Linear Algebra Term Project —The Fast Fourier Transform and its applications

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December 28, 2023

1 Rationale

The Fast Fourier Transform (FFT) is one of the most important breakthrough in contemporary science and technology. It is an algorithm that computes the Discrete Fourier Transform (DFT) really efficiently, by reducing the computational complexity from $O(N^2)$ to $O(N)log_2$.

In mechanical field, FFT help solving the Partial differential equation (PDE) of such a dynamic system behavior in numerical way. When it comes to signal analysis, the Fast Fourier Transform can powerfully change the signal from the time domain into the frequency domain. This help us easier to identify the noises and the main frequencies. Other applications of FFT such as Audio Compression, Image Compression and Communication are also critical in nowadays.

I consider the project is a great opportunity for me to realize how FFT work in the linear algebra thinking.

2 Problem background

2.1 Fourier Series

2.1.1 Hilbert Space

One of the perspective to understand the Fourier Series is starting with the **Hilbert Space**. Hilbert Space is a infinite Dimensions space with finite length vectors $\mathbf{v}$. example: $\mathbf{v} = [1,\ 1/2,\ 1/3, \ldots]$ If $\mathbf{v}, \mathbf{w}$ are in Hilbert Space:

$$|\mathbf{v}^T\mathbf{w}| \leq \|\mathbf{v}\|\|\mathbf{w}\|$$
$$|\mathbf{v}^T\mathbf{w}| = 0, \quad if\ \mathbf{v} \perp \mathbf{w}$$

In a finite domain $(-\pi \leq x \leq \pi)$, $f(x) = sin\ x$ has continuous, infinite point. The length square of $f(x) = sin\ x$ is:

$$\|f(x)\|^2 = \int_{-\pi}^{\pi} (sin\ x)^2\ dx = \pi$$

Thus, $sin\,x\ (-\pi \leq x \leq \pi)$ has infinite dimension and finite length. The function space is **Hilbert Space**, and .so do other function $(cos\ x,\ sin\,2x,\ cos\ 2x, \ldots, sin\,kx,\ cos\ ekx, \quad k \rightarrow$

∞). Now, check the inner product of two functions $f(x) = sin\ x$ and $(x) = cos\ x$ in$(-\pi \leq x \leq \pi)$:

$$(f, g) = \int_{-\pi}^{\pi} sin\ x\ *\ cos\ x\ dx = 0$$

This shows that the vectors of the two functions are perpendicular to each other. And this orthogonality can be promoted to other sin,cos functions:
$(sin\ x,\ cos\ x,\ sin\ 2x,\ cos 2x, \ldots, sin\ kx,\ cos\ kx, \quad k \to \infty)$.

2.1.2 Fourier Series

Consider a function $y(x)$ has domain $(-\pi \leq x \leq \pi)$, or a periodic function whose period is 2π. We can project this function $y(x)$ to the coordinate formed by the sin,cos function $(sin\ x,\ cos\ x,\ sin\ 2x,\ cos\ 2x, \ldots, sin\ kx,\ cos\ kx, \quad k \to \infty)$ in infinity dimensions space. Use the projection formula $\hat{\mathbf{x}} = \dfrac{\mathbf{b}^T\mathbf{a}}{\mathbf{a}^T\mathbf{a}}$:

$$\begin{aligned} y(x) &= a_0 + a_1 cos(x) + b_1 sin(x) + a_2 cos(2x) + b_2 sin(2x) + \cdots + a_k cos(kx) + b_k sin(x) \\ &\quad , \quad k \to \infty \\ &= a_0 + \sum_{n=1}^{\infty} [a_n cos(nx) + b_n sin(nx)] \end{aligned}$$

where

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) cos(nx) dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) sin(nx) dx \end{aligned}$$

2.2 Fourier Transform

2.2.1 promote the period 2π to arbitrary $p = 2L$

Consider a periodic function $f(x)$ has a period $p = 2L$. Introduce a new variable v, and let f be a function of v.

$$x = \frac{p}{2\pi} v, \quad v = \frac{2\pi}{p} x = \frac{\pi}{L} x \quad \Rightarrow \begin{cases} v & = \pi \longrightarrow x = L \\ v & = -\pi \longrightarrow x = -L \end{cases}$$

the changing can be written as:

$$\begin{aligned} f(x) = f\left(\frac{L}{\pi} v\right) &= a_0 + \sum_{n=1}^{\infty} [a_n cos(nv) + b_n sin(nv)] \\ \Downarrow \\ f(x) = a_0 + \sum_{n=1}^{\infty} (a_n cos \frac{n\pi}{L} x + b_n sin \frac{n\pi}{L} x) \end{aligned}$$

the coefficients are:

$$\begin{cases} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\frac{L}{\pi}v)dv \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\frac{L}{\pi}v)cos(nv)dv \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\frac{L}{\pi}v)sin(nv)dv \end{cases} \quad \Rightarrow \quad \begin{cases} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)cos(\frac{n\pi x}{L})dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)sin(\frac{n\pi x}{L})dx \end{cases}$$

2.2.2 when $L \to \infty$

Define $w_n = \frac{n\pi}{L}$, and $\Delta w = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$ $f_L(x)$ with $p = 2L$ become:

$$f_L(x) = a_0 + \sum_{n=1}^{\infty}(a_n cos\ w_n x + b_n sin\ w_n x), \quad w_n = \frac{n\pi}{L}$$

When $f(x)$ is not a periodic function, we can write f as $f_L(x)$ with $L \to \infty$ and $\Delta w = \frac{\pi}{L} \to 0$:

$$f(x) = \lim_{L\to\infty} f_L(x)$$

Then the sigma Σ will become the integral $\int$:

$$\begin{aligned} f(x) &= \int_0^{\infty} [A(w)\ cos\ wx\ + B(w)\ sin\ wx]\, dw \\ A(w) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) cos\ wv\ dv, \\ B(w) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) sin\ wv\ dv \end{aligned}$$

2.2.3 Fourier Transform and inverse Fourier Transform

Put A and B into the above formula, we can get:

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(v)[cos\ wv\ cos\ wx\ +\ sin\ wv\ sin\ wx]dv\ dw$$

Since $\int_{-\infty}^{\infty} f(v)[cos\ wv\ cos\ wx + sin\ wv\ sin\ wx]dv$ can be write as $\left[\int_{-\infty}^{\infty} f(v)cos(wx - wv)dv\right]$. and the part in [...] is an even function. The $f(x)$ will change into:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)cos(wx - wv)dv\right] dw$$

If we change cos into sin in the above formula, we will get:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)sin(wx - wv)dv\right] dw\ =\ 0$$

Multiply the above formula i and combine the two formula:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)cos(wx - wv)\ +\ i\ f(v)sin(wx - wv)dv\right] dw$$

Use the Euler formula $e^{ix} = cos\ x\ +\ i\ sin\ x$, form the complex Fourier integral:

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)e^{iw(x-v)}dvdw \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-iwv}dv\right]e^{iwx}dw \end{aligned}$$

The part inside the [...] is a function of w, $f(w)$ called the **Fourier Transform**:

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx = \mathcal{F}(f)$$

The part outside $f(w)$ is the **inverse Fourier Transform**:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}dw = \mathcal{F}^{-1}(\hat{f})$$

2.3 Discrete Fourier Transform (DFT)

2.3.1 DFT

For all practical purposes, people seldom get a whole function $f(x)$. Most of the time, we only have finite sampled points of $f(x)$. The Discrete Fourier Transform (DFT) help people transform a bounch of Discrete data from time domain to frequency domain. Without a continious function, we can't do the integral to get the coefficients of different frequencies. Instead, we get N samples from a funtion f, and evaluate the summation from 0 to $N-1$:

$$\hat{f}_k = \sum_{n=0}^{N-1} f_n \cdot e^{-\dfrac{i2\pi\,kn}{N}}$$

Separate the $e^{-\dfrac{i2\pi}{N}}$ from the above equation and defined it w. The antire part in the summation will become w^{kn}. So the equation will be like :

$$\hat{f}_k = \sum_{n=0}^{N-1} f_n \cdot w^{kn}\ ,\ \ w = e^{-\dfrac{i2\pi}{N}}$$

Turn this equation into the vector expression will be $\hat{\mathbf{f}}_{N\times 1} = \mathbf{F}_{N\times N}\mathbf{f}_{N\times 1}$:

$$\begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{k=N} \end{bmatrix}_{N\times 1} = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 & \cdots & w^0 \\ w^0 & w^1 & w^2 & w^3 & \cdots & w^{N-1} \\ w^0 & w^2 & w^4 & w^6 & \cdots & w^{2(N-1)} \\ w^0 & w^3 & w^6 & w^9 & \cdots & w^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ w^0 & w^{N-1} & w^{2(N-1)} & w^{3(N-1)} & \cdots & w^{(N-1)^2} \end{bmatrix}_{N\times N} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n=N} \end{bmatrix}_{N\times 1}$$

The $\mathbf{F}_{N\times N}$ is called the Fourier matrix $\mathbf{F}_N$

For example 1: If there is function f representing a dynamic system. We take the

samples points f_n each seconds. Thus, each couple of sample $n \ \rightarrow n+1$ will have the time interval $\Delta t = 1$(unit: second). And the $\hat{f}_k$ will represent the magnitude $(\frac{\|\hat{f}_k\|}{N})$ and phase of each frequency F_k , where $\frac{k}{N \cdot \Delta t} \ \Rightarrow F_k$.
For example 2: Set $N = 4$, the sample values are $\mathbf{f} = [0 \ \ 1 \ \ 4 \ \ 9]^T$. The $w = e^{-i2\pi/4} = -i$ and the $\hat{f}$ will become :

$$\hat{\mathbf{f}} = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} \mathbf{f} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ -4+8i \\ -6 \\ -4-8i \end{bmatrix}$$

In this example, we can check the orthogonality of each column in Fourier matrix $\mathbf{F}_{4\times4}$ or $\mathbf{F}_4$. Pick two arbitrary columns and calculate their inner product, just like below:

$$\mathbf{F}_{4_2}^H \mathbf{F}_{4_4} = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}^H \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = 0$$

The result shows that each columns of the Fourier matrix $\mathbf{F}_4$ are orthogonal to each other. And the length of each columns is $\sqrt{\mathbf{F}_{4_n}^H \mathbf{F}_{4_n}} = 2$. It is equal to $\sqrt{N}$, $N = 4$. This feature can be expanded to any arbitrary Fourier matrix $\mathbf{F}_N$. Thus, if we want to find the inverse of Fourier matrix to do the inverse DFT, we will discover that $\mathbf{F}_N^H \mathbf{F}_N = N\mathbf{I}$. The inverse of Fourier matrix is just $\frac{1}{N}\mathbf{F}_N^H = \frac{1}{N}\bar{\mathbf{F}}_N$.

2.3.2 Difficulty

If sampling number N is too small, the equation above will lead to the Aliasing Effect, which mean the signal in different frequencies have been stacked together. Thus, the sampling number N is usually a huge value in applications. This led to a difficult issue. Each $\hat{f}_n$ in $\hat{\mathbf{f}}$ needs $O(N)$ operations. The hole $\hat{\mathbf{f}}$ need $O(N^2)$ operations. For instance, when sampling numbers N is 1000, the operations will up to 10^6. The large number a sampling number is, the more expensive the equation will be.

2.4 Solutions with linear algebra theories and techniques

2.4.1 Thinking w_N as an angle rotation

we can think w_N as a rotation with angle changing $2\pi/N$ in a unit circle. The power of w_N represents the times of angle changing. For example, w_4^3 changes the angle 3 times, the rotation is $3 \cdot \frac{2\pi}{4}$. While w_4^0 changes angle 0 times, so the rotation is zero. If we check the Fourier matrices with N and $M = N/2$ sampling numbers, we will discover there are some connections between those two matrix. First, $(w_N)^2 = w_M$ means the angle changing of w_M is twice of the changing of w_N.
Thus, the Fourier matrix $\mathbf{F}_N$ can be written as the combination of w_N and w_M in each

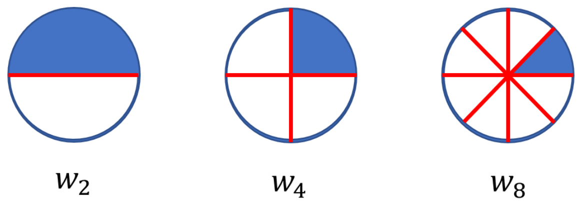

Figure 1: the rotation of different w

element. The elements which do the angle changing $2\pi/N$ even times can be written as the power of w_M. The elements which do the angle changing $2\pi/N$ odd times will be written as the power of w_M and multiply $c \cdot w_N$ to compensating some angle $2\pi/N$.

2.4.2 FFT

What FFT is doing is called "Divide and Conquer". Suppose the sampling number N is even and $M = N/2$. We divide $\mathbf{f}$ into to parts, $\mathbf{f}_{ev}$ and $\mathbf{f}_{od}$,and make:

$$\hat{\mathbf{f}}_{ev} = \mathbf{F}_M \mathbf{f}_{ev} \ , \ \mathbf{f}_{ev} = \begin{bmatrix} f_0 & f_2 & f_4 & \cdots & f_{N-4} & f_{N-2} \end{bmatrix}^T$$
$$\hat{\mathbf{f}}_{od} = \mathbf{F}_M \mathbf{f}_{od} \ , \ \mathbf{f}_{od} = \begin{bmatrix} f_1 & f_3 & f_5 & \cdots & f_{N-3} & f_{N-1} \end{bmatrix}^T$$

Assume each $\hat{f}_k$ in $\hat{\mathbf{f}}$ is a combination of $\hat{f}_{ev,k}$ in $\hat{\mathbf{f}}_{ev}$ and $\hat{f}_{od,k}$ in $\hat{\mathbf{f}}_{od}$, and try to find out the coefficients of those elements. We start from the basic equation:

$$\hat{\mathbf{f}} = \begin{bmatrix} \hat{f}_1 & \hat{f}_2 & \hat{f}_3 & \cdots & \hat{f}_{k=N-1} \end{bmatrix}^T \ , \ \hat{f}_k = \sum_{n=0}^{N-1} w_N^{nk} \cdot f_n$$

And divide the summation into two parts:

$$\begin{aligned} \hat{f}_k &= (w_N^{0k} \cdot f_0 \ + w_N^{2k} \cdot f_2 \ + \cdots + w_N^{N-2} \cdot f_{N-2}) \\ &\quad + (w_N^{1k} \cdot f_1 \ + w_N^{3k} \cdot f_3 \ + \cdots + w_N^{N-1} \cdot f_{N-1}) \\ &= \sum_{n=0}^{M-1} w_N^{2nk} \cdot f_{2n} \ + \sum_{n=0}^{M-1} w_N^{(2n+1)k} \cdot f_{2n+1} \end{aligned}$$

Since $w_m = w_n^2$, we can rewrite the equation:

$$\begin{aligned} \hat{f}_k &= \sum_{n=0}^{M-1} w_M^{nk} \cdot f_{2n} \ + \sum_{n=0}^{M-1} w_N^{nk+k} \cdot f_{2n+1} \\ &= \sum_{n=0}^{M-1} w_M^{nk} \cdot f_{2n} \ + w_N^k \cdot \sum_{n=0}^{M-1} w_N^{nk} \cdot f_{2n+1} \end{aligned}$$

First, we care about the$\hat{f}_k$ from number 0 to $M-1$. In order to realize the equation, we can write it in an expansion:

$$
\begin{aligned}
\hat{f}_k \qquad &= \begin{bmatrix} w_M^0 & w_M^{1k} & w_M^{2k} & \cdots & w_M^{(M-1)k} \end{bmatrix} \begin{bmatrix} f_0 \\ f_2 \\ f_4 \\ \vdots \\ f_{M-2} \end{bmatrix} \\
&+ w_N^k \cdot \begin{bmatrix} w_M^0 & w_M^{1k} & w_M^{2k} & \cdots & w_M^{(M-1)k} \end{bmatrix} \begin{bmatrix} f_1 \\ f_3 \\ f_5 \\ \vdots \\ f_{M-1} \end{bmatrix} \\
&, k = (0, 1, 2, \cdots, M-1)
\end{aligned}
$$

The equation can be written as three matrix multiplication:

$$
\hat{\mathbf{f}}_{k=0|M-1} = \begin{bmatrix} [I] & [D] \end{bmatrix} \begin{bmatrix} \mathbf{F}_M & [0] \\ [0] & \mathbf{F}_M \end{bmatrix} \begin{bmatrix} \mathbf{f}_{ev} \\ \mathbf{f}_{od} \end{bmatrix}
$$

where $[D]$ is a diagonal matrix:

$$
[D] = \begin{bmatrix} w_N^0 & & & & \\ & w_N^1 & & & \\ & & w_N^2 & & \\ & & & \ddots & \\ & & & & w_N^{M-1} \end{bmatrix}
$$

Next, we check the the $\hat{f}_k$ from number M to $2M-1$. We use $\hat{f}_{k'}$ to express those sample points. Here, $k' = k + M$ and $k = (0, 1, 2, \cdots, M-1)$ .The expension can be written as:

$$
\begin{aligned}
\hat{f}_{k'} \qquad &= \begin{bmatrix} w_M^0 & w_M^{1k'} & w_M^{2k'} & \cdots & w_M^{(M-1)k'} \end{bmatrix} \begin{bmatrix} f_0 \\ f_2 \\ f_4 \\ \vdots \\ f_{M-2} \end{bmatrix} \\
&+ w_N^{k'} \cdot \begin{bmatrix} w_M^0 & w_M^{1k'} & w_M^{2k'} & \cdots & w_M^{(M-1)k'} \end{bmatrix} \begin{bmatrix} f_1 \\ f_3 \\ f_5 \\ \vdots \\ f_{M-1} \end{bmatrix} \\
& where \quad k' = k + M \ , \ k = (0, 1, 2, \cdots, M-1)
\end{aligned}
$$

Since $w_N^{k'} = w_N^{k+M} = w_N^k \cdot w_N^{N/2}$. We can consider $w_N = e^{-i\,2\pi/N} \Rightarrow w_N^{N/2} = e^{-i\pi} = -1$. And rewrite the expansion of $\hat{f}_{k'}$ as:

$$
\begin{aligned}
\hat{f}_{k'} \qquad &= \begin{bmatrix} w_M^0 & w_M^{1k'} & w_M^{2k'} & \cdots & w_M^{(M-1)k'} \end{bmatrix} \begin{bmatrix} f_0 \\ f_2 \\ f_4 \\ \vdots \\ f_{M-2} \end{bmatrix} \\
& -w_N^k \cdot \begin{bmatrix} w_M^0 & w_M^{1k'} & w_M^{2k'} & \cdots & w_M^{(M-1)k'} \end{bmatrix} \begin{bmatrix} f_1 \\ f_3 \\ f_5 \\ \vdots \\ f_{M-1} \end{bmatrix} \\
& where \quad k' = k + M \ , \ k = (0, 1, 2, \cdots, M-1)
\end{aligned}
$$

The equation of $\hat{f}_{k'}$ can be written as three matrix multiplication:

$$
\hat{\mathbf{f}}_{k'=M|2M-1} = \begin{bmatrix} [I] & [-D] \end{bmatrix} \begin{bmatrix} \mathbf{F}_M & [0] \\ [0] & \mathbf{F}_M \end{bmatrix} \begin{bmatrix} \mathbf{f}_{ev} \\ \mathbf{f}_{od} \end{bmatrix}
$$

Now, we can combine $\hat{\mathbf{f}}_{k=0|M-1}$ and $\hat{\mathbf{f}}_{k'=M|2M-1}$ together to form $\hat{\mathbf{f}}$:

$$
\hat{\mathbf{f}} = \begin{bmatrix} [I] & [D] \\ [I] & [-D] \end{bmatrix} \begin{bmatrix} \mathbf{F}_M & [0] \\ [0] & \mathbf{F}_M \end{bmatrix} \begin{bmatrix} \mathbf{f}_{ev} \\ \mathbf{f}_{od} \end{bmatrix}
$$

The sampling points vector which has the even number elements above and odd number elements below can be express as a permutation matrix $[P]$ multiplying the vector $\mathbf{f}$:

$$
\begin{bmatrix} \mathbf{f}_{ev} \\ \mathbf{f}_{od} \end{bmatrix} = [P]\,\mathbf{f} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \mathbf{f}
$$

Recall the original DFT equation. We have the expansion of $\mathbf{F}_N$:

$$
\begin{aligned}
\hat{\mathbf{f}} &= \mathbf{F}_N \, \mathbf{f} \\
&= \begin{bmatrix} [I] & [D] \\ [I] & [-D] \end{bmatrix} \begin{bmatrix} \mathbf{F}_M & [0] \\ [0] & \mathbf{F}_M \end{bmatrix} [P] \, \mathbf{f}
\end{aligned}
$$

$$
\Rightarrow \mathbf{F}_N = \begin{bmatrix} [I] & [D_N] \\ [I] & [-D_N] \end{bmatrix} \begin{bmatrix} \mathbf{F}_M & [0] \\ [0] & \mathbf{F}_M \end{bmatrix} [P_N]
$$

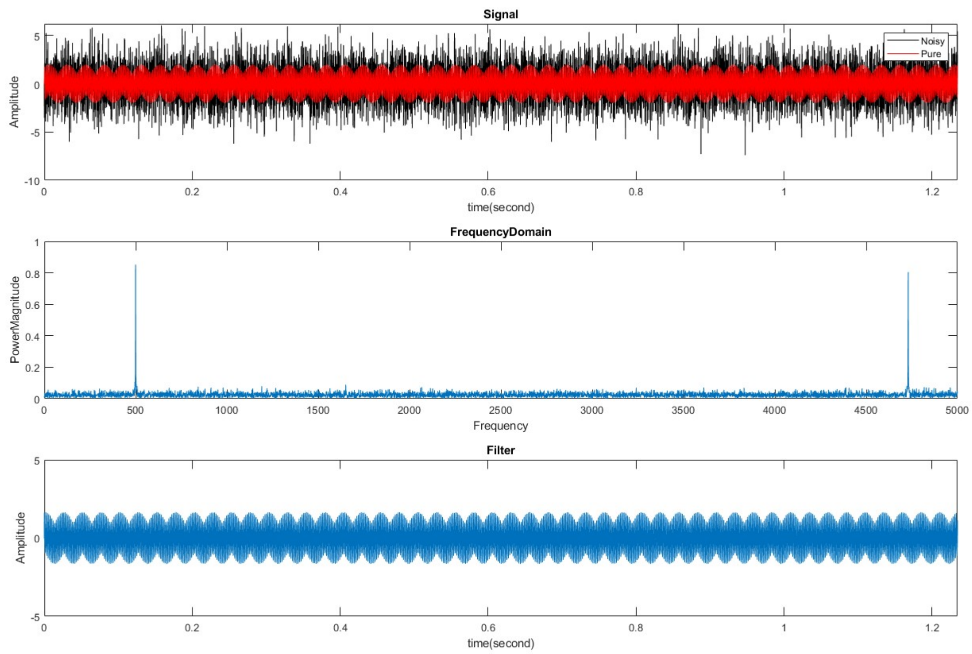

Figure 2: FFT denoising signal

By divide the $\mathbf{F}_N$ into $\mathbf{F}_M$ with some multiplication, we can reduce the operations in a computer from $O(N^2)$ to $O2(N/2)^2 + N/2$. Also, $\mathbf{F}_M$ can be divided into $\mathbf{F}_{M/2}$ with some multiplication by using the same manipulation above. Finally, the operation times of DFT was reduced to $O(N)log_2N$. And the hole algorithm including dividing,calculating and combining the Fourier matrix is call the Fast Fourier Transform (FFT).

2.5 Example (Denoise signal)

Supposed there is a input signal which is a combination of $500Hz$ and $4730Hz$ Sinwave with random noize. The original signal is too messy for the receiver to understand the information. By FFT algorithm, we can extract the specific frequencies from the original signal in the frequency domain. After we know the main frequencies inside the signal, we can easily remove the noizy signal's frequencies and only reserve the frequencies we want. Finally, we can transform the remaining signal from the frequency domain back to the time domain by the inverse FFT.
code: FFT project Example.m(https://github.com/Yi-An-Wang/Linear-Algebra-project-example.git)
result: Figure 2: FFT denoising signal

2.6 Discussions

2.6.1 inverse DFT

Recall the Fourier matrix is a symmetric matrix with each column has a length $\sqrt{N}$ and orthogonal to each other. $\frac{1}{N}\bar{\mathbf{F}}_N$ will be the inverse of Fourier matrix with sampling numbers N. Thus, the transformation equation which changes the $\hat{\mathbf{f}}$ back to $\mathbf{f}$ is:

$$\mathbf{f} = \frac{1}{N}\bar{\mathbf{F}}_N\hat{\mathbf{f}}$$

Because $\bar{\mathbf{F}}_N$ and $\mathbf{F}_N$ almost have the same features, we can separate $\hat{\mathbf{f}}$ into two parts, odd and even. Do the same operation just like what FFT is doing.

$$\mathbf{f} = \frac{1}{N}\begin{bmatrix} \begin{bmatrix} I \\ I \end{bmatrix} & \begin{bmatrix} \bar{D} \\ -\bar{D} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \mathbf{F}_M & [0] \\ [0] & \mathbf{F}_M \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{f}}_{ev} \\ \widehat{\mathbf{f}}_{od} \end{bmatrix}$$

2.6.2 remaining and conclusion

While the sampling number is not the power of 2, people may do Zero padding or Truncatoin on the signal sets. The former add zero terms to the signal set, making the sampling points become the power of two. The latter, on the contrary, cut the signal set to make the sampling points be the power of two. Both of the methods will make the data distortion. Thus, there are some variants of FFT algorithm to deal with this problem. In summary, FFT is a powerful and efficient algorithm with broad applications. For example, in signal processing, image processing, communication and various of engineering computing, FFT all plays a critical role.

2.7 References

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