1}(g_{I_j};\beta) - J_{k-1,1}(g_{I_j};\beta)}{3} \tag{67}$$

$$J_{k,2}(g_{F_j};\gamma) = J_{k,1}(g_{F_j};\gamma) + \frac{J_{k,1}(g_{F_j};\gamma) - J_{k-1,1}(g_{F_j};\gamma)}{3} \tag{68}$$

where $k = 2, 3, \dots, n$. Now, we apply the Richardson extrapolation method and continue this process for $k = 2, 3, \dots, n$ and $m = 2, 3, \dots, k$. Then we have,

$$J_{k,m}(g_{T_j};\alpha) = J_{k,m-1}(g_{T_j};\alpha) + \frac{J_{k,m-1}(g_{T_j};\alpha) - J_{k-1,m-1}(g_{T_j};\alpha)}{4^{m-1}-1} \tag{69}$$

$$J_{k,m}(g_{I_j};\beta) = J_{k,m-1}(g_{I_j};\beta) + \frac{J_{k,m-1}(g_{I_j};\beta) - J_{k-1,m-1}(g_{I_j};\beta)}{4^{m-1}-1} \tag{70}$$

$$J_{k,m}(g_{F_j};\gamma) = J_{k,m-1}(g_{F_j};\gamma) + \frac{J_{k,m-1}(g_{F_j};\gamma) - J_{k-1,m-1}(g_{I_j};\gamma)}{4^{m-1}-1} \tag{71}$$

We generate approximations until the terms $\left|J_{m-1,m-1}(g_{T_j};\alpha) - J_{m,m}(g_{T_j};\alpha)\right|$, $\left|J_{m-1,m-1}(g_{I_j};\beta) - J_{m,m}(g_{I_j};\beta)\right|$ and $\left|J_{m-1,m-1}(g_{F_j};\gamma) - J_{m,m}(g_{F_j};\gamma)\right|$ are within the tolerance.

Not only the above terms but it has also been shown that the terms $\left|J_{m-2,m-2}(g_{T_j};\alpha) - J_{m-1,m-1}(g_{T_j};\alpha)\right|$, $\left|J_{m-2,m-2}(g_{I_j};\beta) - J_{m-1,m-1}(g_{I_j};\beta)\right|$ and $\left|J_{m-2,m-2}(g_{F_j};\gamma) - J_{m-1,m-1}(g_{F_j};\gamma)\right|$ are within the tolerance.

5. Comparative analysis, advantages and limitations of the proposed method

5.1. Comparative analysis

In this Section, a Comparative study of the proposed method with other existing methods has been discussed to show the superiority, reliability, and effectiveness of the proposed method.

In Table 1, it has been seen that Wu [44] applied Riemann–Stieltjes integral to obtain the integration of fuzzy valued function. This is not a numerical method and we cannot find the numerical integration of the fuzzy valued function. Also, this method cannot be applied for neutrosophic valued functions as the falsity and indeterminacy parts are absent here. However, our proposed method is applicable for neutrosophic valued functions. Pedro et al. [45] used fuzzy Riemann integration to find the integration of a fuzzy valued function. Also, this method is not a numerical method. So, we cannot find the numerical integration of a function with the help of this method. Also, this method has been developed for fuzzy functions. So, we cannot apply this method to find the numerical integration of the neutrosophic valued function as the falsity and indeterminacy parts are absent here. But our proposed method is a numerical method and we can apply this method to find the numerical integration of neutrosophic valued functions. In addition, Alhasan et al. [28] used the substitution method to compute the integration of the neutrosophic valued function. This method is an analytical method and we cannot apply this method to compute the numerical integration of neutrosophic valued function. However, our method is applicable to compute the numerical integration of neutrosophic valued functions. Therefore, this study shows the superiority, reliability, and effectiveness of our proposed method.

5.2. Advantages of the proposed method

1. This numerical method gives more accurate results than other existing numerical methods such as the trapezoidal rule and Simpson's rule.
2. The Romberg integration method converges to the exact solution within a few computational steps. So, this method is more efficient compared to the other existing methods.
3. The Romberg integration method provides less error in every computational step. This shows the reliability of the proposed method.
4. The Romberg integration method is easy to use and implement compared to other complex numerical techniques such as adaptive quadrature methods.

5.3. Limitations of the proposed method

1. The major limitation of the proposed method is that the method is difficult to implement for highly complex functions. Also, this method is domain-specific. If the domain is unbounded, then this method cannot be applicable to find the numerical integration of a function.

6. Examples

Some numerical examples are provided in this part to support our suggested methodology. In addition, some error analysis has been done of the proposed method with the help of the following convergence indicators.

- The absolute error: $|\mathbb{E}_{k,m}(g_{M_i};\mu)| = \left|J^{exact}(g_{M_i};\mu) - J_{k,m}(g_{M_i};\mu)\right|$
- The reference error : $e_{k,m}^{ref}(g_{M_i};\mu) = \|\mathbb{E}_{k,m}(g_{M_i};\mu)\|_2$.

Here $M = T, I, F$ and $\mu = \alpha, \beta, \gamma$ where $i = 1, 2$ respectively.

Table 1
Comparative analysis of the proposed method with other existing methods.

Methods	Nature of the environment	Description of the methods	Remarks
Method described by Wu [44].	Fuzzy	Using Riemann–Stieltjes integral to find the integration of fuzzy valued function.	This method is not applicable to find the numerical solution of a function. Also, this method has been developed for fuzzy-valued functions. This method cannot be applicable for neutrosophic valued functions.
Method described by Pedro et al. [45]	Fuzzy	Using Frechet derivative and the Riemann integral to find the integration of fuzzy valued function.	This method cannot be applicable for neutrosophic valued function.
Method described by Alhasan [28]	Neutrosophic	Using substitution method to compute the integration of a neutrosophic valued function.	This is an analytical method. This method cannot be applicable for computing the numerical integration of neutrosophic valued function.
Our method	Neutrosophic	Using Romberg integration method to find the numerical integration of neutrosophic valued function.	This is a numerical method. This method is applicable for computing the numerical integration of neutrosophic valued functions.

Table 2
Absolute errors for truth membership function.

α	$\|E_{11}(g_{T_1};\alpha)\|$	$\|E_{21}(g_{T_1};\alpha)\|$	$\|E_{22}(g_{T_1};\alpha)\|$	$\|E_{11}(g_{T_2};\alpha)\|$	$\|E_{21}(g_{T_2};\alpha)\|$	$\|E_{22}(g_{T_2};\alpha)\|$
0	0.00E−0	0.00E−0	0.00E−0	6.15E−1	1.55E−1	1.16E−3
0.2	1.02E−1	2.58E−2	1.93E−4	5.13E−1	1.29E−1	9.66E−4
0.4	2.05E−1	6.15E−2	3.86E−4	4.10E−1	1.03E−1	7.72E−4
0.6	3.08E−1	7.73E−2	5.79E−4	3.08E−1	7.73E−2	5.79E−4

Example 1. Consider the following neutrosophic integration

$$\int_o^1 \tilde{n}(e^x + x^2 + 1)\, dx \qquad where\ \ \tilde{n} = \langle(0,1,2); 0.6, 0.4, 0.2\rangle.$$

Now, the exact solution of the above equation is given below

$$\left(e+\frac{1}{3}\right)\left\langle\left[\frac{5\alpha}{3}, \frac{6-5\alpha}{3}\right], \left[\frac{5(1-\beta)}{3}, \frac{5\beta+1}{3}\right], \left[\frac{5(1-\gamma)}{4}, \frac{5\gamma+3}{4}\right]\right\rangle$$

where $\alpha \in [0, 0.6], \beta \in [0.4, 1]$ and $\gamma \in [0.2, 1]$. Now, we are going to determine the integral with the help of Romberg integration method. Let us consider $g(x) = \tilde{n}(e^x + x^2 + 1)$. For $h = 1$,

$$J_{1,1}(g_{T_1};\alpha) = \frac{5(4+e)\alpha}{6}, \qquad J_{1,1}(g_{T_2};\alpha) = \frac{(4+e)(6-5\alpha)}{6},$$
$$J_{1,1}(g_{I_1};\beta) = \frac{5(4+e)(1-\beta)}{6}, \quad J_{1,1}(g_{I_2};\beta) = \frac{(4+e)(5\beta+1)}{6},$$
$$J_{1,1}(g_{F_1};\gamma) = \frac{5(4+e)(1-\gamma)}{8}, \quad J_{1,1}(g_{F_2};\gamma) = \frac{(4+e)(5\gamma+3)}{8}$$

For $h = \frac{1}{2}$,

$$J_{2,1}(g_{T_1};\alpha) = \frac{5(13+2e+4\sqrt{e})\alpha}{24},$$
$$J_{2,1}(g_{T_2};\alpha) = \frac{(13+2e+4\sqrt{e})(6-5\alpha)}{24}$$
$$J_{2,1}(g_{I_1};\beta) = \frac{5(13+2e+4\sqrt{e})(1-\beta)}{24},$$
$$J_{2,1}(g_{I_2};\beta) = \frac{(13+2e+4\sqrt{e})(5\beta+1)}{24}$$
$$J_{2,1}(g_{F_1};\gamma) = \frac{5(13+2e+4\sqrt{e})(1-\gamma)}{32},$$
$$J_{2,1}(g_{F_2};\gamma) = \frac{(13+2e+4\sqrt{e})(5\gamma+3)}{32}$$

Then we have,

$$J_{2,2}(g_{T_1};\alpha) = \frac{5(9+e+4\sqrt{e})\alpha}{18}, \qquad J_{2,2}(g_{T_2};\alpha) = \frac{(9+e+4\sqrt{e})(6-5\alpha)}{18},$$
$$J_{2,2}(g_{I_1};\beta) = \frac{5(9+e+4\sqrt{e})(1-\beta)}{18}, \quad J_{2,2}(g_{I_2};\beta) = \frac{(9+e+4\sqrt{e})(5\beta+1)}{18},$$
$$J_{2,2}(g_{F_1};\gamma) = \frac{(9+e+4\sqrt{e})(1-\gamma)}{24}, \quad J_{2,2}(g_{F_2};\gamma) = \frac{(9+e+4\sqrt{e})(5\gamma+3)}{24}$$

In Table 2, 3 and 4, $E_{k,m}(g_{T_i};\alpha)$, $E_{k,m}(g_{I_i};\beta)$ and $E_{k,m}(g_{F_i};\gamma)$ have been investigated respectively. Here $E_{k,m}(g_{T_i};\alpha)$, $E_{k,m}(g_{I_i};\beta)$ and $E_{k,m}(g_{F_i};\gamma)$ denotes Absolute errors for truth indeterminacy and falsity of g respectively. Also, from Figs. 1, 2, 3, 4, 5 and 6, it has been seen that $E_{k,m}(g_{T_i};\alpha)$, $E_{k,m}(g_{I_i};\beta)$ and $E_{k,m}(g_{F_i};\gamma)$ are decrease as (k, m) increase. Also, $e^{ref}_{k,m}(g_{T_i};\alpha)$, $e^{ref}_{k,m}(g_{I_i};\beta)$ and $e^{ref}_{k,m}(g_{F_i};\gamma)$ have been investigated in Table 5. Here $e^{ref}_{k,m}(g_{T_i};\alpha)$, $e^{ref}_{k,m}(g_{I_i};\beta)$ and $e^{ref}_{k,m}(g_{F_i};\gamma)$ denotes Reference errors for truth indeterminacy and falsity of g respectively (see Table 5). From Table 5, it has been noticed that $e^{ref}_{k,m}(g_{T_i};\alpha)$, $e^{ref}_{k,m}(g_{I_i};\beta)$ and $e^{ref}_{k,m}(g_{F_i};\gamma)$ decrease as (k, m) increases. The pictorial representation of $g(x)$ has been shown in Figs. 7, 8, and 9. In $g(x)$, we have considered the parameters as a triangular neutrosophic number. Then, from Figs. 7, 8, and 9, it has been seen that $g(x)$ forms a triangular neutrosophic number.

7. Conclusion

In this research work, the concept of the Romberg integration method is developed for neutrosophic valued function. For the first time, the Romberg integration method has been examined in a neutrosophic environment. The Richardson extrapolation method has been used in this numerical technique to find a better approximation of the given function. Also, for the first time, the Richardson extrapolation method is studied in a neutrosophic environment. In addition, from Theorem 3.1, it can be concluded that $J(g_{T_j};\alpha)$, $J(g_{I_j};\beta)$, $J(g_{F_j};\gamma)$ converge uniformly to $L(g_{T_j};\alpha)$, $L(g_{I_j};\beta)$ and $L(g_{F_j};\gamma)$ respectively. Additionally, it can be deduced from Theorem 3.2 that the integral form of the error $E(g)$ is a well-known result due to Peano. These conclusions demonstrate the usefulness of our suggested approach. An analysis of a numerical example along with a few convergence indicators demonstrates the applicability and effectiveness of the suggested approach. From Tables 1 to 4, it can be inferred that absolute errors and reference errors reduce as the value of (k, m) grows. In Figs. 1 to 6, this finding is depicted graphically. Additionally, we have deduced from Figs. 7, 8, and 9 that the integration in Example 1 yields a triangular neutrosophic number if all the parameters are assumed to be triangular neutrosophic numbers. Triangular neutrosophic numbers are not the only ones to which this result applies. Any neutrosophic number can be included in its extension. This numerical example also demonstrates the validity of the suggested approach. The researchers are encouraged to continue working on the numerical integration for neutrosophic

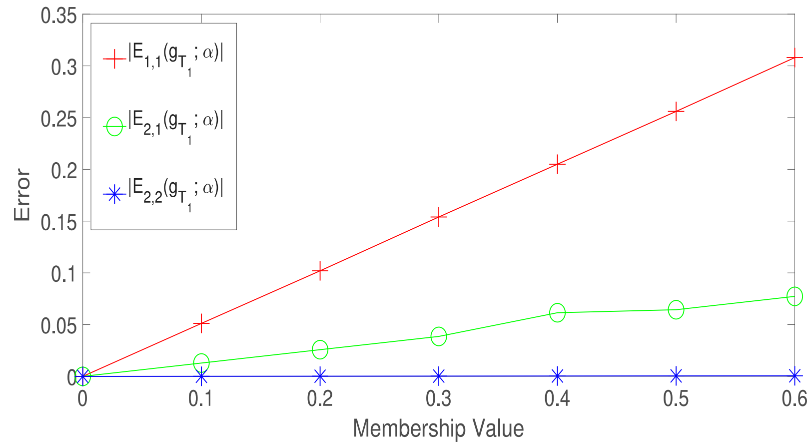

Fig. 1. Absolute error for left branch of truth membership function.

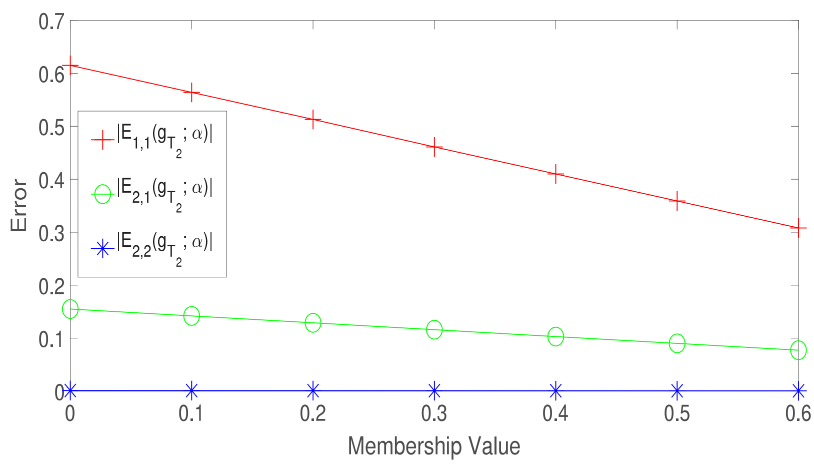

Fig. 2. Absolute error for right branch of truth membership function.

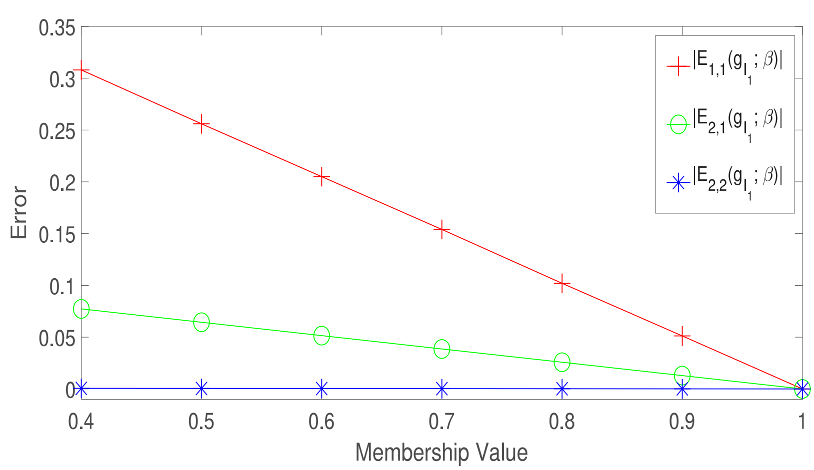

Fig. 3. Absolute error for left branch of indeterminacy membership function.

Table 3
Absolute errors for indeterminacy membership function.

β	$\lvert E_{11}(g_{I_1};\beta)\rvert$	$\lvert E_{21}(g_{I_1};\beta)\rvert$	$\lvert E_{22}(g_{I_1};\beta)\rvert$	$\lvert E_{11}(g_{I_2};\beta)\rvert$	$\lvert E_{21}(g_{I_2};\beta)\rvert$	$\lvert E_{22}(g_{I_2};\beta)\rvert$
0.4	3.08E−1	7.73E−2	5.79E−4	3.08E−1	7.73E−2	5.79E−4
0.6	2.05E−1	5.15E−2	3.86E−4	4.10E−1	1.03E−1	7.72E−4
0.8	1.02E−1	2.58E−2	1.93E−4	5.12E−1	1.29E−1	9.66E−4
1.0	0.00E−0	0.00E−0	0.00E−0	6.15E−1	1.55E−1	1.16E−3

Table 4
Absolute errors for falsity membership function.

γ	$\|E_{11}(g_{F_1};\gamma)\|$	$\|E_{21}(g_{F_1};\gamma)\|$	$\|E_{22}(g_{F_1};\gamma)\|$	$\|E_{11}(g_{F_2};\gamma)\|$	$\|E_{21}(g_{F_2};\gamma)\|$	$\|E_{22}(g_{F_2};\gamma)\|$
0.2	3.07E−1	7.73E−2	5.79E−4	3.07E−1	7.73E−2	5.79E−4
0.6	1.54E−1	3.86E−2	2.90E−4	4.61E−1	1.16E−1	8.69E−4
0.8	7.69E−2	1.93E−2	1.45E−4	5.38E−1	1.35E−1	1.01E−3
1.0	0.00E−0	0.00E−0	0.00E−0	6.15E−1	1.55E−1	1.16E−3

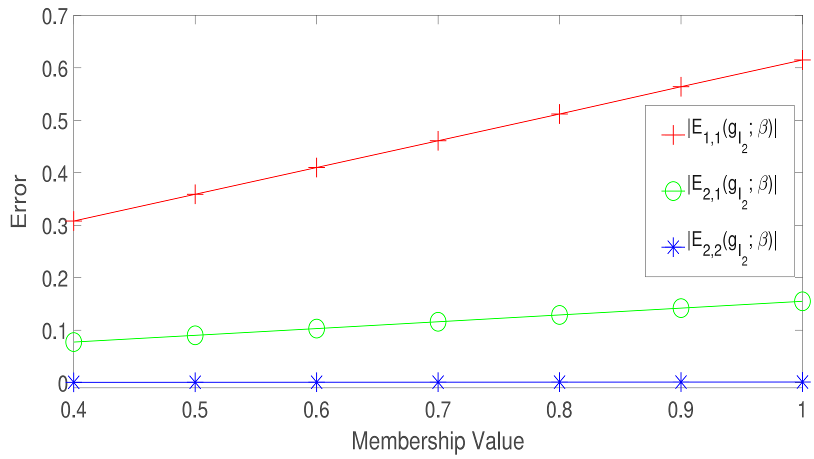

Fig. 4. Absolute error for right branch of indeterminacy membership function.

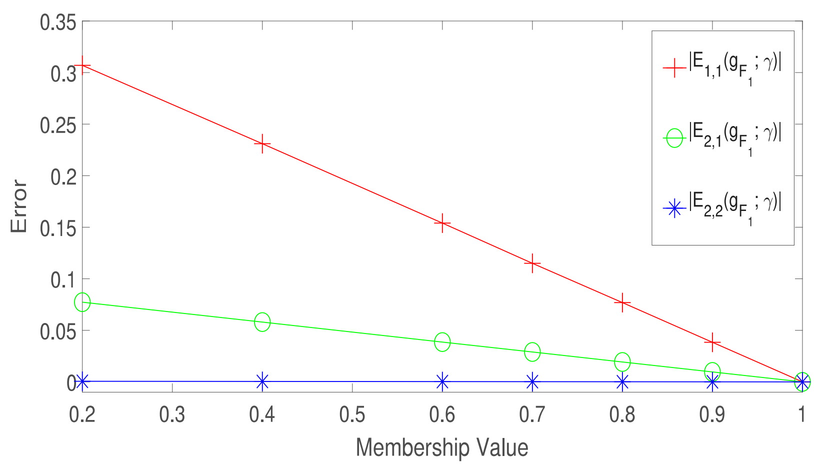

Fig. 5. Absolute error for left branch of falsity membership function.

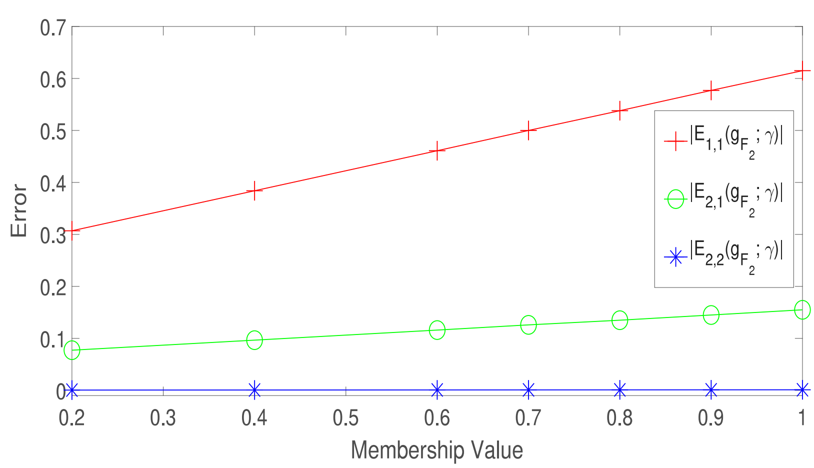

Fig. 6. Absolute error for right branch of falsity membership function.

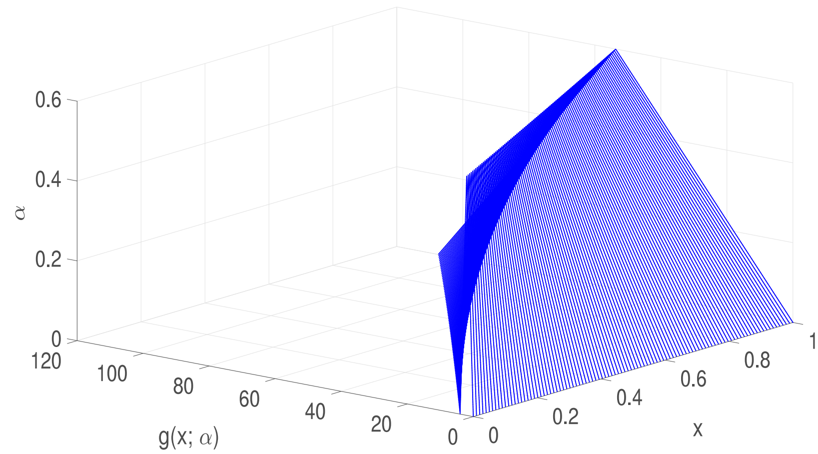

Fig. 7. Graphical representation for truth membership function of g(x).

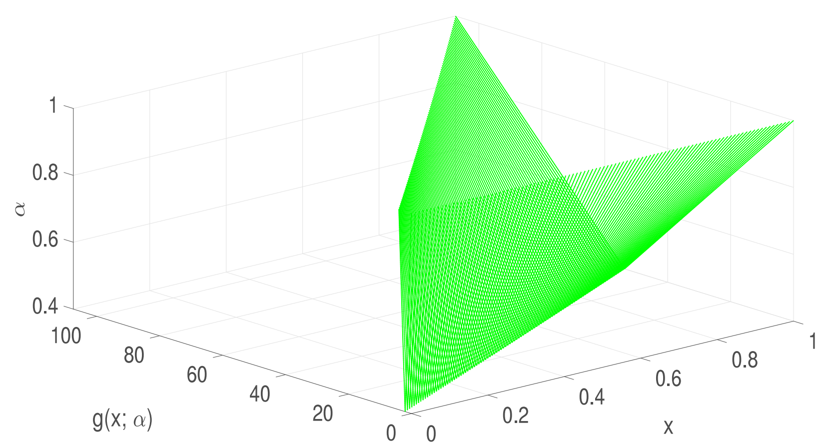

Fig. 8. Graphical representation for indeterminacy membership function of g(x).

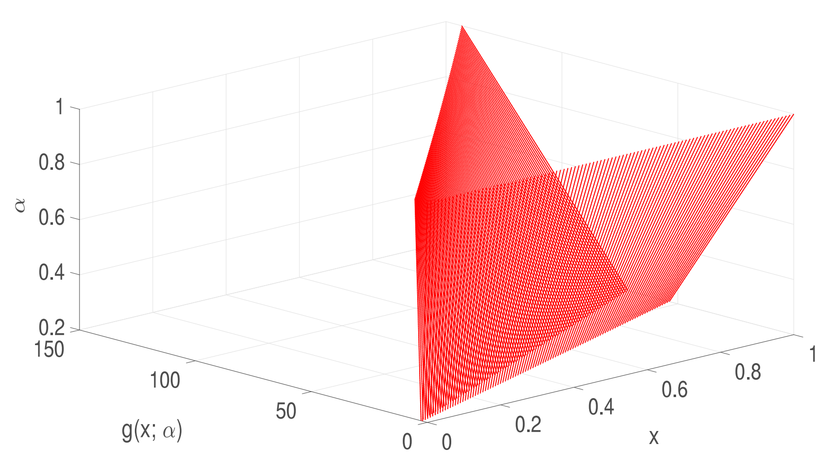

Fig. 9. Graphical representation for falsity membership function of g(x).

valued functions by this research. In addition, Romberg integration has some managerial implications. Romberg integration is an extrapolation formula of the trapezoidal rule for integration. In an uncertain real-world environment, many scientific and engineering problems such as

Table 5
Reference errors for truth, indeterminacy and falsity membership function.

Error	$(k,m)=(1,1)$	$(k,m)=(2,1)$	$(k,m)=(2,2)$
$e^{ref}_{k,m}(g_{T_1};0.4)$	4.89E−1	1.27E−1	9.21E−4
$e^{ref}_{k,m}(g_{T_2};0.4)$	1.25E−0	5.24E−1	2.35E−3
$e^{ref}_{k,m}(g_{I_1};0.6)$	4.89E−0	1.23E−0	9.21E−4
$e^{ref}_{k,m}(g_{I_2};0.6)$	1.25E−0	3.15E−1	2.35E−3
$e^{ref}_{k,m}(g_{F_1};0.8)$	4.38E−0	1.10E−1	8.25E−4
$e^{ref}_{k,m}(g_{F_2};0.8)$	1.30E−0	3.28E−1	2.46E−3

diffraction problems, water waves, scattering in quantum mechanics, the Volterra population model, heat transfer, heat radiation, etc. are determined by neutrosophic integral equations. In these real-world problems, Romberg integration can be used to approximate the integral and find the approximate solution of the equations.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Mr. Sandip Moi reports financial support was provided by Council of Scientific and Industrial Research (CSIR), Government of India.

Data availability

No data was used for the research described in the article.

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