Linear Algebra and its Applications

HW#12

2. Find unitary U to triangularize the following matrices (Schur's Lemma):

$$\begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

3. Show that an upper triangular and normal matrix must be diagonal.

4. Show that all permutation matrices are normal

5. Let $A=\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ and the quadratic function be $\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.
   (i) Factor A into LDU and expand the quadratic function into a summation of two quadratic terms.
   (ii) Diagonalize A into $Q\Lambda Q^T$ and expand the quadratic function into a summation of two quadratic terms.
   (iii) Compare results of (i) and (ii) and use them to determine whether the quadratic function is definite, semi-definite or indefinite.

6. Show that $\boldsymbol{R}^{\mathrm{T}}\boldsymbol{R}$ is positive semidefinite when the columns of $\boldsymbol{R}$ are linearly dependent and is positive definite when the columns are linearly independent.

7. Show that for a symmetric positive definite A,
   (a) we can chose a $R=Q\sqrt{\Lambda}Q^T$ such that $A= R^TR=R^2$ (R is called "symmetric positive definite square root" of A). Let $A=\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$. Find the square root of R.
   (b) we can choose an $R=\sqrt{D}L^T$ and a lower triangular $C=R^T$**,** such that $A= R^TR=CC^T$ (This is called Cholesky Decomposition). Let $A=\begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$. Find C with positive diagonal elements.

8. Decide between a minimum, maximum or saddle point for the following functions:
   (a) $F= -1+4(e^x-x)-5x \sin y +6y^2$ at the point $x=y=0$.
   (b) $F=(x^2-2x) \cos y$, with stationary point at $x=1$, $y=\pi$.

9. Give a quick reason why each of these statements is true:
   (a) Every positive definite matrix is invertible.

(b) The only positive definite projection matrix is $P=I$

(c) A diagonal matrix with positive diagonal entries is positive definite.

10. Let $f=4x^2+\lambda y^2$. Use excel or any other software to plot the 7 contour plots for $\lambda=4$, 2, 1, 0, -1, -2, -4 on the x-y plane with contours $f=0, 2, 4, 6$

View the class video "PCA" of Week 14 and solve the following problems:

1. In three dimensions, $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 1$ represents an ellipsoid when all $\lambda_i>0$. Describe all the different kinds of surfaces that appear in the positive semidefinite case when one or more of the eigenvalues is zero.

2. Find the minimum, if there is one, of $P_1=0.5x^2+xy+y^2-3y$ and $P_2=0.5x^2-3y$.

3. Find the minimum values of
$R(x)=\dfrac{x_1^2 - x_1x_2 + x_2^2}{x_1^2 + x_2^2}$ and $R(x)=\dfrac{x_1^2 - x_1x_2 + x_2^2}{2x_1^2 + x_2^2}$ (hint: let $y=\sqrt{2}x_1$)

4. The ten largest U.S. industrial corporations yield the following data.

Company	Sales (in million)	Profit (in million)
General Motors	126974	4224
Ford	96933	3835
Exxon	86656	3510
IBM	63438	3758
General Electric	55264	3939
Mobil	50976	1809
Philip Morris	39069	2946
Chrysler	36156	359
du Pont	35209	2480
Texaco	32416	2413

(a) Calculate the covariance and correlation matrices for the Sales and Profit

(b) Use the first eigenvector of the **covariance matrix** to find a weighted index of the sales and the profit so that the companies' performance can be best distinguished.

(c) Use the first eigenvector of the **correlation matrix** to find a weighted index so that the companies' performance can be distinguished.

(d) Compare and discuss the difference between the two indices found in (b) and (c).

(e) Use the second eigenvector of the **correlation matrix** to find a second weighted index. Show that this index is uncorrelated to the index in (c) and compare the two indexes.