Chapter 2 Representation of Mechanisms

2-1 Methods of Representing the Topology of Kinematic Chains

In this chapter, we shall introduce several methods of representing the topology of a kinematic chain. Some methods of representation are abstract rather than actual, and they do not necessarily have a one-to-one correspondence. The following assumptions are made for all methods of representation [1]:

1. All parallel redundant paths are shown as one path only. Parallel paths are usually created to increase the load capacity of a mechanism. As an example, the functional representation of a planetary gear train will only show one planet gear regardless of how many planets are actually used in the design.
2. All joints are assumed to be binary. A multiple joint will be substituted by a set of equivalent binary joints. Thus, a ternary joint will be replaced by two coaxial binary joints, etc.
3. Two or more elements rigidly connected together for the ease of manufacturing will be considered and shown as one link. For example, compound gear made of more than one part and keyed together will be treated as one link.

2-2 Functional Schematic Representation

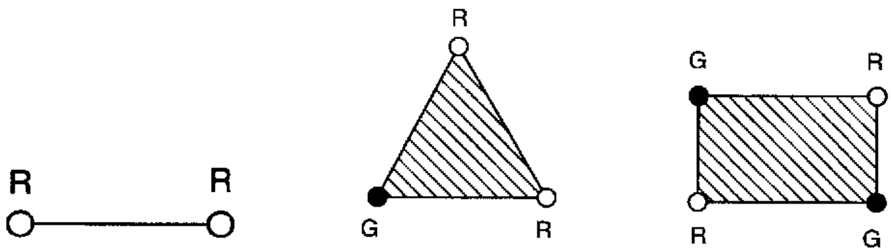
This refers to a conventionally drawn cross-section of the mechanism. Shafts, gears, links and other elements are identified as such. Different functional representations may represent different designs of the same structural topology. For example, the following figure (Fig. 2.1) shows the functional representation of a spur gear drive with external gear mesh, while another figure shows another gear drive with internal gear mesh. The two designs are different in nature, but their structural topologies are identical as will become clear in following section.

(a) (b)

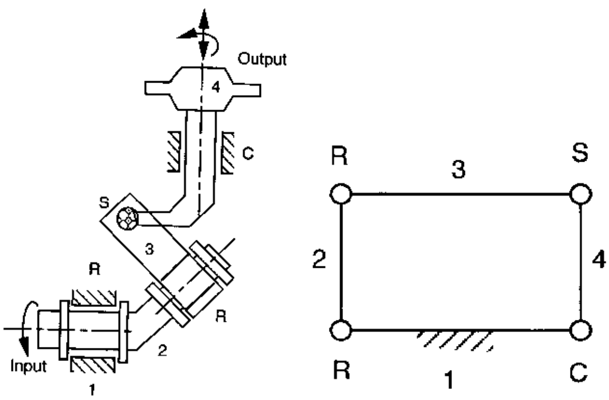
Fig. 2.1 (a) Spur gear drive with external gear mesh (b) internal gear mesh

2-3 Structural Representation

In a structural representation, each link of a mechanism is denoted by a polygon whose vertices represent the kinematic pairs. Specifically, a binary link is represented by a line with two end vertices, a ternary link is represented by a cross-hatched triangle with three vertices, and a quaternary link is represented by a cross-hatched quadrilateral with four vertices, and so on. Figure 2.2 shows the structural representation of a binary, ternary, and quaternary link. The vertices of a structural representation can be colored or labeled for the identification of pair connections. For example, plain vertices shown in Figure 2.2 denote revolute joints, whereas solid vertices denote gear pairs. Some simplified symbols are used to represent the element, such as a joint is represented by a small circle; a binary link is represented by two small circles connected by a line, etc.



(a)



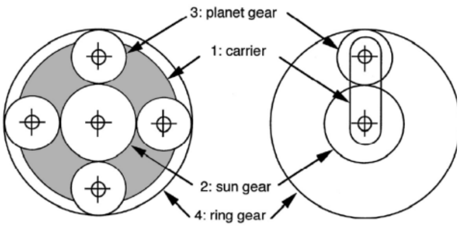
(b)

Fig. 2.2 (a) Structural representation of links (b) Structural representation of an RRSC mechanism

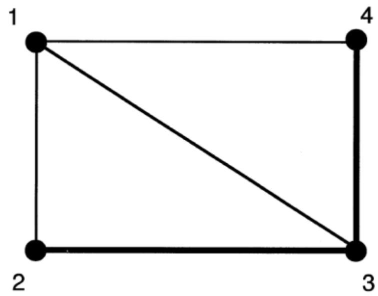
The structural representation of a mechanism is defined similarly, except that the polygon denoting the fixed link is labeled accordingly. Unlike the functional schematic representation, the dimensions of a mechanism, such as the offset distance and twist angle between two adjacent links, are not shown in the structural representation. Figure 2.2(b) shows the structural representation of the RRSC spatial mechanism, where the edge label denotes the link number and the vertex label denotes the joint type. Figure 2.2(b) shows that the four links are connected in a closed loop by revolute, revolute, spherica1, and cylindrical joints.

2-4 Graph Representation

In graph representation, links are denoted by vertices and joints by edges. The edge-connection between vertices corresponds to the pair-connection between links. In order to distinguish the difference among pair connections, the edges of a graph can be colored or labeled. For example, the gear pairs can be represented by thick edges and the turning pairs represented by thin edges in a gear train application. (Draw Fig. 2.3)



(a) Schematic diagram and kinematic representation



(b) Graph representation

Fig. 2.3 Graph representation of the planetary gear set

The advantages of using the graph representation are:

(i) Many network properties of graphs are directly applicable.

(ii) It may be used to assist the development of computer-aided kinematic and dynamic analysis of mechanisms.

(iii) A single atlas of graphs can be used to enumerate an enormous number of mechanisms.

(iv) The structural topology of a kinematic chain can be uniquely identified

(v) Graph theory may be employed for systematic enumeration of mechanisms.

2-5 Matrix Representation

For convenience of computer programming, the kinematic structure of a kinematic chain is represented by a graph and the graph is expressed in matrix form. Matrix representations are particularly useful for computer aided enumeration of kinematic structures of mechanisms. There are several methods of matrix representation. Perhaps, the most frequently used method is the link-to-link form of adjacency matrix.

Link-Link Adjacency Matrix

To define the adjacency matrix, first, the links of a kinematic chain is numbered sequentially from 1 to n. Then, the link-to-link adjacency matrix A is defined as an n×n matrix with its element a_{ij} , defined as follows:

$$a_{ij} = \begin{cases} 1 & \text{if vertex i is connected to j} \\ 0 & \text{otherwise, including } i = j \end{cases} \qquad (2\text{-}1)$$

Therefore, all the diagonal elements are zero and the matrix is symmetric.

Example 1 Write the link-link adjacency matrix for a four-bar linkage. (Draw Fig. E1)

Example 2 Write the link-link adjacency matrix for a simple gear pair. (Draw Fig. E2)

Sol: $$A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$$

The spur-gear drive Link-Link adjacency matrix can be further modified by employing suitable number or alphabets other than 0 and 1; for example:

$$A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & g\\ 1 & g & 0\end{bmatrix} \tag{2-3}$$

Notice that the form of the adjacency matrix is dependent on the order of numbering the links of the kinematic chain. For example, the link adjacency matrix for Fig. 2.4 is (Draw Fig. 2.4)

$$A^{*}=\begin{bmatrix}0 & g & 1\\ g & 0 & 1\\ 1 & 1 & 0\end{bmatrix} \tag{2-4}$$

Permutation of matrix

Let L be a column matrix whose elements denote the link number of a given kinematic chain, and let L^{*} be another column matrix whose elements correspond to a relabeling of the links in the same kinematic chain. Then, the permutation matrix can be defined as the matrix E, satisfying

$$L^{*}=E\,L \tag{2-5}$$

and the adjacency matrix A^{*} can be obtained from A

$$A^{*}=E^{T}A\,E \tag{2-6}$$

The permutation matrix E can be derived by reordering the column of an identity matrix. For example,

$$L=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix} \tag{2-7}$$

$$L^* = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \qquad (2\text{-}8)$$

and
$$E = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad (2\text{-}9)$$

Thus E satisfies $L^* = E L$ (2-10)

and

$$E^T AE = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & g \\ 1 & g & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & g & 1 \\ g & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = A^* \qquad (2\text{-}11)$$

2-6 Structural Isomorphism

Two kinematic chains or mechanisms are said to be *isomorphic* if they share the same topological structure. Equations (2-5) and (2-6) correspond to the definition of isomorphism for kinematic chains, i.e. that there exists a one-to-one correspondence between the numbering of links in the two kinematic chains. In other words, when the links are consistently renumbered, the topology of the two chains is identical.

An important step in structure synthesis of kinematic chains or mechanisms is the identification of *isomorphic* structures. Undetected isomorphic structures lead to duplicate solutions, while falsely identified isomorphisms reduce the number of feasible solutions for new design.

2-7 **Identification of Structural Isomorphism**

(a) Identification by linkage characteristic polynomial

The problem of testing for isomorphism is equivalent to one of determining a permutation matrix E that transforms A into A^* for the two kinematic chains in question. A well known theorem of matrix algebra states that a congruence relation given by Eq.(2-6) can exist only if the characteristic polynomials of two adjacency matrices, A and A^*, are identical, that is, if

$$|xI-A|=|xI-A^*| \qquad (2\text{-}12)$$

holds for all x, where x is a dummy variable, I is an identity matrix of the same order as the adjacency matrix A, and where

$$p(x)=|xI-A|$$

is known as the characteristic polynomial of A. For the example given above, we have

$$|xI - A| = \begin{vmatrix} x & -1 & -1 \\ -1 & x & -g \\ -1 & -g & x \end{vmatrix} = x^3 - (2 + g^2)x - 2g$$

and

$$|xI - A^*| = \begin{vmatrix} x & -g & -1 \\ -g & x & -1 \\ -1 & -1 & x \end{vmatrix} = x^3 - (2 + g^2)x - 2g$$

Since Equation (2-12) is satisfied, the two kinematic structures are most likely isomorphic. It should be noted that the above theorem is a necessary, but not a sufficient condition for two kinematic chains to be isomorphic. Although this condition is not completely discriminatory, it can be successfully distinguish bar linkages with up to eight links. Counter examples are as follows: Figure 2.5 shows two (10, 13) nonisomorphic graphs sharing the characteristic polynomial:

$$p(x)= x^{10}-13x^8+53x^6-8x^5-82x^4+26x^3+39x^2-16x$$

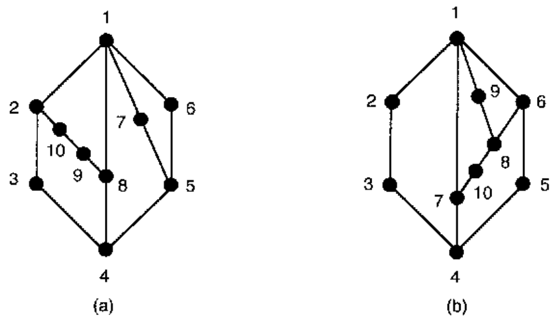


Fig. 2.5 Two nonisomorphic graphs

Example 3 Show the functional, schematic, graph, and matrix representations of the Watt's mechanism.

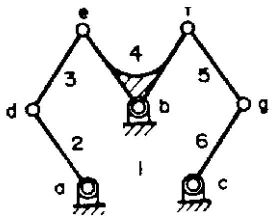


Fig. E3 Watt's mechanism

(b) Optimum code

A binary string can be obtained by concatenating the upper triangular elements of the adjacency matrix row by row, excluding the diagonal element. For example, the adjacency matrix of Fig. 2.6 can be written as

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

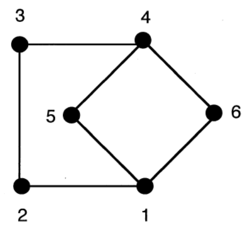


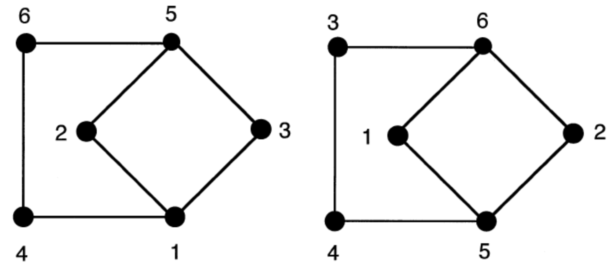
Fig. 2.6 Graph representation

A binary string whose elements are taken from the upper triangular elements can be written as:
Binary string 10011 0010 100 11 0 $_2$

Converting the binary string into decimal number yields 20,006.

If vertices in Fig.2.6 are relabeled as shown in Fig. 2.7, the adjacency matrix of Fig. 2.7 (a) becomes

$$A_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$



(a) (b)
Fig. 2.7 Relabeling of Fig. 2.6 (a) Graph labeled for MAX code(b) Graph labeled for MIN code

Binary string 11100 0010 010 01 1 $_2$

decimal number = 28,819.

It can be shown that among all possible labeling of the graph, the label for A_2 leads to a maximum number. We call the number 28,819 the maximum code of Stephenson chain. Alternatively, we can search for a labeling that minimizes the binary string. We call the number the minimum code (see Fig. 2.7(b)). The method requires n! permutations to arrive at the optimum code.

(c) Degree code

The vertex degrees are used as a constant for labeling the links of a kinematic chain. Links of the same degrees are grouped together and the various group of links are arranged in a descending order according to their vertex degree. Permutation of the links are constrained within each group such that the degrees of all vertices are always kept in a descending order. For example, an n-link chain is divided into 3 groups having p, q, r number of links, where n=p+q+r. Then, the number of permutations will be limited from n! to p! q! r!.

From A_1 (1, 4) (2, 3, 5, 6)

Relabel (1, 2) (3, 4, 5, 6)

1 and 2 are similar; therefore, no need to permute these two vertices

5 and 6 are similar; therefore, no need to permute these two vertices

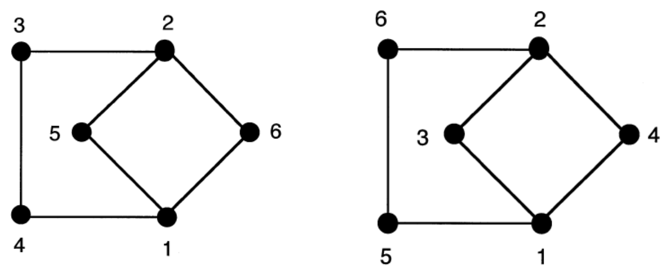
Following the above procedure, it can be shown that among all possible permutations, the permutation shown in Fig. 2.8 produces a maximum number. The adjacency matrix is

$$A_4 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Therefore, the binary string is $01110\ 1101\ 000\ 00\ 1_2$, which is equal to a degree code of (decimal number) = 15,169. (See Fig. 2.8b)

We note that the degree code is smaller than the MAX code, because permutations of the links for the degree code are confined within each group of vertices of the same degree.

Several methods of identification can also be found in the literature [2-6]. Readers are encouraged to study the details of the methods.



(a) (b)

Fig. 2.8 Relabeling for degree code (a) First relabeling (b) Graph labeled for degree code

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