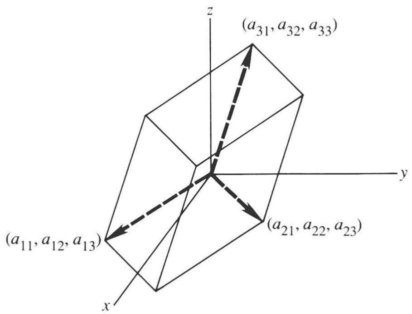
Determinants – An Introduction

- **Determinants were once the "star" of Mathematics – Muir's** *A History of Determinant* **filled four volumes.**
- **For applications, there are only 4 uses most often mentioned:**

1. **The determinants give formulas for pivots:** ***determinant*=±(*product of the pivots*); that is, regardless of the order of elimination, the product of the pivots remains the same in size (apart from sign).**



2. **The determinant of A equals the volume of a parallelepiped P. This is how the *Jacobian* determinant is from: coordinates changed from (x, y, z) to (r, θ, z) then** $V = \iiint f(x,y,z)dV = \iiint f(r\cos\theta, r\sin\theta, z)Jdrd\theta dz$ **where**

$$J=\begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial z \\ \partial y/\partial r & \partial y/\partial\theta & \partial y/\partial z \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\cos\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

3. **The determinant measures the dependence of $A^{-1}b$ on b. If one of the elements in b is changed the influence on the solution $x=A^{-1}b$ is a ratio of determinants.**

4. **Test for invertibility: if the determinant of A is zero, then A is singular and if det$A \neq 0$ then A is invertible. Using this property and determinant's explicit formula, find the eigenvalues by letting det $A-\lambda I$=0 since $A-\lambda I$ is singular.**

- **We are back to *square* matrices only!**
- **There are difficulties to determine the importance and proper place of determinant in the theory of linear algebra. It is also difficult to *define* a determinant.**
- **<u>Determinant is an attempt to summarize a square matrix (no matter how big the matrix is) into one single value!</u>**
- **<u>The simple things about the determinant are not the explicit formulas, but the three properties it possesses</u>.**
- **We start with three basic properties of determinants. In fact, these three properties are sufficient to define the determinants. To your surprise, we will show how the well-known formula of determinants can be derived from the simple properties!**

Three Basic Properties of Determinants (1-3)

$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ **will be used as an example**

1. ***The determinant <u>transforms</u> a matrix of values to one single value <u>linearly on one row</u>.***

Add vectors in one row (one row at a time)

$$\det\begin{bmatrix} a+a' & b+b' \\ c & d \end{bmatrix} = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \det\begin{bmatrix} a' & b' \\ c & d \end{bmatrix}$$ **Note: det*B*+det*C*≠det(*B*+*C*)**

Multiply by *t* in one row (one row at a time)

$$\det\begin{bmatrix} ta & tb \\ c & d \end{bmatrix} = t\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Note: For a square $n\times n$ matrix A: $\det(tA)=t^n \det A \neq t \det A$

tA is like stretching every sides of the edges to t times. The volume should become $t^n\det A$ not $t\det A$.

2. ***The determinant changes sign when two rows are exchanged***

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

3. ***The determinant of the identity matrix is* 1**

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \quad \text{and} \quad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \quad \text{and} \ldots$$

Rules 2 and 3 give the determinant a "value" with "sign". Example: *permutation* matrices

Derived Properties of Determinants (4-6)

4. ***If two rows of A are equal, then* det*A*=0**

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ba = 0$$

From rule 2, let B be the matrix with the equal rows exchanged, then

$\det B = -\det A = \det A \Rightarrow \det A=0$

5. ***The elementary operation of subtracting a multiple of one row from another row leaves the determinant unchanged***

$$\begin{vmatrix} a-lc & b-ld \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

From rule 1: $\begin{vmatrix} a-lc & b-ld \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - l\begin{vmatrix} c & d \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

6. ***If A has a zero row then* det*A*=0**

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0.$$

By rules 5 and 4: $\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = \begin{vmatrix} 0+c & 0+d \\ c & d \end{vmatrix} = \begin{vmatrix} c & d \\ c & d \end{vmatrix} = 0.$

Derived Properties of Determinants (7-8)

7. ***If A is triangular, then*** **det*A*** ***is the product*** $a_{11}a_{22}\dots a_{nn}$ ***of the entries on the main diagonal.***

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = ad, \qquad \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = ad.$$

Proof: For an upper (or lower) triangular matrix *A*, we can go through elementary operations to eliminate the off-diagonal matrix as:

$$D = \begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{bmatrix}$$ **without changing the value of determinant (det*A*=det*D*)**

by rule 5. By rule 1, factor out a_{11} and then a_{22} and finally a_{nn}:

$$\det D = a_{11}a_{22}\cdots a_{nn}\det I = a_{11}a_{22}\cdots a_{nn} \quad \textbf{by rule 3.}$$

8. ***If A is singular,*** **det*A* = 0.** ***If A is invertible,*** **det*A* $\neq 0$**

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is singular if and only if } \quad ad - bc = 0$$

If *A* is singular, elimination leads to matrix *U* with at least a zero row. By rules 5 and 6, det *A* = det *U* = 0. If A is not singular, elimination leads to *U* with nonzero pivots in the diagonal. Let these pivots be $d_1,\dots,d_n$, then by rule 7, det*A*=$\pm d_1 d_2 .. d_n$ (*first formula for determinants*)

Derived Properties of Determinants (9)

9. ***For any two n by n matrices, the determinant of the product AB is the product of the determinants:*** **det*AB*=(det*A*)(det*B*).**

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}\begin{vmatrix} e & f \\ g & h \end{vmatrix} = \begin{vmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{vmatrix}$$

Check: $(ad-bc)(eh-fg) = (ae+bg)(cf+dh) - (af-bh)(ce+dg)$

Special Case:

$$(\det A)(\det A^{-1}) = \det AA^{-1} = \det I = 1 \Rightarrow \det A^{-1} = \frac{1}{\det A}$$

For singular case: If *B* is singular *AB* is singular $\Rightarrow$ det*AB*=det*A* det*B* =0.

For nonsingular case:

Proof

If *A* is a diagonal matrix *D*, by rule 1

$$\det\begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{bmatrix}\begin{bmatrix} -b_1- \\ \vdots \\ -b_n- \end{bmatrix} = \det\begin{bmatrix} a_{11}b_1 \\ \vdots \\ a_{nn}b_n \end{bmatrix}$$
$$= a_{11}\cdots a_{nn}\det B = \det D \det B$$

And any *A* can be factored to *D* to keep det*A*=det*U*

Derived Properties of Determinants (10)

10. ***The transpose of A has the same determinant as A itself:*** **det$A$=detA^T**

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

For the singular case: If A is singular, A^T is singular $\Rightarrow$ det$A$=det A^T

For the nonsingular case: By rule 9 $\det P \det A = \det L \det D \det U$

$\Rightarrow \det A^{\mathrm{T}} \det P^{\mathrm{T}} = \det U^{\mathrm{T}} \det D^{\mathrm{T}} \det L^{\mathrm{T}} = \det U \det D \det L$

since L, U and D are either triangular or diagonal and their determinants are the product of entries in the main diagonal by rule7, which is not changed by taking transpose and because $PP^T=I \Rightarrow$ detPP^T=det$P$ detP^T=det$I$=1 $\Rightarrow$ det P=det P^T

(Both of them must be 1 or –1)

<u>All the properties for rows are now all applicable to columns</u>

Particularly, rules 1, 2, 4, and 5

First Formula for Determinants

- **If A is nonsingular, then $A=P^{-1}LDU$**

det A = det P^{-1}det L det D det U = ± (product of the pivots)

where the sign ± is determined by detP^{-1}(=detP^T=det$P$) and depends on the number of row exchanges is even or odd. Also, det$L$=det$U$=1 and det$D$=$d_1 \ldots d_n$

- **Example: 2×2 case**

Without row exchange: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ c/a & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & (ad-bc)/a \end{bmatrix} \begin{bmatrix} 1 & b/a \\ 0 & 1 \end{bmatrix}$

With row exchange:

$$PA = \begin{bmatrix} c & d \\ a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a/c & 1 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & (cb-da)/c \end{bmatrix} \begin{bmatrix} 1 & d/c \\ 0 & 1 \end{bmatrix}$$

det PA = cb-da = -(bd-bc) = -det$A$

- **Example: finite second-order difference matrix**

$$\begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \cdot & \\ & & \cdot & \cdot & -1 \\ & & & -1 & 2 \end{bmatrix} = L \begin{bmatrix} 2 & & & & \\ & 3/2 & & & \\ & & 4/3 & & \\ & & & \cdot & \\ & & & & (n+1)/n \end{bmatrix} U$$

$$\Rightarrow \det A = 2\left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\cdots\left(\frac{n+1}{n}\right) = n+1.$$

Explicit Expression for Determinants

- **Pivots are products after elimination. Any explicit expression for determinants directly using the values of the entries in *A*?**
- **For *n*=3, we all know:**

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} +a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32} \\ -a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}. \end{matrix}$$

how? From the basic three rules?

- **First step: breaking down every row in the matrix:**

$$\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathrm{c} & \mathrm{d} \end{bmatrix} = \begin{bmatrix} c & 0 \end{bmatrix} + \begin{bmatrix} 0 & d \end{bmatrix}.$$

$$\Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix}.$$

For any $n \times n$ matrix, each row can be split into *n* coordinate directions ⇒ total of n^n terms in expansion. BUT, when two rows are chosen to be in the same coordinate direction, the determinants for these matrices are zero:

$$\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} = 0, \quad \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} = 0.$$

That is, only when the rows point in different directions (or nonzero terms come in different columns) should survive.

Explicit Formula - 3×3 Case

- **For 3×3 case:**

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}.$$

That is, when the first row has a nonzero entry in column α, the second row must have a nonzero entry in column β and so on till finally the *n*th row is nonzero in column ν and all these nonzero column numbers should be different ($\beta \neq \alpha \neq \cdots \neq \nu$) ⇒ *permutation* of numbers 1, 2,…, *n*

- **For $n \times n$ case, there are *n*! ways to permute the numbers**
- **Example: 3×3 case**

$$(\alpha, \beta, \nu) = (1,2,3), (2,3,1), (3,1,2), (1,3,2), (2,1,3)(3,2,1)$$

$$\det A = a_{11}a_{22}a_{33} \begin{vmatrix} 1 & & \\ & 1 & \\ & & 1 \end{vmatrix} + a_{12}a_{23}a_{31} \begin{vmatrix} & 1 & \\ & & 1 \\ 1 & & \end{vmatrix} + a_{13}a_{21}a_{32} \begin{vmatrix} & & 1 \\ 1 & & \\ & 1 & \end{vmatrix}$$

$$+ a_{11}a_{23}a_{32} \begin{vmatrix} 1 & & \\ & & 1 \\ & 1 & \end{vmatrix} + a_{12}a_{21}a_{33} \begin{vmatrix} & 1 & \\ 1 & & \\ & & 1 \end{vmatrix} + a_{13}a_{22}a_{31} \begin{vmatrix} & & 1 \\ & 1 & \\ 1 & & \end{vmatrix}.$$

Explicit Formula - $n\times n$ Case

$$\det A = \sum_{\sigma}\left(a_{1\alpha}a_{2\beta}\cdots a_{n\nu}\right)\det P_{\sigma}.$$

where the sum is over all $n!$ permutations $\sigma=(\alpha, \beta,\ldots,\nu)$ of number (1, 2, …, n) and P_σ is a permutation matrix with entries $(1,\alpha)$, $(2,\beta)$, …,(n,ν) having value 1.

- **$\det P_\sigma = +1$ or -1 *depending on whether the number of exchanges is even or odd.***

$$P_\sigma = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} \text{ has column sequence} (\alpha,\beta,\nu) = (1,3,2) \Rightarrow \det P_\sigma = -1$$

$$P_\sigma = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix} \text{ has column sequence } \sigma = (3,1,2) \Rightarrow \det P_\sigma = (-1)^2 = 1$$

- **Check: 2×2 Case:**

$$\det A = a_{11}a_{22}\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_{12}a_{21}\det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21} (\text{or } ad - bc).$$

Cofactors - $n\times n$ Case

Example:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

$$=(-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3}a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- $\det A = \sum_{\sigma}\left(a_{1\alpha}a_{2\beta}\cdots a_{n\nu}\right)\det P_\sigma = \sum_{\alpha=1}^{n} a_{1\alpha}\sum_{\sigma'}\left(a_{2\beta}\cdots a_{n\nu}\right)\det P_{\sigma'}$

where $\sigma' = (\beta,\ldots,\nu)$ are $(n-1)!$ permutations over the set of numbers (1, 2, …, n)-α and $P_{\sigma'}$ is a $(n-1)\times(n-1)$ permutation matrix with entries $(1,\beta)$, …,$(n-1,\nu)$ having value 1.

- **Let** $A_{1\alpha} = \sum_{\sigma'}\left(a_{2\beta}\cdots a_{n\nu}\right)\det P_{\sigma'}$ **with permutation σ' over (1,2,…,n)-α**

  $\Rightarrow \det A = \sum_{\alpha} a_{1\alpha}A_{1\alpha} = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}.$

Example:

$$a_{11}(-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$=a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

- **$A_{1\alpha}$'s are called *cofactors***
- **If matrix M_{1j} is the matrix formed by throwing away row 1 and column j, then** $A_{1j} = (-1)^{1+j}\det M_{1j}$.

Expansion of det$A$ in Cofactors

- **The expansion can be conducted for any row i:**

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}.$$

where cofactor A_{ij} is the determinant of M_{ij} with correct sign:

$$A_{ij} = (-1)^{i+j} \det M_{ij}$$

where M_{ij} is formed by deleting row i and column j of A

- **Example: 4×4 finite difference matrix**

$$A_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

Cofactor method is most useful for a row with a lot of zeros.

$$M_{11} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = A_3$$

and $A_{12} = (-1)^{1+2} \det M_{12} = (-1)\det\begin{bmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = +\det\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \det A_2$

(column 1 of A_{12} is chosen for the last step) ⇒

$\det\ A_4 = (-1)^{1+1} a_{11}\left(\det\ A_3\right) - \det\ A_2.$

For n×n case, $\det\ A_n = 2\left(\det\ A_{n-1}\right) - \det\ A_{n-2}.$

Since detA_1=2, detA_2=3,…we have detA_n=n+1

Application of Determinants – Computing A^{-1}

$$AA_{\text{cof}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} \det A & 0 & \cdots & 0 \\ 0 & \det A & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \det A \end{bmatrix} = (\det A)I$$

Why?

- **For the main diagonals:** $\det A = a_{i1}A_{i1} + \cdots + a_{in}A_{in}$

- **But why is off-diagonal zero? For example, why the (1, 2)th element**

$a_{11}A_{21} + a_{12}A_{22} + \cdots + a_{1n}A_{2n} = 0$**?**

$$a_{11}A_{21} + a_{12}A_{22} + \cdots + a_{1n}A_{2n} = \det\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{11} & a_{12} & \cdots & a_{1n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = 0$$

- **We have** $(A)(A_{\text{cof}}) = (\det A)I$ **where A_{cof} is the *cofactor* matrix or *adjugate* matrix.**

$\Rightarrow$ $\underline{A^{-1} = \dfrac{1}{\det\ A} A_{\text{cof}}}$ **if det$A$=0 then A is not invertible.**

Examples of A^{-1} Computing

- **Example: 2×2 case**

Cofactors of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **are $A_{11} = d$, $A_{12} = -c$, $A_{21} = -b$, $A_{22} = a$:**

$$(A)(A_{\text{cof}}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = (ad-bc)I$$

Dividing by det$A$=ad-bc:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- **Example: Inverse of** $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is $\dfrac{A_{\text{cof}}}{\det A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$

Application of Determinants – Solution of *Ax=b*

$$x = A^{-1}b = \frac{1}{\det A} A_{\text{cof}} b$$

$$\Rightarrow\ x_j = \frac{\det B_j}{\det A}, \quad \text{where} \quad B_j = \begin{bmatrix} a_{11} & a_{12} & b_1 & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & b_n & a_{nn} \end{bmatrix}.$$

(*Cramer's rule*)

In B_j the vector b replaces the jth column of the original A.

Proof: Expand detB_j in the cofactors of the jth column (which is b now):

$\det B_j = b_1 A_{1j} + b_2 A_{2j} + \cdots + b_n A_{nj}.$ **= jth component of $A_{\text{cof}}b$**

and det B_j/det$A$ is the jth component of $A_{\text{cof}}b$/det A

- **Example: The solution of**

$$x_1 + 3x_2 = 0$$
$$2x_1 + 4x_2 = 6$$

is

$$x_1 = \frac{\begin{vmatrix} 0 & 3 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = \frac{-18}{-2} = 9, \quad x_2 = \frac{\begin{vmatrix} 1 & 0 \\ 2 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = \frac{6}{-2} = -3.$$

Application – Volume of Parallelepiped

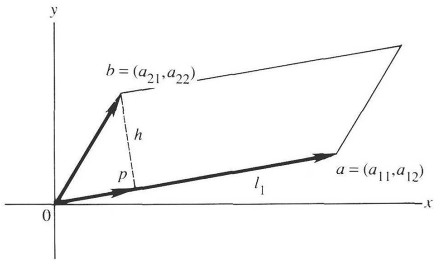
- **If all angles of the parallelepiped are right angle, all the vectors are perpendicular to one another:**

$$\textbf{Volume}=l_1 l_2 \cdots l_n$$

Where $l_1, l_2, \cdots, l_n$ **are the lengths of edges (vectors)**

- **Since** $AA^{\mathrm{T}} = \begin{bmatrix} \text{row }1 \\ \vdots \\ \text{row }n \end{bmatrix} \begin{bmatrix} \mathrm{r} & & \mathrm{r} \\ \mathrm{o} & & \mathrm{o} \\ \mathrm{w} & \cdots & \mathrm{w} \\ 1 & & n \end{bmatrix} = \begin{bmatrix} l_1^2 & & 0 \\ & \ddots & \\ 0 & & l_n^2 \end{bmatrix}$

$$l_1^2 l_2^2 \cdots l_n^2 = \det\left(AA^{\mathrm{T}}\right) = (\det A)\left(\det A^{\mathrm{T}}\right) = (\det A)^2. \Rightarrow |\det A| = l_1 l_2 \cdots l_n$$



$$\left\|\overrightarrow{pb}\right\| \times l_1 = \left\|\vec{b} - \vec{p}\right\| \times l_1 = \left\|\vec{b} - \hat{x}\vec{a}\right\| \times l_1 = \left|\det\begin{bmatrix} a_{21} - \hat{x}a_{11} & a_{22} - \hat{x}a_{12} \\ a_{11} & a_{12} \end{bmatrix}\right| = |\det A| \Rightarrow$$

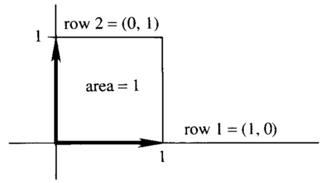
Gram-Schmidt process without dividing the length does not change the determinants

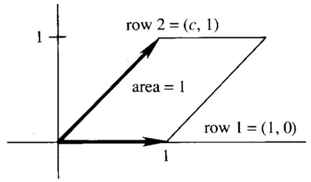
$\Rightarrow$ **det** A **= det** Q **= Volume where** Q **has orthogonal rows**

Example – Volume of Parallelepiped

Example:

$$\det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1, \quad \det\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} = 1.$$





The volume is not affected by the "shearing" factor c

Application – Formula for Pivots

$$A=\begin{bmatrix} a & b & e \\ c & d & f \\ g & h & i \end{bmatrix} \to \begin{bmatrix} a & b & e \\ 0 & (ad-bc)/a & (af-ec)/a \\ g & h & i \end{bmatrix} \Rightarrow$$

$$A=LDU=\begin{bmatrix} 1 & & \\ c/a & 1 & \\ * & * & 1 \end{bmatrix}\begin{bmatrix} a & & \\ & (ad-bc)/a & \\ & & * \end{bmatrix}\begin{bmatrix} 1 & b/a & * \\ & 1 & * \\ & & 1 \end{bmatrix}.$$

The first pivot depends only on [a] while the second pivot (ad-bc)/a depends only on A_2= $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- **If A is factored into LDU, then the upper left corners satisfy $A_k=L_kD_kU_k$**
- **For every k, the submatrix A_k is going through a Guassian elimination of its own**
- ***Block multiplication of matrices*:**

$$\begin{bmatrix} L_k & 0 \\ B & C \end{bmatrix}\begin{bmatrix} D_k & 0 \\ 0 & E \end{bmatrix}\begin{bmatrix} U_k & F \\ 0 & G \end{bmatrix}=\begin{bmatrix} L_kD_kU_k & L_kD_kF \\ BD_kU_k & BD_kF+CEG \end{bmatrix}$$

- **$A_k=L_kD_kU_k \Rightarrow$** $\det A_k=\det L_k \det D_k \det U_k=\det D_k=d_1d_2\cdots d_k$.

$$\Rightarrow \frac{\det A_k}{\det A_{k-1}}=\frac{d_1d_2\cdots d_k}{d_1d_2\cdots d_{k-1}}=d_k$$

Formula for Pivots - Implication

Since $\dfrac{\det A_k}{\det A_{k-1}}=\dfrac{d_1d_2\cdots d_k}{d_1d_2\cdots d_{k-1}}=d_k$

$$\Rightarrow d_1d_2\cdots d_n=\frac{\det A_1}{\det A_0}\frac{\det A_2}{\det A_1}\cdots\frac{\det A_n}{\det A_{n-1}}=\frac{\det A_n}{\det A_0}=\det A.$$

- **Implication: pivot entries are all nonzero whenever the numbers detA_k's are all nonzero.**

⇒ We don't need to exchange rows or multiply a permutation matrix during Gaussian elimination if and only if the leading submatrices A_1, A_2, A_3, …, A_n are all nonsingular where

A_1, A_2, A_3, …, A_n are upper-left corner submatrices

$$\begin{bmatrix} \begin{bmatrix} \begin{bmatrix} [A_1] & \\ & A_2 \end{bmatrix} & \\ & A_3 \end{bmatrix} & \\ & \ddots \end{bmatrix}$$