Basic mathematical question: *existence* and *uniqueness*

For *Ax=b*, one solution, no solution or infinitely many?

All start with "Elimination"

Gaussian Elimination

Example:

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

<u>Subtracting multiples</u> of the first equation from the others, so as to eliminate u from the last two equations.

$$\begin{aligned} \boxed{2}u + v + w &= 5 \\ \boxed{-8}v - 2w &= -12 \\ 8v + 3w &= 14 \end{aligned}$$

2: the *first <u>pivot</u>*

–8: the *second pivot*

We continue to eliminate *v* from the third equation.

$$\begin{aligned} \boxed{2}u + v + w &= 5 \\ \boxed{-8}v - 2w &= -12 \\ \boxed{1}w &= 2 \end{aligned}$$

***Pivots:* 2, –8, 1**

<u>Back-substitution</u>: solve *w* then *v* then *u*.

Breakdown of Gaussian Elimination

- ***n* equations →*n* pivots: <u>nonsingular</u>, only one solution**
- **a zero appears in a pivot position →some problem! May be curable or incurable.**
- **Curable case (cured by row exchange): <u>nonsingular</u> case**

$$\begin{matrix} u+v+w= & & u+v+w= & & u+v+w= \\ 2u+2v+5w= & \rightarrow & 3w= & \rightarrow & 2v+4w= \\ 4u+6v+8w= & & 2v+4w= & & 3w= \end{matrix}$$

- **Incurable case: <u>singular</u> case**

$$\begin{matrix} u+v+w= & & u+v+w= \\ 2u+2v+5w= & \rightarrow & 3w= \\ 4u+4v+8w= & & 4w= \end{matrix}$$

Why is this system singular?

If 3*w*=6 and 4*w*=7 then inconsistent →no solution

If 3*w*=6 and 4*w*=8 then *u*+*v*= constant →many solutions

Cost of Gaussian Elimination

- **How many separate arithmetical operations (cost) does elimination require, for *n* equations in *n* unknowns?**
- **<u>Operations</u>: <u>division</u> and <u>multiply-subtract</u>**
- **To produce a zero in the first column: 1 division and *n*-1 multiply-subtract are needed →total *n* operations needed**
- **There are *n*-1 rows in the first column to be operated on: $n(n-1)=n^2-n$ operations are needed to produce zeros in the first column**
- **For the second column: $(n-1)^2-(n-1)$ …..and so on**
- **There are n columns:**

$$(n^2+\cdots+1^2)-(n+\cdots+1)=\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}=\frac{n^3-n}{3}$$

- **When *n* is large, $n^3/3$ is a pretty good estimate**
- **For the right side: about $n^2/2$ is required. Why?**
- **Operations required in back-substitution:**

$$1+2+\cdots+n=\frac{n(n+1)}{2}\approx\frac{n^2}{2}$$

Why?

- **The right side is responsible for a total of n^2 operations**
- **Minimum operations required? $Cn^{\log_2 7}$ = $Cn^{2.8}$ ($\log_2 7\approx 2.8$). IBM research center has find some the power of *n* below 2.5!!**

Multiplication of a Matrix and a Vector

- **Inner product of two vectors:**

$$\begin{bmatrix}2 & 1 & 1\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix}=\begin{bmatrix}2u+v+w\end{bmatrix}$$

- **<u>By rows</u>:**

$$Ax=\begin{bmatrix}2 & 1 & 1\\ 4 & -6 & 0\\ -2 & 7 & 2\end{bmatrix}\begin{bmatrix}u\\ v\\ w\end{bmatrix}=\begin{bmatrix}2u+v+w\\ 4u-6v\\ -2u+7v+2w\end{bmatrix}$$

- **Let a_{ij} be the entry in the *i*th row and *j*th column and x_j be the jth component of *x***

$$\sum_{j=1}^{n} a_{ij}x_j \text{ is the } i\text{th component of } Ax$$

<u>Each row is the linear combination of components of *x*</u>

- **<u>By columns</u>:**

$$Ax=u\begin{bmatrix}2\\ 4\\ -2\end{bmatrix}+v\begin{bmatrix}1\\ -6\\ 7\end{bmatrix}+w\begin{bmatrix}1\\ 0\\ 2\end{bmatrix}$$

***<u>Ax is a combination of the columns of A</u>*. The coefficients which multiply the columns are the components of *x*.**

- **<u>Identity matrix *I*</u> leaves every vector unchanged.**

$$Ib=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}5\\ -2\\ 9\end{bmatrix}=\begin{bmatrix}5\\ -2\\ 9\end{bmatrix}$$

Matrix Multiplication

- **Most common and least used: Each entry of *AB* is the product of a row and a column: $(AB)_{ij}$ = *inner product of row i of A and column j of B***

Example: Multiplication of matrix $A_{3\times 4}$ and matrix $B_{4\times 2}$

$$\underset{\text{3 by 4 matrix}}{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}} \underset{\text{4 by 2 matrix}}{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix}} = \underset{\text{3 by 2 matrix}}{\begin{bmatrix} * & * \\ * & * \\ * & (AB)_{32} \end{bmatrix}}$$

- **<u>Each column of *AB* is the product of *A* and a column of B</u>**

Column j of AB = A times column j of B

Example:

$$AB = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 2a+3c & 2b+3d \end{bmatrix} \Rightarrow$$

Each column of *AB* is a combination of the columns of *A* with coefficients coming from each column of *B*

- **<u>Each row of *AB* is the product of a row of *A* and *B*</u>:**

Row i of AB = row i of A times B $\Rightarrow$

Each row of *AB* is a combination of the rows of *B* with coefficients coming from each row of *A*

Elimination Steps and Elementary Matrices

- **An Elimination step can be a Matrix: <u>Elementary Matrix</u>**
  HOW? Subtracting 2 times the first row of A from the second row of A

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{bmatrix}$$

The matrix that subtracts a multiple l_{ij} of row *j* from row *i* is the <u>elementary matrix E_{ij}</u> with 1's on the diagonal and the number $-l_{ij}$ in row *i*, column *j*.

Example: $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l_{31} & 0 & 1 \end{bmatrix}$

- **Gaussian elimination for**

$$Ax = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = b$$

Step 1: Subtract 2 times the 1st equation from the 2^{nd} (*E*)

Step 2: Subtract –1 times the 1st equation from the 3rd (*F*)

Step 2: Subtract –1 times the 2nd equation from the 3rd (*G*)

- **Matrix form of Gaussian elimination:** $\boxed{GFEA=U}$

$$GFE = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix Multiplication (Cont'd)

- **Matrix multiplication is associative ($(AB)C=A(BC)$)**
- **Matrix multiplication is distributive**

$$A(B+C)=AB+AC \quad \text{and} \quad (B+C)D=BD+CD$$

- **Matrix multiplication is not commutative. Usually,**

$$AB \neq BA$$

Example: let G be the elementary matrix that adds row 2 to row 3 and E be the matrix that subtracts 2 times row 1 from row 2. Then,

$$EG=\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\-2&1&0\\0&1&1\end{bmatrix}\neq GE=\begin{bmatrix}1&0&0\\-2&1&0\\-2&1&1\end{bmatrix}$$

That is, <u>the order of elimination steps matters</u>

Inverse of Elimination Steps

$$Ax=\begin{bmatrix}2&1&1\\4&-6&0\\-2&7&2\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}5\\-2\\9\end{bmatrix}=b$$

- **Gaussian elimination leads to:**

$$GFEAx=Ux=\begin{bmatrix}2&1&1\\0&-8&-2\\0&0&1\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=GFEb=\begin{bmatrix}5\\-12\\2\end{bmatrix}=c$$

U is called *upper triangular*

After Gaussian elimination $\boxed{Ax=b \Rightarrow Ux=c}$

- **<u>Undo the steps of Gaussian elimination</u>:**

Example: $E=\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}$ **then E^{-1}=?**

$$E^{-1}E=\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

<u>If the elementary matrix E has the number $-l_{ij}$ in the (i,j) position, then its inverse has $+l_{ij}$ in that position</u>

- **The 3rd step (G) was last in going from A to U. Its matrix G must be the first to be inverted in the reverse direction. That is:**

$$E^{-1}F^{-1}G^{-1}U=A$$

Inverse of Elimination Steps in Matrix Form: L

- **The matrix that takes U back to A: $L \Rightarrow LU=A$**

$$E^{-1}F^{-1}G^{-1}=\begin{bmatrix}1&0&0\\2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0\\2&1&0\\-1&-1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}=L$$

$$\begin{bmatrix}1&0&0\\l_{21}&1&0\\0&0&1\end{bmatrix}\bullet\begin{bmatrix}1&0&0\\0&1&0\\l_{31}&0&1\end{bmatrix}\bullet\begin{bmatrix}1&0&0\\0&1&0\\0&l_{32}&1\end{bmatrix}=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}$$

$\Rightarrow$ **Expressing Inverse of Elimination Steps is EASY!**

How about Guassian Elimination in matrix form? It's NOT EASY!

$GFE=L^{-1}=$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&-l_{32}&1\end{bmatrix}\bullet\begin{bmatrix}1&0&0\\-l_{21}&1&0\\0&0&1\end{bmatrix}\bullet\begin{bmatrix}1&0&0\\0&1&0\\-l_{31}&0&1\end{bmatrix}\neq\begin{bmatrix}1&0&0\\-l_{21}&1&0\\-l_{31}&-l_{32}&1\end{bmatrix}$$

Triangular Factorization of A: $A=LU$

- **Example:**

$$\begin{aligned}2u+v+w&=\\4u-6v&=\\-2u+7v+2w&=\end{aligned}$$

Step 1: Subtract 2 times the 1st equation from the 2nd $\Rightarrow l_{21}=2$

Step 2: Subtract –1 times the 1st equation from the 3rd $\Rightarrow l_{31}=-1$

Step 2: Subtract –1 times the 2nd equation from the 3rd $\Rightarrow l_{32}=-1$

$$GFEA=\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&1&1\\4&-6&0\\-2&7&2\end{bmatrix}=\begin{bmatrix}2&1&1\\0&-8&-2\\0&0&1\end{bmatrix}=U$$

$$L=E^{-1}F^{-1}G^{-1}=\begin{bmatrix}1&0&0\\l_{21}&1&0\\l_{31}&l_{32}&1\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\-1&-1&1\end{bmatrix}$$

$$A=\begin{bmatrix}2&1&1\\4&-6&0\\-2&7&2\end{bmatrix}=\begin{bmatrix}1&0&0\\2&1&0\\-1&-1&1\end{bmatrix}\begin{bmatrix}2&1&1\\0&-8&-2\\0&0&1\end{bmatrix}=LU$$

- **Triangular Factorization** $\boxed{A=LU}$

Without row exchanges required, the original matrix A can be written as a product A=LU. The matrix L is lower triangular, with 1's on the diagonal and the multipliers l_{ij} (taken from the elimination) below the diagonal. U is the upper triangular matrix which appears after forward elimination and before back-substitution; its diagonal entries are the pivots.

Examples of Triangular Factorization

- **The steps of elimination are actually L^{-1} that takes A to U and reduce L to I: $L^{-1}A = L^{-1}LU = U$.**

- **Another way of looking at it:**

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} row\,1\,of\,U \\ row\,2\,of\,U \\ row\,3\,of\,U \end{bmatrix} = original\ A$$

The matrix L, applied to U, brings back A!

Example: When U is the identity then L is the same as A

$$A = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} I = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_{31} & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_{32} & 1 \end{bmatrix} I$$

Example:

$$A = \begin{bmatrix} 1 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & & \\ & 1 & -1 & \\ & & 1 & -1 \\ & & & 1 \end{bmatrix}$$

Solving a Linear System

- **c is the right side after Gaussian elimination. That is,**

  $c=L^{-1}b \Rightarrow c$ can be solved by $Lc=b$

- **When $A=LU$ and L and U are known,**

  $$Ax=b \quad \Rightarrow \quad \boxed{Lc=b \quad and \quad Ux=c}$$

- **Given $A=LU$**

  One Linear System = Two Triangular Systems

- **We first solve a lower triangular system for c ($Lc=b$) then the upper triangular system for x ($Ux=c$)**

- **Multiply $Ux=c$ by L. We have $LUx=Lc \Rightarrow Ax=b$.**

  Example: Band matrix in the previous example

- **U can be further rewritten as:**

$$U = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix}\begin{bmatrix} 1 & u_{12}/d_1 & u_{13}/d_1 & \cdot \\ & 1 & u_{23}/d_2 & \cdot \\ & & \ddots & \cdot \\ & & & 1 \end{bmatrix}$$

The triangular factorization is often written A=LDU,
where L and U have 1's on the diagonal
and D is the diagonal matrix of pivots.

- **Some freedom in the elimination steps and calculations, but absolutely no freedom in the final L, D and U. That is,**

  <u>L, D and U are uniquely determined by A</u>

Row Exchanges and Permutation Matrices

- **Remember what we do if a zero appears in the pivot location? Problem may be curable or incurable (singular)**
- **How can the problem be cured?**
  Looking below the zero to seek out nonzero entry lower down in the same column. Then, a row exchange is carried out; the nonzero entry becomes the needed pivot and elimination can get going again.
- **Row exchanges in matrix forms: Permutation matrices**
  Example: an exchange of rows 1 and 3 and exchange of rows 2 and 3:

$$P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

  Why? A trick: $P_{13}I=P_{13}$
- **Do both of the row exchanges at once:**

$$P_{23}P_{13} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = P$$

- **In a nonsingular case, $Ax=b$ has a unique solution**
  (1) It is found by elimination with row exchanges
  (2) With the rows reordered in advance, PA can be factored into LU.
- **In a singular case, no reordering can produce a full set of pivots.**

Inverse of A

The matrix A is <u>invertible</u> if there exists a matrix B such that BA=I and AB=I. There is at most one such B, called the inverse of A and denoted by A^{-1}:

$$\underline{A^{-1}A=I} \quad \text{and} \quad \underline{AA^{-1}=I}$$

- **There could not be two different inverses, because if $BA=I$ and $AC=I$ then**

$$B=BI=B(AC)=(BA)C=IC=C$$

- $(A^{-1})^{-1} = A$
- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- **if** $A = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$ **then** $A^{-1} = \begin{bmatrix} 1/d_1 & & \\ & \ddots & \\ & & 1/d_n \end{bmatrix}$
- **$(A+B)^{-1}$=? Nothing. No quick solution. But $(AB)^{-1}$=?**

$$(AB)^{-1}=B^{-1}A^{-1}$$

Proof:

$(AB)(B^{-1}A^{-1})=ABB^{-1}A^{-1}=AIA^{-1}=AA^{-1}=I$

$(B^{-1}A^{-1})(AB)=BAA^{-1}B^{-1}=BIB^{-1}=BB^{-1}=I$

- **Similarly, $(ABC)^{-1}=C^{-1}B^{-1}A^{-1}$**
- **Remember $GFEA=U$ and $A=E^{-1}F^{-1}G^{-1}U=LU$!**

Gauss-Jordan Method to Find A^{-1}

- **We first go from A to U using Gaussian elimination**

$$[A \quad e_1 \quad e_2 \quad e_3] = \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} U & L^{-1} \end{bmatrix}$$

$$[A \quad I] \to L^{-1}[A \quad I] = [L^{-1}A \quad L^{-1}I] = [U \quad L^{-1}]$$

- **Then from U to I: creating zeros above the pivots**

$$[U \quad L^{-1}] \to \begin{bmatrix} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & 0 & 0 & \frac{12}{8} & -\frac{5}{8} & -\frac{6}{8} \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & \frac{12}{16} & -\frac{5}{16} & -\frac{6}{16} \\ 0 & 1 & 0 & \frac{4}{8} & -\frac{3}{8} & -\frac{2}{8} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} I & U^{-1}L^{-1} \end{bmatrix}$$

$$[U \quad L^{-1}] \to U^{-1}[U \quad L^{-1}] = [I \quad U^{-1}L^{-1}] = [I \quad A^{-1}]$$

Operations required for calculating A^{-1}? $n^3/6+n^3/3+n(n^2/2)$ How?

- **Are Gauss-Jordan method the only order of elimination?**
  We could have produced zeros above and below a pivot!

Invertible = Nonsingular

- **Nonsingular → Invertible**
  - **A matrix has a full set of n pivots → nonsingular by definition. (There exists a unique solution for any b)**
  - **$AA^{-1}=I$ can then be viewed as n separate $Ax_i=e_i$ systems where x_i's are columns of A^{-1}.**
  - **x_i's can be solved because A is nonsingular and therefore A^{-1} can be determined uniquely.**
  - **$A^{-1}A=I$? A^{-1} is actually the multiplication of many elementary matrices formed by the Gauss-Jordan steps.**
- **Invertible → Nonsingular**
  - **If A has an inverse, Gauss-Jordan method must not break down. That is, the following should not occur.**

$$A' = \begin{bmatrix} d_1 & x & x & x & x \\ 0 & d_2 & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}$$

  - **No matter what x's are, the Gauss-Jordan method cannot be carried out. That is, A must have a full set of pivots and is nonsingular.**

A square matrix is invertible if and only if it is nonsingular

Transposes

- **The transpose of $A = A^T$**
- **The ith row of A becomes the ith column of A^T**
- **The entry $(A^T)_{ij}=A_{ji}$**

$$\left(\begin{bmatrix}2 & 1 & 4\\0 & 0 & 3\end{bmatrix}\right)^T=\begin{bmatrix}2 & 0\\1 & 0\\4 & 3\end{bmatrix}$$

- **$(AB)^T=B^TA^T$**

$$AB=\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}3 & 3 & 3\\2 & 2 & 2\end{bmatrix}=\begin{bmatrix}3 & 3 & 3\\5 & 5 & 5\end{bmatrix}$$

$$B^TA^T=\begin{bmatrix}3 & 2\\3 & 2\\3 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}=\begin{bmatrix}3 & 5\\3 & 5\\3 & 5\end{bmatrix}$$

- **$(A^{-1})^T=(A^T)^{-1}$ not trivial!**
  $(A^{-1})^TA^T=(AA^{-1})^T=I^T=I$ and $A^T(A^{-1})^T=(A^{-1}A)^T=I^T=I$
- **A <u>symmetric matrix: $A^T=A$</u> or $a_{ij}=a_{ji}$**
- **A symmetric matrix's inverse, if there is one, is also symmetric**
- **If A is symmetric and can be factored into LDU without row exchanges to destroy the symmetry, then the upper triangular U is the transpose of the lower triangular L. That is, the factorization becomes <u>$A=LDL^T$</u>**

$$A=LDU=A^T=U^TD^TL^T$$

Roundoff Error

- **A typical computer round-off:**

  $.345+.00123 \rightarrow .346$ (keep three digits)
- **Roundoff error: the digits lost (0.00023)**
- **Example:**

$$A=\begin{bmatrix}1. & 1.\\1. & 1.0001\end{bmatrix} \quad \textbf{and} \quad A'=\begin{bmatrix}.0001. & 1.\\1. & 1.\end{bmatrix}$$

- **Some matrices are extremely sensitive to small changes and others are not. The matrix A is ill-conditioned (sensitive) and A' is well-conditioned. In A, if a_{ij} (in A) becomes 1 due to roundoff error the system becomes singular**

Example

Original: $u+v=2$, $u+1.0001v=2.0001$ **Roundoff:** $u+v=2$, $u+1.0001v=2$

solution: $u=v=1$,   **solution: $u=2$ $v=0$**

A change in the fifth digit of b was amplified to a change in the first digit of the solution!!

Partial Pivoting

- **Even a well-conditioned matrix can be ruined by a poor algorithm.** ***Straightforward Gaussian elimination is among the poor algorithms!!***

Example: the well-condition matrix *A'* (symmetric)

$$.0001u+v=1$$
$$u+v=2$$

after elimination: -9999*v* = -9998 → *v* = .99990

original: .0001*u* + .9999 = 1 → *u* = 1.0

round-off: *v* = .99990 → *v* ~ 1.0

.0001*u* + 1 = 1 → *u* ~ 0 !!!

$$A' = \begin{bmatrix} 1 & 0 \\ 10{,}000 & 1 \end{bmatrix} \begin{bmatrix} .0001 & 0 \\ 0 & -9999 \end{bmatrix} \begin{bmatrix} 1 & 10{,}000 \\ 0 & 1 \end{bmatrix} = LDU$$

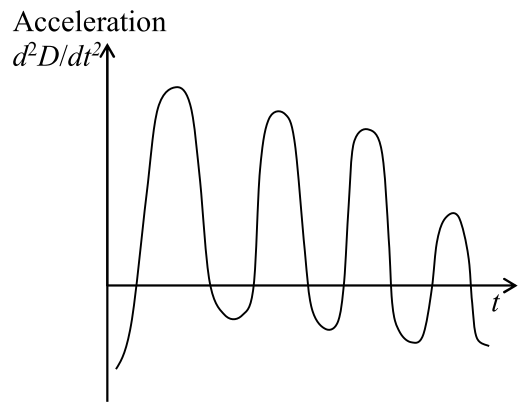
- ***The small pivot 0.0001 brought instability!!***
- **Remedy: row exchanges to remove small pivots.**
- **A computer must compare each pivot with all the other possible pivots in the same column.** ***Choosing the largest of these candidates, and exchanging the corresponding rows so as to make this largest value the pivot is called Partial Pivoting.***

$$A'' = \begin{bmatrix} 1 & 1 \\ .0001 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ .0001 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .9999 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = LDU$$

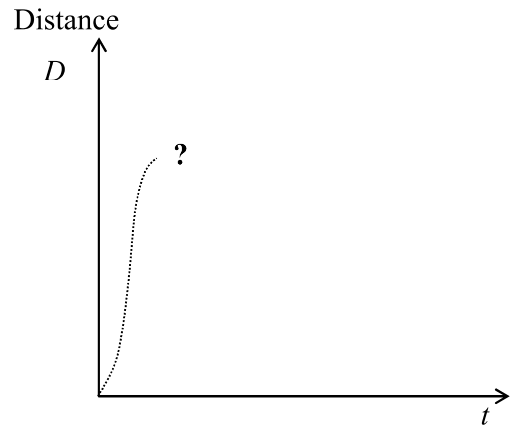
A symmetric A' is not necessarily better for a computer!!

An Application: Differential Equation

- **A continuous problem cannot be solved by computers. Computer can only approximate a continuous problem by a discrete problem**
- **Example: riding motorcycles with only acceleration known as a function of time**



Question: distance as a function of time?



Continuous→Discrete

$$\frac{d^2D}{dt^2} = f(t), \qquad 0 \le t \le 1 \;\Rightarrow \boldsymbol{D(t)=?}$$

- **Uncertainty in the problem since $a+bt$ for any a and b after second derivative contributes nothing to $f(t)$.**
- **Usually, initial conditions, such as $D(0)=0$ (starting from the origin) and $dD(0)/dt=0$ (initial speed is zero), are added to remove the uncertainty.**
- **Discrete approximation of differentiation:**

$$\frac{dD}{dt} \approx \frac{D(t+h)-D(t)}{h} \Rightarrow \frac{d^2D}{dt^2} = \frac{d\left(\frac{dD}{dt}\right)}{dt} \approx \frac{\frac{D(t+h)-D(t)}{h} - \frac{D(t)-D(t-h)}{h}}{h} \approx \frac{D(t+h)-2D(t)+D(t-h)}{h^2} = f(t)$$

Let time interval=h ⇒ $D_{j+1} - 2D_j + D_{j-1} = h^2 f(jh)\ \ j=1,\dots,n$

- **Let n=5, D_0=0 and dD_0/dt=0 ⇒ $(D_1-D_0)/h$=0 ⇒ $D_1=D_0$=0**

$$D_0 - 2D_1 + D_2 = D_2 = h^2 f(h)$$

$$D_1 - 2D_2 + D_3 = -2D_2 + D_3 = h^2 f(2h)$$

$D_{j-1} - 2D_j + D_{j+1} = h^2 f(jh)$ **for j=3,…,5**

$$\begin{bmatrix} 1 & 0 & & & \\ -2 & 1 & 0 & & \\ 1 & -2 & 1 & 0 & \\ & 1 & -2 & 1 & 0 \\ & & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{bmatrix} = h^2 \begin{bmatrix} f(h) \\ f(2h) \\ f(3h) \\ f(4h) \\ f(5h) \end{bmatrix} \Rightarrow \textbf{Lower Triangular}$$

Matrix

Continuous→Discrete

- **Let n=5, D_0=0 and dD_5/dt=0 ⇒ $(D_6-D_5)/h$=0 ⇒**

$$D_0 - 2D_1 + D_2 = -2D_1 + D_2 = h^2 f(h)$$

$D_{j-1} - 2D_j + D_{j+1} = h^2 f(jh)$ **for j=2,…,4**

$$D_4 - 2D_5 + D_6 = D_4 - 2D_5 + D_5 = D_4 - D_5 = h^2 f(5h)$$

$$\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{bmatrix} = h^2 \begin{bmatrix} f(h) \\ f(2h) \\ f(3h) \\ f(4h) \\ f(5h) \end{bmatrix}$$

⇒ Tri-diagonal Matrix

- **Let n=5, D_0=0 and D_6=0 (Boundary Conditions)**

$$D_0 - 2D_1 + D_2 = -2D_1 + D_2 = h^2 f(h)$$

$D_{j-1} - 2D_j + D_{j+1} = h^2 f(jh)$ **for j=2,…,4**

$$D_4 - 2D_5 + D_6 = D_4 - 2D_5 = h^2 f(5h)$$

$$\begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{bmatrix} = h^2 \begin{bmatrix} f(h) \\ f(2h) \\ f(3h) \\ f(4h) \\ f(5h) \end{bmatrix}$$

⇒ Tri-diagonal Matrix

Tridiagonal Matrices

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

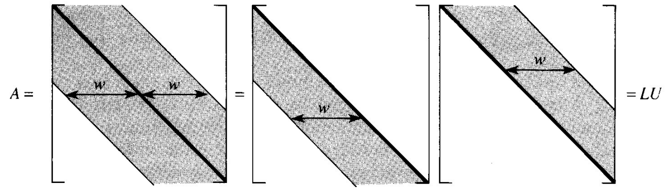
- **Tridiagonal matrix: $a_{ij}=0$ if $|i\text{-}j|>1$; that is, all nonzero entries lie on the main diagonal and the two adjacent diagonals.**
- **Symmetric matrix: $U=L^T$**
- **Positive definite matrix: all pivots are positive. *A symmetric matrix with all positive pivots does not require row exchanges.***
- **For a tridiagonal matrix,**
  (a) only one nonzero entry below the pivot
  (b) operations are carried out on a very short row.

$$A=\begin{bmatrix} 1 & & & & \\ \frac{-1}{2} & 1 & & & \\ & \frac{-2}{3} & 1 & & \\ & & \frac{-3}{4} & 1 & \\ & & & \frac{-4}{5} & 1 \end{bmatrix}\begin{bmatrix} \frac{2}{1} & & & & \\ & \frac{3}{2} & & & \\ & & \frac{4}{3} & & \\ & & & \frac{5}{4} & \\ & & & & \frac{6}{5} \end{bmatrix}\begin{bmatrix} 1 & \frac{-1}{2} & & & \\ & 1 & \frac{-2}{3} & & \\ & & 1 & \frac{-3}{4} & \\ & & & 1 & \frac{-4}{5} \\ & & & & 1 \end{bmatrix}$$

***L* and *U* are *bi-diagonal*. The pivots are converging to +1.**

Operations Required by Band Matrices

- **For a tridiagonal matrix, every forward elimination stage requires 2 multiply-subtract operations and there are n such stages → we need only $2n$ operations instead of the usual $n^3/3$.**
- **Back-substitution: again only $2n$, instead of $n^2/2$, operations are needed.**
- **No. of operations for a tridiagonal matrix is proportional to n instead of to a higher power of n.**



- **In general, a band matrix with nonzero entries only on the band $|i\text{-}j|<w$. A tridiagonal matrix has $w=2$.**
- **Each stage of elimination requires $w(w\text{-}1)$ operations**
- **A total of about w^2n operations are needed.**
- **The exact number of divisions and multiply-subtracts is $w(w\text{-}1)(3n\text{-}2w+1)/3$ ($=n(n\text{-}1)(n+1)/3$ when $w=n$)**
- **In band matrix problems, calculating A^{-1} to solve $x=A^{-1}b$ is not more efficient than calculating L and U to solve $Lc=b$ and $Ux=c$**