# Introduction to Statistical Methods

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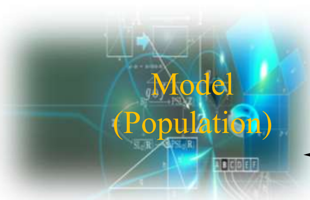
# Probability and Statistics

Probability




*Inference*


Statistics

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# An Example: Taiwan Big Lotto (大樂透)

- 6 winning numbers and 1 special number chosen from 49 numbers
- Mr. Chang chooses numbers randomly and never believes in any historical analysis of number appearance
- Mr. Fang chooses numbers that most frequently appear in the history
- Mr. Wang chooses numbers that most rarely appear in the history
- Mr. Yang chooses meaningful numbers, such as the date of birthday
- **Who is correct?**

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# Probability of Taiwan Big Lotto

- 6 numbers chosen from 49 numbers
- Chance of winning the first prize?
- Answer: $1/C_6^{49}$
- Is this based on probability or statistics?
- What is the model and assumption behind the answer above?

  Every number has the ***identical*** probability to be chosen ***independently*** each time! $\Rightarrow$ ***iid*** assumption
- The event of winning is guessed (inferred) by the "model".

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## Statistics of Taiwan Big Lotto

- What is "Statistics" in the Lotto problem?
- Is the model assumption correct?
- What is the real chance of winning? and the chance for each number to appear?
- "Statistics" is to estimate the appearance probability of each number. How?

  by collecting and observing the data.

  ⇒ "Model" is **inferred** by the "sample data"

  ⇒ Statistical Inference

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## Taiwan Big Lotto (Cont'd)

- Back to Mr. Chang, Fang, and Wang
- Who is correct?
- What does Mr. Chang believe?
- What does Mr. Fang believe?
- What does Mr. Wang believe?
- What does Mr. Yang believe?

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# A note on "Models"

*"All models are wrong, but some are useful."*
- *George E. P. Box* (1979)

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# Definition of "Experiment"

- An **experiment** is the process by which an observation (or measurement) is obtained.
- **Experiment: Record an age**
- **Experiment: Toss a die**
- **Experiment: Record an opinion (yes, no)**
- **Experiment: Toss two coins**
- **Experiment: numbers of a lotto ticket**

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## Definition of "Outcome"

- An outcome is observed on a single repetition of the experiment.
  - Basic element to which probability is applied.
  - One and only one outcome can occur when experiment is performed.
- An outcome is denoted by O with a subscript.

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## Definition of "Outcome Space"

- Each outcome is assigned a probability, measuring "how often" it occurs.
- Set of all outcomes of an experiment is called the **outcome space**, usually denoted by **S**.

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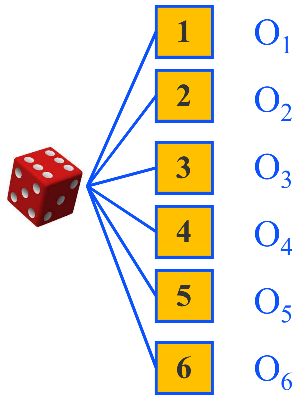
# Example of Experiment and Outcome

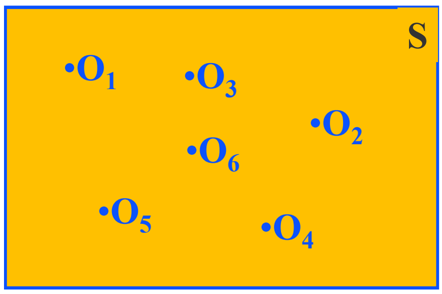
- The die toss experiment:
- Outcomes:




Outcome space:

**$S = \{O_1, O_2, O_3, O_4, O_5, O_6\}$**

**(or $S = \{1, 2, 3, 4, 5, 6\}$)**

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# Example

- Record a person's blood type:
- Outcomes:

$O_1$ A

$O_2$ AB

$O_3$ B

$O_4$ O

Outcome space:

**$S = \{O_1, O_2, O_3, O_4\}$**

**S ={A, B, AB, O}**

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## Example

- Record the numbers of a lotto ticket:
- Outcomes:

**$O_1$=[1, 2, 3, 4, 5, 6]**

**$O_2$=[2, 3, 4, 5, 6, 7]**

**$O_3$=[3, 4, 5, 6, 7, 8]**

**:**

**:**

**$O_?$ =[44, 45, 46, 47, 48]**

Outcome space:

**S ={$O_1$, $O_2$, $O_3$, $O_4$,…, ………, $O_?$}**

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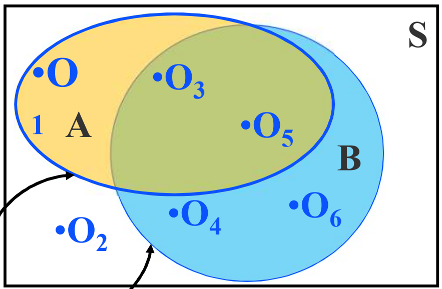
## Definition of "Event"

- An **event** is a collection of one or more **outcomes.**

**•The die toss events:**

- ✓ Event A: an odd number
- ✓ Event B: a number > 2




**A ={$O_1$, $O_3$, $O_5$}**
**B ={$O_3$, $O_4$, $O_5$, $O_6$}**

***Venn Diagram***

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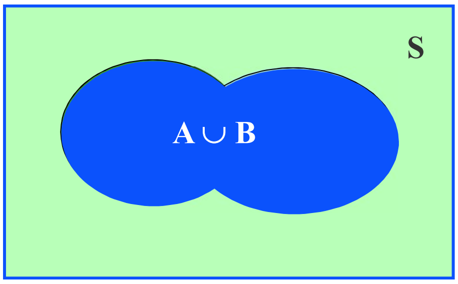
## Event Relations - Union

The **union** of two events, A and B, is the event that either A **or** B **or both** occur when the experiment is performed. We write

**A ∪ B**

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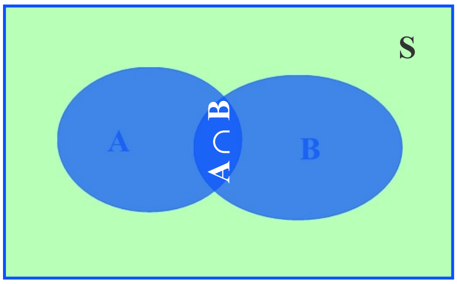
## Event Relations-Intersection

The **intersection** of two events, **A** and **B**, is the event that both A **and** B occur when the experiment is performed. We write **A ∩ B.**




- If two events A and B are **mutually exclusive,** then **A ∩ B =∅.**

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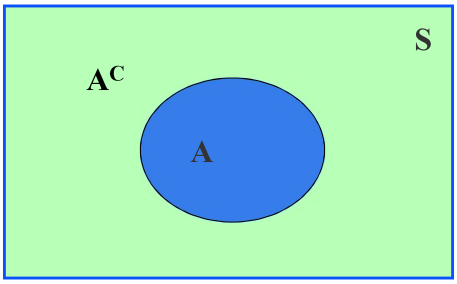
# Event Relations - Complement

The **complement** of an event **A** consists of all outcomes of the experiment that do not result in event A.  We write **$A^C$ ( The event that event A doesn't occur).**

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# Example

Select a student from a college

– **A:** student is colorblind
– **B:** student is biologically female
– **C:** student is biologically male

Mutually exclusive and $B = C^C$

•What is the relationship between events **B** and **C**?

•**$A^C$:** Student is not colorblind

•**$B \cap C$:** Student is both biologically male and female = $\varnothing$

•**$B \cup C$:** Student is either male or female = all students = S

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## Factorial and Sequences

**Example:** How many possible distinct sequences of 5 different numbers among 1, 2, 3, 4 and 5?

The "order" of the choices → $5(4)(3)(2)(1) = 120$

The number of distinct sequences you can arrange ***n*** distinct objects is ***n*** factorial denoted as:

$$n! = n(n-1)(n-2)...(2)(1) \text{ and } 0! \equiv 1.$$

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## Permutations

**Example:** How many 3-digit lock passwords can we make by using 3 **different numbers** among 1, 2, 3, 4 and 5?

The order of the choice is important! → $5(4)(3) = 60$

The number of ways you can arrange ***n*** distinct objects, taking them ***r*** at a time is

$$P_r^n = \frac{n!}{(n-r)!}$$

$$P_3^5 = \frac{5!}{(5-3)!} = \frac{5(4)(3)(2)(1)}{2(1)} = 60$$

20 hen

## Combinations

- The number of distinct combinations of ***n*** **distinct objects** that can be formed, **taking them *r* at a time** is $C_r^n = \frac{n!}{(n-r)!}(\frac{1}{r!}) = \frac{n!}{r!(n-r)!}$ since the number of permutations is $r!$ times that of combinations

**Example:** 3 members of a 5-person faculty must be chosen to form a committee. How many different committees could be formed?

The order of the choice is not important!

$$C_3^5 = \frac{5!}{3!(5-3)!} = \frac{5(4)(3)}{3(2)(1)} = \frac{5(4)}{(2)1} = 10$$





## The Probability of an Event

- ***P***(A) must be between 0 and 1.
  - If event A can never occur, ***P***(A) = 0.
  - If event A always occurs when the experiment is performed, ***P***(A) =1.
- Sum of probabilities for all simple events in S equals 1. ***P***(S)=1.
- **Probability of an event A is found by adding the probabilities of all outcomes contained in A**

## Taiwan Big Lotto

- 6 numbers chosen from 49 numbers
- First prize: all six numbers

  Event probability:
  $$1/C_6^{49} = \frac{6!(49-6)!}{49!}$$
- Second prize: 5 numbers + 1 special number

  Event probability?
  $$C_5^6 \,/\, C_7^{49}$$
  Why? Model assumptions?

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## Mutually Exclusive Events

- When A and B have no outcomes in common, they are said to be **mutually exclusive** or **disjoint** events

•Experiment: Toss a die

–A: observe an odd number

–B: observe a number greater than 2

–C: observe a 6

–D: observe a 3

Not Mutually Exclusive

Mutually Exclusive

**A and C?**
**A and D?**
**B and C?**

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# Conditional Probability

- For any two events $A$ and $B$ with $P(B)>0$, the **conditional probability** of $A$ given that $B$ has occurred is defined by $P(A|B)=P(A\cap B)/P(B)$

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# Independent Events

- Two events $A$ and $B$ are **independent** if $P(A|B)=P(A)$ and are dependent otherwise

- In other words,
  $P(A|B)=P(A\cap B)/P(B) =P(A)$

$\Rightarrow P(A\cap B)=P(A)\ P(B)$ if $A$ and $B$ are independent

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# Basic Probability Rules

- Rule 1: Addition rule for **mutually exclusive** events $P(A \text{ or } B)=P(A)+P(B)$
- Rule 2: Multiplication rule for **independent** events $P(A \text{ and } B)=P(A)P(B)$
- Rule 3: General addition rule

$$P(A \text{ or } B)=P(A)+P(B)-P(A \text{ and } B)$$

- Rule 4: General multiplication rule

$$P(A \text{ and } B)=P(A)P(B|A)$$

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# Sensitivity and Specificity

Test: disease, fire, quality, etc.
Testing performance?

| | | True Condition | | |
|---|---|---|---|---|
| | | **Condition positive (CP)** | **Condition negative (CN)** | |
| **Predicted Condition** | **Test outcome positive (P)** | **True positive** (TP) = 20 {P∩CP} | **False positive** (FP) = 180 {P∩CN} | **Positive predictive value** (**PPV/Precision**) $P$(CP\|P) = TP / (TP + FP) = 20 / (20 + 180) = **10%** |
| | **Test outcome negative (N)** | **False negative** (FN) = 10 {N∩CP} | **True negative** (TN) = 1820 {N∩CN} | **Negative predictive value** (NPV) $P$(CN\|N) = TN / (FN + TN) = 1820 / (10 + 1820)≈ **99.5%** |
| | | **Sensitivity (Recall)** $P$(P\|CP) = TP / (TP + FN) = 20 / (20 + 10) ≈ **67%** | **Specificity** $P$(N\|CN) = TN / (FP + TN) =1820/(180 +1820) = **91%** | |

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## Meanings of Sensitivity and Specificity

Sensitivity (Recall): "I know my patient has the disease. What is the chance that the test will show that my patient has it?“

Specificity: "I know my patient doesn't have the disease. What is the chance that the test will show that my patient doesn't have it?"

## Meanings of PPV and NPV

PPV (Precision/Risk): “I just got a positive test result back on my patient. What is the chance (risk) that my patient actually has the disease?”

NPV: “I just got a negative test result back on my patient. What is the chance that my patient actually doesn't have the disease?”

# Total Probability and Bayes

- Let $A_1,\ldots, A_n$ be **mutually exclusive** and **exhaustive** events. Then for any other event $B$: (**law of total probability**)

$$P(B) = \sum_{i=1}^{n} P(B \mid A_i)P(A_i)$$

- **Bayes Thm**: Let $A_1,\ldots,A_n$ be a collection of n **mutually exclusive** and **exhaustive** events with $P(A_i)>0$ for $i=1,\ldots,n$. Then for any other event $B$ for which $P(B)>0$

$$\boxed{P(A_k \mid B)} = \frac{P(A_k \cap B)}{P(B)} = \frac{\boxed{P(B \mid A_k)P(A_k)}}{\sum_{i=1}^{n} \boxed{P(B \mid A_i)P(A_i)}}$$

**a priori knowledge**

**a posteriori inference**

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# Example 1

It is known that the H1N1 flu, a subtype of influenza type *A* also known as swine flu, prevalence is 10% (the proportion of population infected) in the pandemic period (2009~2010). A testing method has been developed to quickly test if a patient is infected with the influenza virus. This method is called Rapid Influenza Diagnostic Test (RIDT). Only 60% (sensitivity) of patients infected with H1N1 can be tested positive (40% of false negative) and 95% (specificity) of patients without H1N1 infection are tested negative. Only patients tested positive can be treated with Tamiflu administering. If a patient is not treated with Tamiflu administering, what is the probability that the patient is infected with H1N1 without treatment of Tamiflu?

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# Example 1 (Cont'd)

- a priori knowledge:
  - $A_1$: infected with H1N1 $P(A_1)=10\%$
  - $A_0$: without infection with $P(A_0)=1-P(A_1)$
  - $B_1$: tested positive
  - $B_0$: tested negative
  - $P(B_1|A_1)= 0.6$ ; $P(B_0|A_0)=0.95$
- a posteriori question: $P( A_1 \mid B_0 )=?$




# Complete Testing Evaluation Table

| | | True condition | | | | |
|---|---|---|---|---|---|---|
| | Total population | Condition positive | Condition negative | Prevalence = $\frac{\Sigma \text{ Condition positive}}{\Sigma \text{ Total population}}$ | Accuracy (ACC) = $\frac{\Sigma \text{ True positive} + \Sigma \text{ True negative}}{\Sigma \text{ Total population}}$ | |
| Predicted condition | Predicted condition positive | **True positive** | **False positive**, Type I error | Positive predictive value (PPV), Precision = $\frac{\Sigma \text{ True positive}}{\Sigma \text{ Predicted condition positive}}$ | False discovery rate (FDR) = $\frac{\Sigma \text{ False positive}}{\Sigma \text{ Predicted condition positive}}$ | |
| | Predicted condition negative | **False negative**, Type II error | **True negative** | False omission rate (FOR) = $\frac{\Sigma \text{ False negative}}{\Sigma \text{ Predicted condition negative}}$ | Negative predictive value (NPV) = $\frac{\Sigma \text{ True negative}}{\Sigma \text{ Predicted condition negative}}$ | |
| | | True positive rate (TPR), Recall, Sensitivity, probability of detection, Power = $\frac{\Sigma \text{ True positive}}{\Sigma \text{ Condition positive}}$ | False positive rate (FPR), Fall-out, probability of false alarm = $\frac{\Sigma \text{ False positive}}{\Sigma \text{ Condition negative}}$ | Positive likelihood ratio (LR+) = $\frac{TPR}{FPR}$ | Diagnostic odds ratio (DOR) = $\frac{LR+}{LR-}$ | $F_1$ score = $2 \cdot \frac{\text{Precision} \cdot \text{Recall}}{\text{Precision} + \text{Recall}}$ |
| | | False negative rate (FNR), Miss rate = $\frac{\Sigma \text{ False negative}}{\Sigma \text{ Condition positive}}$ | Specificity (SPC), Selectivity, True negative rate (TNR) = $\frac{\Sigma \text{ True negative}}{\Sigma \text{ Condition negative}}$ | Negative likelihood ratio (LR−) = $\frac{FNR}{TNR}$ | | |

# Example 2

Only one in 1000 chips is defective by not conforming to a certain type of electrical test. The test is performed by a tester that detects the failure 99% of the time when the chip is actually defective. The tester misidentified a good chip only 2% of the time. If a randomly selected chip is tested defective, what is the probability that the chip is actually defective.

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# Variables with Random Nature

- **Definition:** For a given sample space of some experiment, a **random variable** is **any rule** that associates a number with each outcome in the sample space. A random variable is always denoted by a **capital letter (e.g. *X*, *Y*, etc)**
- Can you think of anything that is not random in nature? Can you think of two things that are exactly the same?

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# Random Variables

- What are random variables?
  - Life is a never ending experiment
  - **Define the rules of experiment with its outcome as a random variable**
- Types of random variables
  - **discrete** if its set of possible values is a discrete set
  - **continuous** if its set of possible values is an entire interval of numbers
- Examples: defects, c.d., thickness

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# Probability Distribution

- The **probability distribution** or **probability mass function (p.m.f)** of a **discrete** random variable is defined for every number $x$ by $p(x) = P(X=x)$
- Let $X$ be a continous random variable. Then a probability distribution or probability density function (p.d.f) of $X$ is a function $f$(x) such that for any two numbers $a$ and $b$ with $a \leqq b$

$$P(a \le X \le b) = \int_a^b f(x)dx$$

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## Cumulative Distribution Function (c.d.f)

- The **cumulative distribution function (c.d.f)** $F(x)$ of a discrete (continuous) random variable $X$ with p.m.f $p(x)$ (p.d.f. $f(x)$) is defined for every number $x$ by

$$F(x) = P(X \le x) = \sum_{y:y\le x} p(y)$$

$$(= \int_{-\infty}^{x} f(y)dy)$$

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## Expected Value (Mean)

- The **expected (or mean value)** of a discrete (continuous) r.v. $X$ with p.m.f $p(x)$ (p.d.f $f(x)$) is a **weighted average** weighted by $p(x)$ or $f(x)$:

$$E(X) = \mu_X = \sum_{x\in D} x \bullet p(x)$$

$$(= \int_{-\infty}^{\infty} x \bullet f(x)dx)$$

where $D$ represents the set of possible values for discrete r.v. $X$

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# Mean of a Function

- If the discrete (continuous) r.v. $X$ has a set of possible values $D$ and p.m.f (p.d.f) $p(x)$ ($f(x)$), then the **expected value** of any function $h(X)$, denoted by $E[h(X)]$ or, is computed by

$$E[h(X)] = \mu_{h(X)} = \sum_{x \in D} h(x) \cdot p(x) (= \int_{\mathrm{D}} h(x) \cdot f(x) dx)$$

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# Variance

- The **variance** of a discrete (continuous) r.v. $X$ with p.m.f. $p(x)$ (p.d.f $f(x)$) and mean value $\mu$ is

$$\sigma^2 = V(X) = E[(X-\mu)^2] = \sum_{x \in D} (x-\mu)^2 \cdot p(x)$$

$$(= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x) dx)$$

- The standard deviation of $X$ is $\sigma_X = \sqrt{V(X)}$

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# Rules of Mean and Variance

- **Rules of Expected Values:**

$$E(aX+b)=aE(X) + b$$

- **Shortcut Formula for $V(X)$:**

$$V(X)=E(X^2) - [E(X)]^2$$

- **Rules of Variance:**

$$V(aX+b)=a^2V(X)$$

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# Family of Probability Distributions

- Suppose that $p(x)$ depends on a quantity that can be assigned any one of a number of possible values, with each different value determining a different probability distribution. Such a quantity is called a **parameter** of the distribution. The collection of all probability distributions for different values of the **parameter** is called a **family** of probability distributions

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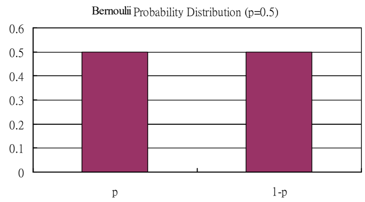
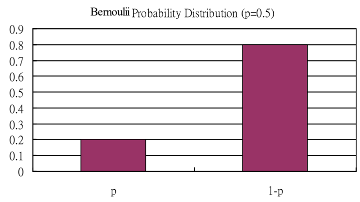
# Discrete Distribution Models

- **Bernoulli** Distribution: tossing coin
  **r.v.** $X$ **= 0 or 1**
  **P(**$X$**=0)=1-**$p$**; P(**$X$**=1)=**$p$

  **One parameter: 0<**$p$**<1**
  Example: tossing coin to get the head where the probability of head is $p$




Mean: $p$ Variance: $p(1-p)$

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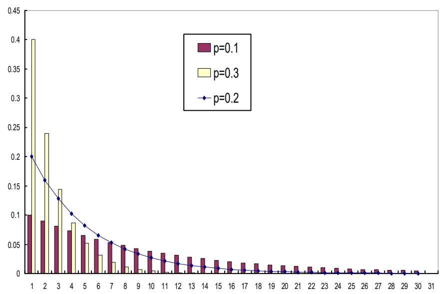
# Geometric Distribution Family

- Tossing coin
  r.v. $X$: number of tosses before first head appears where the probability of head is $p$

$$P(X = x) = p(1-p)^{x-1}$$

$$P(X < x) = 1-(1-p)^{x-1}; P(X \geq x) = (1-p)^{x-1}$$




Mean: ?
Variance:$(1-p)/p^2$

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- Memoryless property: $P(X>m+n|X>n)=P(X>m)$

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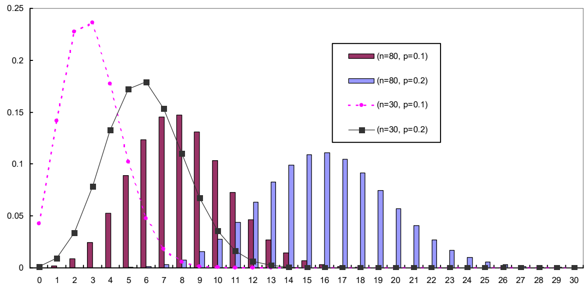
# Binomial Distribution Family

- Tossing coin with **different r.v. definition**
  $X$: number of heads in $n$ tosses

$$b(x;n,p)=\begin{cases} C_x^n p^x(1-p)^{n-x} & \text{x} = 0,1,2,\ldots,\text{n} \\ 0 & \text{otherwise} \end{cases}$$




Mean: *?* Variance: $np(1\text{-}p)$

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# Derivation of Binomial Mean

$$E(X)=\sum_{x=0}^{n} x\binom{n}{x}p^x(1-p)^{n-x}$$

$$=\sum_{x=0}^{n}\frac{n!}{x!(n-x)!}xp^x(1-p)^{n-x}$$

$$=\sum_{x=0}^{n}\frac{np(n-1)!}{(x-1)!(n-x)!}p^{x-1}(1-p)^{n-x} \quad let k = x-1$$

$$=np\sum_{k=0}^{n-1}\binom{n-1}{k}p^k(1-p)^{n-1-k} \quad = np$$

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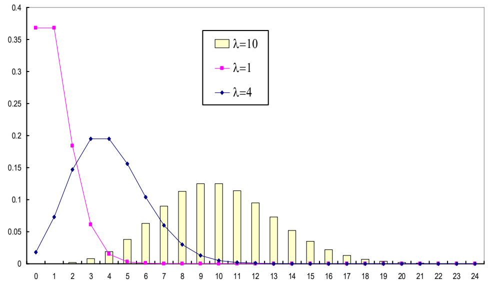
# Poisson Distribution Family

$$p(x;\lambda)=\frac{e^{-\lambda}\lambda^{x}}{x!}\qquad \text{x} = 0, 1, 2, \ldots \text{ for some } \lambda>0$$




- $E(X)=V(X)=\lambda$
- **Poisson ~ binomial with** $n\to\infty$ and $p\to 0$
- Example: defects on a wafer, communication network, car accidents

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# Derivation of Poisson Mean

$$E(X)=\sum_{x=0}^{\infty}x\frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$=\sum_{x=1}^{\infty}\frac{e^{-\lambda}\lambda^{x}}{(x-1)!}$$

$$=\lambda e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}\qquad \textit{let } k=x-1$$

$$=\lambda e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{k}}{(k)!}=\lambda e^{-\lambda}e^{\lambda}=\lambda$$

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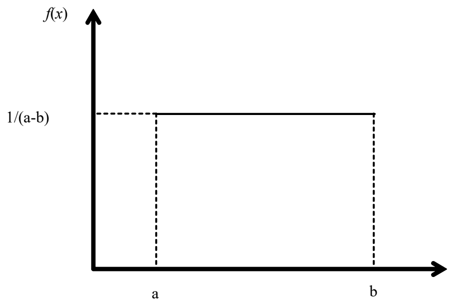
# Continuous Distribution Models

- Uniform Distribution family:

$$f(x)=\begin{cases}\dfrac{1}{b-a} & a<x<b\\ 0 & otherwise\end{cases}$$

$$F(x)=\begin{cases}0 & x\le a\\ \dfrac{x-a}{b-a} & a<x<b\\ 1 & x\ge b\end{cases}$$




- Mean? Variance?

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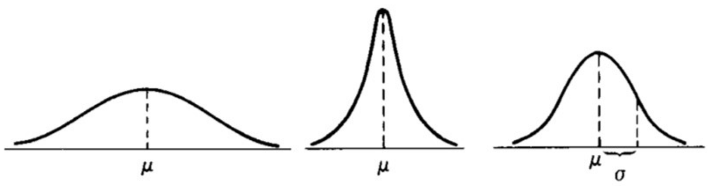
# Normal Distribution Family

$$f(x;\mu,\sigma)=\frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty<x<\infty$$

$$F(X=x;\mu,\sigma)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}\sigma}e^{-(X-\mu)^2/2\sigma^2}dX \text{ (no close form)}$$

- Mean: $\mu$ and Variance: $\sigma^2$

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# Standard Normal Distribution

- Normal distribution with mean 0 and standard deviation 1: $f(z;0,1)$; $F(x)=\Phi(x)$

$$\Phi(x) = F(X = x; 0,1) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-X^2/2} dX \text{ (no close form)}$$

- Proposition: If $X$ follows a normal distribution with mean $\mu$ and standard deviation $\sigma$, then

$$Z = \frac{X-\mu}{\sigma}$$

is a standard normal random variable

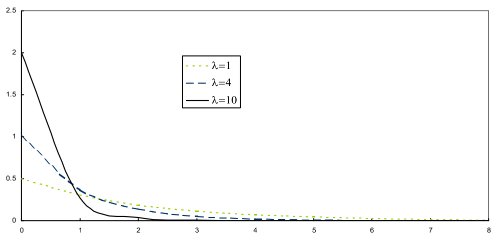
- $F(x;\ \mu,\ \sigma)=\Phi(z=(x-\mu)/\sigma)$ $\Rightarrow$Table of $\Phi(x)$




# Exponential Distribution Family

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{where } \lambda > 0$$

- c.d.f. $F(x;\lambda) = \begin{cases} 0 & \text{X<0} \\ 1-e^{-\lambda x} & \text{X} \geq 0 \end{cases}$; $P(X > x) = e^{-\lambda x}$




mean=$1/\lambda=\sigma$

variance=$1/\lambda^2$

- Memoryless property: $P(X>t+s|X>t)=P(X>s)$

# Gamma Function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \qquad \alpha > 0$$

- For $\alpha$>1, $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$
- For any positive integer $n$, $\Gamma(n)=(n-1)!$
- $\Gamma(1/2) = \sqrt{\pi}$

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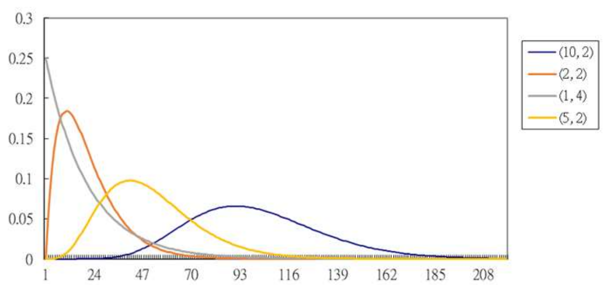
# Gamma Distribution Family

$$f_\Gamma(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & \text{x} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- $E(X)=\alpha\beta \quad V(X)=\alpha\beta^2$

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# Chi-square ($\chi^2$) Distribution Family

$$f(x;\nu)=\begin{cases}\dfrac{1}{2^{\nu/2}\Gamma(\nu/2)}x^{\nu/2-1}e^{-x/2} & \text{x} \ge 0\\ 0 & \text{otherwise}\end{cases}$$

- $\chi^2(\nu) \equiv f_\Gamma(\alpha=\nu/2,\beta=2)$
- $E(X)$=? $V(X)$=?
- **$X_i$ ~ standard normal $N$(0,1)**
  $\Rightarrow \sum_{i=1}^{\nu} X_i^2 \sim \chi^2(\nu)$

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# Discrete Joint Distribution and R.V.

- Let $X$ and $Y$ be two discrete random variables defined defined on a sample space of an experiment. The joint probability mass function $p(x,y)$ is defined for each pair of numbers $(x, y)$ by $p(x, y)=P(X=x$ and $Y=y)$
- For any discrete set $A$ consisting of $(x,y)$ values,

$$P[(X,Y)\in A]=\sum_{(x,y)\in A}\sum p(x,y)$$

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## Continuous Joint Distribution and R.V.

- Let *X* and *Y* be two continuous random variables. Then *f(x,y)* is the joint probability density function for *X* and *Y* if for any two-dimensional set *A*

$$P[(X,Y) \in A] = \iint_A f(x,y)dxdy$$

- In particular,

$$P[a \le X \le b, c \le Y \le d] = \int_a^b \int_c^d f(x,y)dxdy$$

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## Marginal Probability

- Marginal probability mass (density) functions:

$$p_X(x) = \sum_y p(x,y), \qquad p_Y(y) = \sum_x p(x,y)$$

$$(f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy, \quad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx)$$

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# Independence of Two R.V.'s

- Two discrete (continuous) random variables $X$ and $Y$ are said to be **independent** if for every pair of $x$ and $y$ values,

$$p(x,y) = p_X(x) \cdot p_Y(y)$$

$$f(x,y) = f_X(x) \cdot f_Y(y)$$

- If above is not satisfied for all $(x,y)$, $X$ and $Y$ are said to be **dependent**

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# Covariance of Two R.V.'s

- Variance of two discrete (continuous) random variables $X$ and $Y$ is defined:

$$\begin{aligned}\sigma(X,Y) &= \mathrm{E}[(X-\mathrm{E}[X])(Y-\mathrm{E}[Y])] \\ &= \mathrm{E}[XY - X\mathrm{E}[Y] - \mathrm{E}[X]Y + \mathrm{E}[X]\mathrm{E}[Y]] \\ &= \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y] - \mathrm{E}[X]\mathrm{E}[Y] + \mathrm{E}[X]\mathrm{E}[Y] \\ &= \mathrm{E}[XY] - \mathrm{E}[X]\mathrm{E}[Y]\end{aligned}$$

- Then $X$ and $Y$ are **independent:**

  $\mathrm{E}[XY]=\mathrm{E}[X]\mathrm{E}[Y]$ (why?)$\Rightarrow$ $\boldsymbol{\sigma(X, Y)=0}$

- **However**, if $\sigma(X, Y)=0$, $X$ and $Y$ may not be **independent unless they jointly normally distributed.**

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## Linear Combination of R.V.'s

- Linear Combination of r.v. $X_i$:

$$Y = \sum_{i=1}^{n} a_i X_i$$

- For independent $X_i$'s:

$$E(\sum_{1}^{n} a_i X_i) = \sum_{1}^{n} a_i E(X_i)$$

$$V(\sum_{1}^{n} a_i X_i) = \sum_{1}^{n} a_i^2 V(X_i)$$

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## Average of Random Variables

- For independent and identically distributed (*iid*) $X_i$'s with $\mu$ and $\sigma$

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

- Mean and Variance:

$$E(\overline{X}) = \mu \text{ and } V(\overline{X}) = \frac{\sigma^2}{n}$$

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# Sum of R.V.'s

- Sum of r.v. $X_i$:

$$Y = \sum_{i=1}^{n} X_i$$

- For independent $X_i$'s:

$$E(\sum_{1}^{n} X_i) = \sum_{1}^{n} E(X_i)$$

$$V(\sum_{1}^{n} X_i) = \sum_{1}^{n} V(X_i)$$

- How about distribution of $Y$? Not an easy answer!

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# Sum of Ind. Geometric RVs

- Sum of independent r.v. $X_i$: $Y = \sum_{i=1}^{n} X_i$

$$E(\sum_{1}^{n} X_i) = \sum_{1}^{n} E(X_i) = n/p$$

$$V(\sum_{1}^{n} X_i) = \sum_{1}^{n} V(X_i) = n(1-p)/p^2$$

- How about distribution of $Y$? probability distribution function=?
  - It's a Negative Binomial Distribution!
    r.v. $X$=number of trials to $n^{th}$ head

$$\boldsymbol{P(X = x) = \binom{x-1}{n-1}(1-p)^{x-n} p^n \quad x = n, n+1, \ldots}$$

Why?

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## Convolution: Sum of Two Independent R.V.'s

- We already know that for two independent r.v. $X$ and $Y$

$$E(X+Y)=E(X)+E(Y)$$
$$V(X+Y)=V(X)+V(Y)$$

- How about the distribution of $X+Y$ ?

  That is, pmf or pdf $X+Y$: $p_{x+y}(X+Y)$ or $f_{x+y}(X+Y)$?

- In general, there is no easy answer for this even given the known distribution of $X$ and $Y$! In fact, the solution is to solve the convolution of functions:

$$p_{x+y}(X+Y=a)=\sum_{\forall y} p_x(X=a-y)p_y(Y=y)$$
$$f_{x+y}(X+Y=a)=\int_{-\infty}^{\infty} f_x(a-y)f_y(y)dy$$

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## Distribution of the Sum of Two Independent R.V.'s

- The answer could be as complicated as solving the convolution of functions. But could be quite simple too…..
- Example:

  r.v. X+Y takes possible values: 2,3,4,5

  $p_x(0)=.3 \qquad p_x(1)=.2 \qquad P_x(2)=.5$

  $p_y(2)=.5 \qquad p_y(3)=.5$

  Convolution:

$$p(x+y=2)=p(x=0)p(y=2)=0.15$$
$$p(x+y)=3)=p(x=0)p(y=3)+p(x=1)p(y=2)=0.25$$
$$p(x+y=4)=p(x=1)p(y=3)+p(x=2)p(y=2)=0.35$$
$$p(x+y=5)=p(x=2)p(y=3)=0.25$$
$$or \quad in \quad general \quad p(x+y=5)=\sum_{y=2}^{3} p(x=5-y)p(Y=y)$$

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## Distribution of the Sum of Two Independent R.V.'s

- The answer could be as complicated as solving the convolution of functions. But could be quite simple too…..
- Example:

  r.v. X+Y takes possible values: 2,3,4,5

  $p_x(0)=.3 \quad p_x(1)=.2 \quad P_x(2)=.5$

  $p_y(2)=.5 \quad p_y(3)=.5$

  Convolution:

  $p(x+y=2)=p(x=0)p(y=2)=0.15$

  $p(x+y)=3)=p(x=0)p(y=3)+p(x=1)p(y=2)=0.25$

  $p(x+y=4)=p(x=1)p(y=3)+p(x=2)p(y=2)=0.35$

  $p(x+y=5)=p(x=2)p(y=3)=0.25$

  $or \quad in \quad general \quad p(x+y=5)=\sum_{y=2}^{3} p(x=5-y)p(Y=y)$

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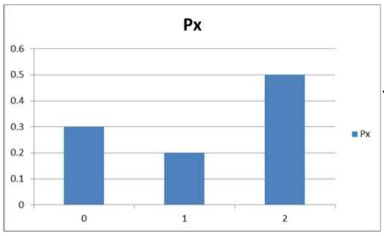
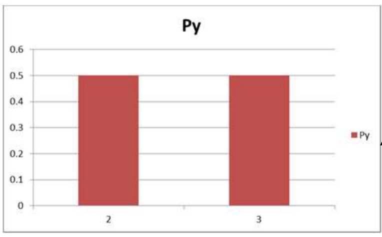
## More on Sum of RV's

$p_x(0)=.3 \; p_x(1)=.2 \; p_x(2)=.5$

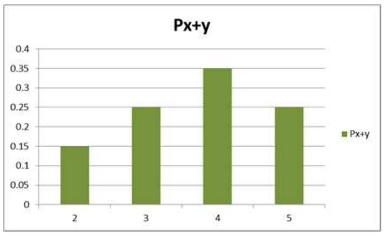



$p_y(2)=.5 \; p_y(3)=.5$




$p_{x+y}(2)=0.15 \quad p_{x+y}(3)=0.25$

$p_{x+y}(4)=0.35 \quad p_{x+y}(5)=0.25$

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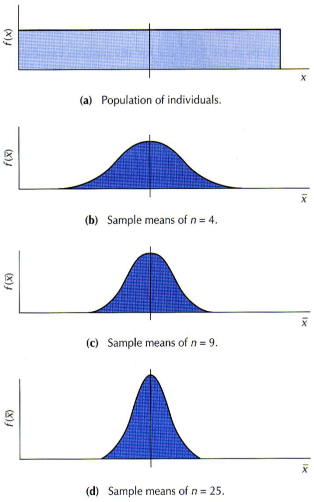
# Central Limit Theorem

If $X_1$, $X_2$, ... $X_n$ are outcomes of a sample of $n$ independent observations of a random variable $X$ with mean $\mu_x$ and variance $\sigma_x^2$, then

$\Sigma X_i \sim N(n\mu_x, n\sigma_x^2)$

and

$\Sigma X_i /n \sim N(\mu_x, \sigma_x^2/n)$




(a) Population of individuals.

(b) Sample means of n = 4.

(c) Sample means of n = 9.

(d) Sample means of n = 25.

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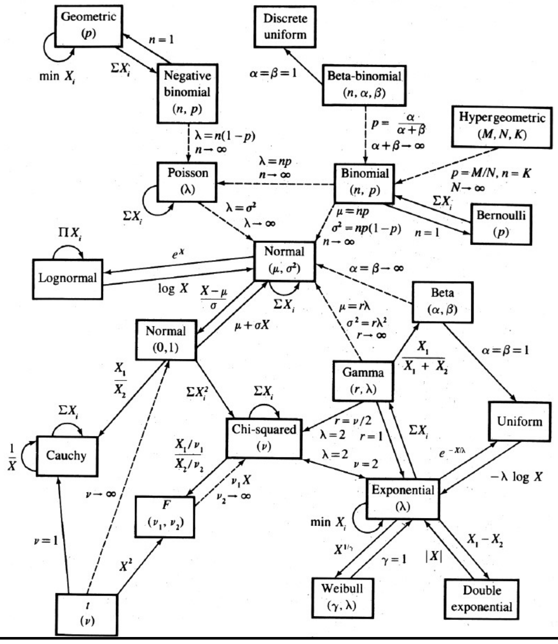
# Relationships of Distribution Models

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