Linear Algebra

@ NTU, 2023/12/18
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HW#9_#2

2. What is the volume of the parallelepiped with the four of its vertices at (1, 1, 1), (0, 3, 3), (3, 0, 3) and (3, 3, 0)?

- $Let\ A = (1, 1, 1), B = (0, 3, 3), C = (3, 0, 3), D = (3, 3, 0)$
- $vector\ \overrightarrow{AB} = (-1, 2, 2), vector\ \overrightarrow{AC} = (2, -1, 2), vector\ \overrightarrow{AD} = (2, 2, -1)$
- $K = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}, KK^T = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$
- $det(KK^T) = 9 \times 9 \times 9 = 3^6 = (det\ K)^2$
- $|det\ K| = 3^3 = 27 = volume$

HW#9_#3

3. Let $\boldsymbol{P}$ be the projection matrix that projects any vector in R^3 onto $x_1+x_2+x_3=0$. Find the eigenvalues and eigenvectors of $\boldsymbol{P}$.

- $x_1 + x_2 + x_3 = 0 \rightarrow [1 \quad 1 \quad 1]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \rightarrow x = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$
- $\lambda_1 = 1 \rightarrow Px = \lambda_1 x = x \rightarrow eigenvector\ x_1 = a\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, a, b \in R$
- $\lambda_2 = 0 \rightarrow Px = \lambda_2 x = 0 \rightarrow eigenvector\ x_2 = c\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, c \in R$

HW#9_#4

4. Suppose the matrix A has eigenvalues 0, 1, 2 with eigenvectors v_0, v_1, v_2. Describe the nullspace and the column space. Solve the equation $Ax = v_1 + v_2$. Show that $Ax = v_0$ has no solution.

- $\boldsymbol{When\ \lambda = 0,\ Ax = \lambda x = 0}$

  $\rightarrow$ $\boldsymbol{Nullspace\ of\ A\ is\ an\ eigenspace\ of\ A\ with\ \lambda = 0.}$

- $\boldsymbol{When\ \lambda_1 = 1,\ Ax = \lambda_1 x = x\ ;when\ \lambda_2 = 2,\ Ax = \lambda_2 x = 2x}$

  $\rightarrow$ $\boldsymbol{Column\ space\ of\ A\ is\ an\ eigenspace\ of\ A\ with\ \lambda_1 = 1\ \&\ \lambda_2 = 2.}$

- $\boldsymbol{Ax = v_1 + v_2 \rightarrow Ax = x, Av_1 = v_1\ ;\ Ax = 2x, Av_2 = 2v_2}$

$$\boldsymbol{v_1 = Av_1, v_2 = \frac{1}{2}Av_2}$$

$$\boldsymbol{Ax = Av_1 + \frac{1}{2}Av_2 = Av_1 + A\left(\frac{1}{2}v_2\right)}$$

$$\boldsymbol{x = v_1 + \frac{1}{2}v_2 + cv_0, c \in R}$$

- $\boldsymbol{If\ Ax = v_0\ has\ solution \rightarrow}$

$$\boldsymbol{v_0\ (RHS)\ can\ be\ represented\ by\ linear\ combination\ of\ v_1\ \&\ v_2}$$

$$\boldsymbol{v_0\ can't\ be\ linear\ combination\ of\ v_1\ \&\ v_2}$$

$$\boldsymbol{\rightarrow Ax = v_0\ has\ no\ solution}$$

HW#10_#6

- $\boldsymbol{u} = \begin{bmatrix} v \\ w \end{bmatrix}, \frac{du}{dt} = \begin{bmatrix} w - v \\ v - w \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \boldsymbol{u}$

$$\lambda_1 = 0, x_1 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, c_1 \in R$$

$$\lambda_2 = -2, x_2 = c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}, c_2 \in R$$

- $u(t) = \begin{bmatrix} 20 \\ 20 \end{bmatrix} + e^{-2t} \begin{bmatrix} 10 \\ -10 \end{bmatrix}, u(1) = \begin{bmatrix} 20 \\ 20 \end{bmatrix} + e^{-2} \begin{bmatrix} 10 \\ -10 \end{bmatrix} = \begin{bmatrix} 20 + 10e^{-2} \\ 20 - 10e^{-2} \end{bmatrix} = \begin{bmatrix} v(1) \\ w(1) \end{bmatrix}$
- $t \to \infty, e^{-2t} \to 0, u(t \to \infty) = \begin{bmatrix} 20 \\ 20 \end{bmatrix} = \begin{bmatrix} v(t \to \infty) \\ w(t \to \infty) \end{bmatrix}$
- $\lambda\ changes\ from\ (0\ \&\ -2) to\ (0\ \&\ 2)$
- $u(t) = \begin{bmatrix} 20 \\ 20 \end{bmatrix} + e^{2t} \begin{bmatrix} 10 \\ -10 \end{bmatrix}, u(t \to \infty) = \begin{bmatrix} \infty \\ -\infty \end{bmatrix} = \begin{bmatrix} v(t \to \infty) \\ w(t \to \infty) \end{bmatrix} (diverge\ \&\ unstable)$

6. A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \text{ and } \frac{dw}{dt} = v - w$$

The total $v + w$ is constant (40 people).

(a) Find the matrix in $\frac{du}{dt} = Au$, and its eigenvalues and eigenvectors.

(b) What are v and w at $t = 1$?

(c) what are v and w as t approaches infinity?

(d) Reverse the diffusion of people to $du/dt = -Au$:

$$\frac{dv}{dt} = v - w \text{ and } \frac{dw}{dt} = w - v$$

The total $v + w$ still remains constant. How are the λ's changed now that A is changed to $-A$? What is v as t approaches infinity?

HW#11_#1

1. Find the eigenvalues and eigenvectors for

$$\frac{du}{dt} = Au = \begin{bmatrix} 0 & 3 & 0 \\ -3 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix} u.$$

Why do you know, without computing, that e^{At} will be an orthogonal matrix and $\|u(t)\|^2 = u_1^2 + u_2^2 + u_3^2$ will be constant?

- $A = \begin{bmatrix} 0 & 3 & 0 \\ -3 & 0 & 4 \\ 0 & -4 & 0 \end{bmatrix}$

$$\lambda_1 = 0, x_1 = c_1 \begin{bmatrix} 1 \\ 0 \\ 3/4 \end{bmatrix}, c_1 \in R$$

$$\lambda_2 = 5i, x_2 = c_2 \begin{bmatrix} 1 \\ 5i/3 \\ -4/3 \end{bmatrix}, c_2 \in R$$

$$\lambda_3 = -5i, x_3 = c_3 \begin{bmatrix} 1 \\ -5i/3 \\ -4/3 \end{bmatrix}, c_3 \in R$$

- $A^T = -A, A \text{ is skew} - \text{symmetric}, (e^{At})^T = e^{-At}, (e^{At})(e^{At})^T = \left(e^{At}\right)\left(e^{-At}\right) = I$

e^{At} *is an orthogonal matrix.*

$\left\|e^{At}u(t)\right\| = \|u(t)\|, \|u(t)\|^2 = {u_1}^2 + {u_2}^2 + {u_3}^2$ *will be a constant.*

HW#11_#2

2. (a) What matrix M changes the basis V_1=(1, 1), V_2=(1, 4) to the basis v_1=(2, 5), v_2=(1, 4)? (Hint: the columns of M come from expressing V_1 and V_2 as combinations $\Sigma m_{ij} v_i$ of the v's.)
(b) For the same two bases, express the vector (3, 9) as a combination $c_1V_1+c_2V_2$ and also as $d_1v_1+d_2v_2$. Check numerically that M connects c to d: $Mc=d$.

- $\boldsymbol{V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = v_1 - v_2, V_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = v_2}$

$$\boldsymbol{M = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}$$

- $\boldsymbol{\begin{bmatrix} 3 \\ 9 \end{bmatrix} = V_1 + 2V_2 = v_1 + v_2}$

$$\boldsymbol{c = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, d = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$\boldsymbol{Mc = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = d}$$

HW#11_#3

3. If the transformation T is a reflection across the 45° line in the plane, find its matrix with respect to the standard basis v_1=(1, 0), v_2=(0, 1) and also with respect to V_1=(1, 1,), V_2=(1, -1). Show that those matrices are similar.

- $T_1 v_1 = v_2, T_1 v_2 = v_1 \rightarrow T_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- $T_2 V_1 = V_1, T_2 V_2 = -V_2 \rightarrow T_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
- $M = [I]_{V\ to\ v} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
- $M^{-1} = [I]_{v\ to\ V} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix}$
- $M^{-1} T_1 M = T_2$

$$\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = T_2$$

HW#12_#2(1/2)

2. Find unitary U to triangularize the following matrices (Schur's Lemma):

$$\begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- $A = \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}, det(A - \lambda I) = 0$
- $\lambda_1 = 1, x_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}; \quad \lambda_2 = 2, x_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$
- $x_2' = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} - \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \times \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} 4\sqrt{2} \\ -3\sqrt{2} \end{bmatrix} \rightarrow take\ x_2' = \begin{bmatrix} \frac{4}{5} \\ \frac{-3}{5} \end{bmatrix}$
- $U^{-1}AU = T, U = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{-3}{5} \end{bmatrix}, U^{-1} = U^T = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{-3}{5} \end{bmatrix}, \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{-3}{5} \end{bmatrix} \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{-3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 0 & 2 \end{bmatrix}$
- $U^{-1}AU = T, U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}, U^{-1} = U^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 0 & 1 \end{bmatrix}$

HW#12_#2(2/2)

2. Find unitary U to triangularize the following matrices (Schur's Lemma):

$$\begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

- $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, det(A - \lambda I) = 0$
- $\lambda_1 = 0, x_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
- $Take\ U_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, U^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow B = U_1{}^{-1} A U_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
- $Consider \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow \lambda_2 = 0, x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow Take\ U_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U_2{}^{-1}$

$$T = U_2{}^{-1} B U_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = U_2{}^{-1} U_1{}^{-1} A U_1 U_2$$

$$U = U_1 U_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

HW#12_#5

5. Let $A=\begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix}$ and the quadratic function be $\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$.
   (i) Factor A into LDU and expand the quadratic function into a summation of two quadratic terms.
   (ii) Diagonalize A into $Q\Lambda Q^T$ and expand the quadratic function into a summation of two quadratic terms.
   (iii) Compare results of (i) and (ii) and use them to determine whether the quadratic function is definite, semi-definite or indefinite.

- $A=\begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix}=\begin{bmatrix}1 & 0\\ 4/5 & 1\end{bmatrix}\begin{bmatrix}5 & 0\\ 0 & 9/5\end{bmatrix}\begin{bmatrix}1 & 4/5\\ 0 & 1\end{bmatrix}$
- $\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}1 & 0\\ 4/5 & 1\end{bmatrix}\begin{bmatrix}5 & 0\\ 0 & 9/5\end{bmatrix}\begin{bmatrix}1 & 4/5\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=$

  $\begin{bmatrix}x+\frac{4}{5}y & y\end{bmatrix}\begin{bmatrix}5 & 0\\ 0 & 9/5\end{bmatrix}\begin{bmatrix}x+\frac{4}{5}y\\ y\end{bmatrix}=5\left(x+\frac{4}{5}y\right)^2+\frac{9}{5}y^2$

- $A=\begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix}=\begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 9\end{bmatrix}\begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}$
- $\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}5 & 4\\ 4 & 5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}x & y\end{bmatrix}\begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 9\end{bmatrix}\begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=$

  $\begin{bmatrix}\frac{-1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y & \frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 9\end{bmatrix}\begin{bmatrix}\frac{-1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y\\ \frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y\end{bmatrix}=\left(\frac{-1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y\right)^2+9\left(\frac{1}{\sqrt{2}}x+\frac{1}{\sqrt{2}}y\right)^2$

- $x^TAx>0$ *for all nonzero vectors* $x\rightarrow$ ***positive definite***

HW#12_#9

9. Give a quick reason why each of these statements is true:
   (a) Every positive definite matrix is invertible.
   (b) The only positive definite projection matrix is $P=I$
   (c) A diagonal matrix with positive diagonal entries is positive definite.

- $\boldsymbol{det(A) \neq 0 \rightarrow full\ rank\ \&\ invertible}$
- $\boldsymbol{All\ projection\ matrix\ except\ I\ are\ singular.}$
- $\boldsymbol{Diagonal\ entries\ of\ a\ diagonal\ matrix = eigenvalues = pivots}$

HW#12_PCA #2

2. Find the minimum, if there is one, of P_1=0.5x^2+xy+y^2−3y and P_2=0.5x^2−3y.

- $\boldsymbol{P_1 = 0.5x^2 + xy + y^2 - 3y = \frac{1}{2}x^T Ax - x^T b}$
- $\boldsymbol{P_1 = \frac{1}{2}[x \quad y]\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} - [x \quad y]\begin{bmatrix}0\\ 3\end{bmatrix}, A = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}, A^{-1} = \begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}}$
- $\boldsymbol{P_{min} = -\frac{1}{2}b^T A^{-1} b = -\frac{1}{2}[0 \quad 3]\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}\begin{bmatrix}0\\ 3\end{bmatrix} = \frac{-9}{2} = -4.5}$

- $\boldsymbol{P_2 = 0.5x^2 - 3y = \frac{1}{2}[x \quad y]\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} - [x \quad y]\begin{bmatrix}0\\ 3\end{bmatrix}, A = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, A^{-1}\ doesn't\ exist}$
- $\boldsymbol{Minimum\ of\ P_2\ doesn't\ exist.}$