Fundamentals of Multi-Objective Optimization

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Consider the following optimization problem (1) with more than one objective functions $[f_1, f_2, \cdots, f_k]$ to be optimized simultaneously.

$$\begin{aligned} \min_{\mathbf{x}} \mathbf{f}(\mathbf{x}) &= [f_1, f_2, \cdots, f_k] \\ \text{subject to } \ & \mathbf{x} \in \mathcal{X} \end{aligned} \tag{1}$$

Formulation (1) is also referred to as vector optimization, multicriteria optimization, and multiobjective optimization in the literature. We will use multiobjective optimization throughout this lecture.

Let us define two spaces of reference : the design space with axes being the design variables $\mathbf{x}$ and the objective function space with objective functions $\mathbf{f}$ as axes. In Eq.(1) we are seeking the optimal design within the feasible set $\mathcal{X}$ in the design space. By mapping $\mathcal{X}$ into the objective space, we have the attainable set $\mathcal{A}$. Figure 1 illustrate the concept of design space, objective function space, feasible set and attainable set.

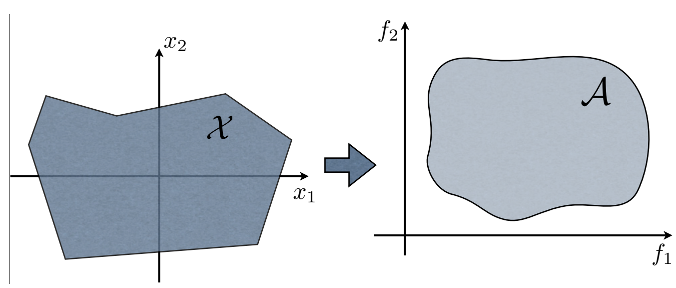

Figure 1: Feasible Set and Attainable Set

If we look at the attainable set $\mathcal{A}$ in Fig.1, common preferences are made to improve at least one objective value without deteriorating the other one. The subset of $\mathcal{A}$ satisfies the criteria is called the *Pareto set*.

> **Pareto Optima** : A point $\mathbf{f}_0$ in the attainable set $\mathcal{A}$ is Pareto optimal if and only if there is not another $\mathbf{f} \in \mathcal{A}$ such that $f_i \leq f_{0,i}$ for all i and $f_i < f_{0,i}$ for at least one i.

Utopia Point : A point $\mathbf{f}_0$ in the objective space is the utopia point if $f_{0,i} = \min_{\mathbf{x}} f_i(\mathbf{x}), \forall \mathbf{x} \in \mathcal{X}, i = 1, \cdots, k$. It is also called the ideal point.

Figure 2 illustrates Pareto set and the utopia design point in the objective function space. As can be see if the utopia point is attainable, the objective functions do not have trade-offs and therefore the optimal can easily be obtained via individual optima.

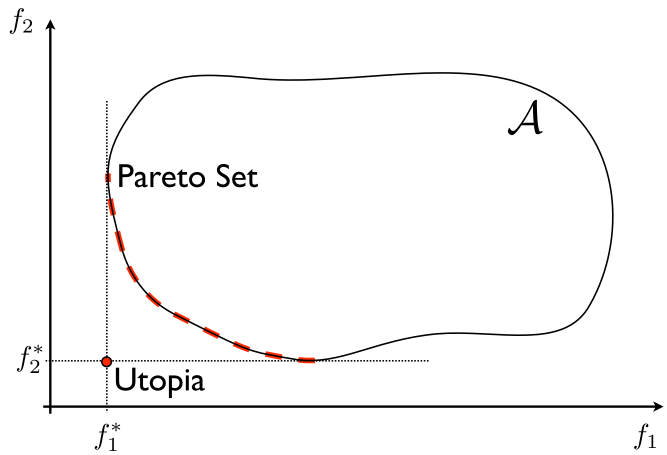

Figure 2: Pareto Set and Utopia Point in the Objective Function Space

The Pareto set provides the trade-off information between objective functions, therefore also called the "trade-off curve". Research in the literature uses the distance between the Pareto set and the utopia as the 'strength' of the trade-offs between objective functions. If the utopia point is far away from the Pareto set, the objective function f_1 has high influence on the f_2, and vise versa.

Pareto Set Generation Pareto set is generated by extracting part of the attainable set that satisfies the Pareto optima definition. Theoretically Pareto set contains infinite design alternatives; however, in practice only finite Pareto points are found to approximate the entire set. Some techniques in finding the Pareto set is described in this section.

Weighted Sum

$$\min_{\mathbf{x}} \sum_{i=1}^{k} w_i f_i(\mathbf{x}) \tag{2}$$
$$\text{subject to} \quad \mathbf{x} \in \mathcal{X}$$
$$\sum_{i=1}^{k} w_i = 1$$

The most common approach in obtaining the Pareto set is using weighted sum as shown in Eq.(2). $\mathbf{w}$ is the weights typically set by the decision. The method is easy to use, and if all the weights are positive, the minimum of Eq.(2) is always Pareto optimal. However, there are a few recognized difficulties with the weighted sum method. First, a satisfactory weights does not necessarily guarantee that the final solution will be acceptable; one may have to re-solve the problem with new weights. That means the weights may not reveal the actual preference on the objective functions. The second problem is that it is impossible to obtain points on nonconvex Pareto set using weighted sum. As a result the Pareto set might be disconnected, difficult for decision-making. The final difficulty is that varying the weights evenly and consistently may not necessarily result in an even distribution of Pareto optimal points. As a result, some region has more information than others and a complete representation of the Pareto set is thus missing.

Constrain Method

$$\min_{\mathbf{x}} \{f_1, f_2\} \tag{3}$$
$$\text{subject to} \quad \mathbf{g}(\mathbf{x}) \leq 0$$

The constrain method treat the multiobjective problem as a single objective and move the other objective functions as constraints at various boundaries. Consider a bi-objective problem with $\{f_1, f_2\}$ being the objective functions as shown in Eq.(3). Let f_1^* and $\mathbf{x}_1^*$ be the optimal results of considering f_1 only as in Eq.(4). Let f_2^* and $\mathbf{x}_2^*$ be the optimum of considering f_2 only as in Eq.(5).

$$\min_{\mathbf{x}} f_1, \text{ subject to } \mathbf{g}(\mathbf{x}) \leq 0 \tag{4}$$

$$\min_{\mathbf{x}} f_2, \text{ subject to } \mathbf{g}(\mathbf{x}) \leq 0 \tag{5}$$

The utopia point is therefore $\{f_1^*, f_2^*\}$. The points $\{f_1^*, f_2(\mathbf{x}_1^*)\}$ and $\{f_1(\mathbf{x}_2^*), f_2^*\}$ are at the boundary of the Pareto set as shown in Fig.3. If Eq.(5) considers the effects of f_1, we can treat f_1 as a constraint with bounds on various levels. Let us use N segments on f_1 as an example. Each segment has the bounds being

$$f_1^i = f_1^* + \frac{f_1(\mathbf{x}_2^*) - f_1^*}{N}$$

The constrain method obtains the Pareto point via solving

$$\begin{aligned} &\min_{\mathbf{x}} f_2 \\ \text{subject to }\ &\mathbf{g}(\mathbf{x}) \le 0 \\ &f_1(\mathbf{x}) \le f_1^i \\ &i = 1, \cdots, N \end{aligned} \tag{6}$$

The solutions of Eq.(6) is an approximation to the Pareto set. With the increase of N, the constrain method is capable of obtaining accurate and nonconvex Pareto set.

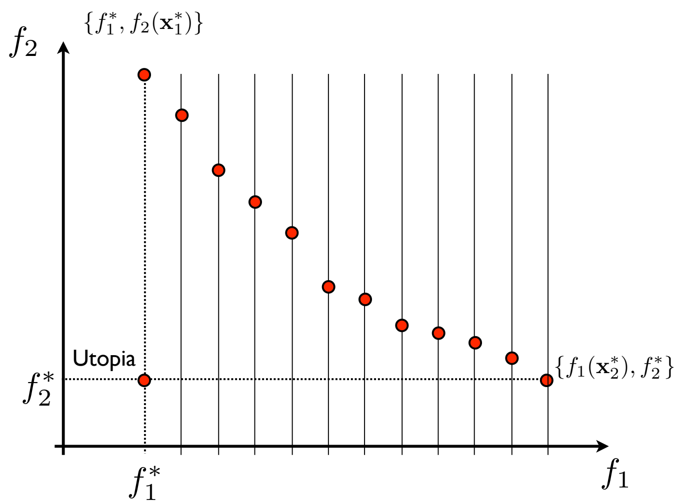

Figure 3: Constrain method for Eq.(3)

The constraint method is also called the ε-constraint method. This method has the following advantages : (1) it focuses on a single objective with limits on others, (2) it always provides a weakly Pareto optimal point, assuming the formulation gives a solution, (3) it is not necessary to normalize the objective function, and (4) it gives Pareto optimal solution if one exists and is unique. The major disadvantage is that the optimization problem may be infeasible if the bounds on the objective function are not appropriate.

In addition to generating the Pareto set, several methods exist for converting the multiobjective formulation into a scalar substitute problem that has a scalar objective and can be solved with the usual single objective optimization methods. The scalar objective has the form $F(\mathbf{f}, \mathbf{M})$, where $\mathbf{M}$ is a vector of preference parameters that can be adjusted to tune the scalarization to the designers' subjective preference.

The simplest scalar substitute objective is obtained by assigning subjective weights to each objective and summing up all objective multiplied by their corresponding weights as shown in Eq.(7).

$$\min F = \sum w_i f_i(\mathbf{x}) \tag{7}$$

However, this subjective information on weights of each objective functions can sometimes be misleading and results in a inferior design. To avoid this, the designer must be careful in tracing the effect of subjective preferences on the decisions suggested by the optimal solution obtained. Such design preferences are rarely known precisely a priori, so preference values are adjusted gradually and trade-offs become more evident with repeated solutions of the substitute problem with different preference parameter values.