Linear Algebra and its Applications

HW#06

1. Show **"geometrically"** that the following transformation are linear transformations:
   (a) Rotation of a vector in $\mathbf{R}^2$ through an angle θ.
   (b) Reflection of a vector in $\mathbf{R}^2$ through the mirror θ-line.
   (c) Projection of a vector in $\mathbf{R}^2$ onto the θ-line.

2. On the 4-dimensional space of cubic (degree 3) polynomials, let the basis consists of four terms: 1, t, t^2, t^3 (i.e., any cubic polynomial is a linear combination of the four basis terms).
   (a) Show that the second derivative of the cubic polynomials is a linear transformation.
   (b) Take the second derivative of each basis term. Suppose there is a cubic polynomial $a_0+a_1t+a_2t^2+a_3t^3$. Express the second derivative of this cubic polynomial as the linear combination of the second derivatives of the four basis terms.
   (c) Find a 4 by 4 matrix that represents taking the second derivative of any cubic polynomial. (a square polynomial is a special case of cubic polynomial with the cubic coefficient equal to zero)
   (d) Find a 2 by 4 matrix A that represents taking the second derivative of any cubic polynomial in the four-dimensional space with basis 1, t, $t^2/2$, $t^3/6$ to become a degree-1 polynomial in the two-dimensional space with 1, t as its basis. Find the second derivative of $4+3t+2t^2+t^3$ by expressing the polynomial by a vector x and take the second derivative by Ax.

3. Show that the following matrices are reflections:

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \quad \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{bmatrix}$$

   Show that the product of two reflections is a rotation. Multiply the above reflection matrices to find the rotation angle.

4. Find the matrix that projects every vector in R^3 onto the intersection of the planes $x_1+x_2+x_3=0$ and x_1-$x_3=0$, which is a line.

5. The least squared approximate of x is to minimize $E^2=\|Ax-b\|^2$ by setting its derivatives with respect to u and v to zero. Let

$$A=\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix},\ x=\begin{bmatrix} u \\ v \end{bmatrix},\ b=\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}.$$

   Compare the resulting least-square equations (the equations that sets the derivatives to zero) with the normal equations. Find the least squared approximate of x and compare it to the b's projection onto the column space of A.

6. Suppose the values $b_1=1$ and $b_2=7$ at times $t_1=1$ and $t_2=5$ are fitted by a line $b=Dt$ through the origin. Find $\hat{D}$ by least square and sketch the observations with the best-fit line. Find $\hat{D}$ by projection and sketch the projection of b onto the column space of t.

7. If P is the projection matrix onto a k-dimensional subspace S of the whole space $\mathbf{R}^n$, what is the column space and nullspace of P and what is its rank?

8. If u is a unit vector, show that $Q=I-2uu^T$ is a reflection transformation. Compute Q when $u^T=(1/2, 1/2, -1/2, -1/2)$ and explain what Q does to x with Qx.