Optimal Design : Model Boundedness

ME 7129

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In modeling an optimization problem, the easiest and most common mistake is to leave something out.

Things you will learn in this chapter

- Bounds
- Optima
- Monotonicity Analysis
- Constraint Activity
- Criticality
- Dominance
- Relaxation

Bounds - Notations

- Lower bound $f(x) \geq l$
- Greatest lower bound (infimum) $g \geq l, \ \forall l \leq f(x)$
- Non-negative domain $\mathcal{N} = \{x : 0 \leq x \leq \infty\}$
- Positive finite domain $\mathcal{P} = \{x : 0 < x < \infty\}$

Greatest Lower Bounds of Different Domains

- Let g be the greatest lower bound for $f(x)$ over $\mathcal{R}$
- Let g_0 be the greatest lower bound for $f(x)$ over $\mathcal{N}$
- Let g^+ be the greatest lower bound for $f(x)$ over $\mathcal{P}$

$$\mathcal{R} \supset \mathcal{N} \supset P \longrightarrow g \leq g_0 \leq g^+$$

To represent g^+ is the infimum of $f(x)$ (over $\mathcal{P}$) we write

$$g^+ = \inf_{x \in \mathcal{P}} f(x)$$

Arguments

- Suppose that there is a nonnegative number $\underline{x}$ such that $f(\underline{x}) = g^+$.
- $\underline{x}$ is called an argument of the infimum over $\mathcal{P}$
- In case more than one argument, the set of all arguments is represented by $\underline{\mathcal{X}} = \{\underline{x} : f(\underline{x}) = g^+\}$

Well Boundedness

If all $\underline{x}$ are positive and finite, then $f(x)$ is said to be ***well bounded*** (below) over $\mathcal{P}$

If **all** arguments of g^+ are positive and finite, that is, $\mathcal{P} \supset \underline{\mathcal{X}}$, the infimum will be called the minimum for $f(x)$ over $\mathcal{P}$

Examples

Example 3.1 Consider the following functions:

1. $f(x) = x$: no (finite) g exists, but $g_0 = g^+ = 0$, so $\underline{x} = 0 \notin \mathcal{P}$ and hence $f(x)$ is not well bounded below over $\mathcal{P}$.
2. $f(x) = x^2 + 1$:$g = g_0 = g^+ = 1$. Since the argument $\underline{x} = 0$, $f(x)$ is not well bounded below over $\mathcal{P}$.
3. $f(x) = (x-1)^2$:$g = g_0 = g^+ = 0$, and since $\underline{x} = 1 \in \mathcal{P}$, $f(x)$ is well bounded below over $\mathcal{P}$.
4. $f(x) = -x$:$(finite) g$, g_0, and g^+ do not exist, so no arguments exist, and $f(x)$ is not well bounded below over $\mathcal{P}$.
5. $f(x) = 1/x^2$:$g = g_0 = g^+ = 0$. Although $f(x) = 0$ for x equal to positive or negative infinity, only the positive one qualifies as an argument of the infimum. Since $\underline{x} \notin \mathcal{P}$, $f(x)$ is not well bounded below over $\mathcal{P}$.
6. $f(x) = 1/x$: no (finite) g exists, but $g_0 = g^+ = 0$ for $\underline{x} = \infty$, so $f(x)$ is not well bounded below over $\mathcal{P}$.
7. The infimum itself can be negative, for example, $f(x) = (x-1)^2 - 1$:$g = g_0 = g^+ = -1$ where the argument $\underline{x} = 1$; well bounded over $\mathcal{P}$.
8. $f(x) = \exp(-x)$:$g = g_0 = g^+ = 0$; not well bounded over $\mathcal{P}$ because their arguments are infinite.
9. $f(x) = (x-1)^2(x-2)^2$:$g = g_0 = g^+ = 0$. There are two arguments: $\underline{\mathcal{X}} = \{1, 2\}$; well bounded over $\mathcal{P}$.
10. $f(x_1, x_2) = 3 + (x_2 - 1)^2$: Here the bivariate function does not depend on x_1; consequently $g = g_0 = g^+ = 3 = f(x_1, 1)$, which gives the same value not only in $\mathcal{P}$ but also when $x_1 = 0$ (and ∞). Hence f is well bounded with respect to x_2, although not with respect to x_1.

The argument of a minimum is written as x_* when it is unique. ($\mathcal{X}_*$ represents the set of all x_*)

Existence of Minimum $\neq$ Well Bounded Function

well bounded $\longrightarrow$ minimum exist

well bounded $\not\longleftarrow$ minimum exist

for example :

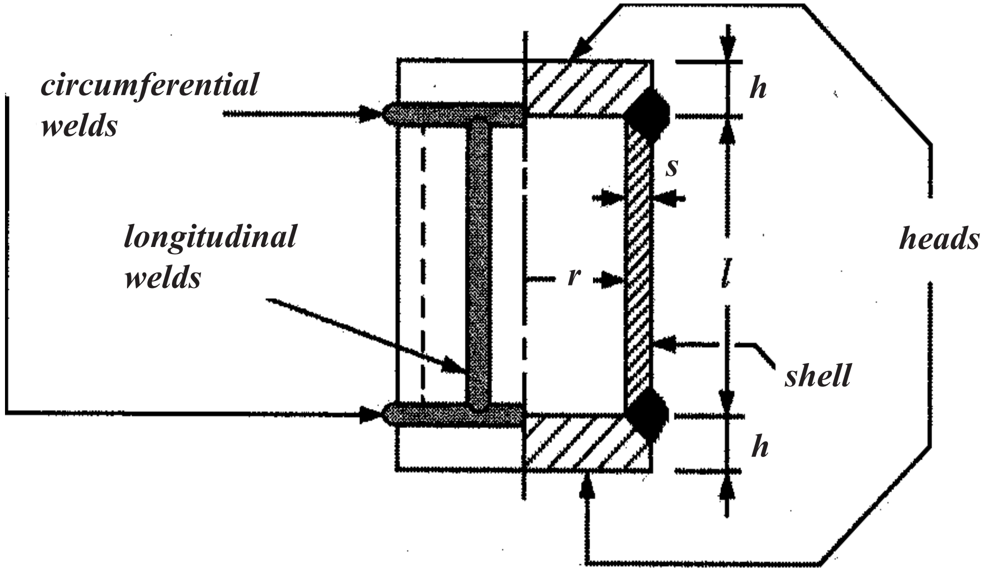
$$f(x) = 5$$

Upper vs. Lower Bounds

- The analogous concepts involving upper instead of lower bounds are given as below.

Bound	Extremum	Arg	Optimum	Arg
Lower	Greatest lb infimum	$\underline{x}$	Minimum	x_*
Upper	Least ub supremum	$\bar{x}$	Maximum	x^*

Air Tank Example : Objective Function



$$m = \pi[(r+s)^2 - r^2]l + 2\pi(r+s)^2 h.$$

$$\begin{aligned} f(\mathbf{x}) &= \pi\{[(x_3+x_4)^2 - x_3^2]x_2 + 2(x_3+x_4)^2 x_1\} \\ &= \pi[(2x_3x_4 + x_4^2)x_2 + 2(x_3+x_4)^2 x_1]. \end{aligned}$$

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Constraint Set

- The domain of $\mathbf{x}$ may be restricted further by constraints, for example, equalities, inequalities, discreteness restrictions, and/or logical conditions, defining a ***constraint set*** $\mathcal{K}$.

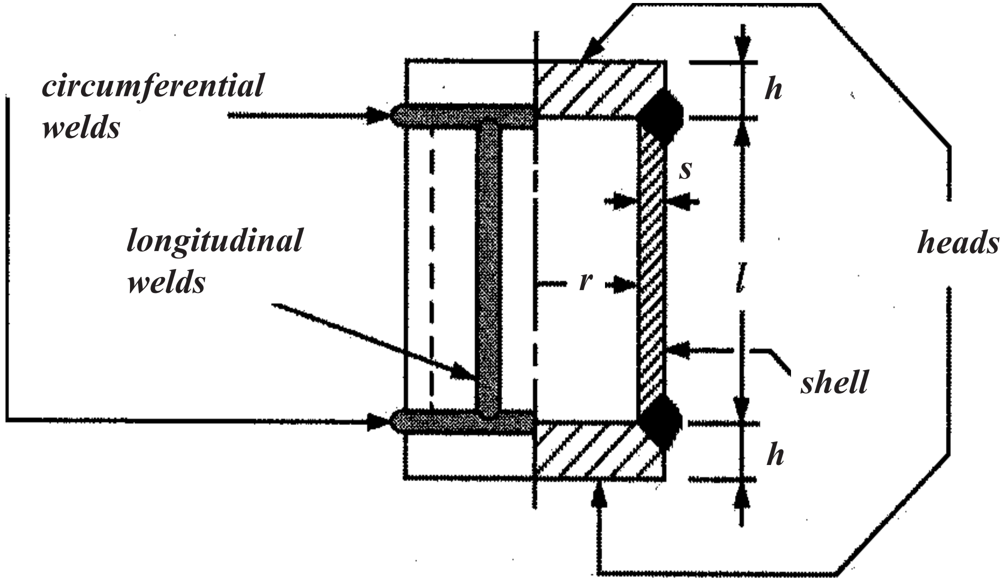
- The set $\mathcal{K}$ is said to be consistent if and only if

$$\mathcal{K} \neq \{\}$$

Feasible Set and Minimum

- The feasible set $\mathcal{F} = \mathcal{K} \cap \mathcal{P}$
- Let $f(\mathbf{x})$ be the objective function defined on $\mathcal{F}$
- Let g^+ be the greatest lower bound (infimum) on $f(\mathbf{x})$, $f(\mathbf{x}) \geq g^+ \ \forall \mathbf{x} \in \mathcal{F}$
- If there exists $\mathbf{x}_* \in \mathcal{F}$ such that $f(\mathbf{x}_*) = g^+$, then $f(\mathbf{x}_*)$ is the constrained minimum of $f(\mathbf{x})$
- $\mathbf{x}_*$ is the minimizer $\mathbf{x}_* = \arg\min f(\mathbf{x})$, for $\mathbf{x} \in \mathcal{F}$

Air Tank Example : Constraints



volume constraint

$$\mathcal{K}_1 = \{\mathbf{x}{:}g_1 = -\pi x_2 x_3^2 + 2.12(10^7) \leq 0\}.$$

ASME : ratio of head thickness to radius

$$\mathcal{K}_2 = \{\mathbf{x}{:}g_2 = -x_1 x_3^{-1} + 130(10^{-3}) \leq 0\}.$$

ASME : shell thickness

$$\mathcal{K}_3 = \{\mathbf{x}{:}g_3 = -x_3^{-1} x_4 + 9.59(10^{-3}) \leq 0\}.$$

shell length for nozzle

$$\mathcal{K}_4 = \{\mathbf{x}{:}g_4 = -x_2 + 10 \leq 0\}$$

$$\mathcal{F} = \left[\bigcap_{j=1}^{m} \mathcal{K}_i\right] \cap \mathcal{P}^n$$

Partial Minimization

- If all variables but one are held constant, it may be easy to see which constraints, if any, bound the remaining variable away from zero.

- In the air tank problem, let $\mathbf{x} = [x_1, x_2, x_3, x_4] = [x_1, \mathbf{X}_1]$

- Formally, define the feasible set for x_1 , given $\mathbf{X}_1$, as

$$\mathcal{F}_1 = \{\mathbf{x} : \mathbf{x} \in \mathcal{F} \text{ and } x_i = X_i \text{ for } i \neq 1\}$$

- Let $\mathbf{x}'$ be any element of $\mathcal{F}_1(\mathbf{X}_1)$

- The function min $f(x_1, \mathbf{X}_1)$ $\forall \mathbf{x}' \in \mathcal{F}_1$ is called the partial minimum of f w.r.t. x_1

Constraint Activity

- When removing a constraint changes the location of the optimum, the constraint is said to be ***active***.

- Otherwise it is termed ***inactive***.

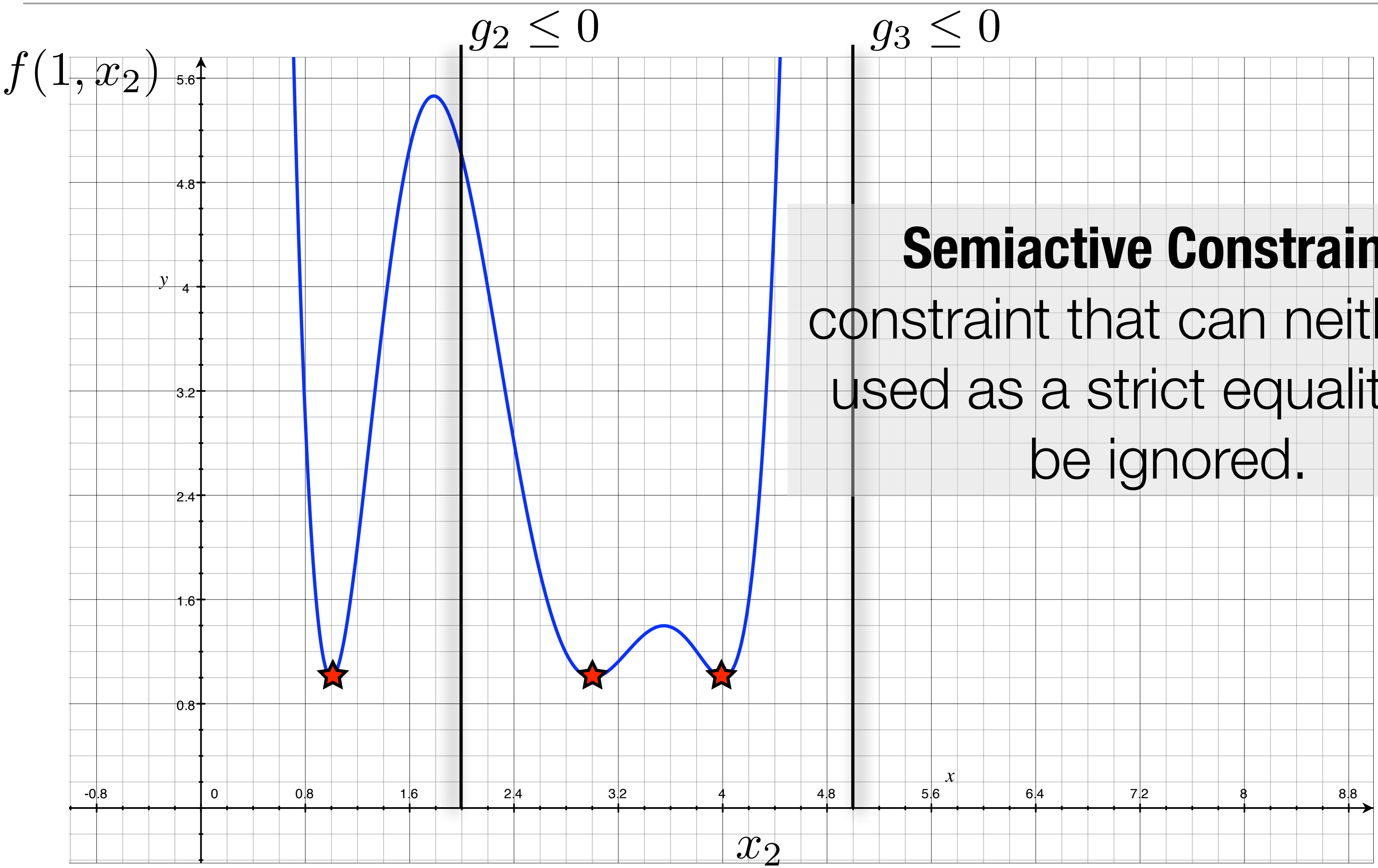
- Let $\mathcal{D}_i = \bigcap_{j \neq i} \mathcal{K}_j$ be the set of all solutions to all constraints except g_i

- The set of all feasible point $\mathcal{F} = \mathcal{D}_i \cap \mathcal{K}_i \cap \mathcal{P}^n$

- Let $\mathcal{X}_* = \arg\min f(\mathbf{x}) \ \forall \mathbf{x} \in \mathcal{F}$
  $\mathcal{X}_i = \arg\min f(\mathbf{x}) \ \forall \mathbf{x} \in \mathcal{D}_i \cap \mathcal{P}^n$

g_i is active if $\mathcal{X}_* \neq \mathcal{X}_i$

Example 3.4



Semiactive Constraint : constraint that can neither be used as a strict equality nor be ignored.

Activity Theorem

Constraint g_i is active if and only if $f(\mathcal{X}_i) < f(\mathcal{X}_*)$

Binding Constraint : If an inequality constraint holds with *equality* at the optimal point, the constraint is said to be **binding**, as the point *cannot* be varied in the direction of the constraint even though doing so would improve the value of the objective function.

Monotonicity

- A function is said to increase with respect to the single positive finite variable $x \in \mathcal{P}$ if

$$\forall x_2 > x_1, f(x_2) > f(x_1)$$

- Such a function will be written as $f(x^+)$
- A function is said to decrease with respect to x and is written as $f(x^-)$
- Functions that are either increasing or decreasing are called **monotonic**.

Monotonicity Theorem

If $f(x)$ and the consistent constraint functions $g_i(x)$ all increase (weakly) or all decrease weakly with respect to x, the minimization problem domain is not well constrained.

First Monotonicity Principle (MP1)

In a well-constrained minimization problem, every increasing variable is bounded below by at least one nonincreasing active constraint.

Second Monotonicity Principle (MP2)

In a well-constrained minimization problem every nonobjective variable is bounded both

below by at least one nonincreasing semiactive constraint

and

above by at least one non-decreasing semiactive constraint.

Criticality

A constraint is critical if its removal results in an unbounded minimization problem

Monotonicity of Composite Functions

- Let f_1 and f_2 be two positive differentiable functions monotonic wrt x_1 over the positive range of $\mathbf{x}_1$

- Then the following functions are all monotonic in the same sense.

$$f_1 + f_2$$

$$f_1 f_2$$

$$f^a, \forall a > 0$$

$$f_1(f_2(\mathbf{x}))$$

- If a function is monotonic, so is its integral

Monotonicity Analysis for Inequality Constraints

- Conditional criticality
- Multiple criticality
- Dominance
- Relaxation
- Uncriticality

Equality Constraints

- An equality constraint can be written as an active inequality constraint in such a way that the optimum is not affected.

- The approach is called '***directing an active equality***'

- Note : NOT all equality constraints are active.

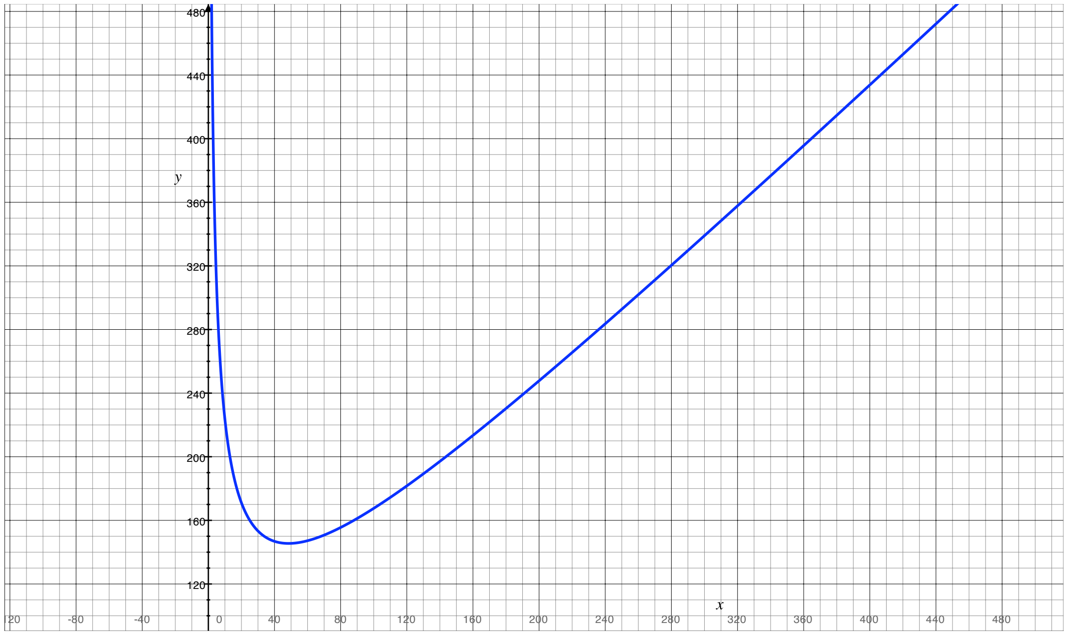
Monotonic Direction Theorem

If h_1 is active, then the inequality-constrained problem is well constrained, and its solution set $\mathcal{X}'_*$ is identical to $\mathcal{X}_*$, the solution set of the equality-constrained problem.

Regional Monotonicity of Nonmonotonic Functions

- A nonmonotonic function can be considered as the combinations of several monotonic functions.

- For example : $g(l) = l + 675.5l^{-1/2}$



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Second Monotonicity Principle (MP2)

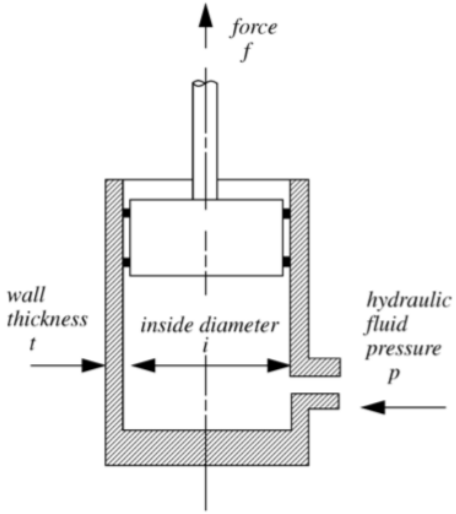
In a well-constrained minimization problem every nonobjective variable is bounded both

below by at least one nonincreasing semiactive constraint

and

above by at least one non-decreasing semiactive constraint.

Example : Hydraulic Cylinder



$$
\begin{aligned}
\text{minimize } g_0&: \quad i + 2t \\
\text{subject to } g_1&: \quad t \geq 0.3, \\
g_2&: \quad f \geq 98, \\
g_3&: \quad p \leq 2.45(10^4), \\
g_4&: \quad s \leq 6(10^5), \\
h_1&: \quad f = (\pi/4)i^2 p, \\
h_2&: \quad s = ip/2t.
\end{aligned}
$$

Table 3.3. Hydraulic Cylinder: Initial Monotonicity Table 0

Function Nu	Variables
(0)	
(1)	
(2)	
(3)	
(4)	
(5)	
(6)	
(7)	

Table 3.6. Hydraulic Cylinder: Monotonicity Table 3 (Final Reduction)

Function Number	Functions	Variables i	t
(0, 1)	$i + 2t$	+	+
(4), (3, 2)	$(4/\pi)F/i^2 - P \leq 0$	−	.
(6, 5), (3, 2)	$(2/\pi)F/it - S \leq 0$	−	−
(7)	$-t + T \leq 0$	.	−

Eliminated Variables and Constraints

Constraint Number	Variable Eliminated
(3, 2)	$p \geqq ((4/\pi)F/i^2)$
(5), (3, 2)	$s \geqq ((2/\pi)F/it)$
(2), (3, 2)[=3]	$f \geq F$
(1)	$d \geqq (i + 2t)$

Number	Eliminated
(5)	$s \geqq ((1/2)ip/t)$
(2)	$f \leq (\pi/4)i^2 p$
(1)	$d \geqq (i + 2t)$

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