Chapter 3 Structural Analysis of Mechanisms

3.1 General

Mechanism is an assemblage of interconnected links with one of its member fixed to ground. The study of the nature of connection among its members is known as the topological analysis or structural analysis of mechanism. The topology, or structure, of a mechanism is a function of the number of links and the nature of connection among them. Various methods of representing the structure of a mechanism have been introduced in previous chapter. An understanding of the kinematic structure can be useful in several ways. It provides a means of recognizing the kinematic essentials of a mechanism; it serves to detect mechanism similarities and differences; it can be used as an aid in conceptual design studies; and can be used in creating new mechanisms systematically. Topological analysis is not concerned with the functional characteristics, such as displacement, velocity, and acceleration. However, topological analysis can provide general statements about a wide variety of mechanisms.

Using the correspondence between a mechanism and its graph, many useful properties of graphs can be transferred to corresponding properties of mechanisms. Although there are mechanisms with non-planar graphs, in what follows, we shall discuss mechanisms whose corresponding graphs are simply connected planar graphs. The following are the summary of the notational correspondence between mechanisms and graphs.

Table 3-1 The correspondence between graphs and mechanisms

Mechanisms	Graphs
Number of links, n	Number of vertices, v
Number of joints, j	Number of edges, e
Number of links with i joints, n_i	Number of vertices of degree i, v_i
Number of joints on link i, d_i	Degree of vertex i, d_i
Number of independent loops L	Number of independent loops L
Total number of loops, $\underline{L}$ (including peripheral loop)	Total number of loops, $\underline{L}$
No. of loops with i joints, L_i	No. of loops with i edges, L_i

Table 3-2 lists some of the basic characteristics of graphs and the corresponding characteristics of mechanisms.

Table 3-2 Characteristics of graphs and the corresponding characteristics of mechanisms

Item	Mechanisms	Graphs
1	$\sum_{i=2} in_i = 2j$	$\sum_{i=2} iv_i = 2e$
2	$\sum_{i=2} n_i = n$	$\sum_{i=2} v_i = v$
3	$\sum_{i=3} \mathrm{L}_i = \underline{\mathrm{L}}$	$\sum_{i=3} \mathrm{L}_i = \underline{\mathrm{L}} = \mathrm{L}+1$
4	$\sum i\mathrm{L}_i = 2\mathrm{j}$	$\sum i\mathrm{L}_i = 2\mathrm{e}$
5	$\sum_{i=1}^{j} d_i = 2j$	$\sum_{i=1} d_i = 2e$
6	$L=j-n+1$	$L=e-v+1$
7	$n_2 \geq 3n-2j$	$v_2 \geq 3v-2e$
8	$j-n+2 \geq d_i \geq 2$	$e-v+2 \geq d_i \geq 2$
9	Isomorphic kinematic chains	Isomorphic graphs

Proof of Item 7:

Multiplying Item 2 by 3, and subtracting Item 1 from the resulting expression yields

$3(v_2+v_3+v_4+\ldots+v_m) - (2v_2+3v_3+\ldots+mv_m)=3v-2e$

$v_2=3v-2e+(v_4+2v_5+\ldots)$

$v_2 \geq 3v-2e$.

Let the number of independent loops, L, be the number of faces minus the peripheral face, then

$$L = \underline{L} - 1 \tag{3-1}$$

The Euler's equation becomes

$$L=j-n+1 \tag{3-2}$$

Recalling the degree-of-freedom equation, Eq. (1-1) is given as

$$F = \lambda(n-j-1) + \sum_i f_i \tag{3-3}$$

and the loop mobility criterion is given as

$$F = \sum f_i - \lambda \cdot L \tag{3-4}$$

We now consider several special cases as follows.

3.2 Single-Loop Mechanisms (L=$\underline{L}$-1=1)

Substituting L=1 into Eq. (3-2), Equation (3-2) becomes

$$j = n \tag{3-5}$$

Substituting L=1 into Equation (3-4),

$$F=\Sigma f_i - \lambda \tag{3-6}$$

Sub. Eq. (3-5) into item 7 of Table.3-2

$$n_2 \geq n,$$

Yet, $\Sigma n_i = n$; Therefore,

$$n_2=n \tag{3-7}$$

All links in a single-loop mechanism are necessarily binary.

3.3 Planar Linkages with Lower-pair Joints Only

For planar linkages with lower-pair joints (i.e. R-joints and P-joints only), we have j_1=j, and

$$\Sigma f_i=j \tag{3-8}$$

Substituting Eq. (3-8) into (3-3), yields

$$F=3n-2j-3 \tag{3-9}$$

Substituting (3-8) into (3-4), yields

$$F=j-3L \tag{3-10}$$

Eliminating j from (3-9) & (3-10), we obtain

$$F=n-(2L+1) \tag{3-11}$$

Since (2L+1) is always an odd number, hence F is odd or even depending on the number of links is even or odd.

Eq. (3-11) can be also expressed in terms of $\underline{L}$

$$\underline{L}=L+1=(n-F+1)/2 \tag{3-12}$$

where $\underline{L}$ is the number of faces including the peripheral face.

A vertex of degree i must be part of i different faces or loops. But, the number of faces is related to the number of links and number of degree-of-freedom by Eq.(3-12). Hence

$$d_i \leq (n\text{-}F+1)/2 \qquad (3\text{-}13)$$

In other words, the maximum number of joints on any link cannot exceed (n-F+1)/2.

Since we are interested in mechanisms with positive mobility, i.e., F≥1, it follows from Eq. (3-9) that

$$j \leq 3n/2 - 2 \qquad (3\text{-}14)$$

or in terms of graphs,

$$e \leq 3v/2 - 2 \qquad (3\text{-}15)$$

Hence, only those graphs which satisfy Eq.(3-15) are applicable to planar linkages with lower-pair joints.

For F=1, Eq.(3-12) becomes

$$\underline{L} = n/2 \qquad (3\text{-}16)$$

Eq. (3-16) states that the number of faces equals one half the number of links, and the number of links must be even. It also implies that the number of joints on a link can not exceed one-half the number of links ($d_i \leq n/2$).

Example 1 Verify the following relations for both one and two degree-of-freedom planar linkages.

F=1

n	j	L
4	4	1
6	7	2
8	10	3

F=2

n	j	L
5	5	1
7	8	2
9	11	3

Example 2 Find the link assortments of n=8, j=10 kinematic chains.

We wish to find all possible link assortments for planar kinematic chains with n=8 and j=10. Applying Equation (3-13), the upper bound on the number of joints on a link is 4. From Item 7 of Table 3-2, the lower bound on the number of binary links is 4. Hence, Items (1) and (2) in Table 3-2 reduce to

$$n_2+n_3+n_4=8$$

$$2n_2+3n_3+4n_4=20$$

Solving the above equations in 3 unknowns, we can obtain

$(n_2, n_3, n_4) = (4, 4, 0)$ or $(5, 2, 1)$ or $(6, 0, 2)$.

3.4 Planar Mechanisms with up to Two-d.o.f. Joints

For planar mechanisms with up to two-d.o.f. joints,

$$\sum_i f_i = j_1 + 2j_2 \qquad (3\text{-}17)$$

Substituting Eq. (3-17) into (3-3) with $\lambda=3$, we have

$$F = 3(n - j - 1) + j_1 + 2j_2 \qquad (3\text{-}18)$$

But, $j=j_1+j_2$ (3-19)

Hence, $F=3n-2j-3+j_2$ (3-20)

Since we are interested in $F\geq 1$, Eq.(3-20) becomes

$$3n-2j-3+j_2 \geq 1 \qquad (3\text{-}21)$$

It is shown that the number of two d.o.f. joints in a planar mechanism cannot exceed the number of independent loops, i.e.

$$j_2 \leq L \qquad (3\text{-}22)$$

or,

$$j_2 \leq j-n+1 \qquad (3\text{-}23)$$

$$j_2 + n-j-1 \leq 0 \qquad (3\text{-}24)$$

Rewriting Eq.(3-21) in the following form

$$(2n-j-3)+(n-j+j_2-1) \geq 0 \qquad (3\text{-}25)$$

Combining Eqs. (3-24) & (3-25), we have

$$2n-j-3 \geq 0 \qquad (3\text{-}26)$$

or

$$j \leq 2n-3 \qquad (3\text{-}27)$$

Hence in terms of graph

$$e \leq 2v-3 \qquad (3\text{-}28)$$

We conclude that only those graphs with their number of edges and vertices satisfying Eq. (3-28) are potentially applicable for planar mechanisms.

3.5 Spatial Linkages- Single Loop

For spatial linkages with only one independent loop, the results derived in Section 3.2 can be directly applied. Hence $j=n$ and $n=n_2$

Eq. (3-6) becomes

$$F+6=\sum_i f_i \tag{3-29}$$

For one d.o.f. mechanism, F=1, we have

$$\sum f_i = 7 \tag{3-30}$$

Furthermore, if all the joints are one-degree-of-freedom then

$$n = j = \sum_i f_i = 7 \tag{3-31}$$

The following are a few examples of one d.o.f. single-loop spatial linkage with revolute and prismatic joints

(a) R-R-R-R-R-R-R mechanism
(b) R-R-R-R-R-R-P mechanism
(c) R-R- R-R-R-P-R mechanism

3.6 General Spatial Mechanisms

In this case, we allow joints with up to three degrees of freedom, so that

$$j=j_1+j_2+j_3 \tag{3-32}$$

and

$$\sum f_i = j_1+2j_2+3j_3 \tag{3-33}$$

Hence

$$\begin{aligned} F&=6(n-j-1)+j_1+2j_2+3j_3 \\ &=6(n-j-1)+3j-2j_1-j_2 \\ &=6n-3j-6-2j_1-j_2 \end{aligned} \tag{3-34}$$

Again, we are interested in $F \geq 1$

$$6n-3j-7-2j_1-j_2 \geq 0 \tag{3-35}$$

Since $2j_1+j_2 \geq 0$, it follows that

$$6n-3j-7 \geq 0 \tag{3-36}$$

or $$j \leq 2n-7/3 \tag{3-37}$$

However, since j is an integer, Eq. (3-37) is equivalent to

$$j \leq 2n-3 \tag{3-38}$$

or, in terms of graphs

$$e \leq 2v-3 \tag{3-39}$$

The transition from Equation (3-37) to (3-38) also allows the possibility of at least one turning pair (j_1 =1) in all mechanism, as is desirable from a practical point of view.

We notice that the inequality, Equation (3-39), derived for spatial mechanisms is the same as that for planar mechanisms, Equation (3-28).

3.7 Planar Mechanism with Sliding Pairs Only

We consider a special case of planar mechanisms in which all links perform planar translation without rotation. This type of mechanisms is made of sliding pairs only. In this case, the motion parameters, λ, is equal to two. Hence, Eq. (1-1) becomes

$$F=2n-j-2 \qquad (3\text{-}40)$$

Therefore, a 3-link wedge mechanism has one degree-of-freedom.
For $F \geq 1$, we have

$$2n-j-2 \geq 1 \qquad (3\text{-}41)$$

or,

$$j \leq 2n-3 \qquad (3\text{-}42)$$

or in terms of the graph,

$$e \leq 2v-3 \qquad (3\text{-}43)$$

This inequality equation, (3-43), is again the same as the equation (3-28) for general planar mechanisms.

3.8 The Atlas of Graphs for Admissible Kinematic Chains

The collection of graphs given by Buchsbaum and Freudenstein (1970) can be reduced according to the following guidelines:

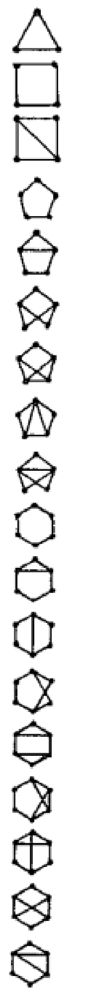
(a) Since each link possesses two or more joints, all graphs must be connected with minimum degree of two for all vertices.
(b) Mechanisms consisting of submechanisms connected by a single joint or link are usually treated as two concatenated mechanisms, and therefore, are omitted. Or, in terms of the graph, there cannot exist any articulation point or cut edges.
(c) Mechanisms corresponding to non-planar graphs are omitted. This is somewhat arbitrary, but, in view of the complexity of such mechanisms, it is reasonable to exclude them. For example, in one d.o.f. linkage with revolute joints, mechanisms with non-planar graphs must have at least 10 links.
(d) All mechanisms must obey the general degree-of-freedom equation. That is, no special proportions are required to ensure the mobility of a mechanism. Those graphs which do not obey the inequality (3-28), are omitted.

Based on the above restrictions, Mayourian and Freudenstein (1984) found that there are

35 simple graphs that represent the kinematic chains of mechanisms with up to six links as shown by Table 3-3.

Table 3-3 Atlas of graphs of the kinematic chains of mechanisms with up to six links

No.	v	e	Graph	No.	v	e	Graph
1	3	3		19	6	8	
2	4	4		20	6	8	
3	4	5		21	6	8	
4	5	5		22	6	8	
5	5	6		23	6	9	
6	5	6		24	6	9	
7	5	7		25	6	9	
8	5	7		26	6	9	
9	5	7		27	6	9	
10	6	6		28	6	9	
11	6	7		29	6	9	
12	6	7		30	6	9	
13	6	7		31	6	9	
14	6	8		32	6	9	
15	6	8		33	6	9	
16	6	8		34	6	9	
17	6	8		35	6	9	
18	6	8					

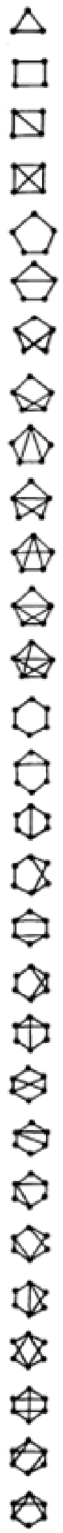
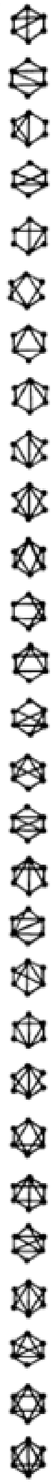


References

1. Tsai, L.W. *Mechanism Design- Enumeration of Kinematic Structures According to Function*, CRC Press, 2001.

2. Freudenstein, F., and Mayourian, M., "The Development of an Atlas of the Kinematic Structures of Mechanisms," ASME *J. of Mechanisms, Transmissions, and Automation in Design*, 1984, Vol. 106, pp.458~461

Appendix I Atlas of graphs with up to six links

Number	v	e	LDS[a]	Graph
1	3	3	222	
2	4	4	2222	
3	4	5	3322	
4	4	6	3333	
5	5	5	22222	
6	5	6	33222-1	
7	5	6	33222-2	
8	5	7	33332	
9	5	7	43322	
10	5	7	44222	
11	5	8	43333	
12	5	8	44332	
13	5	9	44433	
14	6	6	222222	
15	6	7	332222-1	
16	6	7	332222-2	
17	6	7	332222-3	
18	6	8	333322-1	
19	6	8	333322-2	
20	6	8	333322-3	
21	6	8	333322-4	
22	6	8	433222-1	
23	6	8	433222-2	
24	6	8	433222-3	
25	6	8	442222-1	
26	6	8	442222-2	
27	6	9	333333	
28	6	9	433332-1	
29	6	9	433332-2	
30	6	9	433332-3	
31	6	9	443322-1	
32	6	9	443322-2	
33	6	9	443322-3	
34	6	9	443322-4	
35	6	9	443322-5	
36	6	9	444222	
37	6	9	533322	
38	6	9	543222	
39	6	9	552222	
40	6	10	443333-1	
41	6	10	443333-2	
42	6	10	444332-1	
43	6	10	444332-2	
44	6	10	444332-3	
45	6	10	444422	
46	6	10	533333	
47	6	10	543332-1	
48	6	10	543332-2	
49	6	10	544322	
50	6	10	553322	
51	6	11	444433	
52	6	11	544333	
53	6	11	544432	
54	6	11	553333	
55	6	11	554332	
56	6	12	444444	
57	6	12	554433	

[a] Local degree structure.