Linear Algebra and its Applications

HW#9

1. If Q is an orthogonal matrix, so that $Q^TQ = I$, prove that det$Q$ equals +1 or −1. What kind of box is formed from the rows (or columns) of Q?
2. What is the volume of the parallelepiped with the four of its vertices at (1, 1, 1), (0, 3, 3), (3, 0, 3) and (3, 3, 0)?
3. Let $\boldsymbol{P}$ be the projection matrix that projects any vector in R^3 onto $x_1+x_2+x_3=0$. Find the eigenvalues and eigenvectors of $\boldsymbol{P}$.
4. Suppose the matrix $\boldsymbol{A}$ has eigenvalues 0, 1, 2 with eigenvectors v_0, v_1, v_2. Describe the nullspace and the column space. Solve the equation $Ax = v_1 + v_2$. Show that $Ax = v_0$ has no solution.
5. "Suppose $Ax=\lambda x$. If $\lambda=0$, then the eigenvector x is in the nullspace. If $\lambda\neq 0$, then the eigenvector x is in the column space of A. The eigenvectors in the column space has r (rank of A) linearly independent vectors and the eigenvectors in the nullspace has $n-r$ linearly independent vectors. Since $n+(n-r)=n$, any $n\times n$ matrix A must have n linearly independent eigenvectors." What is wrong in the statement to lead to the incorrect conclusion? Find a 2×2 example (from the internet) that shows the statement is incorrect. Is the statement correct when A is a projection matrix?
6. Show that the eigenvalues of $\boldsymbol{A}$ equal the eigenvalues of $\boldsymbol{A^T}$. Show by an example that the eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{A^T}$ are not the same.
7. A 3 by 3 matrix B is known to have eigenvalues 0, 1, 2, Is this information enough to find: (a) the rank of B, (b) the determinant of B^TB, (c) the eigenvalues of B^TB? How?
8. Which of these matrices cannot be diagonalized?
$$A_1 = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 2 & 0 \\ 2 & -2 \end{bmatrix} \quad A_3 = \begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$$
9. Find the matrix A whose eigenvalues are 1 and 4, and whose eigenvectors are $(3, 1)^T$, and $(2, 1)^T$, respectively. (Hint: A = $S\Lambda S^{-1}$)
10. Find all the eigenvalues and eigenvectors of
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$
and write two different diagonalizing matrices S