Introduction to Statistical Methods

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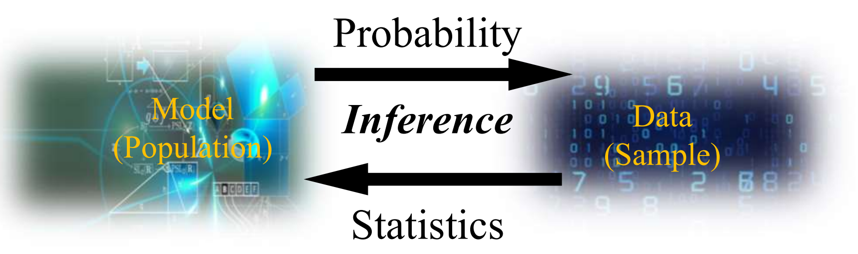
Probability and Statistics



An Example: Taiwan Big Lotto (大樂透)

- 6 winning numbers and 1 special number chosen from 49 numbers
- Mr. Chang chooses numbers randomly and never believes in any historical analysis of number appearance
- Mr. Fang chooses numbers that most frequently appear in the history
- Mr. Wang chooses numbers that most rarely appear in the history
- Mr. Yang chooses meaningful numbers, such as the date of birthday
- **Who is correct?**

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Probability of Taiwan Big Lotto

- 6 numbers chosen from 49 numbers
- Chance of winning the first prize?
- Answer: $1/C_6^{49}$
- Is this based on probability or statistics?
- What is the model and assumption behind the answer above?

  Every number has the ***identical*** probability to be chosen ***independently*** each time! ⇒ ***iid*** assumption
- The event of winning is guessed (inferred) by the "model".

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Statistics of Taiwan Big Lotto

- What is "Statistics" in the Lotto problem?
- Is the model assumption correct?
- What is the real chance of winning? and the chance for each number to appear?
- "Statistics" is to estimate the appearance probability of each number. How?

  by collecting and observing the data.

  $\Rightarrow$ "Model" is **inferred** by the "sample data"

  $\Rightarrow$ Statistical Inference

5

Taiwan Big Lotto (Cont'd)

- Back to Mr. Chang, Fang, and Wang
- Who is correct?
- What does Mr. Chang believe?
- What does Mr. Fang believe?
- What does Mr. Wang believe?
- What does Mr. Yang believe?

6

A note on "Models"

"All models are wrong, but some are useful."

\- *George E. P. Box* (1979)

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Definition of "Experiment"

- An **experiment** is the process by which an observation (or measurement) is obtained.
- **Experiment: Record an age**
- **Experiment: Toss a die**
- **Experiment: Record an opinion (yes, no)**
- **Experiment: Toss two coins**
- **Experiment: numbers of a lotto ticket**

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Definition of "Outcome"

- An outcome is observed on a single repetition of the experiment.
  - Basic element to which probability is applied.
  - One and only one outcome can occur when experiment is performed.
- An outcome is denoted by O with a subscript.

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Definition of "Outcome Space"

- Each outcome is assigned a probability, measuring "how often" it occurs.
- Set of all outcomes of an experiment is called the **outcome space**, usually denoted by **S**.

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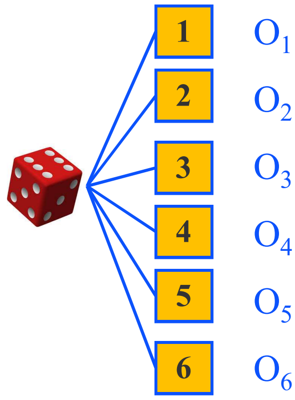
Example of Experiment and Outcome

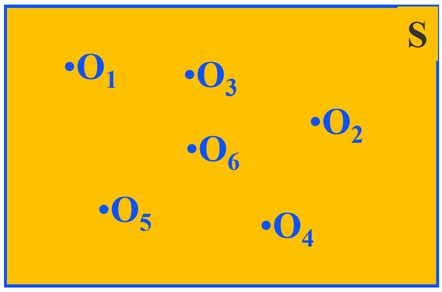
- The die toss experiment:
- Outcomes:



Outcome space:

$S = \{O_1, O_2, O_3, O_4, O_5, O_6\}$

(or $S = \{1, 2, 3, 4, 5, 6\}$)



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Example

- Record a person's blood type:
- Outcomes:

O_1	A
O_2	AB
O_3	B
O_4	O

Outcome space:

$S = \{O_1, O_2, O_3, O_4\}$

$S = \{A, B, AB, O\}$

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Example

- Record the numbers of a lotto ticket:
- Outcomes:

O_1=[1, 2, 3, 4, 5, 6]

O_2=[2, 3, 4, 5, 6, 7]

O_3=[3, 4, 5, 6, 7, 8]

:

:

$O_?$ =[44, 45, 46, 47, 48]

Outcome space:

S ={O_1, O_2, O_3, O_4,…, ………, $O_?$}

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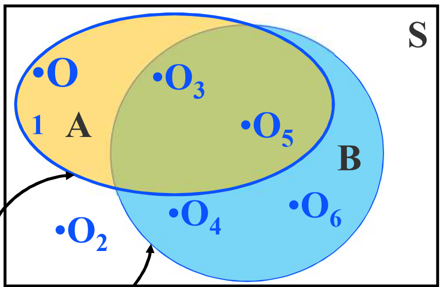
Definition of "Event"

- An **event** is a collection of one or more **outcomes.**

•The die toss events:

- ✓ Event A: an odd number
- ✓ Event B: a number > 2



A ={O_1, O_3, O_5}
B ={O_3, O_4, O_5, O_6}

Venn Diagram

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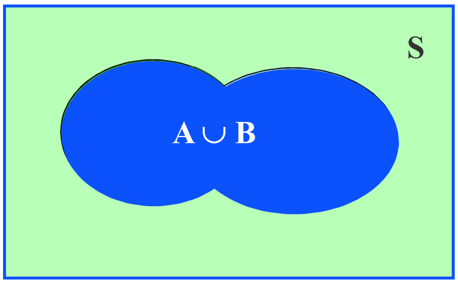
Event Relations - Union

The **union** of two events, A and B, is the event that either A **or** B **or both** occur when the experiment is performed. We write

$A \cup B$



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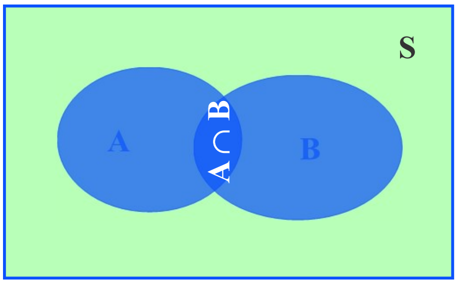
Event Relations-Intersection

The **intersection** of two events, **A** and **B**, is the event that both A **and** B occur when the experiment is performed. We write **$A \cap B$.**



- If two events A and B are **mutually exclusive,** then **$A \cap B = \emptyset$.**

16

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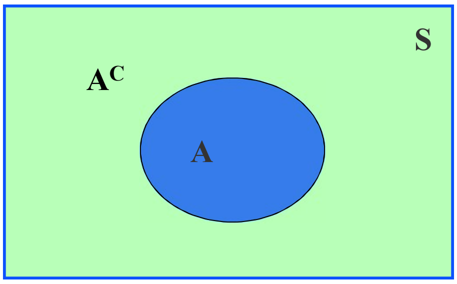
Event Relations - Complement

The **complement** of an event **A** consists of all outcomes of the experiment that do not result in event A. We write $\mathbf{A^C}$ **(The event that event A doesn't occur).**



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Example

Select a student from a college
- **A:** student is colorblind
- **B:** student is biologically female
- **C:** student is biologically male

Mutually exclusive and $B = C^C$

•What is the relationship between events **B** and **C**?

•$\mathbf{A^C}$: Student is not colorblind

•$\mathbf{B \cap C}$: Student is both biologically male and female = $\varnothing$

•$\mathbf{B \cup C}$: Student is either male or female = all students = S

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Factorial and Sequences

Example: How many possible distinct sequences of 5 different numbers among 1, 2, 3, 4 and 5?

The "order" of the choices → $5(4)(3)(2)(1) = 120$

The number of distinct sequences you can arrange ***n*** distinct objects is ***n*** factorial denoted as:

$$n! = n(n-1)(n-2)...(2)(1) \text{ and } 0! \equiv 1.$$

Permutations

Example: How many 3-digit lock passwords can we make by using 3 **different numbers** among 1, 2, 3, 4 and 5?

The order of the choice is important! → $5(4)(3) = 60$

The number of ways you can arrange ***n*** distinct objects, taking them ***r*** at a time is

$$P_r^n = \frac{n!}{(n-r)!}$$

$$P_3^5 = \frac{5!}{(5-3)!} = \frac{5(4)(3)(2)(1)}{2(1)} = 60$$

Combinations

- The number of distinct combinations of ***n*** **distinct objects** that can be formed, **taking them *r* at a time** is $C_r^n = \frac{n!}{(n-r)!}(\frac{1}{r!}) = \frac{n!}{r!(n-r)!}$ since the number of permutations is $r!$ times that of combinations

Example: 3 members of a 5-person faculty must be chosen to form a committee. How many different committees could be formed?

The order of the choice is not important!

$$C_3^5 = \frac{5!}{3!(5-3)!} = \frac{5(4)(3)}{3(2)(1)} = \frac{5(4)}{(2)1} = 10$$

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The Probability of an Event

- ***P***(A) must be between 0 and 1.
  - If event A can never occur, ***P***(A) = 0.
  - If event A always occurs when the experiment is performed, ***P***(A) =1.
- Sum of probabilities for all simple events in S equals 1. ***P***(S)=1.
- **Probability of an event A is found by adding the probabilities of all outcomes contained in A**

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Taiwan Big Lotto

- 6 numbers chosen from 49 numbers
- First prize: all six numbers

Event probability:

$$1/C_6^{49} = \frac{6!(49-6)!}{49!}$$

- Second prize: 5 numbers + 1 special number

Event probability?

$$C_5^6 / C_7^{49}$$

Why? Model assumptions?

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Mutually Exclusive Events

- When A and B have no outcomes in common, they are said to be **mutually exclusive** or **disjoint** events

•Experiment: Toss a die

–A: observe an odd number

–B: observe a number greater than 2

–C: observe a 6

–D: observe a 3

Not Mutually Exclusive

Mutually Exclusive

A and C?
A and D?
B and C?

24 n

Conditional Probability

- For any two events A and B with $P(B)>0$, the **conditional probability** of A given that B has occurred is defined by $P(A|B)=P(A\cap B)/P(B)$

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Independent Events

- Two events A and B are **independent** if $P(A|B)=P(A)$ and are dependent otherwise

- In other words,
  $P(A|B)=P(A\cap B)/P(B) =P(A)$

$\Rightarrow P(A\cap B)=P(A)\ P(B)$ if A and B are independent

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Basic Probability Rules

- Rule 1: Addition rule for **mutually exclusive** events $P(A \text{ or } B) = P(A) + P(B)$
- Rule 2: Multiplication rule for **independent** events $P(A \text{ and } B) = P(A)P(B)$
- Rule 3: General addition rule

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

- Rule 4: General multiplication rule

$$P(A \text{ and } B) = P(A)P(B|A)$$

27

Sensitivity and Specificity

Test: disease, fire, quality, etc.
Testing performance?

		True Condition		
		Condition positive (CP)	**Condition negative (CN)**	
Predicted Condition	**Test outcome positive (P)**	**True positive** (TP) = 20 {P∩CP}	**False positive** (FP) = 180 {P∩CN}	**Positive predictive value (PPV/Precision)** P(CP|P) = TP / (TP + FP) = 20 / (20 + 180) = **10%**
	Test outcome negative (N)	**False negative** (FN) = 10 {N∩CP}	**True negative** (TN) = 1820 {N∩CN}	**Negative predictive value** (NPV) P(CN|N) = TN / (FN + TN) = 1820 / (10 + 1820)≈ **99.5%**
		Sensitivity (Recall) P(P|CP) = TP / (TP + FN) = 20 / (20 + 10) ≈ **67%**	**Specificity** P(N|CN) = TN / (FP + TN) =1820/(180 +1820) = **91%**	

28

Meanings of Sensitivity and Specificity

Sensitivity (Recall): "I know my patient has the disease. What is the chance that the test will show that my patient has it?“

Specificity: "I know my patient doesn't have the disease. What is the chance that the test will show that my patient doesn't have it?"

Meanings of PPV and NPV

PPV (Precision/Risk): “I just got a positive test result back on my patient. What is the chance (risk) that my patient actually has the disease?”

NPV: “I just got a negative test result back on my patient. What is the chance that my patient actually doesn't have the disease?”

Total Probability and Bayes

- Let $A_1, \ldots, A_n$ be **mutually exclusive** and **exhaustive** events. Then for any other event B: (**law of total probability**)

$$P(B)=\sum_{i=1}^{n}P(B\mid A_i)P(A_i)$$

- **Bayes Thm**: Let $A_1,\ldots,A_n$ be a collection of n **mutually exclusive** and **exhaustive** events with $P(A_i)>0$ for $i=1,\ldots,n$. Then for any other event B for which $P(B)>0$

$$\boxed{P(A_k\mid B)}=\frac{P(A_k\cap B)}{P(B)}=\frac{\boxed{P(B\mid A_k)P(A_k)}}{\sum_{i=1}^{n}\boxed{P(B\mid A_i)P(A_i)}}$$

a priori knowledge

a posteriori inference

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Example 1

It is known that the H1N1 flu, a subtype of influenza type *A* also known as swine flu, prevalence is 10% (the proportion of population infected) in the pandemic period (2009~2010). A testing method has been developed to quickly test if a patient is infected with the influenza virus. This method is called Rapid Influenza Diagnostic Test (RIDT). Only 60% (sensitivity) of patients infected with H1N1 can be tested positive (40% of false negative) and 95% (specificity) of patients without H1N1 infection are tested negative. Only patients tested positive can be treated with Tamiflu administering. If a patient is not treated with Tamiflu administering, what is the probability that the patient is infected with H1N1 without treatment of Tamiflu?

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Example 1 (Cont'd)

- a priori knowledge:
  - A_1: infected with H1N1 $P(A_1)$=10%
  - A_0: without infection with $P(A_0)=1-P(A_1)$
  - B_1: tested positive
  - B_0: tested negative
  - $P(B_1|A_1)$= 0.6 ; $P(B_0|A_0)$=0.95
- a posteriori question: $P(A_1 \mid B_0)$=?

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Complete Testing Evaluation Table

		True condition				
	Total population	Condition positive	Condition negative	Prevalence = $\frac{\Sigma \text{ Condition positive}}{\Sigma \text{ Total population}}$	Accuracy (ACC) = $\frac{\Sigma \text{ True positive} + \Sigma \text{ True negative}}{\Sigma \text{ Total population}}$	
Predicted condition	Predicted condition positive	**True positive**	**False positive**, Type I error	Positive predictive value (PPV), Precision = $\frac{\Sigma \text{ True positive}}{\Sigma \text{ Predicted condition positive}}$	False discovery rate (FDR) = $\frac{\Sigma \text{ False positive}}{\Sigma \text{ Predicted condition positive}}$	
	Predicted condition negative	**False negative**, Type II error	**True negative**	False omission rate (FOR) = $\frac{\Sigma \text{ False negative}}{\Sigma \text{ Predicted condition negative}}$	Negative predictive value (NPV) = $\frac{\Sigma \text{ True negative}}{\Sigma \text{ Predicted condition negative}}$	
		True positive rate (TPR), Recall, Sensitivity, probability of detection, Power = $\frac{\Sigma \text{ True positive}}{\Sigma \text{ Condition positive}}$	False positive rate (FPR), Fall-out, probability of false alarm = $\frac{\Sigma \text{ False positive}}{\Sigma \text{ Condition negative}}$	Positive likelihood ratio (LR+) = $\frac{TPR}{FPR}$	Diagnostic odds ratio (DOR) = $\frac{LR+}{LR-}$	F_1 score = $2 \cdot \frac{\text{Precision} \cdot \text{Recall}}{\text{Precision} + \text{Recall}}$
		False negative rate (FNR), Miss rate = $\frac{\Sigma \text{ False negative}}{\Sigma \text{ Condition positive}}$	Specificity (SPC), Selectivity, True negative rate (TNR) = $\frac{\Sigma \text{ True negative}}{\Sigma \text{ Condition negative}}$	Negative likelihood ratio (LR−) = $\frac{FNR}{TNR}$		

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Example 2

Only one in 1000 chips is defective by not conforming to a certain type of electrical test. The test is performed by a tester that detects the failure 99% of the time when the chip is actually defective. The tester misidentified a good chip only 2% of the time. If a randomly selected chip is tested defective, what is the probability that the chip is actually defective.

Variables with Random Nature

- **Definition:** For a given sample space of some experiment, a **random variable** is **any rule** that associates a number with each outcome in the sample space. A random variable is always denoted by a **capital letter (e.g. *X*, *Y*, etc)**
- Can you think of anything that is not random in nature? Can you think of two things that are exactly the same?

Random Variables

- What are random variables?
  - Life is a never ending experiment
  - **Define the rules of experiment with its outcome as a random variable**
- Types of random variables
  - **discrete** if its set of possible values is a discrete set
  - **continuous** if its set of possible values is an entire interval of numbers
- Examples: defects, c.d., thickness

Probability Distribution

- The **probability distribution** or **probability mass function (p.m.f)** of a **discrete** random variable is defined for every number x by $p(x) = P(X=x)$
- Let X be a continous random variable. Then a probability distribution or probability density function (p.d.f) of X is a function f(x) such that for any two numbers a and b with $a \leqq b$

$$P(a \le X \le b) = \int_a^b f(x)dx$$

Cumulative Distribution Function (c.d.f)

- The **cumulative distribution function (c.d.f)** $F(x)$ of a discrete (continuous) random variable X with p.m.f $p(x)$ (p.d.f. $f(x)$) is defined for every number x by

$$F(x) = P(X \le x) = \sum_{y:y\le x} p(y)$$

$$(= \int_{-\infty}^{x} f(y)dy)$$

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Expected Value (Mean)

- The **expected (or mean value)** of a discrete (continuous) r.v. X with p.m.f $p(x)$ (p.d.f $f(x)$) is a **weighted average** weighted by $p(x)$ or $f(x)$:

$$E(X) = \mu_X = \sum_{x\in D} x \bullet p(x)$$

$$(= \int_{-\infty}^{\infty} x \bullet f(x)dx)$$

where D represents the set of possible values for discrete r.v. X

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Mean of a Function

- If the discrete (continuous) r.v. X has a set of possible values D and p.m.f (p.d.f) $p(x)$ ($f(x)$), then the **expected value** of any function $h(X)$, denoted by $E[h(X)]$ or, is computed by

$$E[h(X)] = \mu_{h(X)} = \sum_{x \in D} h(x) \cdot p(x) (= \int_{\mathrm{D}} h(x) \cdot f(x) dx)$$

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Variance

- The **variance** of a discrete (continuous) r.v. X with p.m.f. $p(x)$ (p.d.f $f(x)$) and mean value μ is

$$\sigma^2 = V(X) = E[(X-\mu)^2] = \sum_{x \in D} (x-\mu)^2 \cdot p(x)$$

$$(= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x) dx)$$

- The standard deviation of X is $\sigma_X = \sqrt{V(X)}$

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Rules of Mean and Variance

- **Rules of Expected Values:**

$$E(aX+b)=aE(X) + b$$

- **Shortcut Formula for $V(X)$:**

$$V(X)=E(X^2) - [E(X)]^2$$

- **Rules of Variance:**

$$V(aX+b)=a^2V(X)$$

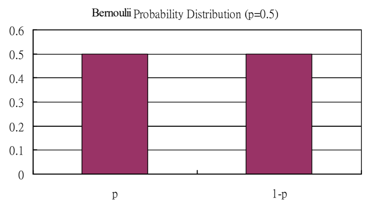
Family of Probability Distributions

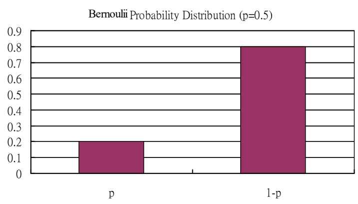
- Suppose that $p(x)$ depends on a quantity that can be assigned any one of a number of possible values, with each different value determining a different probability distribution. Such a quantity is called a **parameter** of the distribution. The collection of all probability distributions for different values of the **parameter** is called a **family** of probability distributions

Discrete Distribution Models

- **Bernoulli** Distribution: tossing coin
  r.v. X = 0 or 1
  P(X=0)=1-p; P(X=1)=p

  One parameter: 0<p<1
  Example: tossing coin to get the head where the probability of head is p





Mean: p Variance: $p(1-p)$

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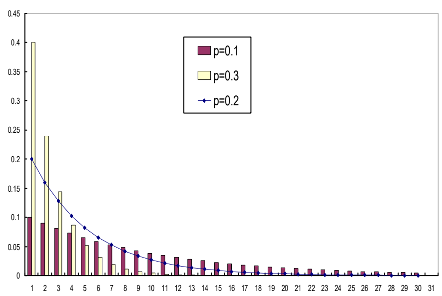
Geometric Distribution Family

- Tossing coin
  r.v. X: number of tosses before first head appears where the probability of head is p

$$P(X = x) = p(1-p)^{x-1}$$

$$P(X < x) = 1-(1-p)^{x-1}; P(X \geq x) = (1-p)^{x-1}$$



Mean: ?
Variance: $(1-p)/p^2$

- Memoryless property: $P(X>m+n|X>n)=P(X>m)$

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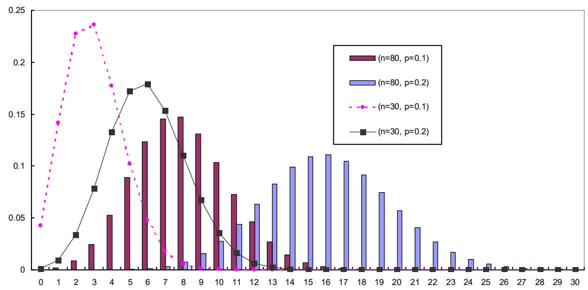
Binomial Distribution Family

- Tossing coin with **different r.v. definition**
  X: number of heads in n tosses

$$b(x; n, p) = \begin{cases} C_x^n p^x (1-p)^{n-x} & \text{x = 0,1,2,...,n} \\ 0 & \text{otherwise} \end{cases}$$



Mean: ? Variance: $np(1-p)$

Derivation of Binomial Mean

$$E(X) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

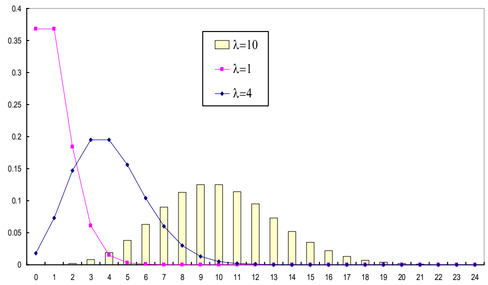
$$= \sum_{x=0}^{n} \frac{n!}{x!(n-x)!} x p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \frac{np(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x} \, let k = x-1$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} \quad = np$$

Poisson Distribution Family

$$p(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad x = 0, 1, 2, \ldots \text{ for some } \lambda > 0$$



- $E(X)=V(X)=\lambda$
- **Poisson ~ binomial with** $n \to \infty$ and $p \to 0$
- Example: defects on a wafer, communication network, car accidents

Derivation of Poisson Mean

$$E(X) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!}$$

$$= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \qquad let\ k = x-1$$

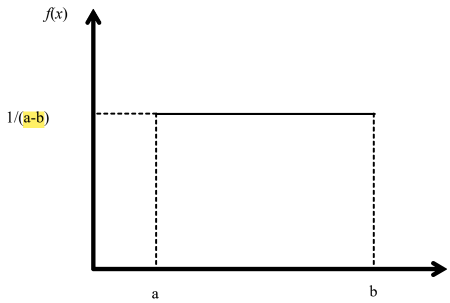
$$= \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{(k)!} = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

Continuous Distribution Models

- Uniform Distribution family:

$$f(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & otherwise \end{cases}$$

$$F(x) = \begin{cases} 0 & x \le a \\ \dfrac{x-a}{b-a} & a < x < b \\ 1 & x \ge b \end{cases}$$



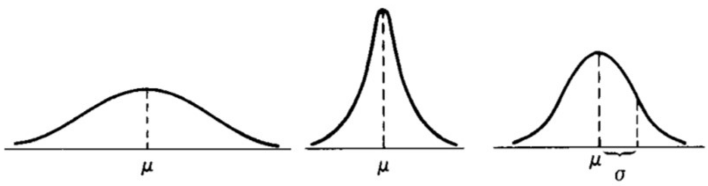
- Mean? Variance?

Normal Distribution Family

$$f(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty < x < \infty$$

$$F(X = x;\mu,\sigma) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-(X-\mu)^2/2\sigma^2} dX \text{ (no close form)}$$

- Mean: μ and Variance: σ^2



Standard Normal Distribution

- Normal distribution with mean 0 and standard deviation 1: $f(z;0,1)$; $F(x)=\Phi(x)$

$$\Phi(x)=F(X=x;0,1)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-X^2/2}dX \text{ (no close form)}$$

- Proposition: If X follows a normal distribution with mean μ and standard deviation σ, then

$$Z=\frac{X-\mu}{\sigma}$$

is a standard normal random variable

- $F(x;\mu,\sigma)=\Phi(z=(x-\mu)/\sigma)$ $\Rightarrow$Table of $\Phi(x)$

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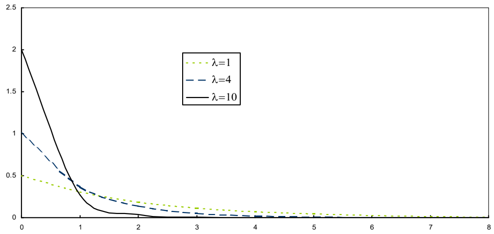
Exponential Distribution Family

$$f(x;\lambda)=\begin{cases}\lambda e^{-\lambda x} & x\ge 0\\ 0 & \text{otherwise}\end{cases} \quad \text{where } \lambda>0$$

- c.d.f. $F(x;\lambda)=\begin{cases}0 & \text{X<0}\\ 1-e^{-\lambda x} & \text{X}\ge 0\end{cases}$; $P(X>x)=e^{-\lambda x}$



mean=$1/\lambda=\sigma$

variance=$1/\lambda^2$

- Memoryless property: $P(X>t+s|X>t)=P(X>s)$

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Gamma Function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \qquad \alpha > 0$$

- For α>1, $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$
- For any positive integer n, $\Gamma(n)=(n-1)!$
- $\Gamma(1/2) = \sqrt{\pi}$

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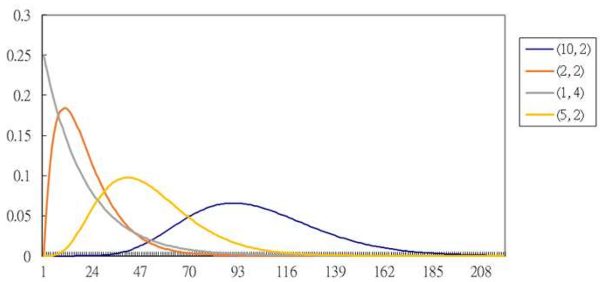
Gamma Distribution Family

$$f_\Gamma(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & \text{x} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- $E(X)=\alpha\beta \quad V(X)=\alpha\beta^2$



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Chi-square (χ^2) Distribution Family

$$f(x;\nu)=\begin{cases}\dfrac{1}{2^{\nu/2}\Gamma(\nu/2)}x^{\nu/2-1}e^{-x/2} & x\geq 0\\ 0 & \text{otherwise}\end{cases}$$

- $\chi^2(\nu) \equiv f_\Gamma(\alpha=\nu/2,\beta=2)$
- $E(X)$=? $V(X)$=?
- **X_i ~ standard normal N(0,1)**
  $\Rightarrow \sum_{i=1}^{\nu} X_i^2 \sim \chi^2(\nu)$

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Discrete Joint Distribution and R.V.

- Let X and Y be two discrete random variables defined defined on a sample space of an experiment. The joint probability mass function $p(x,y)$ is defined for each pair of numbers (x, y) by $p(x, y)=P(X=x$ and $Y=y)$
- For any discrete set A consisting of (x,y) values,

$$P[(X,Y)\in A]=\sum_{(x,y)\in A}\sum p(x,y)$$

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Continuous Joint Distribution and R.V.

- Let X and Y be two continuous random variables. Then $f(x,y)$ is the joint probability density function for X and Y if for any two-dimensional set A

$$P[(X,Y) \in A] = \iint_A f(x,y)dxdy$$

- In particular,

$$P[a \le X \le b, c \le Y \le d] = \int_a^b \int_c^d f(x,y)dxdy$$

Marginal Probability

- Marginal probability mass (density) functions:

$$p_X(x) = \sum_y p(x,y), \qquad p_Y(y) = \sum_x p(x,y)$$

$$(f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx)$$

Independence of Two R.V.'s

- Two discrete (continuous) random variables X and Y are said to be **independent** if for every pair of x and y values,

$$p(x,y) = p_X(x) \cdot p_Y(y)$$

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

- If above is not satisfied for all (x,y), X and Y are said to be **dependent**

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Covariance of Two R.V.'s

- Variance of two discrete (continuous) random variables X and Y is defined:

$$\begin{aligned}\sigma(X,Y) &= \mathrm{E}[(X-\mathrm{E}[X])(Y-\mathrm{E}[Y])] \\ &= \mathrm{E}[XY - X\mathrm{E}[Y] - \mathrm{E}[X]Y + \mathrm{E}[X]\mathrm{E}[Y]] \\ &= \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y] - \mathrm{E}[X]\mathrm{E}[Y] + \mathrm{E}[X]\mathrm{E}[Y] \\ &= \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y]\end{aligned}$$

- Then X and Y are **independent:**

$\mathrm{E}[XY]=\mathrm{E}[X]\mathrm{E}[Y]$ (why?)$\Rightarrow$ $\boldsymbol{\sigma(X, Y)=0}$

- **However**, if $\sigma(X, Y)=0$, X and Y may not be **independent unless they jointly normally distributed.**

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Linear Combination of R.V.'s

- Linear Combination of r.v. X_i:

$$Y = \sum_{i=1}^{n} a_i X_i$$

- For independent X_i's:

$$E(\sum_{1}^{n} a_i X_i) = \sum_{1}^{n} a_i E(X_i)$$

$$V(\sum_{1}^{n} a_i X_i) = \sum_{1}^{n} a_i^2 V(X_i)$$

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Average of Random Variables

- For independent and identically distributed (*iid*) X_i's with μ and σ

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

- Mean and Variance:

$$E(\bar{X}) = \mu \text{ and } V(\bar{X}) = \frac{\sigma^2}{n}$$

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Sum of R.V.'s

- Sum of r.v. X_i:

$$Y = \sum_{i=1}^{n} X_i$$

- For independent X_i's:

$$E(\sum_{1}^{n} X_i) = \sum_{1}^{n} E(X_i)$$

$$V(\sum_{1}^{n} X_i) = \sum_{1}^{n} V(X_i)$$

- How about distribution of Y? Not an easy answer!

Sum of Ind. Geometric RVs

- Sum of independent r.v. X_i: $Y = \sum_{i=1}^{n} X_i$

$$E(\sum_{1}^{n} X_i) = \sum_{1}^{n} E(X_i) = n / p$$

$$V(\sum_{1}^{n} X_i) = \sum_{1}^{n} V(X_i) = n(1-p) / p^2$$

- How about distribution of Y? probability distribution function=?
  - It's a Negative Binomial Distribution!

    r.v. X=number of trials to n^{th} head

$$\boldsymbol{P(X = x) = \binom{x-1}{n-1}(1-p)^{x-n} p^n \quad x = n, n+1, \ldots}$$

Why?

Convolution: Sum of Two Independent R.V.'s

- We already know that for two independent r.v. X and Y

$$E(X+Y) = E(X) + E(Y)$$
$$V(X+Y) = V(X) + V(Y)$$

- How about the distribution of $X+Y$?

  That is, pmf or pdf $X+Y$: $p_{x+y}(X+Y)$ or $f_{x+y}(X+Y)$?
- In general, there is no easy answer for this even given the known distribution of X and Y! In fact, the solution is to solve the convolution of functions:

$$p_{x+y}(X+Y=a) = \sum_{\forall y} p_x(X=a-y)p_y(Y=y)$$
$$f_{x+y}(X+Y=a) = \int_{-\infty}^{\infty} f_x(a-y)f_y(y)dy$$

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Distribution of the Sum of Two Independent R.V.'s

- The answer could be as complicated as solving the convolution of functions. But could be quite simple too…..
- Example:

  r.v. X+Y takes possible values: 2,3,4,5

  $\boldsymbol{p_x(0) = .3 \qquad p_x(1) = .2 \qquad P_x(2) = .5}$

  $\boldsymbol{p_y(2) = .5 \qquad p_y(3) = .5}$

  Convolution:

  $\boldsymbol{p(x+y=2) = p(x=0)p(y=2) = 0.15}$

  $\boldsymbol{p(x+y)=3) = p(x=0)p(y=3) + p(x=1)p(y=2) = 0.25}$

  $\boldsymbol{p(x+y=4) = p(x=1)p(y=3) + p(x=2)p(y=2) = 0.35}$

  $\boldsymbol{p(x+y=5) = p(x=2)p(y=3) = 0.25}$

  $\boldsymbol{or \quad in \quad general \quad p(x+y=5) = \sum_{y=2}^{3} p(x=5-y)p(Y=y)}$

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Distribution of the Sum of Two Independent R.V.'s

- The answer could be as complicated as solving the convolution of functions. But could be quite simple too…..
- Example:

  r.v. X+Y takes possible values: 2,3,4,5

  $p_x(0)=.3 \qquad p_x(1)=.2 \qquad P_x(2)=.5$

  $p_y(2)=.5 \qquad p_y(3)=.5$

  Convolution:

  $$p(x+y=2)=p(x=0)p(y=2)=0.15$$
  $$p(x+y)=3)=p(x=0)p(y=3)+p(x=1)p(y=2)=0.25$$
  $$p(x+y=4)=p(x=1)p(y=3)+p(x=2)p(y=2)=0.35$$
  $$p(x+y=5)=p(x=2)p(y=3)=0.25$$

  $$or \quad in \quad general \quad p(x+y=5)=\sum_{y=2}^{3} p(x=5-y)p(Y=y)$$

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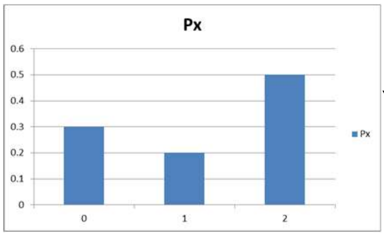
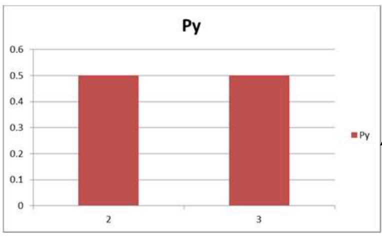
More on Sum of RV's

$p_x(0)=.3\ p_x(1)=.2\ p_x(2)=.5$

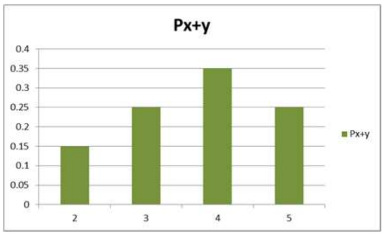


$p_y(2)=.5\ p_y(3)=.5$



$p_{x+y}(2)=0.15 \quad p_{x+y}(3)=0.25$

$p_{x+y}(4)=0.35 \quad p_{x+y}(5)=0.25$



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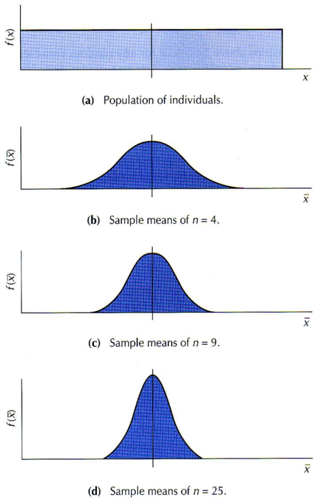
Central Limit Theorem

If X_1, X_2, ... X_n are outcomes of a sample of n independent observations of a random variable X with mean μ_x and variance σ_x^2, then

$\Sigma X_i \sim N(n\mu_x, n\sigma_x^2)$

and

$\Sigma X_i /n \sim N(\mu_x, \sigma_x^2/n)$



(a) Population of individuals.

(b) Sample means of $n = 4$.

(c) Sample means of $n = 9$.

(d) Sample means of $n = 25$.

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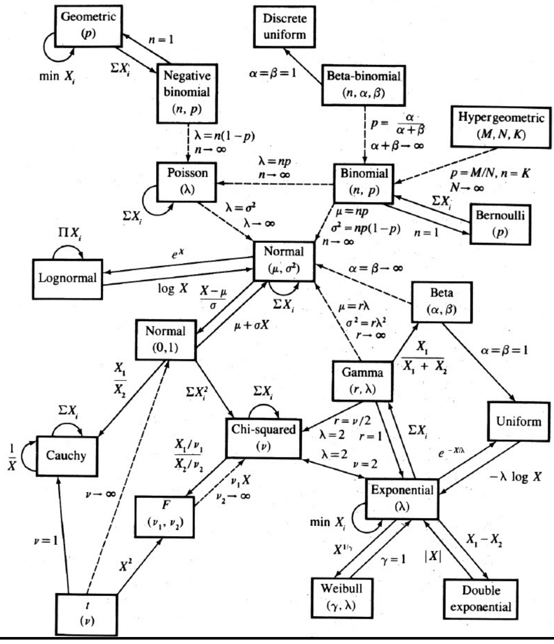
Relationships of Distribution Models



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