Practical Issues of SPC Charting

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Concerns of Using Control Charts

- Interpretation of SPC chart:
  - within limits: in control
  - out-of-limits: out of control

  HOW SIMPLE!!
- Issues of using a control chart:
  - Chart Robustness (Specificity/False alarm rate/Type I error prob.): How often does it give false alarms?
  - Chart Sensitivity (False negative rate/Type II error prob.): How quickly does it detect problems?
- SPC in practice
  - Charts for internal control: over-sensitive
  - Charts for external auditing: insensitive

Questions of Using Control Charts

- Why $\bar{X}$ chart? Why not X?
- Why $\pm 3\hat{\sigma}$? Why not $2.5\hat{\sigma}$ or $3.5\hat{\sigma}$?
- Sample size n? $n\uparrow \Rightarrow \beta\downarrow \Rightarrow$Sensitivity$\uparrow$(why?)
- How to group a sample?
- How often do you take the sample?
- What action should be taken when the chart signals?

An Unrealistic Solution

- Economic design of $\bar{X}$ control chart?
  - cost of sampling
  - cost of false alarm
  - cost of out of control
  - special cause inter-arrival time
  - determine control limits, sample size and sampling interval

  sounds good? NO...

  useless and **dangerous**

More Realistic Answers

- Why $\bar{X}$?
  - A normal distribution?
    - central limit theorem
  - **Improve control chart sensitivity (ARL_1)**
  - **Rational Subgrouping**
- Rational subgrouping
  - revisit the chance (common) and assignable (special) causes definitions
  - what causes do you want to detect?

Heartbeat Rate/Body Weight/Body Temp.

- Example: measuring Heartbeat Rate/Weight/Temp.
  - Measurements taken after/before three meals
  - Measurements taken from right ear, left ear and forehead
- Forming a group of heartbeat observations to compute $\bar{X}$:
  - Group data from a **day**? Data **after/before a meal**? From the same **body site**
- Rational subgrouping: an effective way to detect particular special causes (ex. heartbeat rate)
  - **Arrhythmia at night? influence of alcohol, caffeine, nicotine? heart diseases? COVID? sleep apnea? ...**

Rational Subgrouping

- To effectively detect special causes
  - distinguish chance (common) causes and assignable (special) causes
  - what special causes really concern you?
- Rational subgroups or samples: collections of individual measurements whose variation is subject only to a constant system of common causes.

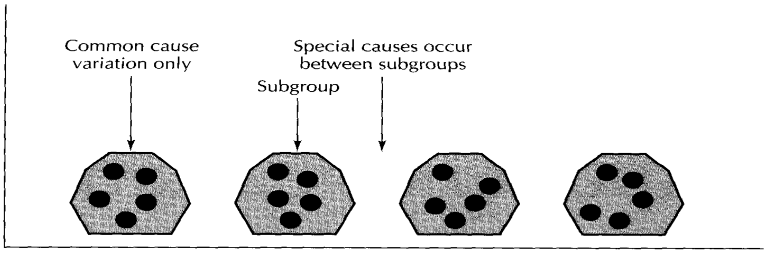
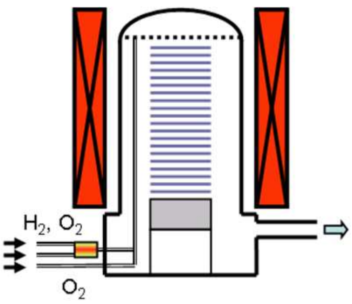
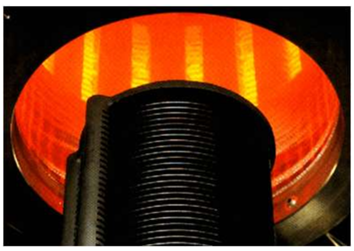


Figure 7.1 Graphical Depiction of the Rational Subgroup

Semiconductor Oxidation Processes

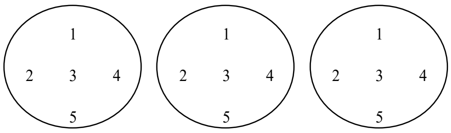
- Creating a thin layer of oxide by exposing the silicon to an oxidizing agent, such as oxygen, at high temperatures. The oxide layer serves as a critical insulating layer between different layers of the device. The thickness of the oxide layer can be controlled by adjusting the time and temperature of the oxidation process.
- The oxidation furnace is a large cylindrical chamber made of high-temperature resistant materials such as quartz, ceramic, or silicon carbide. Inside the chamber, the semiconductor material is heated to high temperatures ranging from 800˚C to 1200˚C.
- The furnace provides a controlled environment, including precise temperature control, the flow of the oxidizing agent, and the removal of any by-products of the reaction. The oxygen or water vapor is introduced into the furnace chamber through a gas inlet at a controlled flow rate.
- The oxidation furnace is designed to provide uniform heating of the semiconductor material to ensure that the oxide layer is created evenly across the entire surface.



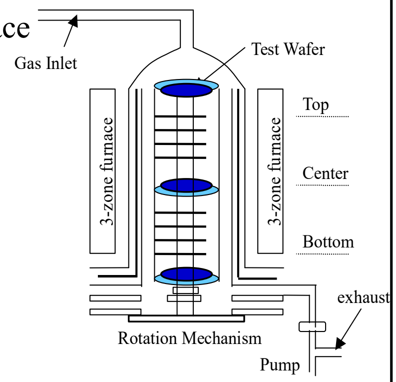


Rational Subgrouping for Oxidation Process

- Oxidation process with 3-zone furnace
- Sampling of thickness data



wafer 1(Top) wafer 2(Center) wafer 3(Bottom)



- Forming a group of thickness data to compute $\bar{X}$:
  - group data from a **lot**? from a **wafer**? from the same **site** on wafers
- Rational subgrouping: an effective way to detect particular special causes

Control Chart with Rational Samples

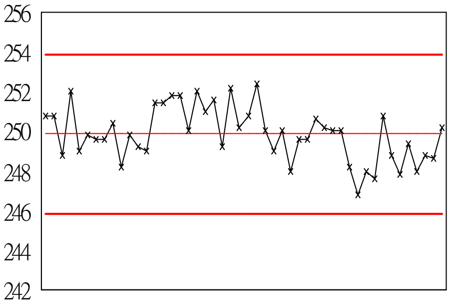
- Subgroup Selection:
  - common causes are included in a sample
  - special causes can be captured between samples
  - sampling should be economical
- Estimation of σ:
  - σ_{lot}: run-to-run lot variation
  - σ_{wafer}: run-to-run wafer variation
  - σ_{site}: run-to-run site variation

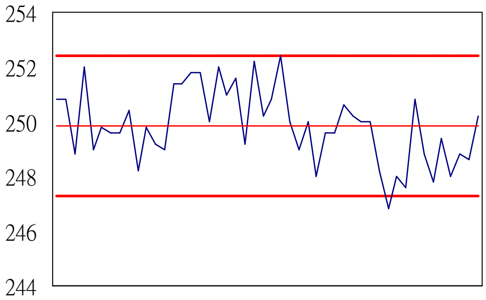
Monitoring Lot-to-lot Trend for Eeach Zone

- Estimating $\sigma_{\text{top-zone}}$:

$$\bar{\bar{X}} = 249.85$$
$$\hat{\sigma}_{\bar{X}} = S_{\bar{X}} = 1.35$$
$$\text{Control limits} = 249.85 \pm 3 \times 1.35$$

$$\hat{\sigma}_{\bar{X}} = \frac{S_X}{\sqrt{n}} = \frac{1.895}{\sqrt{5}} = 0.848$$
$$\text{Control limits} = 249.85 \pm 3 \times 0.848$$



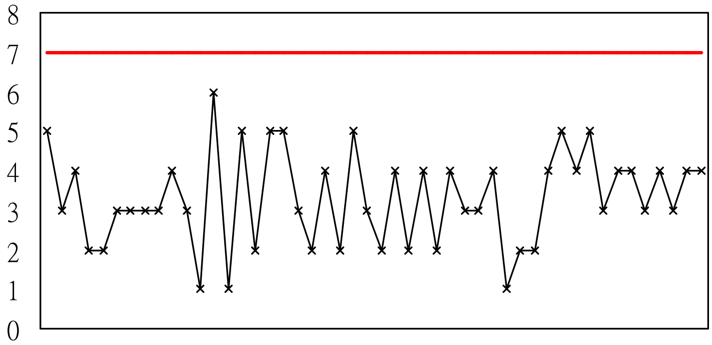


Which control chart is correct?

Monitoring Within-Wafer Variation

- Estimating σ for variation within wafer

$$\hat{\sigma}_R = s_R$$
$$\text{Control limits} = \bar{R} + 3s_R \left(\max(0, \bar{R} - 3s_R\right)$$



Monitoring Lot-to-Lot Trend for Each Site

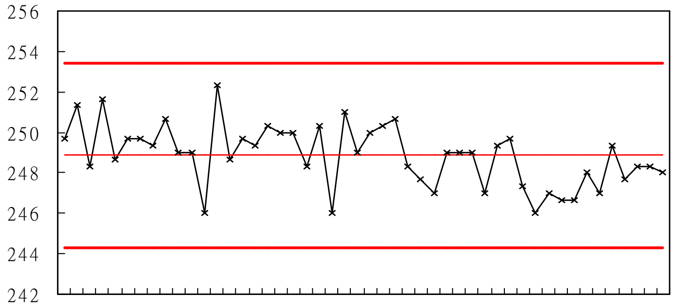
- Estimating $\sigma_{\text{center-site}}$

$$\text{sample statistic} = \frac{\text{site3 of wafer1} + \text{site3 of wafer2} + \text{site3 of wafer3}}{3}$$

$$\overline{\overline{X}} = 248.86$$

$$\hat{\sigma}_{\overline{X}} = S_{\overline{X}} = 1.52$$

$$\text{Control limits} = 248.86 \pm 3 \times 1.52$$



Run-to-Run $\overline{X}$-R Chart

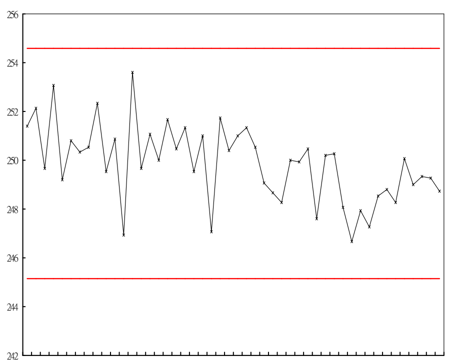
$$\overline{\overline{X}} = 249.87$$

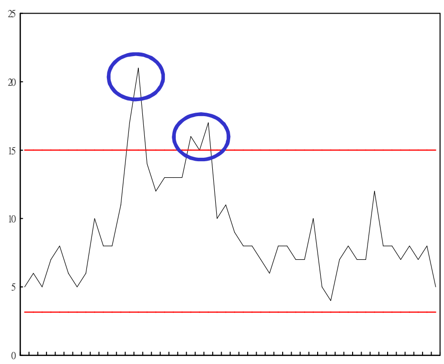
$$\hat{\sigma}_{\overline{X}} = S_{\overline{X}} = 1.572$$

$$\text{Control limits} = 249.87 \pm 3 \times 1.572$$

$$\hat{\sigma}_R = s_R$$

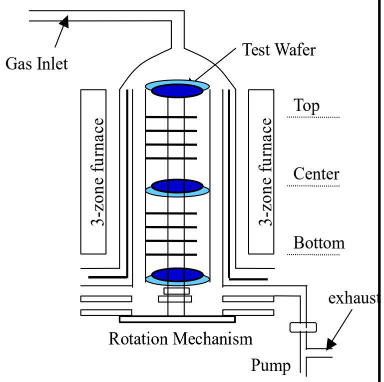
$$\text{Control limits} = \overline{R} + 3s_R (\max(0, \overline{R} - 3s_R)$$





Sampling Size - Example

- Example: sampling body temperature
  - How many times a day?
  - How many measurements each time?
  - How often? every day? every other day? every week?
- Example: sampling thickness data
  - how many wafers from one run?
  - how many measurements from a wafer?
  - sampling from every lot? every other lot? every 3 lots?



Sample Size Considerations

- should be able to detect special causes
- should ensure a normal distribution for the sample mean
- should ensure good sensitivity to the detection of special causes
- should be small enough to be economically appealing
- Example: body-temperature control

Sampling Frequency Considerations

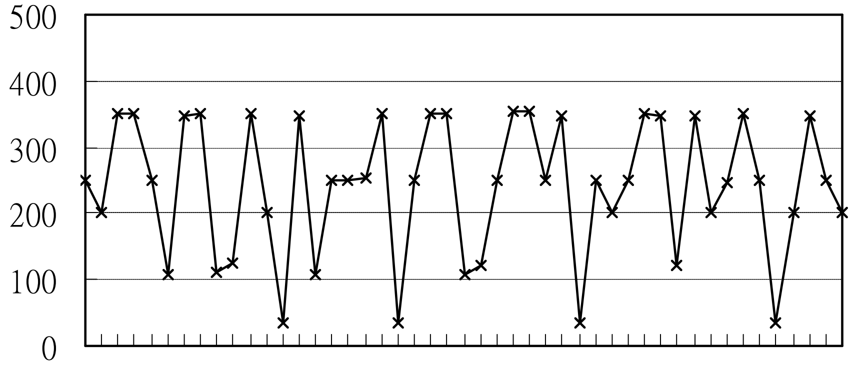
- General nature of process stability
  - Erratic process: more frequent
  - Stable process: less frequent
- Frequency of events:
  - events: ambient condition fluctuations, raw material changes, process adjustment, shift changes, etc
- Cost of sampling:
  - should not be overemphasized
  - considered only when sampling and testing are destructive (expensive) and time-consuming
- Example: body-temperature control

Short Run Production

- Process with high product mix:

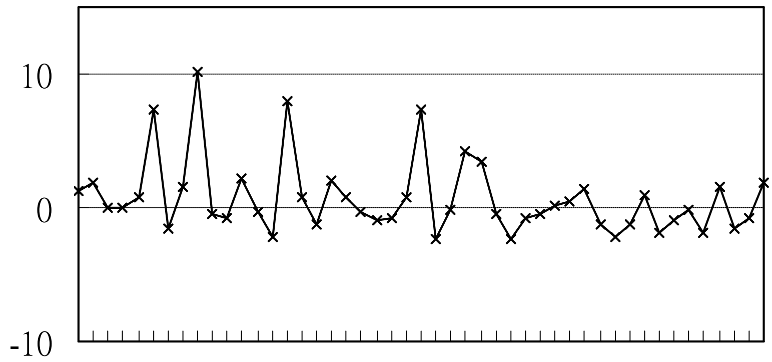
	Target	Obs.		Target	Obs.		Target	Obs.
1	250	251.2	21	250	249	41	350	349
2	200	201.8	22	350	349.2	42	250	249.8
3	350	350	23	350	350.8	43	35	33.18
4	350	350	24	100	107.4	44	200	201.6
5	250	250.8	25	125	122.6	45	350	348.4
6	100	107.4	26	250	249.8	46	250	249.2
7	350	348.4	27	350	354.2	47	200	201.8
8	350	351.6	28	350	353.4	48	250	249
9	100	110.2	29	250	249.6	49	35	33.06
10	125	124.6	30	350	347.6	50	350	355.6
11	350	349.2	31	35	34.16			
12	200	202.2	32	250	249.6			
13	35	34.66	33	200	200.2			
14	350	347.8	34	250	250.4			
15	100	108	35	350	351.4			
16	250	250.8	36	350	348.8			
17	250	248.8	37	125	122.8			
18	250	252	38	350	348.8			
19	350	350.8	39	200	201			
20	35	34.76	40	250	248.2			

Mixed Production Data



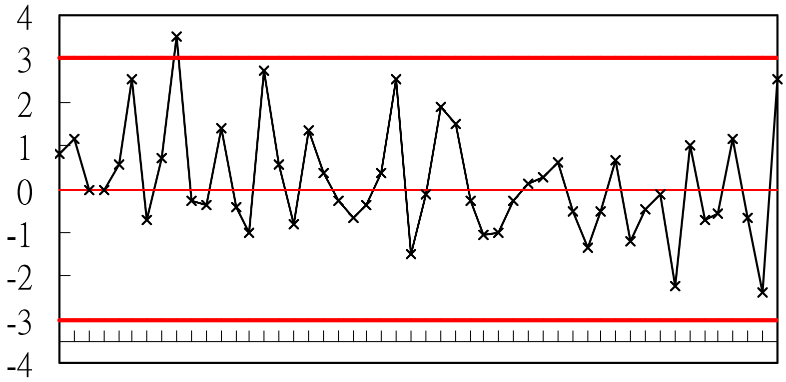
Mixed Production Data Adjusted By Target (Mean)Values

Observation Adjusted By Target (Mean)
= Observation - Corresponding Target (Mean)



Model-based Control Chart

- $Z_i=(X_i\text{-Target or Mean})/\sigma_{\text{target}} \sim N(0, 1)$
- Control limits = 0.0 ± 3.0



EWMA and CUSUM Control Charts

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Limitation of Shewhart Control Chart

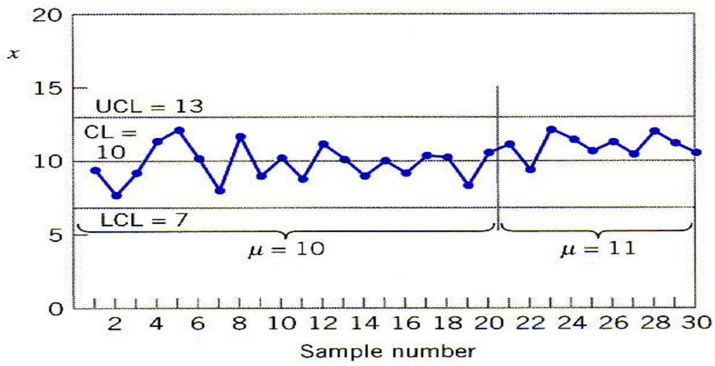
- Without using the entire sets of data points, Shewhart chart is relatively insensitive to small shifts in the process, say on the order of about 1.5σ, or less.
- Runs rules are added to increase the sensitivity (reduce β error) but can dramatically reduce the robustness (increase α error) of the control chart.
- Two very effective alternatives to the Shewhart chart may be used for small shifts detection:
  - Cumulative-sum (CUSUM) control chart
  - Exponentially weighted moving average (EWMA) control chart

Idea of Cumulated Sum

- CUSUM: CUmulative SUM
- First 20 points, $N(10, 1)$, last 10 points, $N(11, 1)$
- Cumulated Sum: $C_i = \sum_{j=1}^{i}(x_j - 10) = (x_i - 10) + C_{i-1}$ (where 10 is the original mean)

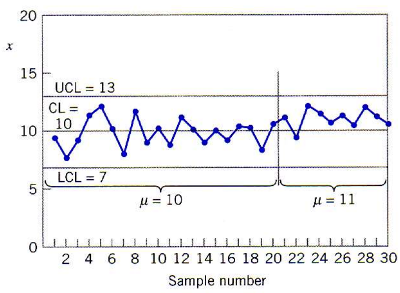
Sample, i	(a) x_i	(b) $x_i - 10$	(c) $C_i = (x_i - 10) + C_{i-1}$
1	9.45	−0.55	−0.55
2	7.99	−2.01	−2.56
3	9.29	−0.71	−3.27
4	11.66	1.66	−1.61
5	12.16	2.16	0.55
6	10.18	0.18	0.73
7	8.04	−1.96	−1.23
8	11.46	1.46	0.23
9	9.20	−0.80	−0.57
10	10.34	0.34	−0.23
11	9.03	−0.97	−1.20
12	11.47	1.47	0.27
13	10.51	0.51	0.78
14	9.40	−0.60	0.18
15	10.08	0.08	0.26
16	9.37	−0.63	−0.37
17	10.62	0.62	0.25
18	10.31	0.31	0.56
19	8.52	−1.48	−0.92
20	10.84	0.84	−0.08
21	10.90	0.90	0.82
22	9.33	−0.67	0.15
23	12.29	2.29	2.44
24	11.50	1.50	3.94
25	10.60	0.60	4.54
26	11.08	1.08	5.62
27	10.38	0.38	6.00
28	11.62	1.62	7.62
29	11.31	1.31	8.93
30	10.52	0.52	9.45

Small Shift on Shewhart Chart

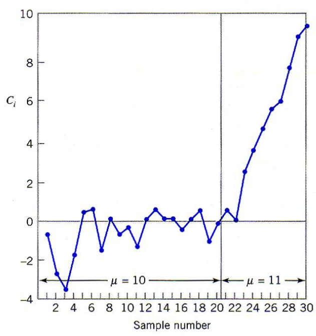


- Shewhart chart is not sensitive to a small shift.

X vs. Cumulated Sum of $X – \mu_0$



Shewhart chart

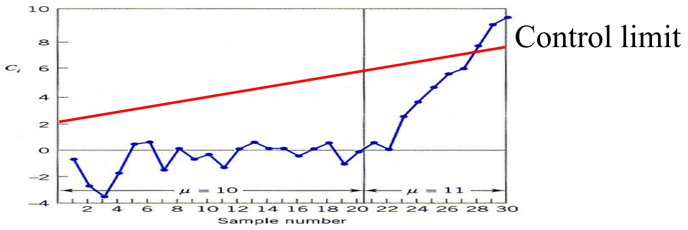


Cumulated Sum

- Shewhart chart is not sensitive to a small shift.
- Cumulated Sum is more sensitive to a small shift.

Problem with Cumulated Sum

- Cumulated Sum: $\Sigma_i(x_i - \mu_0)$
- To construct control limits: $E(y) \pm k \cdot \sigma_y$
- Variance of cumulated sum is required: $Var(\Sigma_i(x_i - \mu_0)) = Var(\Sigma_i x_i) = \Sigma_i Var(x)$
  $\Rightarrow \Sigma_i Var(x) \uparrow$ as $i \uparrow$
  $\Rightarrow$ variance increases as sample size increases
  $\Rightarrow$ control limit $\uparrow$ as $i \uparrow$



Sequential Likelihood Ratio Test

- CUSUM control chart is based on the sequential likelihood ratio test, where, as each new sample becomes available, a test is conducted to determine if the process mean deviates from the target value.
- Sequential probability ratio test:

  H_0: $\Theta = \Theta_0$ vs. $\mathbf{H_1}$: $\Theta = \Theta_1$

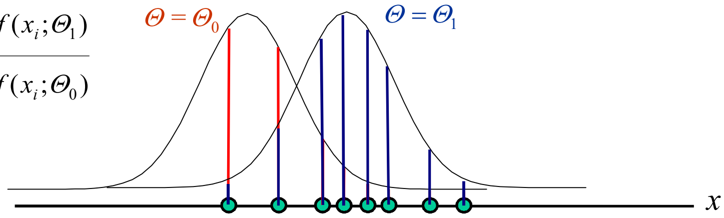
- Test statistic: $R = \dfrac{\prod_{i=1}^{t} f(x_i; \Theta_1)}{\prod_{i=1}^{t} f(x_i; \Theta_0)}$
- Accept H_1 (Reject H_0): $R \geq \dfrac{1-\beta}{\alpha}$
- Accept H_0 (Reject H_1): $R \leq \dfrac{\beta}{1-\alpha}$

where α and β are probabilities of type I and type II errors.

Intuition on Sequential Likelihood Ratio Test

- If x_i is from the dis. of $\Theta=\Theta_1$
  - $\prod_{i=1}^{t} f(x_i;\Theta_1)$ will be large
  - $\prod_{i=1}^{t} f(x_i;\Theta_0)$ will be small
  - then R will be large
  - if $R \geq \frac{1-\beta}{\alpha}$, then accept H_1
- If x_i is from the dis. of $\Theta=\Theta_0$
  - $\prod_{i=1}^{t} f(x_i;\Theta_1)$ will be small
  - $\prod_{i=1}^{t} f(x_i;\Theta_0)$ will be large
  - then R will be small
  - if $R \leq \frac{\beta}{1-\alpha}$, then accept H_0

$$R = \frac{\prod_{i=1}^{t} f(x_i;\Theta_1)}{\prod_{i=1}^{t} f(x_i;\Theta_0)}$$



Testing Normal Mean by Sequential Likelihood Ratio Test

- Test $H_0: \mu=\mu_0$ vs. $H_1: \mu=\mu_1 \quad (\mu_1 > \mu_0)$

  x_i's are assumed to be normally distributed with variance σ^2

$$R=\frac{\prod_{i=1}^{t}\frac{1}{\sqrt{2\pi}\sigma}e^{-(x_i-\mu_1)^2/2\sigma^2}}{\prod_{i=1}^{t}\frac{1}{\sqrt{2\pi}\sigma}e^{-(x_i-\mu_0)^2/2\sigma^2}}=\frac{(\frac{1}{\sqrt{2\pi}\sigma})^t e^{-\sum_{i=1}^{t}(x_i-\mu_1)^2/2\sigma^2}}{(\frac{1}{\sqrt{2\pi}\sigma})^t e^{-\sum_{i=1}^{t}(x_i-\mu_0)^2/2\sigma^2}}=e^{\frac{\Delta}{\sigma^2}\sum_{i=1}^{t}(x_i-(\mu_0+\mu_1)/2)} \quad (\text{here, } \Delta=\mu_1-\mu_0)$$

- Reject H_0 if $R \geq \frac{1-\beta}{\alpha} \Rightarrow \ln(R) \geq \ln(\frac{1-\beta}{\alpha})$

$$\Rightarrow C_t = \sum_{i=1}^{t}(x_i - \frac{\mu_0+\mu_1}{2}) \geq \frac{\sigma^2}{\Delta}\ln(\frac{1-\beta}{\alpha}) \Rightarrow \text{CL is constant}$$

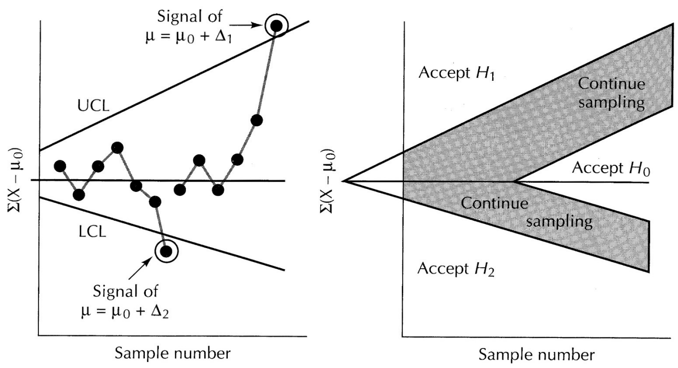
Sample statistic **Control limit (reject region)** $\Rightarrow$ Tabular Form

How about $S_t = \sum_{i=1}^{t}(x_i-\mu_0) \geq \frac{\sigma^2}{\Delta}\ln(\frac{1-\beta}{\alpha}) + \frac{\Delta}{2}\boxed{t} \Rightarrow$ CL $\uparrow$ as $t \uparrow$

(follow the same procedure for testing H_1)

Two-sided V-Sharp CUSUM Control Charts

$$S_t = \sum_{i=1}^{t}(x_i - \mu_0) \geq \frac{\sigma^2}{\Delta}\ln(\frac{1-\beta}{\alpha}) + \frac{\Delta}{2}t \Rightarrow \text{CL} \uparrow \text{ as } t \uparrow \Rightarrow \text{V mask CUSUM}$$



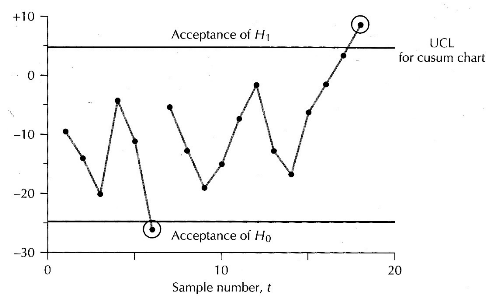
In practice, difficult to use!

Tabular Form of Sequential Likelihood Ratio Test

- Recall that $C_t = \sum_{i=1}^{t}(x_i - \frac{\mu_0 + \mu_1}{2})$ (here, $\mu_1 > \mu_0$)
- Accept H_1 when $C_t \geq \frac{\sigma^2}{\Delta}\ln(\frac{1-\beta}{\alpha})$ or accept H_0 when $C_t \leq \frac{\sigma^2}{\Delta}\ln(\frac{\beta}{1-\alpha})$

$$E(C_t) = \sum_{i=1}^{t}\frac{\mu_0 - \mu_1}{2} \text{ when } E(x_i) = \mu_0 \Rightarrow E(C_t) \text{ decreases } \Rightarrow \text{accept } H_0$$

$$= \sum_{i=1}^{t}\frac{\mu_1 - \mu_0}{2} \text{ when } E(x_i) = \mu_1 \Rightarrow E(C_t) \text{ increases } \Rightarrow \text{accept } H_1$$



Tabular Form of CUSUM Scheme

- To test the likelihood ratio for a same population:

$$C_i^+ = \max[0, x_i - (\mu_0 + K) + C_{i-1}^+]$$

where $K = \frac{\mu_1 - \mu_0}{2}$ since $x_i - (\mu_0 + K) = x_i - \frac{\mu_0 + \mu_1}{2}$

- When the process is *in-control* ($E(x_i) = \mu_0$) the cumulated sum will tend to be negative
- Reset the testing once the CUSUM statistic becomes negative.
- Only when the CUSUM statistic is positive the testing continues to cumulate.

Tabular (Algorithmic) CUSUM for Monitoring Process Mean

- Tabular CUSUM

  $C_i^+ = \max[0, x_i - (\mu_0 + K) + C_{i-1}^+]$

  $C_i^- = \min[0, x_i - (\mu_0 - K) + C_{i-1}^-] (C_i^- = \max[0, (\mu_0 - K) - x_i + C_{i-1}^-])$

  (the starting values $C_0^+ = C_0^- = 0$)

  (μ_0 is the target value)

- Design parameters
  - K (or $k\sigma$) is the reference value
  - H (or $h\sigma$) is the decision interval

- Decision Rule

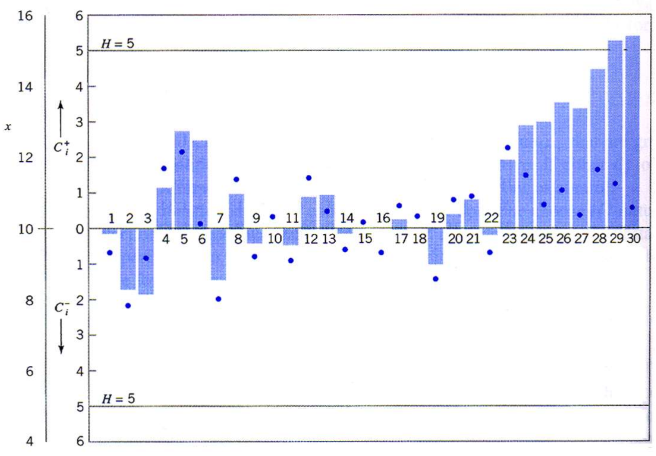
  If $C_i^+ > H$ or $C_i^- < -H$ ($C_i^- > H$), the process is out-of-control.

Tabular CUSUM Example

- Target value $\mu_0 = 10$
- Shifted value $\mu_1 = 11$
- $K = (\mu_1 - \mu_0) / 2 = 0.5$
- $H = 5\sigma = 5$
- $C_i^+ = \max[0, x_i - 10.5 + C_{i-1}^+]$
- $C_i^- = \max[0, 9.5 - x_i + C_{i-1}^-]$

Period		a			b		
i	x_i	$x_i - 10.5$	C_i^+	N^+	$9.5 - x_i$	C_i^-	N^-
1	9.45	−1.05	0	0	0.05	0.05	1
2	7.99	−2.51	0	0	1.51	1.56	2
3	9.29	−1.21	0	0	0.21	1.77	3
4	11.66	1.16	1.16	1	−2.16	0	0
5	12.16	1.66	2.82	2	−2.66	0	0
6	10.18	−0.32	2.50	3	−0.68	0	0
7	8.04	−2.46	0.04	4	1.46	1.46	1
8	11.46	0.96	1.00	5	−1.96	0	0
9	9.20	−1.3	0	0	0.30	0.30	1
10	10.34	−0.16	0	0	−0.84	0	0
11	9.03	−1.47	0	0	0.47	0.47	1
12	11.47	0.97	0.97	1	−1.97	0	0
13	10.51	0.01	0.98	2	−1.01	0	0
14	9.40	−1.10	0	0	0.10	0.10	1
15	10.08	−0.42	0	0	−0.58	0	0
16	9.37	−1.13	0	0	0.13	0.13	1
17	10.62	0.12	0.12	1	−1.12	0	0
18	10.31	−0.19	0	0	−0.81	0	0
19	8.52	−1.98	0	0	0.98	0.98	1
20	10.84	0.34	0.34	1	−1.34	0	0
21	10.90	0.40	0.74	2	−1.40	0	0
22	9.33	−1.17	0	0	0.17	0.17	1
23	12.29	1.79	1.79	1	−2.79	0	0
24	11.50	1.00	2.79	2	−2.00	0	0
25	10.60	0.10	2.89	3	−1.10	0	0
26	11.08	0.58	3.47	4	−1.58	0	0
27	10.38	−0.12	3.35	5	−0.88	0	0
28	11.62	1.12	4.47	6	−2.12	0	0
29	11.31	0.81	5.28	7	−1.81	0	0
30	10.52	0.02	5.30	8	−1.02	0	0

CUSUM Status Chart



Estimate of the New Process Mean

- Change Point Estimate: when out-of-control is signaled, the point the accumulation starts is said to be the point where the change approximately begins.
- New process mean is: $\dfrac{\sum x_i}{N^+} = \mu_0 + K + \dfrac{C_i^+}{N^+}$ if $C_i^+ > H$

$$\frac{\sum x_i}{N^+} = \mu_0 - K - \frac{C_i^-}{N^-} \quad \text{if } C_i^- > H$$

- For example, $\bar{\mu} = \mu_0 + K + \dfrac{C_{29}^+}{N^+}$

$$= 10 + 0.5 + \frac{5.28}{7}$$

$$= 11.25$$

CUSUM Design Based on ARL

- Define $K = k\sigma$, $H = h\sigma$
- Choose $k = 0.5\delta$
  ($\delta = \Delta/\sigma$ is the size of the shift in standard deviation units)
- Choose h to minimize ARL_1 for fixed ARL_0

ARL performance
($k = 0.5$, $h = 4$ or $h = 5$)

Shift in Mean (multiple of σ)	$h = 4$	$h = 5$
0	168	465
0.25	74.2	139
0.50	26.6	38.0
0.75	13.3	17.0
1.00	8.38	10.4
1.50	4.75	5.75
2.00	3.34	4.01
2.50	2.62	3.11
3.00	2.19	2.57
4.00	1.71	2.01

ARL_0

ARL_1

$\text{ARL}_0 = 370$

k	h
0.25	8.01
0.5	4.77
0.75	3.34
1.0	2.52
1.25	1.99
1.5	1.61

CUSUM Chart ARL Calculation

$$ARL_{1|\mu_1=\mu_0+\delta\sigma} =$$

$$[\frac{2(\delta-k)^2}{e^{-2(\delta-k)h'}-1+2(\delta-k)h'}+\frac{2(\delta+k)^2}{e^{2(\delta+k)h'}-1-2(\delta+k)h'}]^{-1}$$

where $h' = h + 1.166$

$$ARL_0 = ARL_{1|\delta=0} = \frac{e^{2kh'}-1-2kh'}{(2k)^2}$$

Srivastava and Wu, "Evaluation of Optimum Weights and Average Run Lengths in EWMA Control Schemes," *Commun. Statist. – Theory Meth.* 26(5), 1253-1267 (1997)

Optimal CUSUM Chart Design

Given $\delta *$ and ARL_0

$k* = \delta * /2$

Find $h*$ from $ARL_0 = \frac{e^{2k*h'}-1-2k*h'}{(2k*)^2}$

where $h' = h * + 1.166$

$$ARL^*_{1|\delta *} = [\frac{\delta *^2 /2}{e^{-\delta * h'}-1+\delta * h'}+\frac{(3\delta *)^2 / 2}{e^{3\delta * h'}-1-3\delta * h'}]^{-1}$$

Srivastava and Wu, "Evaluation of Optimum Weights and Average Run Lengths in EWMA Control Schemes," *Commun. Statist. – Theory Meth.* 26(5), 1253-1267 (1997)

Shewhart Scheme with Runs Rules

- Runs Rule ***T(k, m, a, b)***: Signal out-of-control when ***k*** out of ***m*** points are in the interval ***($a\sigma_x$, $b\sigma_x$)***.
- Example 1: T(1, 1, 3, ∞) signals when one data point are above the $3\sigma_x$ limit.
- Example 2: T(2, 3, 2, 3) signals when two out of three points fall in the interval ($2\sigma_x$, $3\sigma_x$).
- Common runs rules:
  - C_1={T(1, 1, - ∞, -3), T(1, 1, 3, ∞)}
  - C_2={T(2, 3, -3, -2), T(2, 3, 2, 3)}
  - C_3={T(4, 5, -3, -1), T(4, 5, 1, 3)}
  - C_4={T(8, 8, -3, 0), T(8, 8, 0, 3)}
- Joint Rules:

$$\boldsymbol{C}_{1234} = \boldsymbol{C}_1 \cup \boldsymbol{C}_2 \cup \boldsymbol{C}_3 \cup \boldsymbol{C}_4$$

ARL's for Shewhart Scheme with Runs Rules

Control chart

Shift d	C_1	C_7	C_{12}	C_{78}	C_{15}	C_{13}	C_{14}	C_{79}	C_{16}	C_{123}	C_{156}	C_{124}	C_{789}	C_{134}	C_{1456}	C_{1234}
.0	370.40	499.62	225.44	239.75	278.03	166.05	152.73	170.41	349.38	132.89	266.82	122.05	126.17	105.78	133.21	91.75
.2	308.43	412.01	177.56	185.48	222.59	120.70	110.52	120.87	279.53	97.86	208.44	89.14	91.19	76.01	96.37	66.80
.4	200.08	262.19	104.46	106.15	134.17	63.88	59.76	63.80	165.48	52.93	119.47	48.71	49.19	40.95	51.94	36.61
.6	119.67	153.86	57.92	57.80	75.27	33.99	33.64	35.46	89.07	28.70	63.70	27.49	27.57	23.15	29.01	20.90
.8	71.55	90.41	33.12	32.75	42.96	19.78	21.07	22.09	48.40	16.93	34.96	17.14	17.14	14.62	17.94	13.25
1.0	43.89	54.55	20.01	19.70	25.61	12.66	14.58	15.26	27.74	10.95	20.43	11.73	11.71	10.19	12.19	9.22
1.2	27.82	34.03	12.81	12.62	16.06	8.84	10.90	11.42	17.05	7.68	12.83	8.61	8.59	7.66	8.90	6.89
1.4	18.25	21.97	8.69	8.58	10.60	6.62	8.60	9.05	11.28	5.76	8.65	6.63	6.62	6.08	6.84	5.41
1.6	12.38	14.68	6.21	6.16	7.36	5.24	7.03	7.44	7.98	4.54	6.22	5.27	5.27	5.01	5.42	4.41
1.8	8.69	10.15	4.66	4.64	5.36	4.33	5.85	6.24	5.97	3.73	4.71	4.27	4.27	4.24	4.39	3.68
2.0	6.30	7.25	3.65	3.65	4.07	3.68	4.89	5.25	4.67	3.14	3.72	3.50	3.52	3.65	3.61	3.13
2.2	4.72	5.36	2.96	2.98	3.22	3.18	4.08	4.41	3.78	2.70	3.04	2.91	2.94	3.17	3.01	2.70
2.4	3.65	4.08	2.48	2.51	2.64	2.78	3.38	3.67	3.14	2.35	2.55	2.47	2.50	2.77	2.54	2.35
2.6	2.90	3.20	2.13	2.17	2.22	2.43	2.81	3.05	2.64	2.07	2.19	2.13	2.16	2.43	2.19	2.07
2.8	2.38	2.59	1.87	1.91	1.93	2.14	2.35	2.54	2.26	1.85	1.91	1.87	1.91	2.14	1.91	1.85
3.0	2.00	2.15	1.68	1.71	1.70	1.89	1.99	2.14	1.95	1.67	1.70	1.68	1.71	1.89	1.70	1.67

ARL_0

ARL_1

Shewhart and CUSUM Comparison

Ratios of ARL: CUSUM Chart to the corresponding Shewhart chart with supplementary runs rules (shown in parentheses):

ARL(CUSUM)/ARL(Shewhart)

Where k=0.5 and h is adjusted to match the ARL_0 of Shewhart Charts

CUSUM decision interval H

Shift d	4.78 (C_1)	5.07 (C_7)	4.29 (C_{12})	4.35 (C_{78})	4.50 (C_{15})	4.00 (C_{13})	3.91 (C_{14})	4.02 (C_{79})	4.72 (C_{16})	3.78 (C_{123})	4.46 (C_{156})	3.70 (C_{124})	3.73 (C_{789})	3.56 (C_{134})	3.78 (C_{1456})	3.42 (C_{1234})
.0	1.01	1.00	1.00	1.00	1.01	1.01	1.00	1.00	1.00	1.01	1.01	1.01	1.01	1.00	1.00	1.00
.2	.53	.49	.65	.65	.61	.77	.79	.78	.56	.81	.63	.83	.83	.88	.82	.89
.4	.27	.24	.42	.43	.36	.60	.62	.61	.32	.66	.40	.68	.69	.81	.67	.79
.6	.21	.17	.37	.38	.30	.57	.56	.55	.27	.63	.35	.64	.64	.72	.62	.76
.8	.20	.17	.39	.40	.32	.60	.55	.54	.29	.66	.38	.64	.65	.72	.63	.76
1.0	.23	.19	.45	.46	.37	.66	.56	.55	.35	.73	.46	.66	.67	.74	.65	.78
1.2	.27	.23	.53	.55	.44	.73	.58	.56	.44	.79	.55	.70	.70	.76	.69	.81
1.4	.33	.29	.64	.65	.54	.79	.59	.58	.53	.86	.66	.73	.74	.77	.72	.84
1.6	.41	.36	.75	.76	.66	.83	.61	.59	.63	.92	.77	.78	.78	.79	.77	.87
1.8	.51	.45	.86	.87	.78	.87	.63	.61	.73	.97	.88	.83	.84	.81	.82	.91
2.0	.61	.56	.97	.98	.90	.91	.67	.64	.82	1.02	.98	.90	.90	.84	.89	.95
2.2	.73	.68	1.07	1.08	1.02	.94	.72	.68	.90	1.06	1.08	.97	.97	.87	.95	.99
2.4	.86	.81	1.16	1.16	1.13	.98	.80	.75	.99	1.11	1.16	1.04	1.04	.91	1.03	1.03
2.6	.99	.94	1.24	1.24	1.24	1.04	.88	.83	1.08	1.17	1.25	1.12	1.11	.95	1.11	1.09
2.8	1.12	1.08	1.32	1.30	1.32	1.09	.98	.93	1.17	1.22	1.32	1.19	1.17	1.01	1.18	1.14
3.0	1.25	1.22	1.41	1.39	1.44	1.18	1.11	1.08	1.29	1.29	1.40	1.27	1.26	1.10	1.27	1.24

ARL_0

ARL_1

EWMA Control Chart

- The performance of the EWMA chart is approximately equivalent to that of CUSUM chart.
- The exponentially weighted moving average is defined as:

$$Z_i = \lambda x_i + (1-\lambda) Z_{i-1}$$
$$0 < \lambda \le 1$$

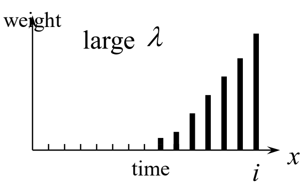
- EWMA Z_i is a weighted average of all previous sample means:

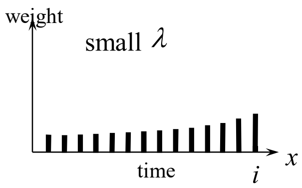
$$\begin{aligned} Z_i &= \lambda x_i + (1-\lambda) Z_{i-1} \\ &= \lambda x_i + (1-\lambda)[\lambda x_{i-1} + (1-\lambda) Z_{i-2}] \\ &= \sum_{j=0}^{i-1} \lambda (1-\lambda)^j x_{i-j} + (1-\lambda)^i Z_0 \end{aligned}$$

- $Z_0 = \mu_0$:zero state; $Z_0 = \overline{X}$:steady state.

Effect of λ on EWMA Weighting Scheme

- A higher λ assigns higher weights to the latest data points.
  - If $\lambda = 1$, then EWMA is like a Shewhart scheme (roughly speaking).
- A lower λ assigns equal weights to all data points.
  - If λ approaches 0, then EWMA is like a CUSUM scheme (roughly speaking).





The EWMA Control Chart

- If the observations x_i are iid with μ_0 and variance σ^2, $E(Z_i) = \mu_0$
- If the observations x_i are iid with variance σ^2, then variance of Z_i is

$$\sigma_{Z_i}^2 = \sigma^2\left(\frac{\lambda}{2-\lambda}\right)[1-(1-\lambda)^{2i}]$$

- The control limits are

$$\text{UCL} = \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}[1-(1-\lambda)^{2i}]}$$

$$\text{LCL} = \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}[1-(1-\lambda)^{2i}]}$$

- If i gets larger, then control limits become

$$\text{UCL} = \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}}$$

$$\text{LCL} = \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}}$$

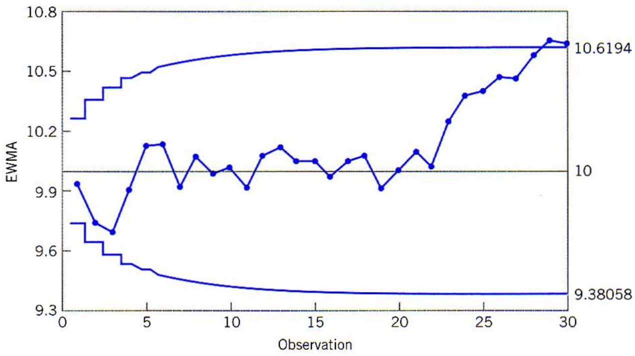
EWMA Example

$\lambda = 0.10, L = 2.7$

* = Beyond Limits

Subgroup, i	x_i	EWMA, z_i
1	9.45	9.945
2	7.99	9.7495
3	9.29	9.70355
4	11.66	9.8992
5	12.16	10.1253
6	10.18	10.1307
7	8.04	9.92167
8	11.46	10.0755
9	9.2	9.98796
10	10.34	10.0232
11	9.03	9.92384
12	11.47	10.0785
13	10.51	10.1216
14	9.4	10.0495
15	10.08	10.0525
16	9.37	9.98426
17	10.62	10.0478
18	10.31	10.074
19	8.52	9.91864
20	10.84	10.0108
21	10.9	10.0997
22	9.33	10.0227
23	12.29	10.2495
24	11.5	10.3745
25	10.6	10.3971
26	11.08	10.4654
27	10.38	10.4568
28	11.62	10.5731
29	11.31	10.6468*
30	10.52	10.6341*

EWMA Chart



EWMA Design Based on ARL

- Design parameters: λ and L
- Choose λ and L to minimize ARL_1 for fixed ARL_0
- In general, $L = 3$ and λ between 0.05 and 0.25 work well.

ARL for several EWMA schemes

Shift or Mean (multiple of σ)	L = 3.054 λ = 0.40	2.998 0.25	2.962 0.20	2.814 0.10	2.615 0.05
0	500	500	500	500	500
0.25	224	170	150	106	84.1
0.50	71.2	48.2	41.8	31.3	28.8
0.75	28.4	20.1	18.2	15.9	16.4
1.00	14.3	11.1	10.5	10.3	11.4
1.50	5.9	5.5	5.5	6.1	7.1
2.00	3.5	3.6	3.7	4.4	5.2
2.50	2.5	2.7	2.9	3.4	4.2
3.00	2.0	2.3	2.4	2.9	3.5
4.00	1.4	1.7	1.9	2.2	2.7

ARL_0

ARL_1

- Smaller λ works better for smaller shifts while larger λ works better for larger shifts

EWMA Chart ARL Calculation

Let $g = L(\frac{\lambda}{2\delta^2})^{1/2}$ and $w = L + 1.166(\delta\lambda)^{1/2} - (\frac{2\delta^2}{\lambda})^{1/2}$. Then,

(a) For $g < 1$ and $0 < \lambda \le 0.75$

$$ARL_{1|\delta} \approx -(\frac{1}{\lambda})\ln(1-g) - \frac{g}{4(1-g)\delta^2} + \frac{3}{4}$$

(b) For $g > 1$, $0 < \lambda \le 0.75$ and $\delta \le 1$ (e.g. ARL_0)

$$ARL_{1|\delta} \approx \frac{1}{\lambda w}[\phi(w)]^{-1}\Phi(w)$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are $N(0,1)$ p.d.f. and c.d.f., respectively

Relationship among ARL_0, L and λ:

$$L \approx [a - \ln(a-1)]^{1/2} + \frac{1}{2}(1-\lambda) \qquad \text{where } a = 2\ln[(\frac{2}{\pi})^{1/2} ARL_0\lambda]$$

Srivastava and Wu, "Evaluation of Optimum Weights and Average Run Lengths in EWMA Control Schemes," *Commun. Statist. – Theory Meth.* 26(5), 1253-1267 (1997)

Optimal EWMA Chart Design

Given δ^* and ARL_0

$$\lambda^* \approx \frac{1.0234\delta^{*2}}{b - \ln b}$$

where $b = 2\ln[1.0234(\frac{2}{\pi})^{1/2}\delta^{*2} ARL_0]$

$$L^* \approx (b - \ln b)^{1/2} - \lambda^*$$

Then ARL_1* can be found:

$$ARL^*_{1|\delta^*} \approx \frac{1}{\delta^{*2}}[1.2277(L^*)^2 - 2.835 + 9.740(L^*)^{-2}] + \frac{1}{2}(1-\lambda^*)$$

Srivastava and Wu, "Evaluation of Optimum Weights and Average Run Lengths in EWMA Control Schemes," *Commun. Statist. – Theory Meth.* 26(5), 1253-1267 (1997)

ARL Comparisons between EWMA and CUSUM Charts

EWMA_1: L=2.856, λ=0.133
EWMA_2: L=2.866, λ=0.139
CUSUM: k=0.5, h=5

	ARL			
Shift	EWMA_1	EWMA_2	CUSUM	
.00	465	465	465	ARL_0
.25	116	118	139	
.50	33.3	33.8	38.0	
.75	16.0	16.1	17.0	
1.00	10.1	10.0	10.4	
1.50	5.71	5.67	5.75	ARL_1
2.00	4.04	3.99	4.01	
2.50	3.16	3.12	3.11	
3.00	2.62	2.59	2.57	
4.00	2.05	2.03	2.01	
5.00	1.77	1.74	1.69	