Two-level Factorial Design of Experiments

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Dealing with the Noises

- Almost impossible to eliminate the noises
- Four attempts:
  1. Design the experiment such that the noise is well controlled in the analysis
  2. Randomize the experimental trials such that the noises are uniformly and randomly distributed across trials
  3. Replication in the experiments to include the noise effect in the analysis
  4. Confirmatory testing to verify the analysis results

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Unwanted Noises in the Experimental Environment

- Factors of interest in the planned experiments could be subject to unwanted noises. How to minimize the effect of the unwanted noises?
  - ***Include the noise effect in the objective function, i.e., SN ratio***

  Example 1: In LPCVD, gases travel from one end of the reactor to the other end causing ***concentration gradient*** along the length of the reactor and differences in ***flow pattern***. There are also variation in ***temperature*** along the length and cross the tube section, ***wafer topography***, ***pumping speed***, and ***gas supply***.
  - ***Randomize the unwanted noises in the experiments***

  Example 2: Nork Tool company claims that their new cutting tools, called Nork-V provide a Longer life than the cutgo-T tools for similar jobs. To check the claim, nine Nork tools and nine Cutgo-T tools will be used to machine 1048 steel bars. What are the unwanted noises?

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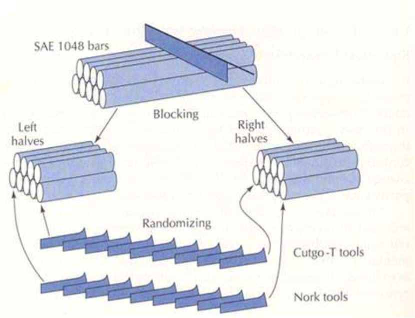
Blocking in Experimental Designs



Test	y_c (Cutgo-T)	y_n (Nork-V)	$d = y_n - y_c$
1 (Bar 1)	18 L	20 R	+2
2 (Bar 2)	16 R	14 L	−2
3 (Bar 3)	10 L	10 R	0
4 (Bar 4)	17 R	20 L	+3
5 (Bar 5)	17 L	19 R	+2
6 (Bar 6)	20 R	20 L	0
7 (Bar 7)	16 R	15 L	−1
8 (Bar 8)	28 L	28 R	0
9 (Bar 9)	24 L	29 R	+5

* L and R refer to the left half and right half of a bar, respectively.

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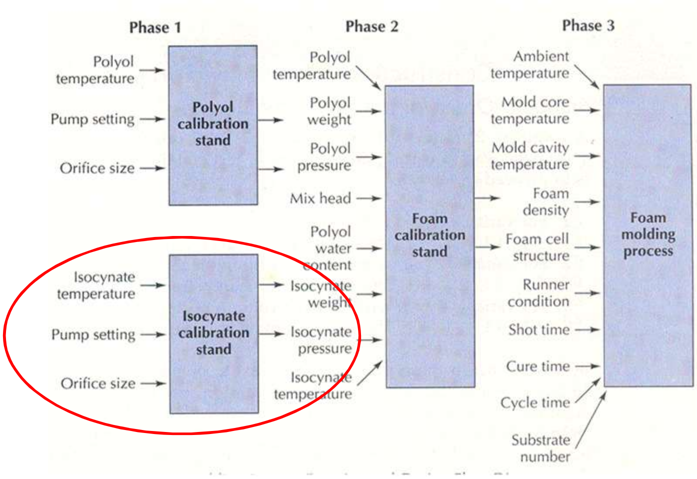
Specific Randomization Schemes for Positive and Negative Autocorrelation Nuisances

- Positive correlation nuisance: learning curve
  - Randomizing adjacent runs within pairs (arrangement 1)
- Negative correlation (alternate/oscillation) nuisance: PM and AM
  - Running the pair both in AM or both in PM (arrangement 2)

	Arrangement 1		Arrangement 2	
Bar	Cutgo-T	Nork-V	Cutgo-T	Nork-V
1	La	Rb	LA	RA
2	Ld	Rc	RP	LP
3	Rf	Le	RP	LP
4	Lh	Rg	LA	RA
5	Lj	Ri	RA	LA
6	Rk	Ll	LP	RP
7	Rm	Ln	RP	LP
8	Rp	Lo	RA	LA
9	Lr	Rq	LP	RP

* L, left half; R, right half; A, A.M., P, P.M.; a to q are the time order of runs.

Foam Process Experimental Design Flow Diagram



Variable Levels for the Isocynate Calibration Experiment

Variable	Unit	Low Level	High Level
Orifice size, *O*	mm	1.30	1.50
Pump setting, *p*		4.00	4.50
Isocynate temperature, *T*	℃	22	30

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Coded and Uncoded Test Conditions in Standard Order

Coded Test Conditions				Actual Test Conditions		
Test	X_1	X_2	X_3	*O* (mm)	*P*	*T* (℃)
1	-1	-1	-1	1.30	4.0	22
2	+1	-1	-1	1.50	4.0	22
3	-1	+1	-1	1.30	4.5	22
4	+1	+1	-1	1.50	4.5	22
5	-1	-1	+1	1.30	4.0	30
6	+1	-1	+1	1.50	4.0	30
7	-1	+1	+1	1.30	4.5	30
8	+1	+1	+1	1.50	4.5	30

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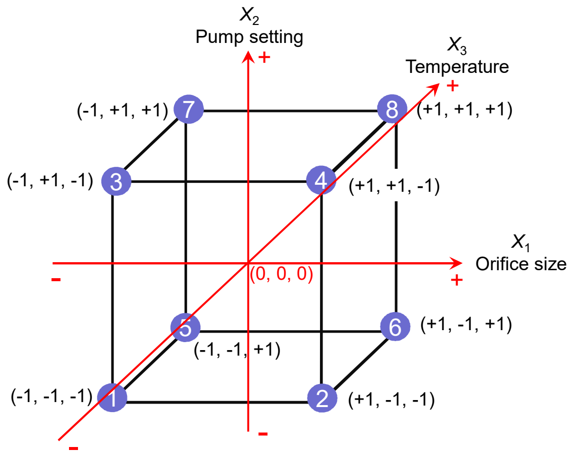
Geometric Representation of the 2^3 Factorial Design



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Test Results for Isocynate Calibration Experiment

Test	X_1	X_2	X_3	Test Order	Response, y (psi)
1	-1	-1	-1	6	1550
2	+1	-1	-1	8	1925
3	-1	+1	-1	1	2150
4	+1	+1	-1	2	2350
5	-1	-1	+1	5	1525
6	+1	-1	+1	3	1800
7	-1	+1	+1	4	2175
8	+1	+1	+1	7	2200

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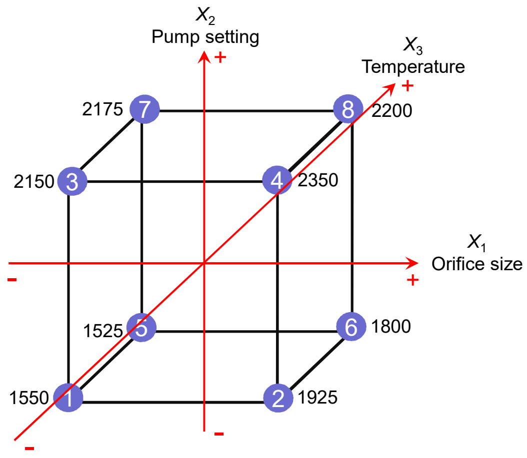
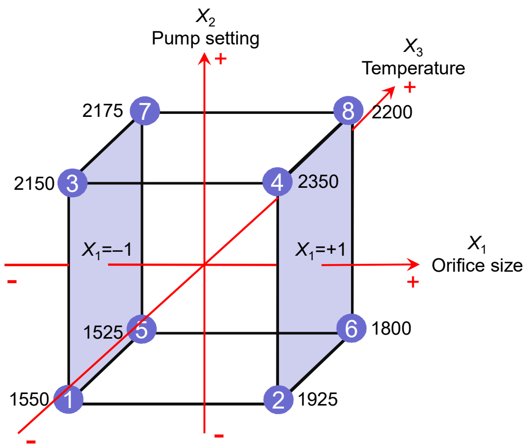
Geometrical Representation of the Test Results



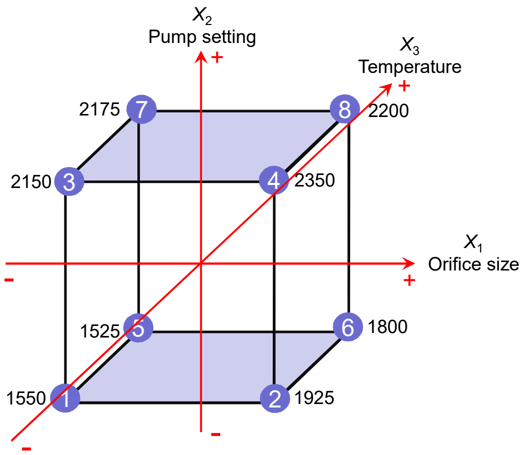
Geometric Representation of the Main Effect of Orifice Size



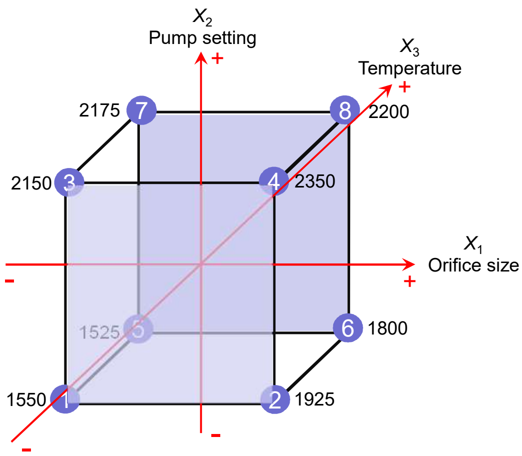
$$E_1 = \frac{y_2 + y_4 + y_6 + y_8}{4} - \frac{y_1 + y_3 + y_5 + y_7}{4}$$

$$= 2068.75 - 1850 = 218.75$$

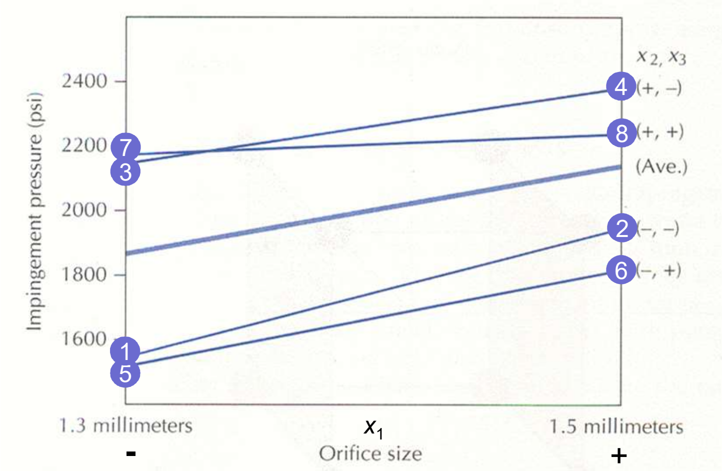
Geometric Representation of the Main Effect of Pump Setting



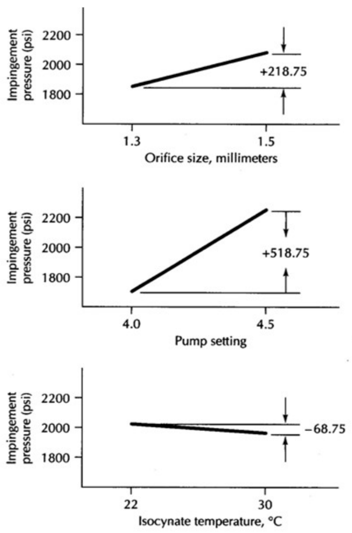
Geometric Representation of the Main Effect of Temperature



Individual Contrasts and Main Effect of Orifice Size (Heavy Line)



Main Effects of Orifice Size, Pump Setting, and Isocynate Temperature



Pump Effects under Different Orifice Sizes and Temperatures

Orifice size = 1.5 mm(+)

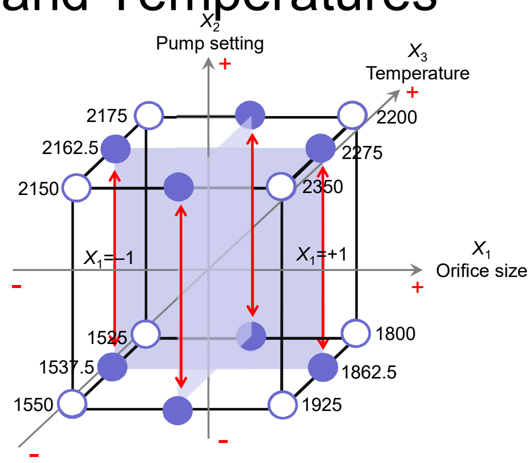
$$E_{pump|\text{orifice size}=+1} = \frac{(2350+2200)}{2} - \frac{(1925+1800)}{2}$$

=2275-1862.5=412.50psi

Orifice size = 1.3 mm(-)

$$E_{pump||\text{orifice size}=-1} = \frac{2150+2175}{2} - \frac{1550+1525}{2}$$

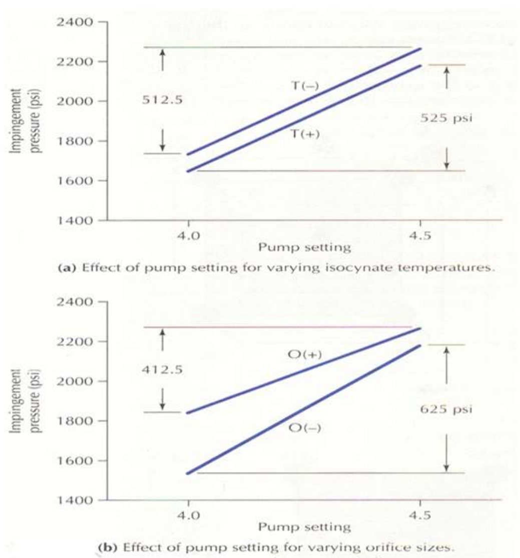
$$= 2162.5 - 1537.5 = 625\text{psi}$$



Temperature = 30(+): $E_{pump|\text{temp}=+1}$=525psi

Temperature = 22(-): $E_{pump|\text{temp}=-1}$=512.5psi

Two-Way Diagrams of Interaction Effects



(a) Effect of pump setting for varying isocynate temperatures.

(b) Effect of pump setting for varying orifice sizes.

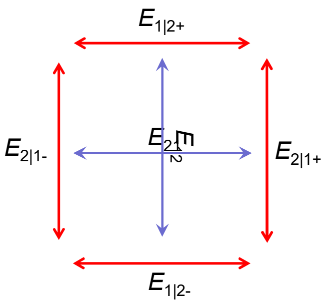
Tow-Factor Interaction: Orifice Size and Pump Setting

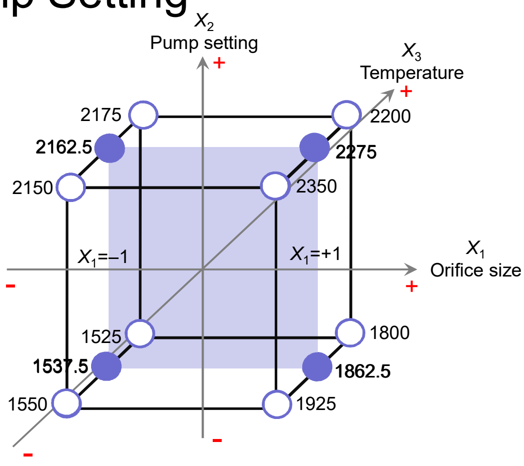
$$E_{21} = \frac{E_{2|1+} - E_{2|1-}}{2} = \frac{412.5 - 625}{2}$$

$$= E_{12} = \frac{E_{1|2+} - E_{1|2-}}{2}$$

$$= \frac{(2275 - 2162.5) - (1862.5 - 1537.5)}{2}$$

$$= \frac{112.5 - 325}{2} = -106.25\text{psi}$$





Other Interaction Effects

$$E_{13} = [(\text{effect of orifice size at high level for temperature}) - (\text{effect of orifice size at low level for temperature})]/2$$

$$= \frac{(2000 - 1850) - (2137.5 - 1850)}{2}$$

$$= \frac{150 - 287.5}{2}$$

$$= -68.75\text{psi.}$$

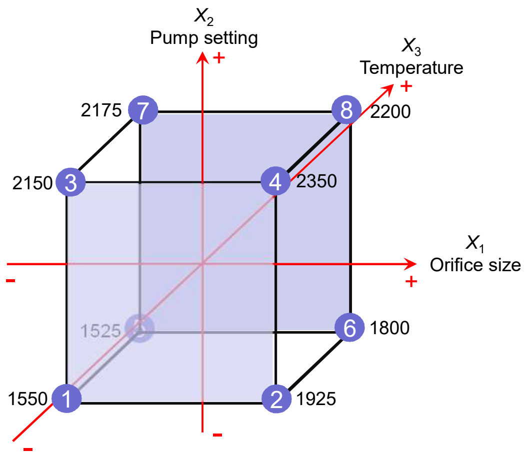
$$E_{23} = [(\text{effect of temperature at high level for pump setting}) - (\text{effect of temperature at low level for pump setting})]/2$$

$$= \frac{(2187.5 - 2250) - (1662.5 - 1737.5)}{2}$$

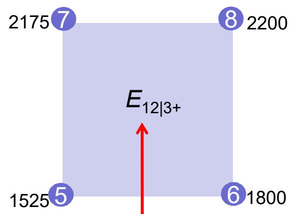
$$= \frac{-62.5 - (-75.0)}{2}$$

$$= 6.25\text{psi.}$$

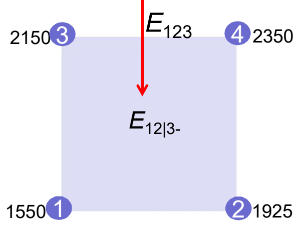
Three-Factor Interaction



$$E_{123} = \frac{E_{12|3+} - E_{12|3-}}{2} = \frac{(-125)-(-87.5)}{2} = -18.75\text{psi}$$



$$E_{12|3+} = \frac{1}{2}\left[(2200-2175)-(1800-1525)\right] = -125.0\text{psi}$$



$$E_{12|3-} = \frac{1}{2}\left[(2350-2150)-(1925-1515)\right] = -87.5\text{psi}$$

Generalized Method for the Calculation of Effects

Test	X_1	X_2	X_3	Test Order	Response, *y* (psi)
1	-1	-1	-1	6	1550
2	+1	-1	-1	8	1925
3	-1	+1	-1	1	2150
4	+1	+1	-1	2	2350
5	-1	-1	+1	5	1525
6	+1	-1	+1	3	1800
7	-1	+1	+1	4	2175
8	+1	+1	+1	7	2200

Main Effect Calculation

$$
\begin{array}{ccc}
x_1 & & y \\
-1 & \times & 1550 \\
+1 & \times & 1925 \\
-1 & \times & 2150 \\
+1 & \times & 2350 \\
-1 & \times & 1525 \\
+1 & \times & 1800 \\
-1 & \times & 2175 \\
+1 & \times & 2200 \\
\hline
\end{array}
$$

$$
\begin{aligned}
\text{Sum} &= 875 \\
\text{Sum}/4 &= 875/4 \\
E_1 &= 218.75.
\end{aligned}
$$

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Two-Factor Interaction Calculation

$$
\begin{array}{ccccc}
x_1 x_2 & & \text{y} & & \\
(+1) & \times & (1550) & & +1550 \\
(-1) & \times & (1925) & & -1925 \\
(-1) & \times & (2150) & & -2150 \\
(+1) & \times & (2350) & = & +2350 \\
(+1) & \times & (1525) & & +1525 \\
(-1) & \times & (1800) & & -1800 \\
(-1) & \times & (2175) & & -2175 \\
(+1) & \times & (2200) & & +2200 \\
\hline
 & \text{Sum} & & = & -425
\end{array}
\qquad E_{12} = \frac{-425}{4}
$$

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Calculation Matrix for 2^3 Design

Test	I	Main Effects x_1	x_2	x_3	Interactions x_1x_2	x_1x_3	x_2x_3	$x_1x_2x_3$	y (psi)
1	+	-1	-1	-1	+1	+1	+1	-1	1550
2	+	+1	-1	-1	-1	-1	+1	+1	1925
3	+	-1	+1	-1	-1	+1	-1	+1	2150
4	+	+1	+1	-1	+1	-1	-1	-1	2350
5	+	-1	-1	+1	+1	-1	-1	+1	1525
6	+	+1	-1	+1	-1	+1	-1	-1	1800
7	+	-1	+1	+1	-1	-1	+1	-1	2175
8	+	+1	+1	+1	+1	+1	+1	+1	2200

Mathematical Empirical Model

$$y = b_0 + b_1x_1 + b_2x_2 + b_3x_3 + b_{12}x_1x_2 + b_{13}x_1x_3 + b_{23}x_2x_3 + b_{123}x_1x_2x_3 + \varepsilon$$

$$\hat{b}_0 = \left(\tfrac{1}{8}\right)\left(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8\right)$$

Effects and Model Coefficients

$$y = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_3 + b_{12} x_1 x_2 + b_{13} x_1 x_3 + b_{23} x_2 x_3 + b_{123} x_1 x_2 x_3 + \varepsilon$$

$$\hat{b}_1 = \frac{E_1}{2} = \frac{218.75}{2} = 109.375$$

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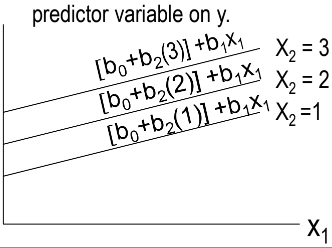
Interactions in Fitted Model

– First order model

$y = b_0 + b_1 x_1 + b_2 x_2 + \varepsilon$

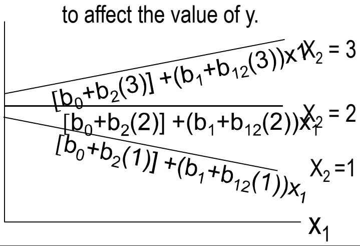
The effect of one predictor variable on y is independent of the effect of the other predictor variable on y.



– First order model with interaction

$y = b_0 + b_1 x_1 + b_2 x_2 + b_{12} x_1 x_2 + \varepsilon$

The two variables interact to affect the value of y.



Other Model Parameters

$$\hat{b}_2 = \frac{E_2}{2} = \frac{518.75}{2} = 259.375$$

$$\hat{b}_3 = \frac{E_3}{2} = \frac{-68.75}{2} = -34.375$$

$$\hat{b}_{12} = \frac{E_{12}}{2} = \frac{-106.25}{2} = -53.125$$

$$\hat{b}_{13} = \frac{E_{13}}{2} = \frac{-68.75}{2} = -34.375$$

$$\hat{b}_{23} = \frac{E_{23}}{2} = \frac{6.25}{2} = 3.125$$

$$\hat{b}_{123} = \frac{E_{123}}{2} = \frac{-18.75}{2} = -9.375$$

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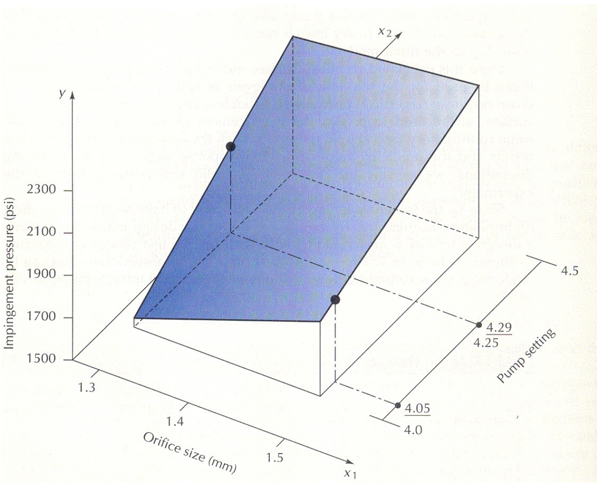
Predicted Response Surface

- Assume E_3, E_{13}, E_{23}, and E_{123} are not significant
- Final fitted model: $\hat{y} = 1959 + 109x_1 + 259x_2 - 53x_1x_2$



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Linear Regression

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Introduction

- A technique to examine the relationship among quantitative variables.
- The technique is used to predict the value of one variable (the dependent variable - y) based on the value of other variables (independent variables x_1, x_2,…x_k.)

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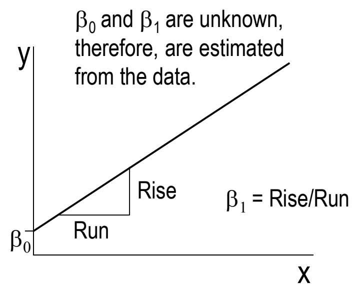
The Simple Linear Regression Model

- The first order linear model

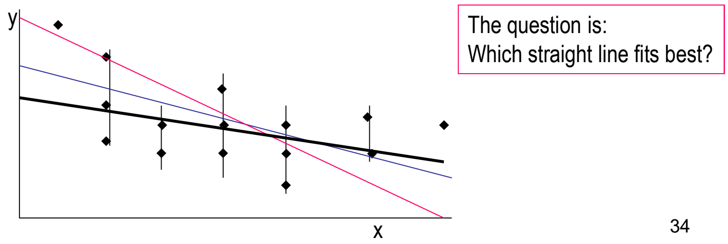
$$y = \beta_0 + \beta_1 x + \varepsilon$$

y = dependent variable
x = independent variable
β_0 = y-intercept
β_1 = slope of the line
ε = error variable



Estimating the Coefficients

- The estimates are determined by
  - drawing a sample from the population of interest,
  - calculating sample statistics.
  - producing a straight line that cuts into the data.



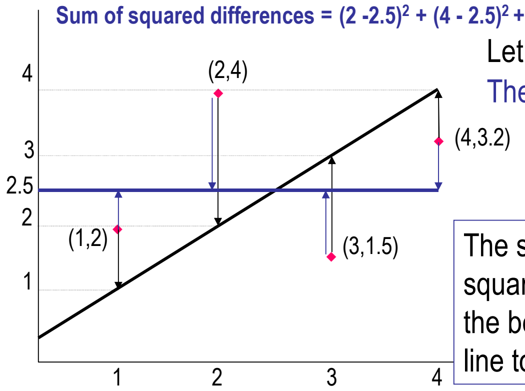
The best line is the one that minimizes the sum of squared vertical differences between the points and the line.

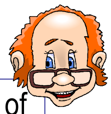
Sum of squared differences = $(2 - 1)^2 + (4 - 2)^2 + (1.5 - 3)^2 + (3.2 - 4)^2 = 6.89$

Sum of squared differences = $(2 - 2.5)^2 + (4 - 2.5)^2 + (1.5 - 2.5)^2 + (3.2 - 2.5)^2 = 3.99$

Let us compare two lines

The second line is horizontal





The smaller the sum of squared differences the better the fit of the line to the data.

Example: Relationship between Orifice Size and Pressure

- Foam processing experiments
- 8 runs of experiments are conducted and corresponding pressures are measured
- Find the regression line.

Independent variable		Dependent variable
O (mm)	X1	Response, y (psi)
1.3	-1	1550
1.5	1	1925
1.3	-1	2150
1.5	1	2350
1.3	-1	1525
1.5	1	1800
1.3	-1	2175
1.5	1	2200

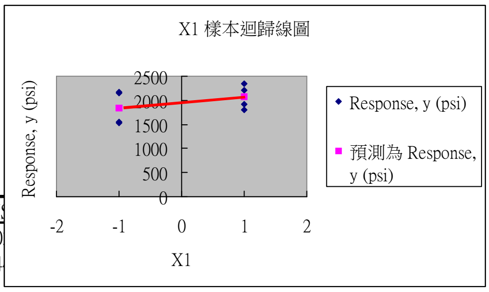
- Using the computer (Excel)

Tools > Data analysis > Regression > [Shade the y range and the x range] > OK

迴歸統計	
R 的倍數	0.376214
R 平方	0.141537
調整的 R	-0.00154
標準誤	311.0382
觀察值個	8

ANOVA

	自由度	SS	MS
迴歸	1	95703.13	9570
殘差	6	580468.8	9674
總和	7	676171.9	



	係數	標準誤	t 統計	P-值	下限 95%	上限 95%	下限 95.0%	上限 95.0%
截距	1959.375	109.9686	17.81758	2.01E-06	1690.291	2228.459	1690.291	2228.459
X1	109.375	109.9686	0.994602	0.358333	-159.709	378.4587	-159.709	378.4587

$$\boxed{\hat{y} = 1959.375 + 109.375\,x}$$

Multiple Regression Model

- We allow for k independent variables and interactions to potentially be related to the dependent variable

Coefficients　　Random error variable

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k + \varepsilon$$

Dependent variable　　Independent variables

Example: Relationship between Three Factors and Pressure

- Foam processing experiments
- 8 experimental runs
- Estimate the regression model

Independent variable							Dependent variable
X1	X2	X3	X1X2	X1X3	X2X3	X1X2X3	Response, y (psi)
-1	-1	-1	1	1	1	-1	1550
1	-1	-1	-1	-1	1	1	1925
-1	1	-1	-1	1	-1	1	2150
1	1	-1	1	-1	-1	-1	2350
-1	-1	1	1	-1	-1	1	1525
1	-1	1	-1	1	-1	-1	1800
-1	1	1	-1	-1	1	-1	2175
1	1	1	1	1	1	1	2200

Using the computer

Tools > Data analysis > Regression > [Shade the y range and the x range] > OK

迴歸統計	
R 的倍數	1
R 平方	1
調整的 R	65535
標準誤	0
觀察值個	8

ANOVA

	自由度	SS	MS	F	顯著值
迴歸	7	676171.9	96595.98	#NUM!	#NUM!
殘差	0	0	65535		
總和	7	676171.9			

	係數	標準誤	t 統計	P-值	下限 95%	上限 95%	下限 95.0%	上限 95.0%
截距	1959.375	0	65535	#NUM!	1959.375	1959.375	1959.375	1959.375
X1	109.375	0	65535	#NUM!	109.375	109.375	109.375	109.375
X2	259.375	0	65535	#NUM!	259.375	259.375	259.375	259.375
X3	-34.375	0	65535	#NUM!	-34.375	-34.375	-34.375	-34.375
X1X2	-53.125	0	65535	#NUM!	-53.125	-53.125	-53.125	-53.125
X1X3	-34.375	0	65535	#NUM!	-34.375	-34.375	-34.375	-34.375
X2X3	3.125	0	65535	#NUM!	3.125	3.125	3.125	3.125
X1X2X3	-9.375	0	65535	#NUM!	-9.375	-9.375	-9.375	-9.375

$$\boxed{\begin{aligned}\hat{y} &= 1959.375 + 109.375\,x_1 + 259.375\,x_2 - 34.375\,x_3 \\ &- 53.125\,x_1x_2 - 34.375\,x_1x_3 + 3.125\,x_2x_3 - 9.375\,x_1x_2x_3\end{aligned}}$$

Experiments with Replicates

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Glove Box Door Alignment Experiment

	Variable	Low(-)	High(+)
x_1:	RH cowl fore/aft movement	Nominal	-5 mm
x_2:	Center brace attachment sequence	Before	After
x_3:	Plenum gasket	No	Yes
x_4:	Evaporator case setup, fore/aft	Nominal	-5 mm

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2^4 Design and Results of the Glove Box Door Experiment (Test Order in Parentheses)

					Parallelism (mm)	
					Run 1	Run 2
Test	x_1	x_2	x_3	x_4	y_{i1}	y_{i2}
1	−	−	−	−	−1.44 (7)	−0.08 (28)
2	+	−	−	−	−1.79 (10)	−1.01 (24)
3	−	+	−	−	0.39 (14)	0.17 (32)
4	+	+	−	−	−0.50 (2)	−0.24 (21)
5	−	−	+	−	−0.20 (9)	0.17 (27)
6	+	−	+	−	−0.79 (6)	−0.64 (30)
7	−	+	+	−	1.22 (13)	0.28 (20)
8	+	+	+	−	0.21 (8)	0.28 (18)
9	−	−	−	+	−0.40 (1)	−0.65 (31)
10	+	−	−	+	−0.63 (15)	−1.19 (25)
11	−	+	−	+	0.47 (3)	0.44 (17)
12	+	+	−	+	−0.01 (5)	−0.03 (23)
13	−	−	+	+	1.29 (12)	0.64 (29)
14	+	−	+	+	−1.17 (4)	0.14 (19)
15	−	+	+	+	0.48 (16)	1.06 (22)
16	+	+	+	+	0.40 (11)	0.34 (26)

 # tests=m=16
 # replicates=n=2
Total # =N=mn=32

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Calculation Matrix of the Glove Box Door Experiment

Test	I	1	2	3	4	12	13	14	23	24	34	123	124	134	234	1234	$\bar{y}_i$	d_i
1	+	−	−	−	−	+	+	+	+	+	+	−	−	−	−	+	−0.76	−1.36
2	+	+	−	−	−	−	−	−	+	+	+	+	+	+	−	−	−1.40	−0.78
3	+	−	+	−	−	−	+	+	−	−	+	+	+	−	+	−	0.28	0.22
4	+	+	+	−	−	+	−	−	−	−	+	−	−	+	+	+	−0.37	−0.26
5	+	−	−	+	−	+	−	+	−	+	−	+	−	+	+	−	−0.02	−0.37
6	+	+	−	+	−	−	+	−	−	+	−	−	+	−	+	+	−0.72	−0.15
7	+	−	+	+	−	−	−	+	+	−	−	−	+	+	−	+	0.75	0.94
8	+	+	+	+	−	+	+	−	+	−	−	+	−	−	−	−	0.25	−0.07
9	+	−	−	−	+	+	+	−	+	−	−	−	+	+	+	−	−0.53	0.25
10	+	+	−	−	+	−	−	+	+	−	−	+	−	−	+	+	−0.91	0.56
11	+	−	+	−	+	−	+	−	−	+	−	+	−	+	−	+	0.46	0.03
12	+	+	+	−	+	+	−	+	−	+	−	−	+	−	−	−	−0.02	0.02
13	+	−	−	+	+	+	−	−	−	−	+	+	+	−	−	+	0.97	0.65
14	+	+	−	+	+	−	+	+	−	−	+	−	−	+	−	−	−0.52	−1.31
15	+	−	+	+	+	−	−	−	+	+	+	−	−	−	+	−	0.77	−0.58
16	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	0.37	0.06

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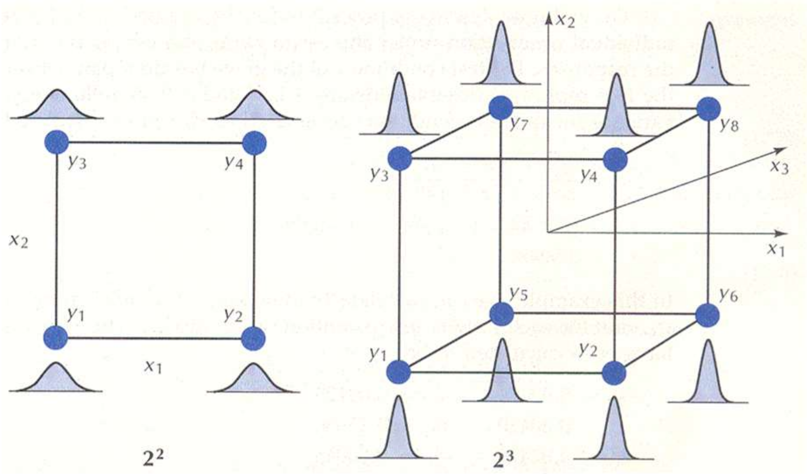
Distribution of Responses on Factorial Designs



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Estimating Variance of Noise (ε) within the Same Experiment Test by Replicates

$$s_1^2 = \frac{(y_{11} - \bar{y}_1)^2 + (y_{12} - \bar{y}_1)^2}{2-1}$$
$$= [-1.44 - (-0.76)]^2 + [-0.08 - (-0.76)]^2$$
$$= 0.9248$$

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Var(ε) Estimated from each Experiment Test

$$
\begin{aligned}
s_1^2 &= 0.92480 & s_9^2 &= 0.03125 \\
s_2^2 &= 0.30420 & s_{10}^2 &= 0.15680 \\
s_3^2 &= 0.02420 & s_{11}^2 &= 0.00045 \\
s_4^2 &= 0.03380 & s_{12}^2 &= 0.00020 \\
s_5^2 &= 0.06845 & s_{13}^2 &= 0.21125 \\
s_6^2 &= 0.01125 & s_{14}^2 &= 0.85805 \\
s_7^2 &= 0.44180 & s_{15}^2 &= 0.16820 \\
s_8^2 &= 0.00245 & s_{16}^2 &= 0.00180
\end{aligned}
$$

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Pool the Replicate Noise to Estimate Overall Variance of Noise

- Pooled Sample Variance with different sample sizes of replicates ($v_1, v_2, \ldots, v_{\mathrm{m}}$)

$$\hat{Var}(\varepsilon) = \hat{\sigma}_\varepsilon^2 = s_p^2 = \frac{v_1 s_1^2 + v_2 s_2^2 + \cdots + v_m s_m^2}{v_1 + v_2 + \cdots + v_m} = \frac{\sum_{i=1}^{m} v_i s_i^2}{\sum_{i=1}^{m} v_i}$$

- When $v_1 = v_2 = \ldots = v_m$

$$\hat{\sigma}_\varepsilon^2 = s_p^2 = \frac{s_1^2 + s_2^2 + \cdots + s_m^2}{m} = \frac{0.9248 + \cdots + 0.0018}{16} = 0.20242$$

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Estimating Effects with Replicates

$$E_1 = \left(\tfrac{1}{8}\right)\left[\left(\bar{y}_2 - \bar{y}_1\right) + \left(\bar{y}_4 - \bar{y}_3\right) + \cdots + \left(\bar{y}_{14} - \bar{y}_{13}\right) + \left(\bar{y}_{16} - \bar{y}_{15}\right)\right]$$

$$E_1 = \frac{\frac{y_{2,1} + y_{2,2}}{2} - \frac{y_{1,1} + y_{1,2}}{2} + \cdots + \frac{y_{16,1} + y_{16,2}}{2} - \frac{y_{15,1} + y_{15,2}}{2}}{8}$$

$$E_1 = \frac{y_{2,1} + y_{2,2} - y_{1,1} - y_{1,2} + \cdots + y_{16,1} + y_{16,2} - y_{15,1} + y_{15,2}}{16}$$

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Effect Error

- Assuming the observations y_{ij} are only subject to "ε" (with common variance σ^2_ε) and all effects of X_i are "null", i.e. $Y_i = \varepsilon_i$. Then, variance

$$Var(E) = Var\left[\frac{(\pm y_1 \pm y_2 \pm \cdots \pm y_N)}{(m/2)n}\right] = \frac{4}{(mn)^2}\left(N\sigma_\varepsilon^2\right) = \frac{4\sigma_\varepsilon^2}{mn}$$

$$Var\left(E_1\right) = Var\left(\frac{y_{2,1} + y_{2,2} - y_{1,1} - y_{1,2} + \cdots + y_{16,1} + y_{16,2} - y_{15,1} - y_{15,2}}{16}\right)$$

$$= \left(\frac{1}{16^2}\right)\left[Var\left(y_{2,1}\right) + Var\left(y_{2,2}\right) + \cdots Var\left(y_{15,1}\right) + Var\left(y_{15,2}\right)\right]$$

$$= \left(\frac{1}{256}\right)\left(\sigma_\varepsilon^2 + \sigma_\varepsilon^2 + \cdots + \sigma_\varepsilon^2 + \sigma_\varepsilon^2\right)$$

$$= \frac{32}{256}\sigma_\varepsilon^2 = \frac{\sigma_\varepsilon^2}{8}$$

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Estimating Effect Estimate Error

$$\begin{aligned}Var(E_1) &= Var(E_2) = Var(E_3) = Var(E_4) = Var(E_{12}) = Var(E_{13})\\ &= Var(E_{14}) = Var(E_{23}) = Var(E_{24}) = Var(E_{34}) = Var(E_{123})\\ &= Var(E_{124}) = Var(E_{134}) = Var(E_{234}) = Var(E_{1234})\\ &= \frac{\sigma_\varepsilon^2}{8}\end{aligned}$$

- Estimating the effect estimate error through the pool sample variance:

$$\Rightarrow \quad \hat{Var}(E_i) = s_{effect}^2 = \frac{\hat{\sigma}_\varepsilon^2}{8} = \frac{s_p^2}{8}$$

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Estimating Effect Errors for Glove Box Door Alignment Experiment

$$\begin{aligned}s_{effect}^2 &= \frac{4s_p^2}{32} = \frac{s_p^2}{8} = \frac{0.20242}{8}\\ &= 0.0253 \Rightarrow \text{s.e.} = s_{effect} = 0.159\text{mm}\end{aligned}$$

- Estimating the variance of response average b_0 when there are no effect from factors X_i

$$\text{Var}(\text{average}) = \text{Var}\left(\frac{y_{1,1} + y_{1,2} + \cdots + y_{16,2}}{N}\right) = \frac{1}{N^2}\left(N\sigma_\varepsilon^2\right) = \frac{\sigma_\varepsilon^2}{N}$$

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Effect Statistical Significance

- Contrasting the size of the effect to the size of the estimate error – *t* statistic:

$$t = \frac{E_i - \mu_{effect}}{s_{effect}}$$

$$= \frac{E_i - 0.0}{0.159} \sim t_{\nu=16}$$

- Degrees of freedom for *t* =16?

$$\nu = \sum_{test}(\#\,\text{Replicates} - 1) = m(n-1) = 16 \times (2-1) = 16$$

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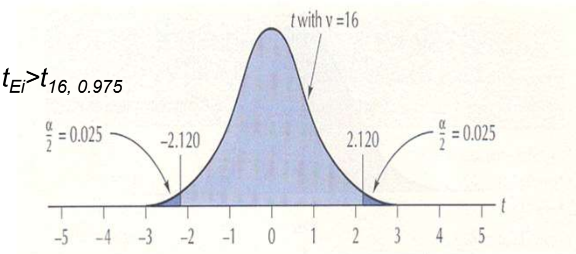
Confidence Intervals for Variable Effects of the Glove Box Door Alignment Study

H_0: E_i=0
H_1: $E_i \neq 0$
Reject H_0 if $t_{Ei} < t_{16,\,0.025}$ or $t_{Ei} > t_{16,\,0.975}$

$$t_{16,0.025} = -2.120$$

$$t_{16,0.975} = 2.120$$



95% Confidence Intervals of E_i:

$$E_i \pm t_{16,0.975} s_{effect}$$

$$E_i \pm (2.120)(0.159)$$

$$E_i \pm 0.337$$

Effect	95% Confidence Interval	Effect	95% Confidence Interval
Mean	−0.087 ± 0.169	E_{23}	−0.191 ± 0.337
E_1	**−0.654 ± 0.337**	E_{24}	−0.154 ± 0.337
E_2	**0.794 ± 0.337**	E_{34}	0.009 ± 0.337
E_3	**0.638 ± 0.337**	E_{123}	0.172 ± 0.337
E_4	0.322 ± 0.337	E_{124}	0.101 ± 0.337
E_{12}	0.147 ± 0.337	E_{134}	−0.138 ± 0.337
E_{13}	−0.117 ± 0.337	E_{234}	−0.104 ± 0.337
E_{14}	−0.031 ± 0.337	E_{1234}	0.121 ± 0.337

95% Confidence Intervals for True Mean Effects of the Glove Box Door Alignment Study Based on Replicated Experiment

Main Effects	95% Confidence Interval
RH cowl fore/aft (E_1)	-0.654 ± 0.337 mm*
Center brace (E_2)	0.795 ± 0.337 mm*
Plenum gasket (E_3)	0.638 ± 0.337 mm*
Evaporator case (E_4)	0.322 ± 0.337 mm
Two-Variable Interactions	*95% Confidence Interval*
RH cowl × center brace (E_{12})	0.147 ± 0.337 mm
RH cowl × plenum gasket (E_{13})	-0.117 ± 0.337 mm
RH cowl × evaporator case (E_{14})	-0.031 ± 0.337 mm
Center brace × plenum gasket (E_{23})	-0.191 ± 0.337 mm
Center brace × evaporator case (E_{24})	-0.154 ± 0.337 mm
Plenum gasket × evaporator case (E_{34})	0.009 ± 0.337 mm
Three-Variable Interaction	*95% Confidence Interval*
Cowl × brace × plenum (E_{123})	0.172 ± 0.337 mm
Cowl × brace × evaporator (E_{124})	0.101 ± 0.337 mm
Cowl × plenum × evaporator (E_{134})	-0.138 ± 0.337 mm
Brace × plenum × evaporator (E_{234})	-0.104 ± 0.337 mm
Four-Variable Interaction	*95% Confidence Interval*
Cowl × brace × plenum × evaporator (E_{1234})	0.121 ± 0.337 mm

* Confidence interval shows significant effect.

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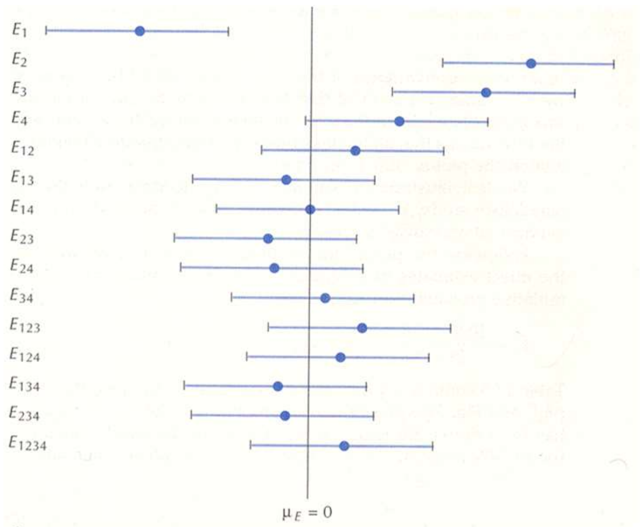
95% Confidence Intervals for True Mean Effects (Only the Intervals Based on E_1, E_2, and E_3 Do Not Include Zero)



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Assuming No Effects (Null Hypothesis)

- E_i : $E(E_i)=0$ $Var(E_i)=\sigma^2_\varepsilon/8$
- E_i~N(0, 0.159)
- We can plot Q-Q plot for the estimated effects

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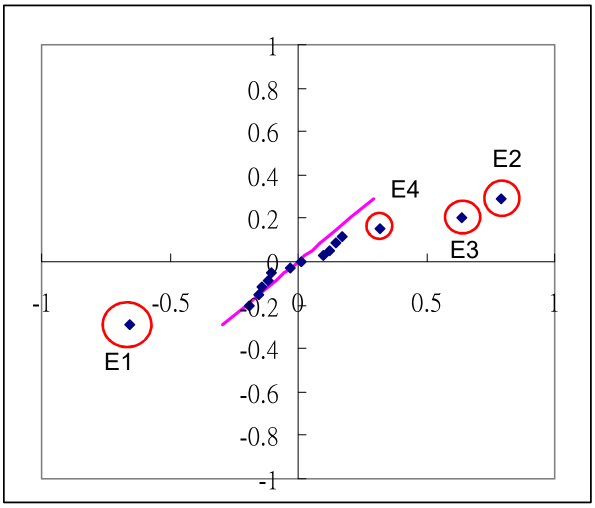
Normal Q-Q Plot of the Sample Effect: Glove Box Door Parallelism Experiment



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Use of Higher-Order Interaction Effects to Estimate Error

- Third- and higher-order interactions effects are often found insignificant (see the probability plot)
- If the higher-order effects are insignificant and are caused by errors, they can be used to estimate the errors

$$s_{effect}^2 = \sum_{\substack{higher-order \\ interactions}} \frac{\left(E_i - \mu_{E_i}\right)^2}{\text{no. of high-order interactions}}$$

$$s_{effect}^2 = \frac{\left[(0.172-0)^2 + (0.101-0)^2 + (-0.104-0)^2 + (-0.138-0)^2 + (0.121-0)^2\right]}{5}$$

$$= 0.0168572 \Rightarrow s_{effect} = \text{s.e.} = 0.1298$$

$$t_{5,0.975} = 2.571 \qquad \Rightarrow \boxed{\begin{array}{c}\text{Effect estimate} \pm (2.571)(0.1298) \\ E_i \pm 0.334\end{array}}$$

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Empirical Modeling with Significant Effects

$$\begin{aligned} y =\ & b_0 + b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_{12}x_1x_2 + b_{13}x_1x_3 \\ & + b_{14}x_1x_4 + b_{23}x_2x_3 + b_{24}x_2x_4 + b_{34}x_3x_4 + b_{123}x_1x_2x_3 \\ & + b_{124}x_1x_2x_4 + b_{134}x_1x_3x_4 + b_{234}x_2x_3x_4 \\ & + b_{1234}x_1x_2x_3x_4 + \varepsilon \end{aligned}$$

$$\begin{aligned} \hat{y} =\ & \hat{b}_0 + \hat{b}_1x_1 + \hat{b}_2x_2 + \hat{b}_3x_3 + \hat{b}_4x_4 + \hat{b}_{12}x_1x_2 + \hat{b}_{13}x_1x_3 \\ & + \hat{b}_{14}x_1x_4 + \hat{b}_{23}x_2x_3 + \hat{b}_{24}x_2x_4 + \hat{b}_{34}x_3x_4 + \hat{b}_{123}x_1x_2x_3 \\ & + \hat{b}_{124}x_1x_2x_4 + \hat{b}_{134}x_1x_3x_4 + \hat{b}_{234}x_2x_3x_4 \\ & + \hat{b}_{1234}x_1x_2x_3x_4 \end{aligned}$$

$$\text{where } \hat{b}_i = \frac{E_i}{2}$$

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Data for Linear Regression

	X1	X2	X3	X4	X1X2	X1X3	X1X4	X2X3	X2X4	X3X4	X1X2X3	X1X2X4	X1X3X4	X2X3X4	X1X2X3X4	Response, *y* (mm)
Run 1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1	-1.44
	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1.79
	-1	1	-1	-1	-1	1	1	-1	-1	1	1	1	-1	1	-1	0.39
	1	1	-1	-1	1	-1	-1	-1	-1	1	-1	-1	1	1	1	-0.5
	-1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	-0.2
	1	-1	1	-1	-1	1	-1	-1	1	-1	-1	1	-1	1	1	-0.79
	-1	1	1	-1	-1	-1	1	1	-1	-1	-1	1	1	-1	1	1.22
	1	1	1	-1	1	1	-1	1	-1	-1	1	-1	-1	-1	-1	0.21
	-1	-1	-1	1	1	1	-1	1	-1	-1	-1	1	1	1	-1	-0.4
	1	-1	-1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	1	-0.63
	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	0.47
	1	1	-1	1	1	-1	1	-1	1	-1	-1	1	-1	-1	-1	-0.01
	-1	-1	1	1	1	-1	-1	-1	-1	1	1	1	-1	-1	1	1.29
	1	-1	1	1	-1	1	1	-1	-1	1	-1	-1	1	-1	-1	-1.17
	-1	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1	1	-1	0.48
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0.4
Run 2	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1	-1	1	-0.08
	1	-1	-1	-1	-1	-1	-1	1	1	1	1	1	1	-1	-1	-1.01
	-1	1	-1	-1	-1	1	1	-1	-1	1	1	1	-1	1	-1	0.17
	1	1	-1	-1	1	-1	-1	-1	-1	1	-1	-1	1	1	1	-0.24
	-1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	1	-1	0.17
	1	-1	1	-1	-1	1	-1	-1	1	-1	-1	1	-1	1	1	-0.64
	-1	1	1	-1	-1	-1	1	1	-1	-1	-1	1	1	-1	1	0.28
	1	1	1	-1	1	1	-1	1	-1	-1	1	-1	-1	-1	-1	0.28
	-1	-1	-1	1	1	1	-1	1	-1	-1	-1	1	1	1	-1	-0.65
	1	-1	-1	1	-1	-1	1	1	-1	-1	1	-1	-1	1	1	-1.19
	-1	1	-1	1	-1	1	-1	-1	1	-1	1	-1	1	-1	1	0.44
	1	1	-1	1	1	-1	1	-1	1	-1	-1	1	-1	-1	-1	-0.03
	-1	-1	1	1	1	-1	-1	-1	-1	1	1	1	-1	-1	1	0.64
	1	-1	1	1	-1	1	1	-1	-1	1	-1	-1	1	-1	-1	0.14
	-1	1	1	1	-1	-1	-1	1	1	1	-1	-1	-1	1	-1	1.06
	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0.34

Linear Regression By Excel

迴歸統計	
R 的倍數	0.901206
R 平方	0.812172
調整的 R	0.636084
標準誤	$\boxed{0.449927}$
觀察值個	32

s_p

ANOVA

	自由度	SS	MS	F	顯著值
迴歸	15	14.0053	0.933686	4.612292	0.00211678
殘差	16	3.23895	$\boxed{0.202434}$		
總和	31	17.24425			

s^2_p

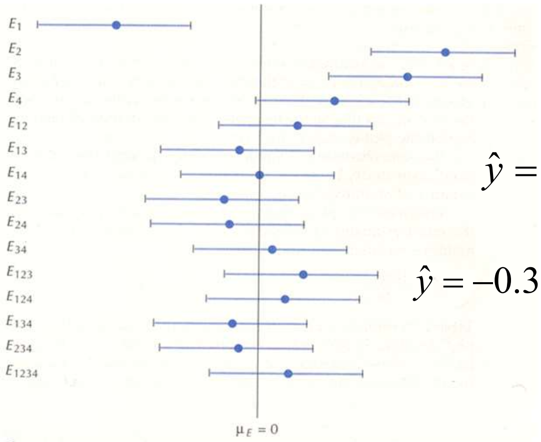
$b_i = E_i/2$

$S_{effect}/2$

	係數	標準誤	t 統計	P-值	下限 95%	上限 95%
截距	-0.08719	0.079537	-1.09619	0.28922	-0.2557976	0.08142257
X1	-0.32719	0.079537	-4.11367	0.000813	-0.4957976	-0.1585774
X2	0.397188	0.079537	4.993769	0.000133	0.22857743	0.56579757
X3	0.319063	0.079537	4.011517	0.001007	0.15045243	0.48767257
X4	0.160938	0.079537	2.023439	0.060064	-0.0076726	0.32954757
X1X2	0.073438	0.079537	0.923317	0.369557	-0.0951726	0.24204757
X1X3	-0.05844	0.079537	-0.73472	0.473139	-0.2270476	0.11017257
X1X4	-0.01531	0.079537	-0.19252	0.849756	-0.1839226	0.15329757
X2X3	-0.09531	0.079537	-1.19835	0.248231	-0.2639226	0.07329757
X2X4	-0.07719	0.079537	-0.97046	0.346258	-0.2457976	0.09142257
X3X4	0.004688	0.079537	0.058935	0.953734	-0.1639226	0.17329757
X1X2X3	0.085938	0.079537	1.080477	0.295948	-0.0826726	0.25454757
X1X2X4	0.050313	0.079537	0.63257	0.535951	-0.1182976	0.21892257
X1X3X4	-0.06906	0.079537	-0.86831	0.398062	-0.2376726	0.09954757
X2X3X4	-0.05219	0.079537	-0.65614	0.521056	-0.2207976	0.11642257
X1X2X3X	0.060313	0.079537	0.758298	0.459297	-0.1082976	0.22892257

$\pm t_{16,\,0.975}\,(S_{effect}/2) = \pm 0.169$

Empirical Modeling with Significant Effects



$$\hat{y} = \hat{b}_1 x_1 + \hat{b}_2 x_2 + \hat{b}_3 x_3$$

$$\hat{y} = -0.327 x_1 + 0.397 x_2 + 0.319 x_3$$

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Model Predictions and Residuals of the Parallelism Prediction Model (Glove Box Door Study)

- Model residuals: $e_{ij} = (y_{ij} - \hat{y}_i)$

Test	x_1	x_2	x_3	x_4	Predicted Response, $\hat{y}_i$	Observed Response, y_{i1}	Run Order	Model Residual, e_{i1}	Observed Response, y_{i2}	Run Order	Model Residual, e_{i2}
1	−	−	−	−	−0.389	−1.440	(7)	−1.051	−0.080	(28)	0.309
2	+	−	−	−	−1.043	−1.790	(10)	−0.747	−1.010	(24)	0.033
3	−	+	−	−	0.405	0.390	(14)	−0.015	0.170	(32)	−0.235
4	+	+	−	−	−0.249	−0.500	(2)	−0.251	−0.240	(21)	0.009
5	−	−	+	−	0.249	−0.200	(9)	−0.449	0.170	(27)	−0.079
6	+	−	+	−	−0.405	−0.790	(6)	−0.385	−0.640	(30)	−0.235
7	−	+	+	−	1.043	1.220	(13)	0.177	0.280	(20)	−0.763
8	+	+	+	−	0.389	0.210	(8)	−0.179	0.280	(18)	−0.109
9	−	−	−	+	−0.389	−0.400	(1)	−0.011	−0.650	(31)	−0.261
10	+	−	−	+	−1.043	−0.630	(15)	0.413	−1.190	(25)	−0.147
11	−	+	−	+	0.405	0.470	(3)	0.065	0.440	(17)	0.035
12	+	+	−	+	−0.249	−0.010	(5)	0.239	−0.030	(23)	0.219
13	−	−	+	+	0.249	1.290	(12)	1.041	0.640	(29)	0.391
14	+	−	+	+	−0.405	−1.170	(4)	−0.765	0.140	(19)	0.545
15	−	+	+	+	1.043	0.480	(16)	−0.563	1.060	(22)	0.017
16	+	+	+	+	0.389	0.400	(11)	0.011	0.340	(26)	−0.049

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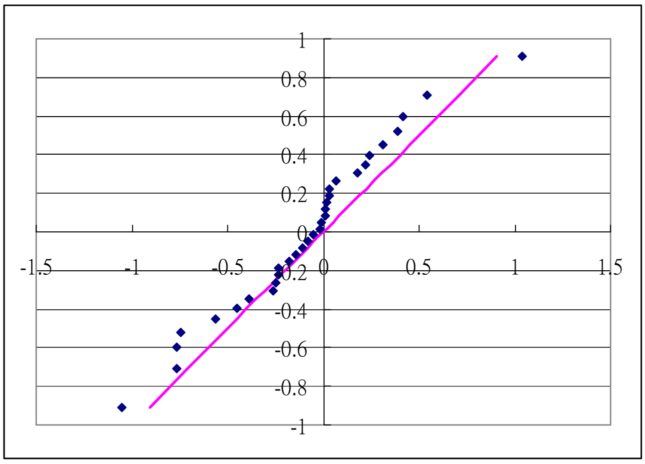
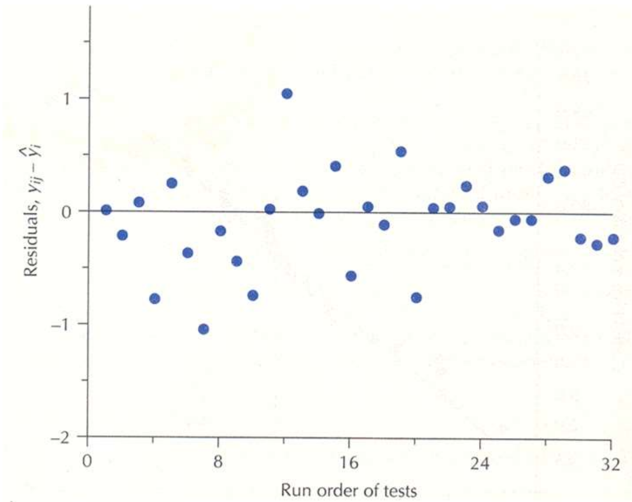
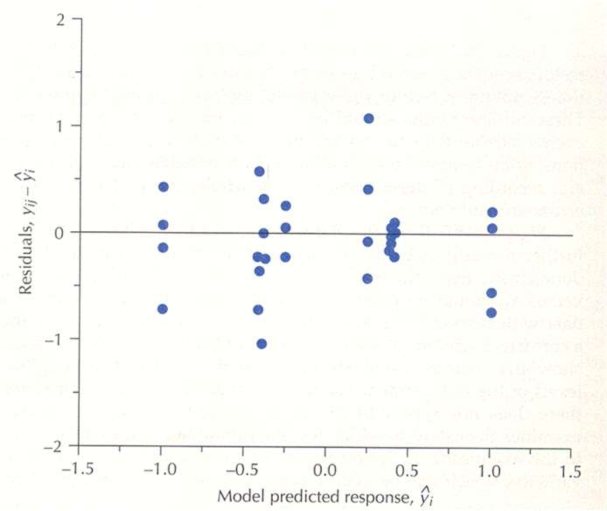
QQ Plot of the Residuals



Residuals Plotted Against the Run Order of the Tests



Residuals Plotted Against the Predicted Responses



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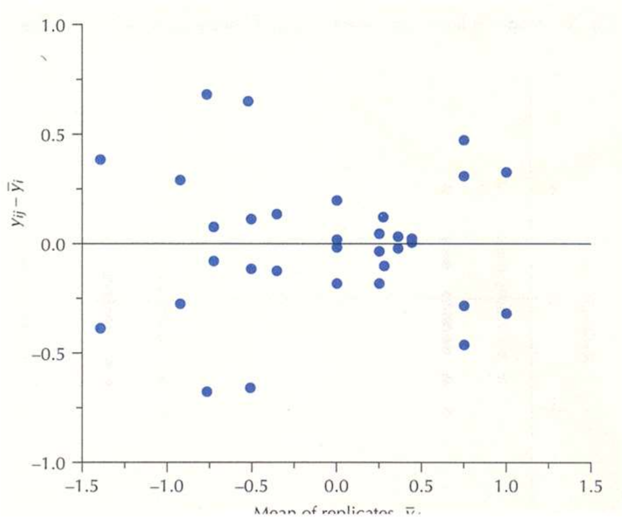
Difference Between the Replicates and the Mean of the Replicates against Replicate Mean



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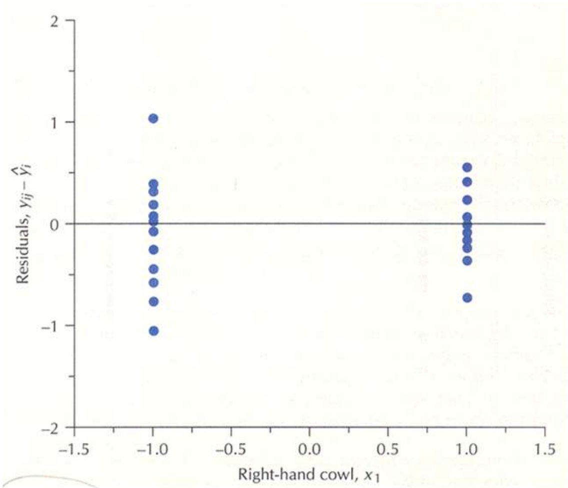
Residuals Versus Right-Hand Cowl Movement Level : Nominal (-1), and -5 Millimeters (+1)



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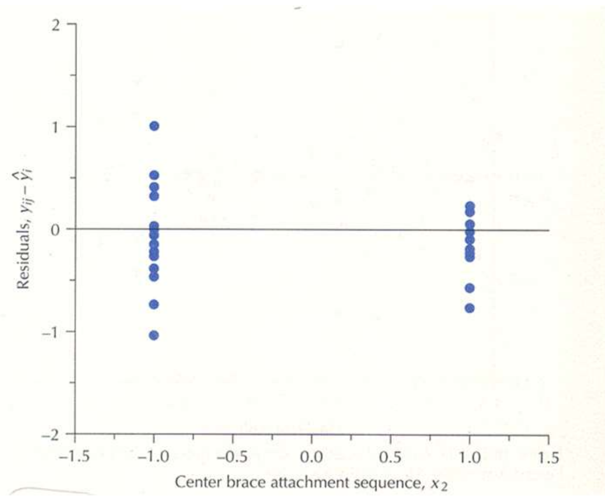
Residuals Versus Center Brace Attachment Level: Before (-1), After (+1)



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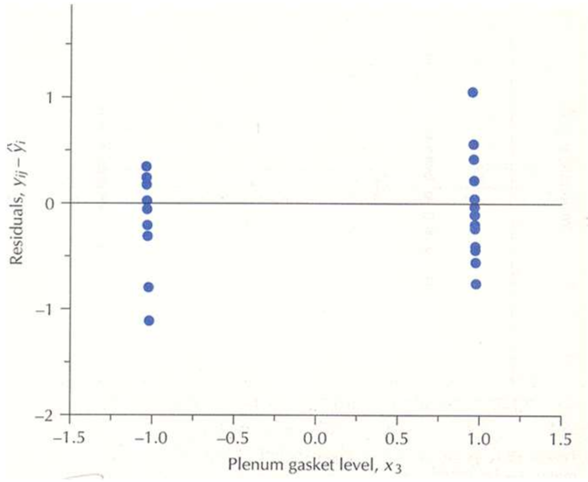
Residuals Versus Plenum Gasket Level : No (-1), Yes (+1)



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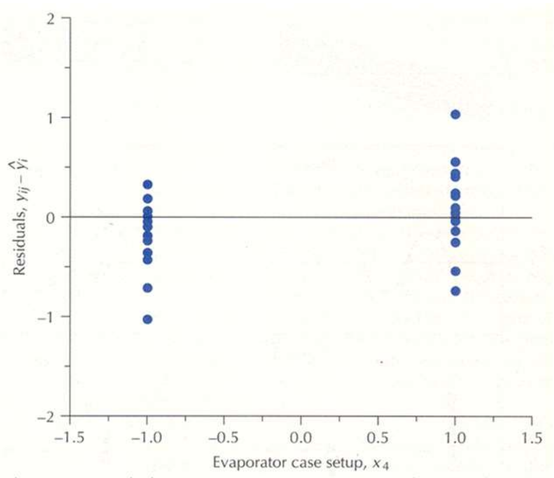
Residuals Versus Evaporator Case Setup Level: Nominal (-1), -5 Millimeters (+1)



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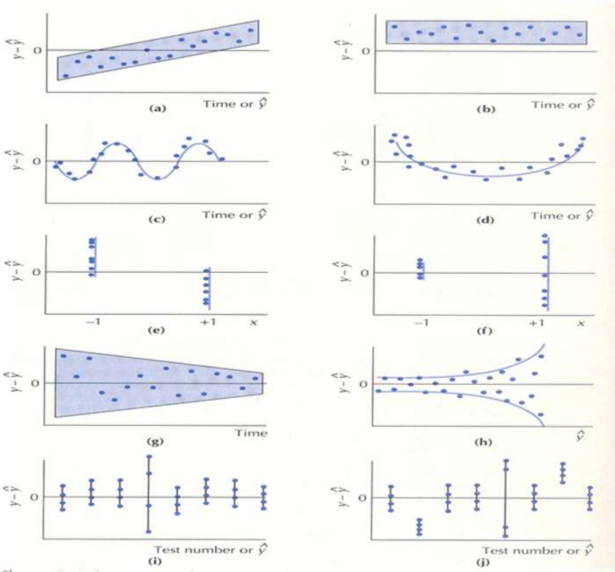
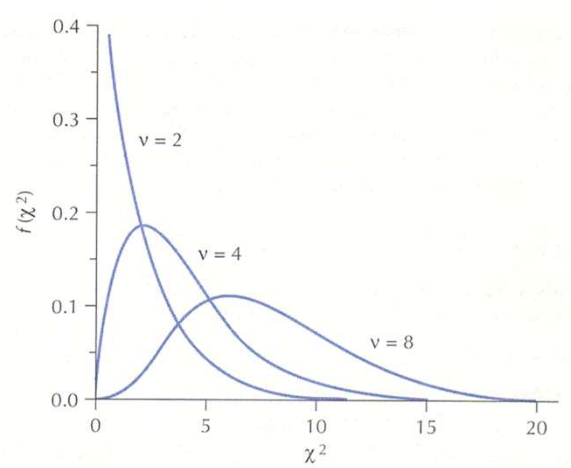
Some Nonrandom Patterns of Residual Plots



Sample Variances and Chi-Square Distributions

$$\frac{(n-1)s_y^2}{\sigma_y^2} \sim \chi^2$$



Testing the Homogeneity of Variance: Bartlett's Test

$$\mathrm{H}_0 : \sigma_1^2 = \sigma_2^2 = \cdots = \sigma_m^2 = \sigma_\varepsilon^2$$

$$\mathrm{H}_1 : \text{at least one } \sigma_{\mathrm{i}}^2 \neq \sigma_{j'}^2 \qquad i \neq j$$

- Test statistic: $\chi^2_{calc} = \dfrac{M}{c} \sim \chi^2_{m-1}$ where

$$M = (N-m)\ln s_p^2 - \sum_{i=1}^{m}(n_i - 1)\ln s_i^2 ; s_p^2 = \frac{\sum_{i=1}^{m}(n_i - 1)s_i^2}{N-m}$$

$$c = 1 + \frac{1}{3(m-1)}\left[\left(\sum_{i=1}^{m}\frac{1}{n_i - 1}\right) - \frac{1}{N-m}\right]$$

Reject H_0 if

$\chi^2_{calc} \rangle \chi^2_{m-1,\alpha}$

Bartlett's Test for Glove Box Door Alignment Study

$$s_p^2 = \frac{(2-1)0.92480 + (2-1)0.30420 + \cdots + 2(2-1)0.00180}{32-16} = 0.20243$$

$$\begin{aligned} M &= (32-16)\ln(0.20243) \\ &\quad - [(2-1)\ln 0.92480 + (2-1)\ln 0.30420 + \cdots + (2-1)\ln 0.00180] \\ &= 28.1695 \end{aligned}$$

$$\begin{aligned} c &= 1 + \frac{1}{3(16-1)}\left(\frac{1}{2-1} + \frac{1}{2-1} + \cdots + \frac{1}{2-1} - \frac{1}{32-16}\right) \\ &= 1.3542 \end{aligned}$$

$$\chi^2_{calc} = \frac{28.1695}{1.3542} = 20.8016 < \chi^2_{15,0.05} = 25.0$$

Using the Fitted Model to Improve Quality

- The ideal door parallelism is ***zero***
- We can use the fitted model to try to achieve the zero parallelism
- For example: when the center brace is attached after (x_2=+1) without Plenum gasket (x_3=–1), we can find the value of RH cowl movement (x_1) to achieve the best parallelism:

$$\hat{y} = -0.327x_1 + 0.397\times(+1) + 0.319\times(-1) = 0$$

$$\Rightarrow x_1 = \frac{-0.078}{-0.327} = 0.239$$

Transforming the Variable Value to and from the Coded Variable Space

- 0.239 is the value for the coded variable x_1
- The real RH cowl movement value should be translated back from the coded value:

$$\text{Best RH cowl movement} = \frac{0+(-5)}{2} + 0.239\left(\frac{-5-0}{2}\right) = -3.10\text{mm}$$

when x_2=+1 and x_3=–1.

Contour Plots for the Fitted Model

- When x_2=+1 (center brace is attached after) and redefine x_3 as the plenum gasket thickness from 0mm (x_3=–1) to 2mm (x_3=+1):

$$\hat{y} = 0.397 - 0.327x_1 + 0.319x_3$$

- Contour Plot