Vector Space

- **Space R^n consists of all column vectors with n components.**

  Ex: R^2 : x-y plane

- **A real vector space is a set of vectors together with 8 rules for vector *addition* and *scalar multiplication*. <u>A vector produced by addition and scalar multiplication must be within the space.</u>**

- **8 rules to be satisfied:**

  1. $x+y=y+x$

  2. $x+(y+z)=(x+y)+z$

  3. $x+0=x$ ***for all x, where 0 is a unique zero vector***

  4. ***For each x, there exists a unique –x such that*** $x+(-x)=0$

  5. $1x=x$

  6. $(c_1c_2)x=c_1(c_2x)$

  7. $c(x+y)=cx+cy$

  8. $(c_1+c_2)x=c_1x+c_2x$

Vector Subspace

- **Any plane that contains the origin in the R^3 space is itself a space. Why? This plane is a *subspace inside* the *original* space R^3**

Definition:

A subspace of a vector space is a nonempty subset that satisfies:

(i) <u>x and y are in the subspace. Then, $x+y$ is in the subspace</u>

(ii) <u>x is in the subspace. Then, cx is in the subspace</u>

- **A subspace is *closed* under addition and scalar multiplication**
- ***Zero vector must be contained in every subspace*: rule (ii) with scalar c=0.**
- **The smallest possible vector space: *zero vector* (zero-dimensional space)**
- **The largest possible: the original space.**

Example: Is the first quadrant ($x\geq 0$, $y\geq 0$) a subspace?

Example: Sets of lower triangular and symmetric matrices

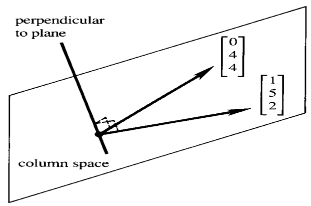
Vector Subspace and Column Space of *A*

$$\begin{bmatrix}1 & 0\\5 & 4\\2 & 4\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

- **3 equations and 2 unknowns ($m>n$): usually no solution**
- **The system $Ax=b$ is solvable if and only if the vector b can be expressed as a combination of the columns of A.**

$$u\begin{bmatrix}1\\5\\2\end{bmatrix}+v\begin{bmatrix}0\\4\\4\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

- **<u>The set of all combination of the columns of *A*: the *column space* of *A* denoted by $\mathfrak{R}$(A)</u> (the plane spanned by the two columns in our example)**



The equation $Ax=b$ can be solved if and only if b lies in $\mathfrak{R}(A)$.

For an m by n matrix A this will be a subspace of R^m since the columns have m components.

<u>Rule (i): $b=Ax$ and $b'=Ax'$ then $b+b'=A(x+x')$</u>

<u>Rule (ii): $b=Ax$ then $cb=A(cx)$</u>

Nullspace of *A*

- **The nullspace of a matrix consists of all vectors x such that Ax=0. It is denoted by $\mathcal{N}(A)$. It is a subspace of R^n, just as the column space was a subspace of R^m.**
- **Requirement <u>(i): If Ax=0 and Ax'=0 then $A(x+x')$=0</u>**
- **Requirement <u>(ii): If Ax=0 then $A(cx)$=0</u>**
- **When $Ax=b$ and $b\neq 0$, vectors x cannot form a subspace. (why?)**
- **Ax=0 is called *homogeneous* equation.**

$$\begin{bmatrix}1 & 0\\5 & 4\\2 & 4\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

- **Only $u=v=0$, that is, zero vector space is the nullspace**

$$B=\begin{bmatrix}1 & 0 & 1\\5 & 4 & 9\\2 & 4 & 6\end{bmatrix}$$

- **B has the same column space as A, but it has the following nullspace:**

$$\begin{bmatrix}1 & 0 & 1\\5 & 4 & 9\\2 & 4 & 6\end{bmatrix}\begin{bmatrix}c\\c\\-c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

where c ranges from $-\infty$ to ∞.

Note: the nullspace is a line passing through (0, 0, 0).

Elimination on a *m* by *n* Matrix *A*

ax=b

Nonsingular: $a \neq 0$, $x = b/a$, **unique. (Ex. 3*x*=4)**

Undetermined: *a*=0 and *b*=0, infinitely many solutions

Inconsistent: *a*=0 and $b \neq 0$, **no solution**

- **For *Ax=b* with square matrices, *a*≠0 ≡ *A* is invertible.**
- **For rectangular matrices, existence with uniqueness is impossible.**

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \textit{Gaussian Elimination} \Rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

- ***Echelon Form*:**

$$U = \begin{bmatrix} \circledast & * & * & * & * & * & * & * & * \\ 0 & \circledast & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \circledast & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \circledast \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **To any *A* (*m* by *n*) with a corresponding permutation matrix *P*, *PA=LU*, where *L* is *m* by *m* with unit diagonal and *U* is a *m* by *n* echelon matrix.**

Nullspace: *Ax*=0 ≡ *Ux*=0

- $Ax=0 \Rightarrow L^{-1}Ax=0 \Rightarrow Ux=0$; $Ux=0 \Rightarrow LUx=0 \Rightarrow Ax=0$

$$Ux = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} u + 3v + 3w + 2y = 0 \\ 3w + 3y = 0 \end{matrix} \rightarrow \begin{matrix} u = \text{-}3v\text{-}3w\text{-}2y \\ w = \quad\quad \text{-}y \end{matrix} \rightarrow \begin{matrix} u = \text{-}3v + y \\ w = \quad \text{-}y \end{matrix}$$

$$\rightarrow \begin{matrix} u = & \text{-}3v + y \\ v = & v \\ w = & \text{-}y \\ y = & y \end{matrix} \Rightarrow x = \begin{bmatrix} -3v + y \\ v \\ -y \\ y \end{bmatrix} = v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

- **The result can be obtained exactly by back-substitution.**
- ***Pivot variables*: *u* and *w* corresponding to pivots 1 and 3.**
- ***Free variables*: *v* and *y* corresponding to zero pivots**
- **Nullspace of *A* is a two-dimensional subspace in** R^4
- ***If a homogeneous system Ax=0 has more unknowns than equations (n>m), it has a nontrivial solution: There is a solution x other than the trivial solution x=0.***
- **If *n>m*, number of free variables ≥ *n–m*. (*m* ≥ # of pivots) The nullspace dimension = no. of free variables ≥ *n–m*.**

Complete Solution of *Ax=b*

When $b \neq 0$, $Ux = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{bmatrix}$

- **If the system is solvable, $b_3 - 2b_2 + 5b_1$ must be 0, i.e. the last equation can be omitted, and the system becomes a 2 by 4 system (2 equations and 4 unknowns)**
- **By columns, b must lie in the plane by columns of A and this plane is (b_1, b_2, b_3) satisfying $5b_1 - 2b_2 + b_3 = 0$ or the plane with a perpendicular vector (5, -2, 1), geometrically.**
- **Let b=(1, 5, 5),**

$$Ux = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \rightarrow \begin{array}{l} u = 1 - 3v - 3w - 2y \\ w = 1 \qquad\quad -y \end{array} \rightarrow \begin{array}{l} u = -2 - 3v + y \\ w = 1 \qquad\quad -y \end{array}$$

$$\rightarrow \begin{array}{ll} u = -2 - 3v + y & \\ v = & v \\ w = 1 & -y \\ y = & y \end{array} \Rightarrow x = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

- $x_{complete} = x_{particular} + x_{nullspace}$ **where $x_{particular}$ can be found by setting all free variables to be zero.**

Ax=b ⇒ *Ux=c* back-substitution⇒ *Rx=d*

$$U = \begin{bmatrix} \bullet & * & * & * & * & * & * & * & * \\ 0 & \bullet & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow R = \begin{bmatrix} 1 & 0 & * & 0 & * & * & * & * & 0 \\ 0 & 1 & * & 0 & * & * & * & * & 0 \\ 0 & 0 & 0 & 1 & * & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Example:**

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

- **Nullspace: *Ax*=0≡*Ux*=0≡*Rx*=0**

$$Rx = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{array}{r} u + 3v - y = 0 \\ w + y = 0 \end{array} \rightarrow \begin{array}{l} u = -3v + y \\ w = \qquad -y \end{array}$$

- ***Ux=c*⇒ *Rx=d*:**

$$Rx = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = d = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \rightarrow \begin{array}{r} u + 3v - y = -2 \\ w + y = 1 \end{array} \rightarrow \begin{array}{l} u = -2 - 3v + y \\ w = 1 \qquad -y \end{array}$$

$$\begin{array}{ll} u = -2 - 3v + y & \\ v = & v \\ w = 1 & -y \\ y = & y \end{array} \Rightarrow x = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Complete Solution of *Ax=b*: Summary

- ***If there are r pivots, there are r pivot variables and n−r free variables.***
- **The number of pivots *r* is also called the *rank* of the $m \times n$ matrix *A*.**

Summary: *Ax=b* (*A* is $m \times n$) reduced to *Ux=c* and *Rx=d*

1. **If *r<m*, last *m−r* rows of *U* are zero and the last *m−r* components of *c* must be zero for the system to be solvable.**
2. **If *r=m*, there is always a solution.**
3. **The complete solution is the sum of a particular solution (with all free variables zero) and a nullspace solution (with the *n-r* free variables as independent parameters).**
4. **If *r=n*, there are no free variables and the nullspace contains only *x*=0.**
5. **The number *r* is called the *rank* of the matrix *A***

- **If *r=n*, the only solution is $x_{particular}$.**
- **If *r=m*, no constraints on *b* and the column space is R^m.**

Linear Independence

- **The rank *r* counts the number of *linearly independent* rows in matrix *A*.**
- ***If only the trivial combination gives zero, so that***

$c_1 v_1 + \cdots + c_k v_k = 0$ ***only happens when*** $c_1 = c_2 = \cdots = c_k = 0$***,***

then the vectors $v_1,\ldots,v_k$ are linearly independent. Otherwise they are linearly dependent and <u>one of them is a linear combination of the others</u>.

- **A random choice of three vectors in R^3, without any special accident, should produce linear independence.**
- **Columns of the triangular matrix must be linearly independent.**

Example: $A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \to c_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By columns, the c_3 must first be zero, then c_2=0, then c_1=0.

- **The *r* nonzero row of an echelon matrix *U* are linearly independent, and so are the *r* columns that contain pivots.**
- ***A set of n vectors in R^m must be linearly dependent if n>m***

Spanning a Subspace by Basis

- **If a vector space V consists of all linear combinations of the particular vectors $w_1, w_2, \ldots, w_l$, then these vectors span the space. In other words, every vector v in V can be expressed as some combination of the w's:**

  $v = c_1 w_1 + \cdots + c_l w_l$ **for some coefficients c_i.**

- **Column space of A = the space spanned by the columns**
- **$e_1, e_2, \ldots, e_n$ are not the only vectors that span R^n!**
- ***Independence involves the nullspace of A, and spanning involves the column space of A.***
- **A *basis* for a vector space is a set of vectors: (1) *linearly independent* (2) *spanning the space*.**
- **If $v = a_1 v_1 + \cdots + a_k v_k$ and $v = b_1 v_1 + \cdots + b_k v_k$, then $0 = (a_1 - b_1)v_1 + \cdots + (a_k - b_k)v_k$. Every coeff. $a_i - b_i$ must be zero due to independence. A vector can uniquely expressed by a linear combination of a basis.**
- **A vector space has *infinitely many bases.***
- ***For U, the columns that contain pivots are a basis for the column space of U. But it is not the column space of A.***

Dimension of a Vector Space

- **All possible bases contain the same number of vectors. The number of vectors in bases expresses the number of degree of freedom of the space and is called the *dimension* of the space.**

***Proof*: Let $v_1, \cdots, v_m$ and $w_1, \cdots, w_n$, where $m<n$, be the bases for space V. Then, w_j can be expressed by the combination of v_i: $w_j = \sum_{i=1}^{m} a_{ij} v_i$. That is, $W=VA$ where w's are columns of W and v's are columns of V. Since $m<n$, $A_{m\times n} c = 0$ must have nontrivial solutions. This leads to $VAc=0$ or $Wc=0$. Since $c \neq 0$, columns in W are dependent! Contradiction.**

- **The dimension of the space R^n is n.**
- **A basis is a *maximal independent* set and also a *minimal spanning* set of vectors.**
- **"basis of a matrix", "rank of a space", "dimension of a basis" are meaningless in linear algebra.**
- **Now, what is the relationship between the "*dimension of the column space*" and the "*rank of the matrix*"?**

Four Fundamental Subspaces

Two ways of describing subspaces:

1. Space spanned by a given set of vectors (column space)

2. Space subject to a list of constraints (nullspace)

- **Four subspaces of matrix A:**

  1. Column space, $\mathcal{R}(A)$

  2. Nullspace, $\mathcal{N}(A)$

  3. Row space or column space of A^T, $\mathcal{R}(A^T)$

  4. Left nullspace or nullspace of A^T, $\mathcal{N}(A^T)$

- **$\mathcal{N}(A)$ and $\mathcal{R}(A^T)$ are subspaces of R^n**
- **$\mathcal{R}(A)$ and $\mathcal{N}(A^T)$ are subspace of R^m**
- **It is easier to find subspaces of U instead of A.**
- **Problem: connect space for U to spaces for A.**

Row Space of A

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Nonzero rows of U are independent (why?), and its row space has dimension r.**
- ***The row space of A has the same dimension r as the row space of U, and it has the same bases, because <u>the two row spaces are the same</u>.***

***Reason*: The rows in U are just combinations of the original rows in A. (Remember what Gaussian elimination does) And it is those combinations that make up the row space! The row space of U contains nothing new.**

- **Dimension of row space for A = rank of $A = r$**
- **$m-r$ rows should be discarded from A. But it is easier to discard rows in U than in A.**
- **For row space, we don't work with A^T. We work with the rows of A.**

Nullspace of A

- **Elimination process is to simplify the equations without changing any of the solutions even if b=0.**
- **Nullspace of A = solution space of Ax=0**

  = solution space of Ux=0 = nullspace of U
- **Dimension of nullspace for A = n–r = no. of free variables**
  - **Free variables are variables corresponding to the columns of U that do not contain pivots.**
  - **We give to each free variable the value 1, to the other free variables the value 0, and solve Ux=0.**
  - **We can therefore find n–r vectors. The solution space is then form by the combinations of these n–r vectors.**
  - **These n-r vectors are the basis for $\mathcal{N}(A)$**

$$Ux = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad x = \begin{bmatrix} -3v + y \\ v \\ -y \\ y \end{bmatrix} = v \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

- **The nullspace is also called the *kernel* of A.**
- **Its dimension n–r is called the *nullity*.**

Column Space of A

- **<u>Column space of A: *range* of A.</u> x is in the domain and $f(x)$ is in the range. Here, $f(x)$=Ax. The range of A = $\Re(A)$**
- **<u>Range is the collection of all combinations of columns</u>.**
- **Problem: $\Re(U) \neq \Re(A)$ but how to derive $\Re(A)$ from $\Re(U)$**
- **<u>Dimension of $\Re(U)$=Dimension of $\Re(A)$</u>**
- **<u>Basis of $\Re(U)$ = columns with pivots → the corresponding columns in A = Basis of $\Re(A)$</u>**

***Reason*: Solutions of Ux=0 ≡ Solutions of Ax=0**

If $x = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$ **is the solution then** $[u_1] - [u_3] + [u_4] = 0$ **and**

$[a_1] - [a_3] + [a_4] = 0$

<u>These are the dependence relationships among columns</u>.

<u>Dependence of columns of U ≡ Dependence of columns of A</u>

- **<u>Independent columns of U ⇔ corresponding independent columns of A</u>**

Row Rank = Column Rank

- **No. of independent columns = no. of independent rows**

$$U = \begin{bmatrix} d_1 & * & * & * & * & * \\ 0 & 0 & 0 & d_2 & * & * \\ 0 & 0 & 0 & 0 & 0 & d_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- **Rank = $r \Rightarrow m-r$ rows are zero rows $\Rightarrow$ only r nonzero components in columns $\Rightarrow$ only r columns are indep.**

$$c_1\begin{bmatrix} d_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} * \\ d_2 \\ 0 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} * \\ * \\ d_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow c_3,\ c_2 \text{ and } c_1 \text{ must be zero}$$

The three columns with pivots must be independent.

- ***$Ux=0$ if and only if $Ax=0$*** **$\Rightarrow$ The corresponding columns in A are also the basis for $\mathcal{R}(A)$**
- **For both $\mathcal{R}(A^T)$ and $\mathcal{R}(A)$, we only work with A and perform elimination on A.**
- **$\mathcal{R}(A^T)$ and $\mathcal{R}(A)$ have the same dimension and can be found at the same time from U.**

Left Nullspace of A

- **$\mathcal{N}(A^T)$ → nullspace of A^T → $A^Ty=0$ → $y^TA=0$ → left nullspace of A**

$$y^T A = \begin{bmatrix} y_1 & \cdots & y_m \end{bmatrix} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}$$

- **y^T is an operation performed on rows of A to give zeros**
- **Dimension of column space + dimension of nullspace = number of columns**
- **For A^T, there are m columns:**

  Dimension of $\mathcal{N}(A^T)$ + dimension of $\mathcal{R}(A^T)$ = $m \Rightarrow$
- **Dimension of $\mathcal{N}(A^T)$ = $m-r$**
- **Find y?**

  1. $PA=LU$ 2. $L^{-1}PA=U$

  3. The last m-r rows of $L^{-1}P$ multiply A to give m-r zero rows in U. These last m-r rows of $L^{-1}P$ are the basis for $\mathcal{N}(A^T)$

$$L^{-1}PAx = Ux = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = L^{-1}Pb = \begin{bmatrix} b_1 \\ b_2 - 2b_2 \\ b_3 - 2b_2 + 5b_1 \end{bmatrix} \rightarrow \begin{matrix} m - r = 3 - 2 = 1 \\ y = \begin{bmatrix} 5 & -2 & 1 \end{bmatrix} \end{matrix}$$

Summary of Subspaces

Fundamental Theorem of Linear Algebra, Part I

1. $\mathcal{R}(A)$ **= column space of** A**; dimension** r

2. $\mathcal{N}(A)$ **= nullspace of** A**; dimension** $n-r$

3. $\mathcal{R}(A^T)$ **= row space of** A**; dimension** r

4. $\mathcal{N}(A^T)$ **= left nullspace of** A**; dimension** $m-r$

Example:

$A=\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}$, $m=n=2$, $r=1$ →

$$L^{-1}A=\begin{bmatrix}1 & 0\\ -3 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 6\end{bmatrix}=\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}=U$$

1. column space: $c\begin{bmatrix}1\\ 3\end{bmatrix}$ **2. nullspace:** $c\begin{bmatrix}-2\\ 1\end{bmatrix}$

3. row space: $c\begin{bmatrix}1\\ 2\end{bmatrix}$ **4. left nullspace:** $c\begin{bmatrix}-3\\ 1\end{bmatrix}$

Existence of Inverses

- **If** A **has a left inverse and a right inverse, then**

$$B=BI=B(AC)=(BA)C=IC=C$$

- ***An inverse exists only when the rank is as large as possible***
- **What we like to explain (prove):**

  1. $r=m$ $(m\le n)$ → **a right-inverse exist** → $A_{m\times n}C_{n\times m}=I_{m\times m}$

  There exists *at least* one solution for $Ax=b$

  2. $r=n$ $(n\le m)$ → **a left-inverse exist** → $B_{n\times m}A_{m\times n}=I_{n\times n}$

  If there exist solution for $Ax=b$**, the solution is unique**

- **One proof:**

  Let $B=(A^TA)^{-1}A^T$ **and** $C=A^T(AA^T)^{-1}$**.**

  We will prove $(A^TA)^{-1}$ **exists if the rank=**n **and** $(AA^T)^{-1}$ **exists if the rank=**m **in Chapter 3.**

- **Another approach:**

$$AC=I \quad \text{or} \quad A\begin{bmatrix}x_1 & x_2 & \cdots & x_m\end{bmatrix}=\begin{bmatrix}e_1 & e_2 & \cdots & e_m\end{bmatrix}$$

Look at every $Ax_i=e_i$ **: To have solutions** x_i**'s, all** e_i**'s must be in the column space of** A**. But** $e_1, e_2, \ldots, e_m$ **fill up the entire** R^m **space. That is, column space of** A **must fill up the space of** R^m →$r=m$**.**

Existence of Inverses – Example 1

Example 1: $A=\begin{bmatrix}4&0&0\\0&5&0\end{bmatrix}$ → ***r*=*m*=2**

$$AC=\begin{bmatrix}4&0&0\\0&5&0\end{bmatrix}\begin{bmatrix}1/4&0\\0&1/5\\c_{31}&c_{32}\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

where c_{31} and c_{32} can be chosen arbitrarily.

→ There are many right-inverses!

- **For solution of $Ax=b$: Substitute $x=Cb \Rightarrow Ax=ACb=Ib=b$**

$$x=Cb=\begin{bmatrix}1/4&0\\0&1/5\\c_{31}&c_{32}\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}=\begin{bmatrix}b_1/4\\b_2/5\\c_{31}b_1+c_{32}b_2\end{bmatrix}$$

Solutions exist but are not unique

$$\boldsymbol{C=A^T(AA^T)^{-1}}=\begin{bmatrix}\mathbf{4}&\mathbf{0}\\\mathbf{0}&\mathbf{5}\\\mathbf{0}&\mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{1/16}&\mathbf{0}\\\mathbf{0}&\mathbf{1/25}\end{bmatrix}=\begin{bmatrix}\mathbf{1/4}&\mathbf{0}\\\mathbf{0}&\mathbf{1/5}\\\mathbf{0}&\mathbf{0}\end{bmatrix}$$

- **In this formula, $c_{31}=c_{32}=0$ → *pseudoinverse***

Existence of Inverses – Example 2

Example 2: $A=\begin{bmatrix}4&0\\0&5\\0&0\end{bmatrix}$ → ***r*=*n*=2**

$$BA=\begin{bmatrix}1/4&0&\beta_{13}\\0&1/5&\beta_{23}\end{bmatrix}\begin{bmatrix}4&0\\0&5\\0&0\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

where β_{13} and β_{23} can be chosen arbitrarily.

→ many left-inverses

- **For $Ax=b$,** $Ax=\begin{bmatrix}4&0\\0&5\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$

  → solvable only if $\underline{b_3=0}$ → $\begin{bmatrix}4&0\\0&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}b_1\\b_2\end{bmatrix}$

- **Solution: $BAx=Bb \Rightarrow Ix=Bb \Rightarrow x=Bb$**

$$x=Bb=\begin{bmatrix}1/4&0&\beta_{13}\\0&1/5&\beta_{23}\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=\begin{bmatrix}\frac{1}{4}b_1+\beta_{13}b_3\\\frac{1}{5}b_2+\beta_{23}b_3\end{bmatrix}=\begin{bmatrix}\frac{1}{4}b_1\\\frac{1}{5}b_2\end{bmatrix}$$

→ If the solution exists, it must be unique!

Inverse of Square Matrix A

- ***Existence implies uniqueness and uniqueness implies existence, when the matrix is square → r = m = n***
- **Square matrix A invertible (nonsingular): sufficient and necessary test list**

1. The columns span R^m, so $Ax=b$ has at least on solution for every b.

2. The columns are independent, so $Ax=0$ has only the solution $x=0$.

3. The rows of A span R^n.

4. The rows are linearly independent.

5. Elimination can be completed: $PA=LDU$, with all $d_i \neq 0$

6. There exists a matrix A^{-1} such that $AA^{-1}=A^{-1}A=I$.

7. The determinant of A is not zero.

8. Zero is not an eigenvalue of A.

9. A^TA is positive definite

Vandermonde Matrix

- **For any unknown function $f(t)$, if we can make n observations: $f(t_1)=b_1, f(t_2)=b_2, \ldots, f(t_n)=b_n$, then we can find exactly one polynomial function of degree $n-1$ to fit these observations.**

That is, $\begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ **has only one**

solution $\Rightarrow \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix}$ **must be nonsingular**

(called *Vandermonde's matrix.*)

- **Another perspective: A polynomial $P(t)$ of degree n-1 can have at most n-1 roots ($P(t)=0$). If there exist n points t_1, $t_2, \ldots, t_n$ that make $P(t_i)=0$, then this polynomial must be** $0+0t+0t^2+\cdots+0t^{n-1}$.

If $\begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^{n-1} \\ 1 & t_2 & t_2^2 & \cdots & t_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ **then** $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Matrices of Rank One

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix}$$

$\Rightarrow$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \\ 8 & 4 & 4 \\ -2 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \end{bmatrix}$$

- **For any matrix of rank one:**

  <u>Every matrix of rank one has the simple form $A=uv^T$</u>

- **The rows of *A* are all multiples of the same vector v^T**
- **The columns of *A* are all multiples of the same vector *u***
- **The row space and column space are lines.**