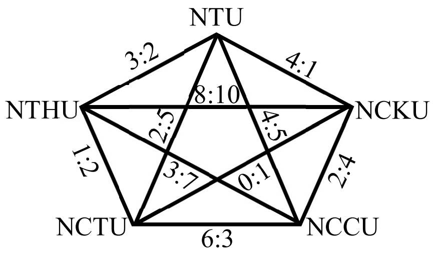
Ranking with Paired Comparisons

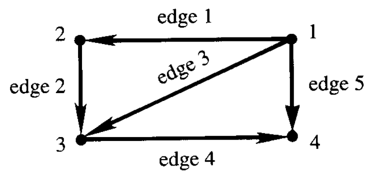
- **Survey questionnaire: ranking A, B, C, D, E,….all together? Difficult!**
- **Instead, A vs. B, B vs. C, C vs. D….Relatively easy!**
- **Ranking of athletes and teams via games, e.g. Tennis, Basketball, Baseball, etc.**
- **Example: NTU, NTHU, NCTU, NCKU, NCCU baseball teams playing games as follows:**



- **Which team is the best? 2nd best? and the rest?**

Application – Graphs

- **Graph: *nodes* and *edges***



- ***Edge-Node Incidence (connectivity or topology) matrices***

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

Number of nodes = number of columns

Number of edges = number of rows

- **Edge *i* goes from node *j* to node *k* = *i*th row has –1 in column *j* and +1 in column *k*.**
- **Rows hold information about Edges and columns hold information about nodes**
- **Transpose of edge-node matrix = node-edge matrix**

Nullspace of Incidence Matrix

- **Meaning of $Ax=b$: x_i's are *potentials* at nodes; Ax gives the potential differences across five edges; $Ax=b \Rightarrow$ given the potential differences b_i's, find the actual node potentials.**

$$Ax = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

- **Nullspace $\Rightarrow$ combination of columns of A that gives zero? All columns add up to zero $\Rightarrow$ if $x=(c, c, c, c)$ then $Ax=0 \Rightarrow$ nontrivial solution $\Rightarrow A$: not full rank**
- **If $Ax=b$ has a solution at all, it is not unique**
- **$Ax=0$: equal potentials across every edge $\Rightarrow x=(c, c, c, c)$**
- **Complete sol = particular sol + nullspace sol: any "constant vector" $x = (c, c, c, c)$ can be added to any particular solution of $Ax = b$. The complete solution has this arbitrary constant c**

<u>Raise or lower all the potentials by the same constant c, the potential differences will not change</u>.

Column Space of Incidence Matrix

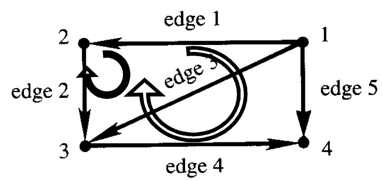
$$Ax = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

- **For what b_i's, we can solve $Ax=b$? First find the constraints for the system to be solvable. These constraints can be found by elimination**
- **Elimination without permutation:**

$$Ax = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \\ b_3 - b_1 - b_2 \\ b_4 \\ b_5 - b_1 - b_2 - b_4 \end{bmatrix}$$

- **If b is in the column space then:**

$b_3 - b_1 - b_2 = 0$ and $b_5 - b_1 - b_2 - b_4 = 0$ what are these?



- **constraints = rule of *potential differences around a loop must add to zero***

Left Nullspace of Incidence Matrix

- **If b is in the column space then:**

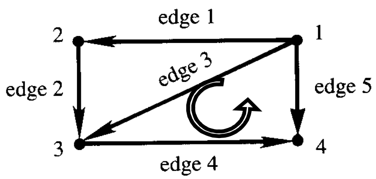
  $b_3 - b_1 - b_2 = 0$ **and** $b_5 - b_1 - b_2 - b_4 = 0 \Rightarrow$

  $y^T A = 0 \Rightarrow y^T$**: combination of rows in A that gives zero row**

  $y_1^T = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 \end{bmatrix}$ **and** $y_2^T = \begin{bmatrix} -1 & -1 & 0 & -1 & 1 \end{bmatrix}$

- **Linear combinations of y_1 and y_2 are also in the left nullspace:** $y_1 - y_2 = (0, 0, 1, 1, -1) \Rightarrow b_3 + b_4 - b_5 = 0 \Rightarrow$

  the lower right loop!



- **Column space and left nullspace are closely related:**

For a system $Ax=b$ to be solvable: $y^T b = 0$ whenever $y^T A = 0$

- ***Kirchhoff's Voltage Law*:**

  <u>The sum of potential differences around a loop must be zero.</u>

Row Space of Incidence Matrix

- **Here, we look at $A^T y = f$**

$$A^T y = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} y_1 + \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} y_2 + \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} y_3 + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} y_4 + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} y_5 = f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} \Rightarrow$$

$-y_1 - y_3 - y_5 = f_1$, $y_1 - y_2 = f_2$, $y_2 + y_3 - y_4 = f_3$, $y_4 + y_5 = f_4$

"flow in = flow out"; $f = (f_1, f_2, f_3, f_4)$ is the outside source

$$A^T y = \begin{bmatrix} -1 & 0 & -1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} y = \begin{bmatrix} -1 & 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} y = \begin{bmatrix} f_1 \\ f_2 + f_1 \\ f_3 + f_2 + f_1 \\ f_4 + f_3 + f_2 + f_1 \end{bmatrix}$$

$$\Rightarrow f_1 + f_2 + f_3 + f_4 = 0$$

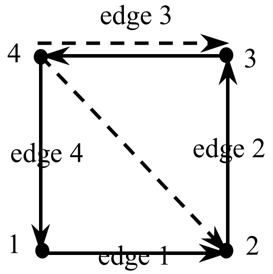
The total sum of the outside current entering the nodes is zero

- ***Kirchhoff's Current Law:***

<u>The net current into every node is zero.</u>

This law can only be satisfied if the total current entering the nodes from outside is zero.

Row Independence and Spanning Tree



$$A=\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

- **Elimination step 1: edge 1 + edge 4 = 4→2 edge**
- **Elimination step 2: 4→2 edge + edge 2 = 4→3 edge**
- **Elimination step 3: 4→3 edge + edge 3 = zero row**
- **Three independent rows left: edges 1, 2 and 3**
- ***Rows are independent if the corresponding edges are without a loop***
- **Edges without loop: *Tree***
- **A tree that touches every node of the graph: *spanning tree***
- **A spanning tree's edges span the graph ⇒ its rows span the row space**
- **A graph with n nodes has a spanning tree with n-1 edges**

⇒ *There are n-1 independent rows in the incident matrix of a spanning tree*

Dimensions of Subspaces for Incident Matrix

- **m ($\geq n-1$) edges and n nodes ⇒ $n-1$ independent rows**

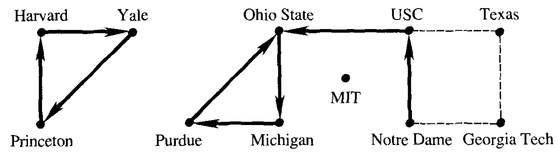
⇒ $n-1$ = rank = dimension of column space

- **nullspace dimension: $n-r = n-(n-1)=1$; contains x=(1,…,1)**
- **dimension of left nullspace = number of independent loops**

$= m-r = m-(n-1)= m-n+1$

Rank of Football Teams

- **US college football teams play one another without a systematic competition scheme**
- **Ranking is based on vote polls: an average opinions, such as result of sports journalists' vote**
- **How can we find the true potentials of football teams and rank them correctly based on game results?**



- **Teams are nodes; games are edges; an edge goes from the visiting team to the home team.**

Ranking the Football Teams

- $Ax=b$: A is the incident matrix; b is the score difference
- Two difficulties:

1. There a few hundred teams (unknows) and a few thousand games (equations). b must be in the column space of A to have solutions.
2. If there is a solution, the solution is not unique.

- Dimension of nullspace = degree of freedom of x = number of pieces of the graph = 3 $\Rightarrow$ arbitrarily assign the potential to one team in each piece.
- For $Ax=b$ to be solvable: $b_{HY} + b_{YP} + b_{PH} = 0$

$\Rightarrow$ Kirchhoff's voltage law.

- In reality, b is almost certainly not in the column space $\Rightarrow$ no solution
- Solution: *least squares* to make Ax as close as possible to b (Chapter 3)
- We also need to give the winner team extra bonus points to distinguish winners and loosers.

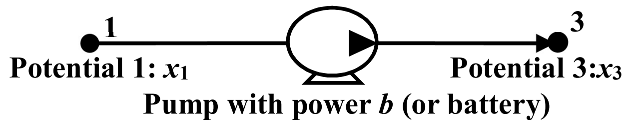
Networks

- A graph becomes a network when numbers $c_1, \ldots, c_m$ are assigned to the edges.
- Example of c_i's: length, capacity, conductance…
- Property matrix C: $C = \begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & c_m \end{bmatrix}$
- A and C are two fundamental matrix of network theory.
- On edge i: conductance c_i; resistance $1/c_i$.
- Ohm's law: V=IR$\Rightarrow$I=V/R: $y_i=c_ie_i$

current = voltage drop/resistance = conductance×voltage drop

$\Rightarrow y=Ce$, where e is the voltage drop across the resistor.

- Current = conductance × (battery: external energy source + potential drop)



Current 1$\rightarrow$3 $y= C(b+(x_1-x_3))= C(b-(x_3-x_1))$

$\Rightarrow y = Ce = C\,(b - Ax)$ or $C^{-1}y + Ax = b$

Equations of Equilibrium

- **KCL: the currents into a node add to zero ($A^Ty=f$)**
- **The fundamental equations of equilibrium (KVL+KCL):**

$$\begin{aligned} C^{-1}y + Ax &= b \\ A^T y \qquad &= f \end{aligned} \;\Rightarrow\; \begin{bmatrix} C^{-1} & A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} b \\ f \end{bmatrix}$$

- **Apply elimination:**

$$\begin{bmatrix} C^{-1} & A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} b \\ f \end{bmatrix} \rightarrow \begin{bmatrix} C^{-1} & A \\ 0 & -A^TCA \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} b \\ f - A^TCb \end{bmatrix}$$

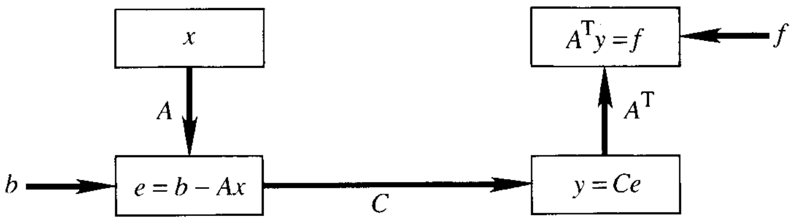
- **Equation to be solved: (or substitute $y=C(b-Ax)$ into $A^Ty=f$)**

$$A^TCAx = A^TCb - f$$

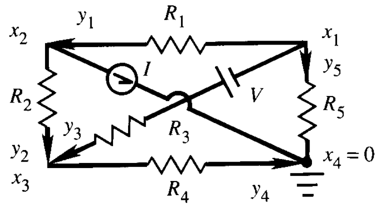
- **Ground the nth node: set potential of node n (x_n)=0**

$\Rightarrow$ A is an $m\times(n\text{-}1)$ matrix $\Rightarrow$

$$\overset{n-1\times m}{\begin{bmatrix} \\ A^T \\ \\ \end{bmatrix}}\overset{m\times m}{\begin{bmatrix} \\ C \\ \\ \end{bmatrix}}\overset{m\times n-1}{\begin{bmatrix} \\ A \\ \\ \end{bmatrix}} = \overset{n-1\times n-1}{\begin{bmatrix} \\ A^TCA \\ \\ \end{bmatrix}}$$



Example of Network - Circuit



- **Current law: $A^Ty=f$**

$$\begin{aligned} -y_1-y_3-y_5&=0 \\ y_1-y_2&=I \\ y_2+y_3-y_4&=0 \end{aligned} \quad \text{has } A^T = \begin{bmatrix} -1 & 0 & -1 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix} \text{ and } f = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}$$

- **Together with Ohm's law: $C^{-1}y+Ax=b$:**

$$\begin{bmatrix} C^{-1} & A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix} = \left[\begin{array}{ccccc|ccc} \multicolumn{5}{c|}{C^{-1}} & \multicolumn{3}{c}{A} \\ R_1 & & & & & -1 & 1 & 0 \\ & R_2 & & & & 0 & -1 & 1 \\ & & R_3 & & & -1 & 0 & 1 \\ & & & R_4 & & 0 & 0 & -1 \\ & & & & R_5 & -1 & 0 & 0 \\ \hline -1 & 0 & -1 & 0 & -1 & & & \\ 1 & -1 & 0 & 0 & 0 & & & \\ 0 & 1 & 1 & -1 & 0 & & & \end{array}\right]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{array}{l} \boxed{\begin{matrix} 0 \\ 0 \\ V \\ 0 \\ 0 \end{matrix}}\; \boldsymbol{b} \\ \boxed{\begin{matrix} 0 \\ I \\ 0 \end{matrix}}\; \boldsymbol{f} \end{array}$$

- **A^TCA:**

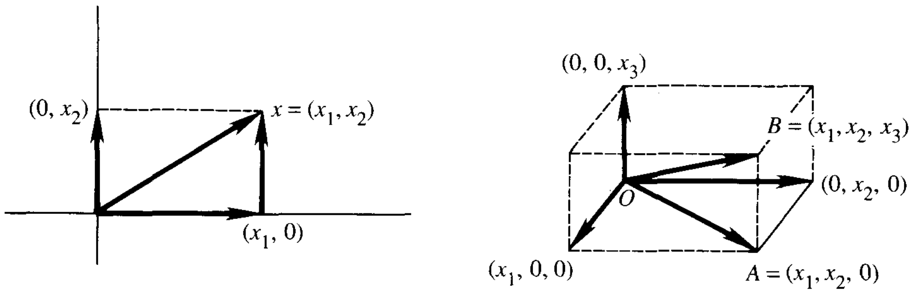
$$A^TCA = \begin{array}{ccccl} c_1+c_3+c_5 & -c_1 & -c_3 & -c_5 & \text{(node1)} \\ -c_1 & c_1+c_2 & -c_2 & 0 & \text{(node 2)} \\ -c_3 & -c_2 & c_2+c_3+c_4 & -c_4 & \text{(node 3)} \\ -c_5 & 0 & -c_5 & c_4+c_5 & \text{(node 4)} \end{array}$$

Note: the 4th node is grounded so as 4th row and column.

- **A^TCA: invertible, symmetric and positive definite**

Length of a Vector

- **Length of vector x: $\|x\|$ Ex: two-dimensional $\|x\|^2 = x_1^2 + x_2^2$**



- **Pythagoras formula:** $\|x\|^2 = \overline{OA}^2 + \overline{AB}^2 = x_1^2 + x_2^2 + x_3^2$
- **Length of a vector** $x = (x_1, \ldots, x_n)$**:**

$$\|x\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2 = x^T x$$

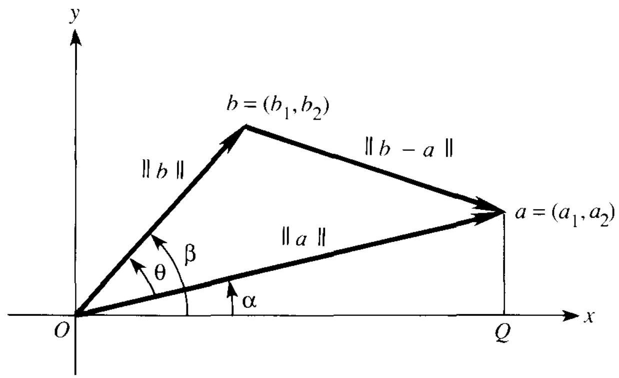
Example: x=(1, 2, -3) $\|x\|^2 = x^T x = 14$; $\|x\| = \sqrt{14}$

Inner Products and Angles

- **$x^T y$ = inner product of x and y =**

$$\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

- **Inner product is directly related to the cosine of the angle.**



$$\sin\alpha = \frac{a_2}{\|a\|}, \quad \cos\alpha = \frac{a_1}{\|a\|} \quad \sin\beta = \frac{b_2}{\|b\|} \quad \textbf{and} \quad \cos\beta = \frac{b_1}{\|b\|}$$

$$\Rightarrow \cos\theta = \cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha = \frac{a_1 b_1 + a_2 b_2}{\|a\|\|b\|}$$

$$\Rightarrow \cos\theta = \frac{a^T b}{\|a\|\|b\|} \Rightarrow a^T b = \|a\|\|b\|\cos\theta$$

- **Using another law of trigonometry:**

$$\|b - a\|^2 = (b-a)^T (b-a) = b^T b + a^T a - 2a^T b = b^T b + a^T a - 2\|b\|\|a\|\cos\theta$$

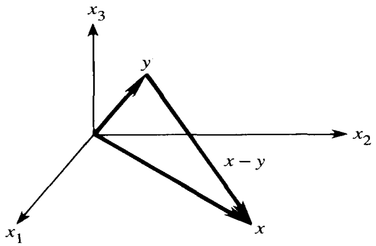
$$\|b - a\|^2 = \|b\|^2 + \|a\|^2 - 2\|b\|\|a\|\cos\theta$$

When θ is 90° $\Rightarrow$ Pythagoras formula $\|b - a\|^2 = \|b\|^2 + \|a\|^2$

Orthogonal (Perpendicular) Vectors

- **Pythagoras formula again:**

  ***x* and *y* are perpendicular if** $\|x\|^2 + \|y\|^2 = \|x-y\|^2$



- **Substituting the length formula:**

$$(x_1^2 + \cdots + x_n^2) + (y_1^2 + \cdots + y_n^2) = (x_1 - y_1)^2 + \cdots (x_n - y_n)^2$$
$$= (x_1^2 + \cdots + x_n^2) + (y_1^2 + \cdots + y_n^2) - 2(x_1 y_1 + \cdots + x_n y_n)$$

$\Rightarrow$ $x_1 y_1 + \cdots + x_n y_n = x^T y = 0$ ***if x and y are orthogonal***

- **The only vector orthogonal to itself ($x^T x$=0) is zero vector**
- ***If the nonzero vectors v_1, …, v_k are mutually orthogonal then they are linearly independent.***

Proof: **Suppose** $c_1 v_1 + \cdots + c_k v_k = 0 \Rightarrow$

$v_1^T (c_1 v_1 + \cdots + c_k v_k) = c_1 v_1^T v_1 = 0 \Rightarrow$ **c_1=0, same for every c_i**

Orthogonal Subspaces

- **Two subspaces *V* and *W* of the same space R^n are *orthogonal* if every vector *v* in *V* is orthogonal to every vector *w* in *W*: $v^T w$=0 for all *v* and *w*.**

Example: *V* is a plane (2-dimension) spanned by $v_1=(1, 0, 0, 0)^T$ and $v_2=(1, 1, 0, 0)^T$ and *W* is a line (1-dimension) spanned by $w=(0, 0, 4, 5)^T$. There is room for a third subspace (4-2-1=1 dimension) *L* spanned by $z=(0, 0, 5, -4)^T$ perpendicular to both *V* and *W*

- **A line can be orthogonal to another line or to a plane, BUT *a plane cannot be orthogonal to a plane* in R^3**
- ***The row space is orthogonal to the nullspace (in R^n) and the column space is orthogonal to the left nullspace (in R^m)***

1st *Proof*: $Ax = \begin{bmatrix} \cdots & \text{row } 1 & \cdots \\ \cdots & \vdots & \cdots \\ \cdots & \text{row } m & \cdots \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

$\Rightarrow$ *x* is orthogonal to every row in *A* and thus to every combination of the rows. $\Rightarrow$ $\aleph(A) \perp \Re(A^T)$

2nd *Proof*:

***x* in** $\aleph(A)$ **and *v* in** $\Re(A^T) \Rightarrow v^T x = (A^T z)^T x = z^T Ax = z^T 0 = 0$

Orthogonal Complement of Subspaces

Example: $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{bmatrix}$

row space spanned by (1, 3) and nullspace contains (-3, 1)

⇒ row space and nullspace together fill up R^2

column space spanned by (1, 2, 3) and left nullspace must be a plane perpendicular to the line: $y_1+2y_2+3y_3=0$ ($y^TA=0$)

⇒ column space and left nullspace together fill up R^3

- **Definition: Given a subspace *V* of R^n, the space formed by all vectors orthogonal to *V* is called the orthogonal complement of *V* and denoted by $V^\perp$**
- ***Fundamental Theorem of Linear Algebra, Part 2***

  $\aleph(A) = (\Re(A^T))^\perp$ **since $r+(n-r)=n$**

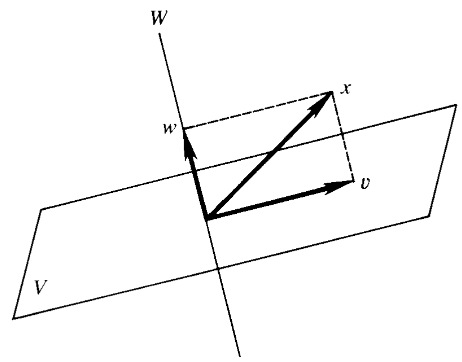
  $\aleph(A^T) = (\Re(A))^\perp$ **since $r+(m-r)=m$**

⇒ For $Ax=b$ to be solvable: *b* must be in the column space and *b must be perpendicular to the left nullspace*

- ***<u>The equation Ax=b is solvable if and only if $b^Ty=0$, where $A^Ty=0$ for all y</u>***
- ***V* and *W* can be orthogonal without being complement**

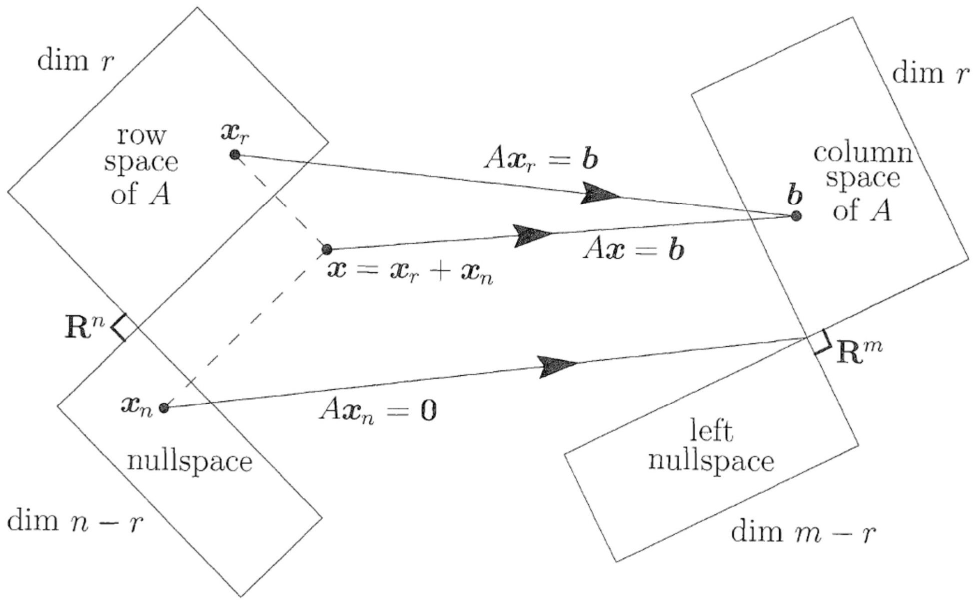
Interpretation of Fundamental Subspaces

- **Orthogonal complements in R^3: a plane and a line**



- **If $W=V^\perp$ then $V=W^\perp$; $V^{\perp\perp}=V$**
- **Every vector *x* in a space can be split into *v* and *w* ($x=v+w$), where *v* and *w* are vectors in *V* and *W* and $V=W^\perp$**

⇒ $x=x_r+x_n$; $Ax=Ax_r+Ax_n$, where $Ax_n=0$ and $Ax_r=Ax$



Mapping from Row Space to Column Space

- ***The mapping from row space to column space is actually invertible. Every vector b in the column space comes from one and only one vector x_r in the row space***

Proof: Let x_r' **be another vector** $Ax_r' = b$**. Then** $A(x_r' - x_r) = b - b = 0 \Rightarrow x_r' - x_r$ **is in** $\aleph(A)$ **and** $\Re(A^T) \Rightarrow$ $x_r' - x_r$ **is orthogonal to itself** $\Rightarrow$ $x_r' - x_r$ **is zero vector (=0)**

- **Matrix *A* "*transforms*" *x* to its column space**
- $A^T(Ax)=A^Tb \Rightarrow$ **Space R**^{m} **→Space R**^{n}**. But** A^T **is not to recover original vector *x* in R**^{n}**.**
- **Zero vectors in R**^{m} **cannot be recovered to a nonzero vector in R**^{n}**!**
- **If** A^{-1} **exists then it can recover all the vectors** $A^{-1}(Ax)=x=A^{-1}b$ **since dim(nullspace)=0**
- **When** A^{-1} **fails to exist, *pseudoinverse*** A^+**:** $A^+Ax=x_r$ **for *x* in the row space and not for *x* in nullspace.**
- A^+ **is to invert *A* where it is invertible (column space *Ax*). For the left nullspace,** A^+ **cannot do anything (**$A^+y=0$**).**