ME7129 Optimization in Engineering : from uncertainty point of view

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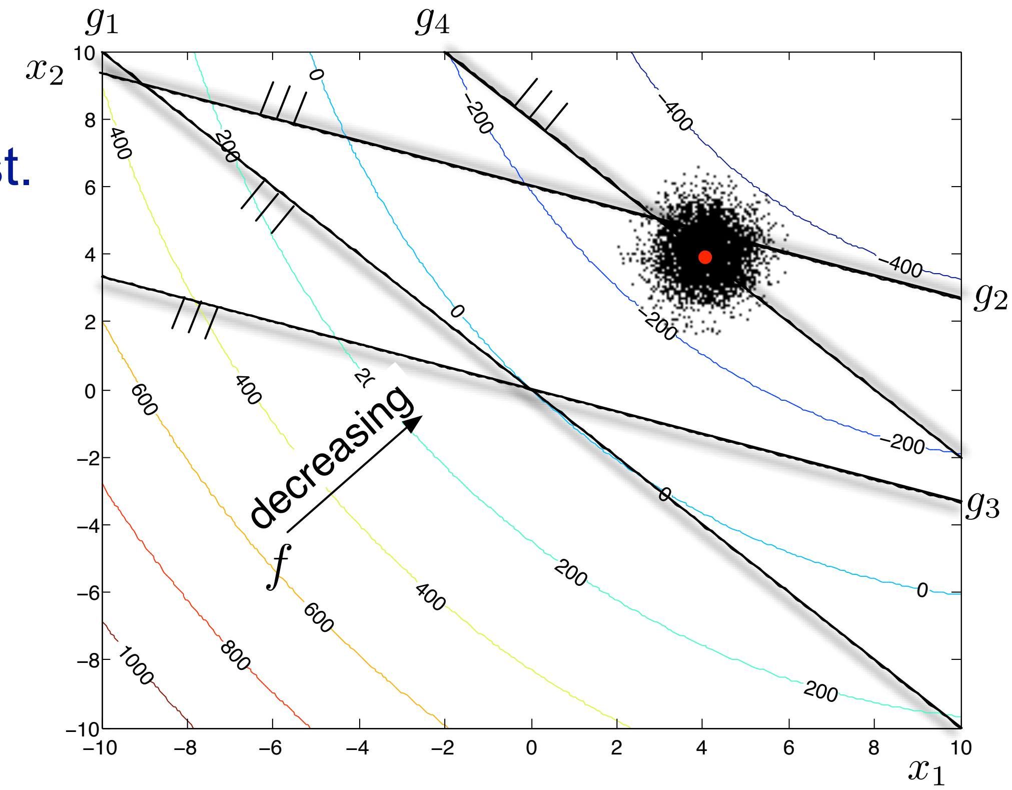
So far as the laws of mathematics refer to reality, they are uncertain, and so far as they are certain, they do not refer to reality.

-Albert Einstein

Optimization

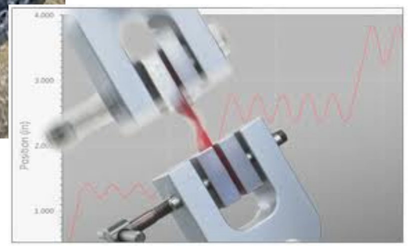
- A design with the best performance within the prescribed limitations is the optimal design.
- A good algorithm can solve a rather complex problem with reasonable cost.
- Most optimal design results lie on the boundary of some constraints, hence defined constraint activity.
- When a constraint is active, any disturbance or uncertainty source could result in infeasible outcome.

$$\min_{\mathbf{x}} f(\mathbf{x})$$
$$\text{s.t. } \mathbf{g}(\mathbf{x}) \leq 0$$



Impacts of Uncertainty

- Uncertainty from the environment could results in bad function performances or unpredictable failure.
- Uncertainty from the manufacturing processes, in the form of tolerances, results in product variation and quality issues.
- Uncertainty from human results in unsafe operations, sometimes catastrophic.
- Uncertainty from vague expressions or imprecise descriptions results in undesirable product attributes.
- Uncertainty from lack of data measurements might be misleading, resulting in completely faulty judgements.



Major Uncertainty Categories

- Aleatory Uncertainty : uncertainty associated with randomness

aleatory | ˈeɪlɪət(ə)ri, ˈal- |(also **aleatoric** | ˌeɪlɪəˈtɒrɪk, ˌal- |)
adjective
depending on the throw of a die or on chance; random.
• relating to or denoting music or other forms of art involving elements of random choice (sometimes using statistical or computer techniques) during their composition, production, or performance.

ORIGIN late 17th cent.: from Latin ***aleatorius***, from ***aleator 'dice player'***, from ***alea 'die'***, + -Y[1].

- Epistemic Uncertainty : uncertainty associated with imperfect knowledge

epistemic | ˌɛpɪˈstiːmɪk, -ˈstɛm- |
adjective
relating to knowledge or to the degree of its validation.

DERIVATIVES
epistemically adverb

ORIGIN 1920s: from Greek ***epistēmē 'knowledge'*** (see EPISTEMOLOGY) + -IC.

Major Uncertainty Models

- Interval Model : uncertainty is known to be within a prescribed region.

- Fuzzy Model : uncertainty at a certain level is known to have a specific preference.

- Probabilistic Model : uncertainty at a certain value is known to have a specific probability of occurrence.

Interval Uncertainty

- Computers might produce wrong answers due to the fact that they are discrete and finite machines, and they cannot cope with some of the continuous and infinite aspects of mathematics.

- On February 25, 1991, a Patriot missile battery assigned to protect a military installation in Dhahran, Saudi Arabia, failed to intercept a Scud missile, and the malfunction was blamed on an error in computer arithmetic

- This error originated from an innocent-looking number - 1/10

The Patriot Missile

- The Patriot's control system kept track of time by counting **tenth of a second**.
- To convert the count into full seconds, the computer multiply by 1/10.
- Mathematically the procedure is unassailable, but computationally it was disastrous.
- Because the decimal fraction **1/10 has no exact finite representation** in binary notation, the computer has to approximate.
- In 24-bit binary fraction, 1/10 = 0.00011001100110011001100, the difference is about one ten-millionth.
- Over 4 days, this discrepancy built up **about 1/3 second**.
- In addition to other control software, this error caused a almost **700 meter miscalculation**.
- **28 soldiers died**.

Interval Variables

- Interval variables use upper and lower bounds to describe uncertainty quantities.

- Interval variables do not improve the accuracy, instead, they provide certificate of accuracy (or lack of it)

- The mathematical model can be written as

upper bound

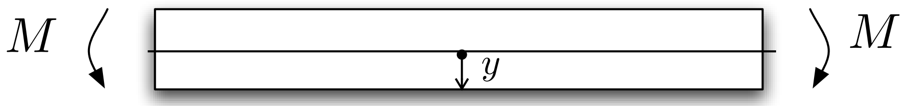
$$X = [\underline{x}, \bar{x}] : \underline{x} \leq \boxed{X} \leq \bar{x}, \ \forall X \in \mathbb{D}$$

lower bound uncertainty variable domain

Interval Arithmetic

- $X_1 + X_2 = [\underline{x}_1 + \underline{x}_2, \bar{x}_1 + \bar{x}_2]$
- $X_1 - X_2 = [\underline{x}_1 - \bar{x}_2, \bar{x}_1 - \underline{x}_2]$
- $X_1 \cdot X_2 = [\min\{\underline{x}_1\underline{x}_2, \underline{x}_1\bar{x}_2, \bar{x}_1\underline{x}_2, \bar{x}_1\bar{x}_2\}, \max\{\underline{x}_1\underline{x}_2, \underline{x}_1\bar{x}_2, \bar{x}_1\underline{x}_2, \bar{x}_1\bar{x}_2\}]$
- $X_1/X_2 = [\underline{x}_1, \bar{x}_1] \cdot [1/\bar{x}_2, 1/\underline{x}_2]$, if $\underline{x}_2 > 0$ or $\bar{x}_2 < 0$.
- $X_1 = X_2$, iff $\underline{x}_1 = \underline{x}_2$ and $\bar{x}_1 = \bar{x}_2$
- $X_1 < X_2$, iff $\bar{x}_1 < \underline{x}_2$
- $X_1 > X_2$, iff $\underline{x}_1 > \bar{x}_2$

Function of Interval Variables



$$\sigma = \frac{My}{I}$$

➡ Pure bending stress is function of inertia I and moment M .

➡ Let both I and M be interval variables

$$I = [1800 \times 10^6 \mathrm{mm}^4, 1850 \times 10^6 \mathrm{mm}^4]$$

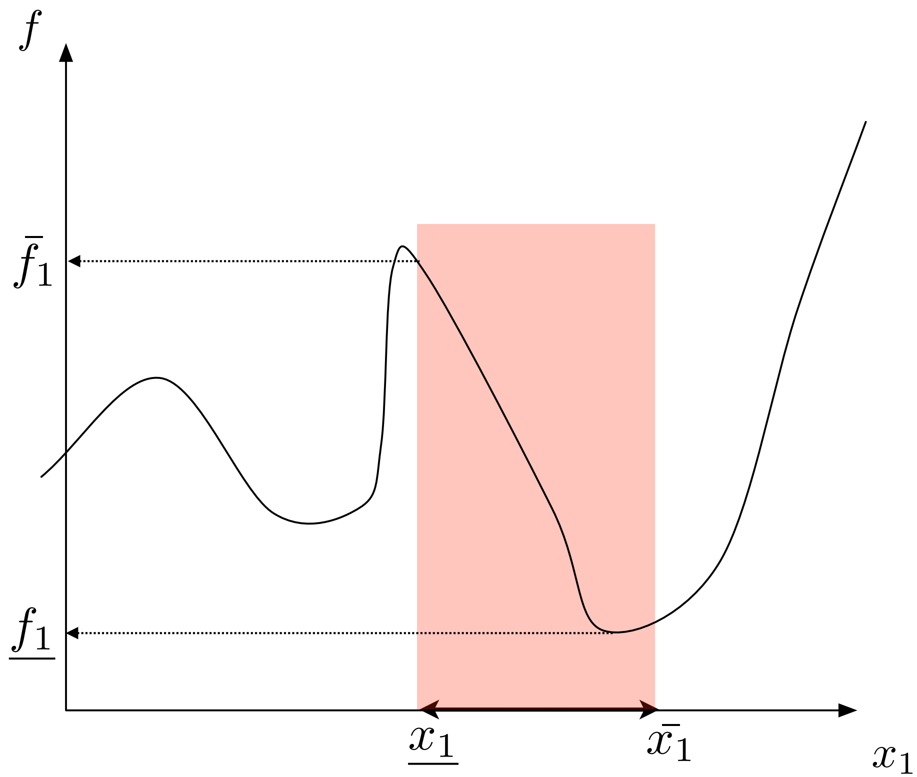
$$M = [4 \times 10^4 \mathrm{mm} \cdot \mathrm{N}, 4.5 \times 10^4 \mathrm{mm} \cdot \mathrm{N}]$$

➡ What is the bending stress at

$$y = 0.2cm$$

Complex Functions

- Consider the following function of one interval variable, the resulting upper and lower limits of function outcomes need to be obtained via optimization.



Analysis for Functions with Interval Variables

Mathematically, for a function of interval variable, to find the resulting interval, we should find

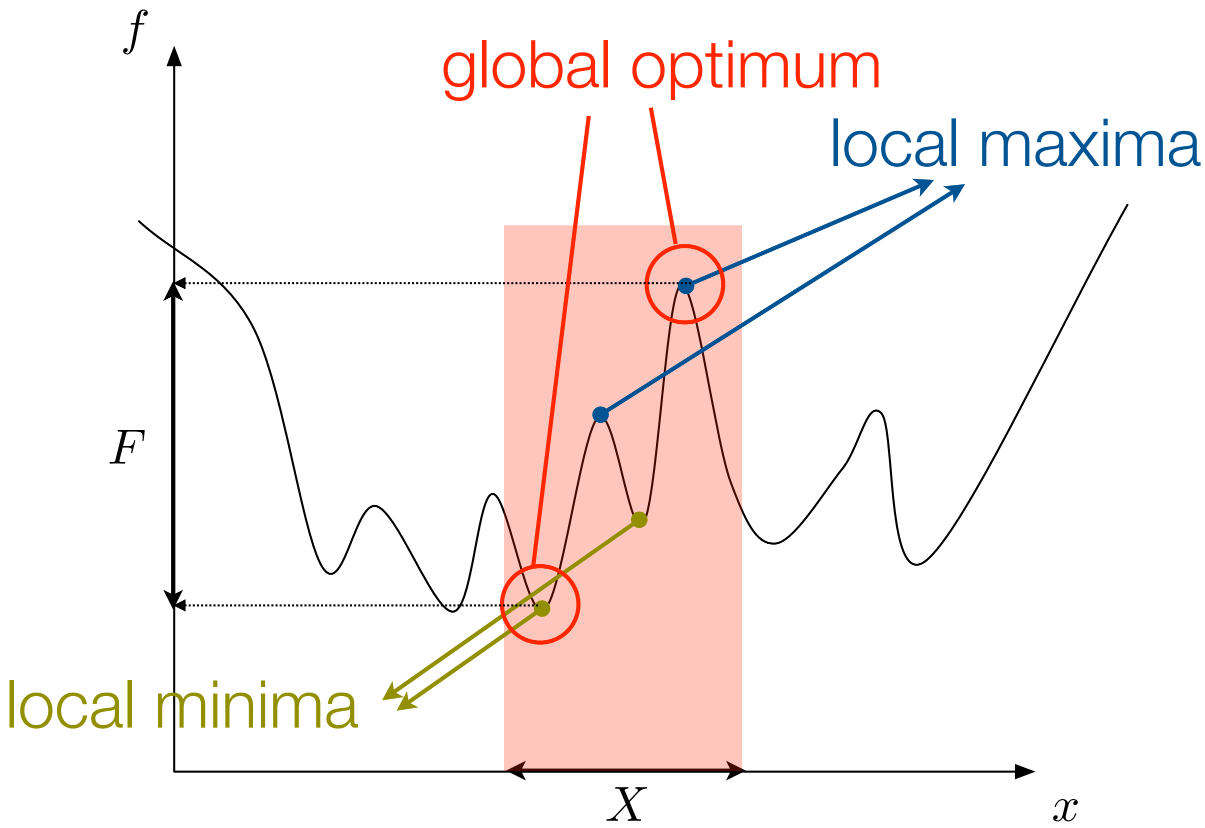
$$F = [\underline{f}, \bar{f}]$$

$$\underline{f} = \min_{x \in [\underline{x}, \bar{x}]} f(x)$$

$$\bar{f} = \max_{x \in [\underline{x}, \bar{x}]} f(x)$$

global optimum

The Need of a Global Optimization Tool

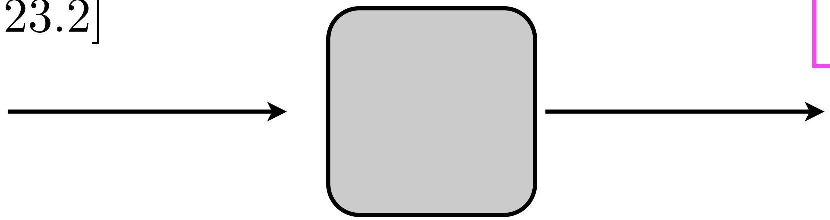


Algorithm Selection

- A good algorithm is one that obtains the correct region with narrower bounds.

select the one with correct narrower bounds

$X = [21.4, 23.2]$



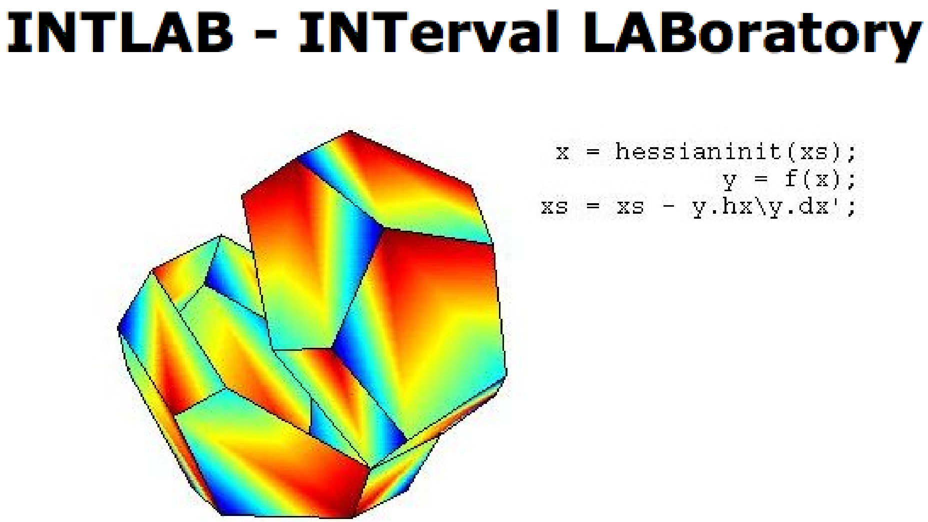
$\boxed{F = [123, 128]}$

$F = [122, 129]$

$F = [110, 150]$

Interval Analysis Resource

- http://www.ti3.tu-harburg.de/rump/intlab/

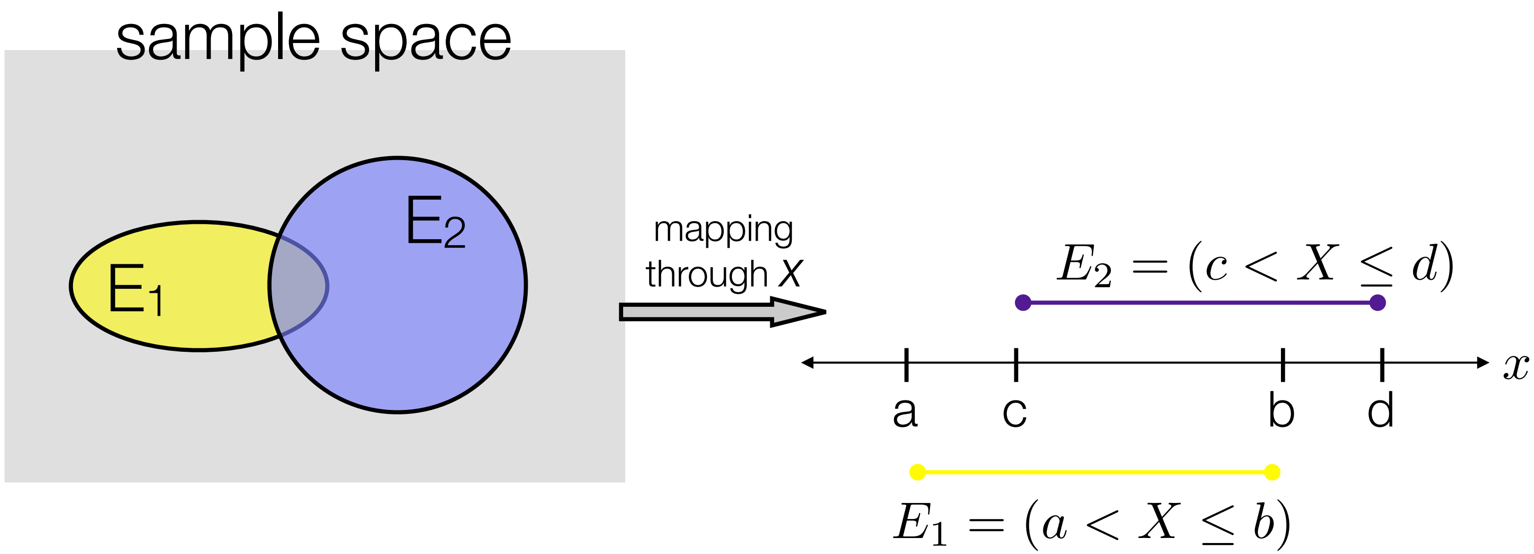


Random Variables

- A random variable X is a quantity that takes on various values of x between $[-\infty, \infty]$
- Depending on the values x can take, we have discrete random variables and continuous random variables.

Random Variables : Mapping Function

- Mathematically, a random variable may be defined as a mapping function that transforms (or maps) the event in a possibility space into the number system (the real line)



Probability Distributions of Random Variables

- As the values or ranges of values of a random variable represent events, the numerical values of the random variables, therefore, are associated with specific probability or probability measure.

- If X is a random variable, its probability distribution can be always be described by its cumulative distribution function (CDF), which is denoted as

$$F_X(x) \equiv \Pr(X \leq x)$$

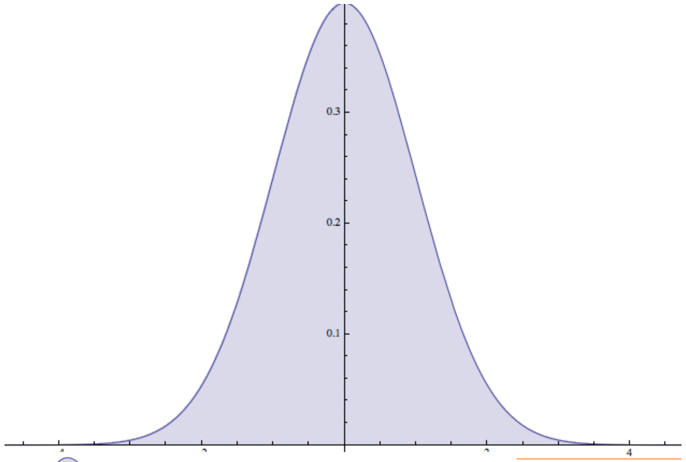
Properties of a CDF

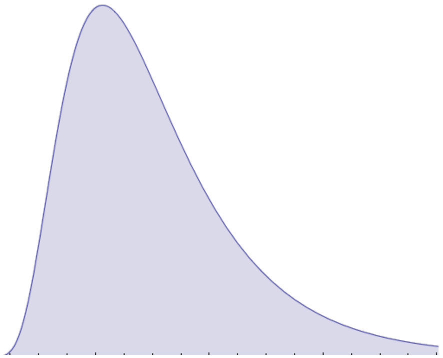
$$F_X(-\infty) = 0 \quad \& \quad F_X(\infty) = 1$$

$F_X(x) \geq 0, \forall x$ and is <u>nondecreasing</u> with x

$F_X(x)$ is <u>continuous</u> to the right with x

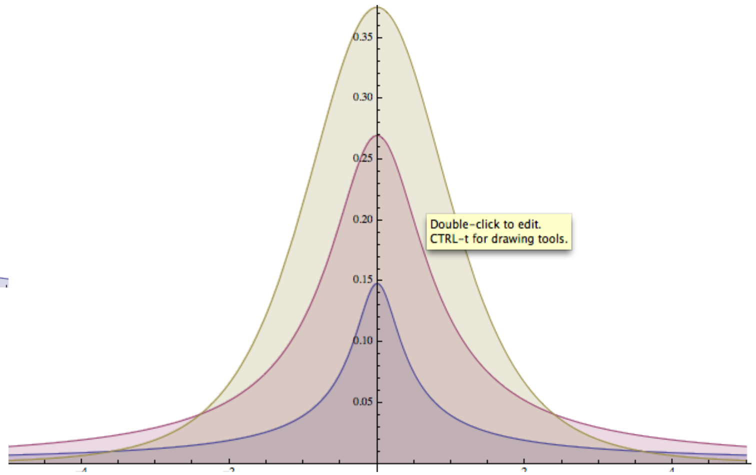
Probability Density Function





$$\Pr(a \le X \le b) = \int_a^b f_X(x)\,dx$$

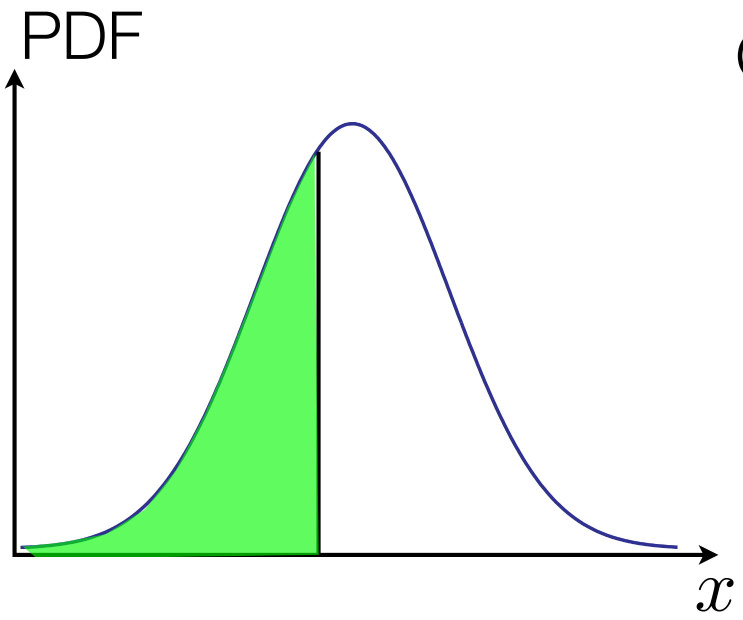
PDF

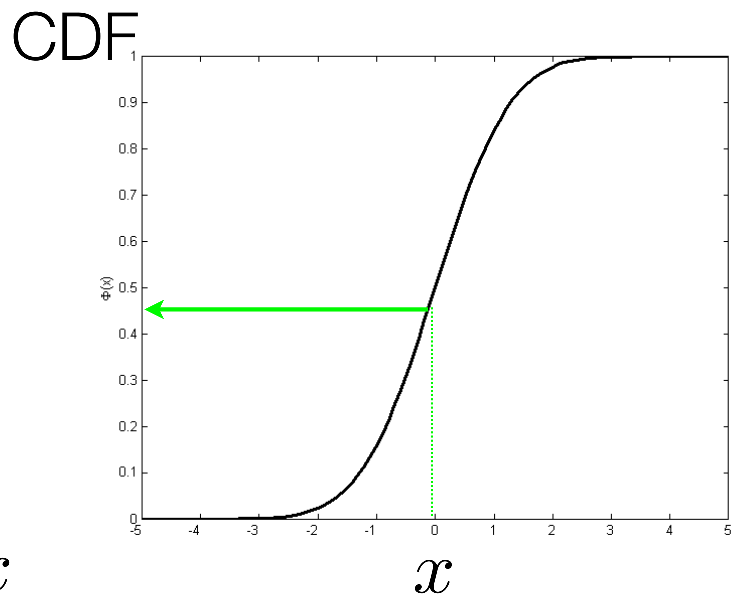


PDF and CDF

- The integration of PDF to a certain level is the CDF value of the random variable at that level.

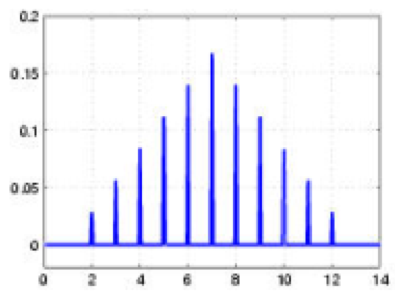
$$f_X(x) = \frac{dF_X(x)}{dx}$$





Probability Mass Function

- Distribution random variables can only take on certain discrete values.
- Therefore their distribution will not be continuous.
- In stead of PDF, their distributions are described via a PMF.



discrete random variables

$$\Pr(X \leq a) = \sum_{\forall x \leq a} f_X(x)$$

Survival Function

- Consider the life of a product a random variable T, its distribution is of great interest for reliability engineers.

- The survival function is the function that describe the survivability of a product at time t

$$S(t) = \Pr(\underbrace{T}_{\text{random variable}} > t) = 1 - F_T(t) = \int_t^{\infty} f_X(x)dx$$

- This survival function is also called reliability function.

- Survival function can be analogous to the compliment of CDF.

$$S(x) = \Pr(X > x) = 1 - F_X(x) = \int_x^{\infty} f_X(x)dx$$

Hazard Function

Hazard function, also known as hazard rate, is the instantaneous failure rate function.

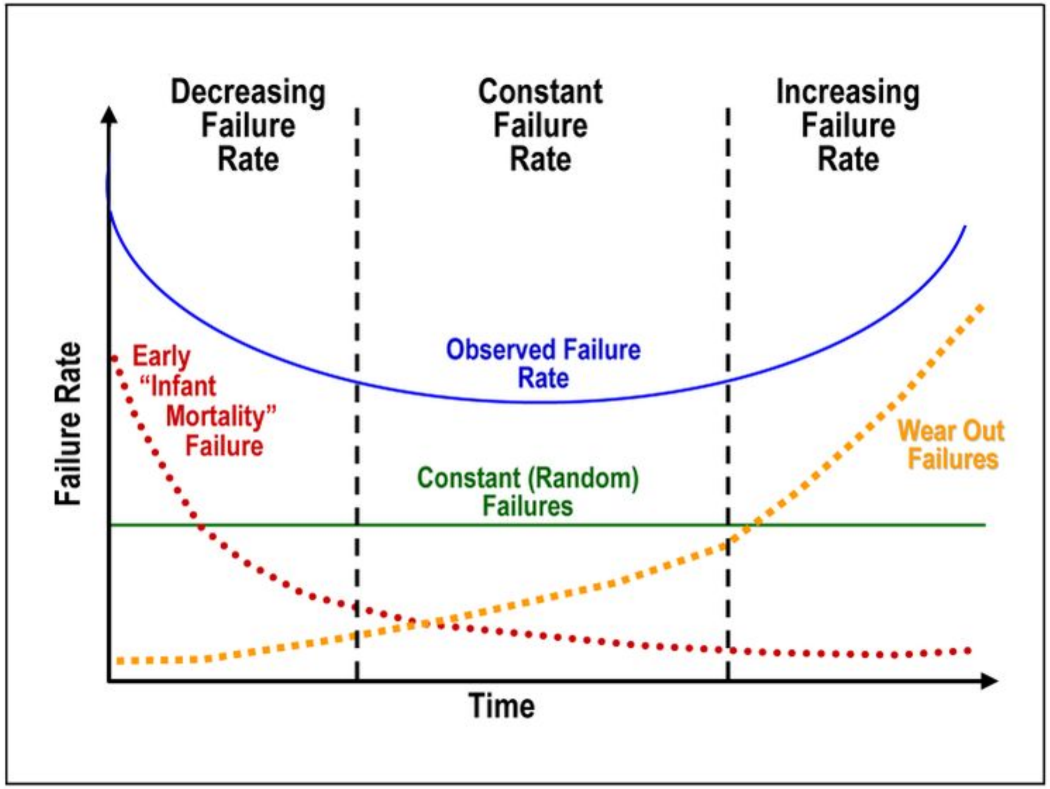
$$h(t) = \lim_{\Delta \to 0} \frac{\Pr(t < T \leq t + \Delta t)|T > t}{\Delta t} = \frac{f_T(t)}{1 - F_T(t)}$$

$$\Big\downarrow \text{ analogous}$$

$$h(x) = \frac{f_X(x)}{1 - F_X(x)}$$

Bathtub Curve

$h(t)$



t

Example

A fleet consists 3 different vehicles. Each of these three vehicles is equally likely to be operating or non-operating after 15 years. Let X be a random variable whose values represent the number of operating vehicles in this fleet after 15 years.

- Show the PMF of X.
- Show the CDF of X.

Example

- The life of a machine is distributed with the following PDF

$$f_T(t) = \lambda e^{-\lambda t}$$

  - find the CDF
  - What is the probability the machine can last for at least 3 years if
  - find the survival function
  - find the hazard function

Metrics of a Random Variable

- Mean value, also called expected value, indicate the central tendency of a random variable

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

- Variance indicates how closely or widely the values of the variate are clustered around a central value

$$\mathrm{Var}[X] = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx$$

Standard Deviation and Coefficient of Variation

- Standard Deviation

$$\sigma_X = \sqrt{\mathrm{Var}[X]}$$

- Coefficient of Variation

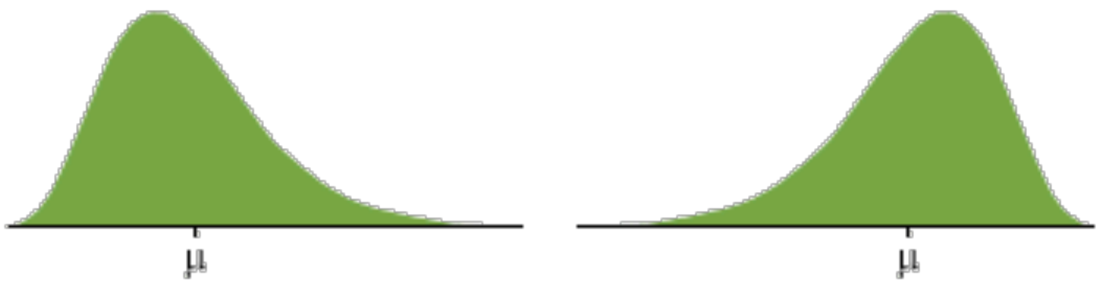
$$\delta_X = \frac{\sigma_X}{\mu_X}$$

Skewness

- measure the symmetry or asymmetry of a PDF

- the third central moment

$$E(X-\mu_X)^3 = \int_{-\infty}^{\infty} (x-\mu_X)^3 f_X(x)dx$$

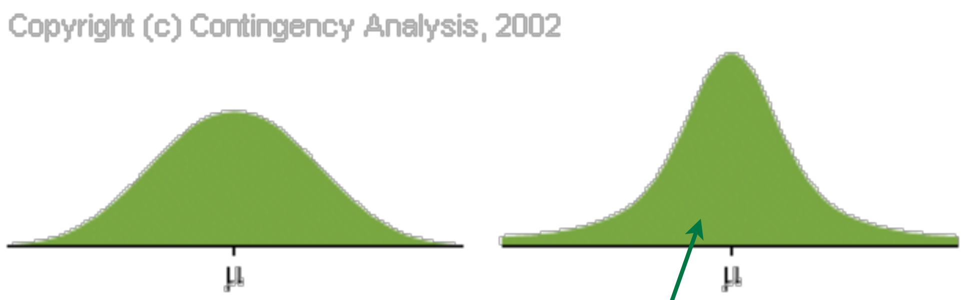


Positive Skewed   Negative Skewed

Kurtosis

- measure the peakedness of a PDF
- the fourth central moment of a random variable

$$E(X-\mu_X)^4 = \int_{-\infty}^{\infty} (x-\mu_X)^4 f_X(x)dx$$

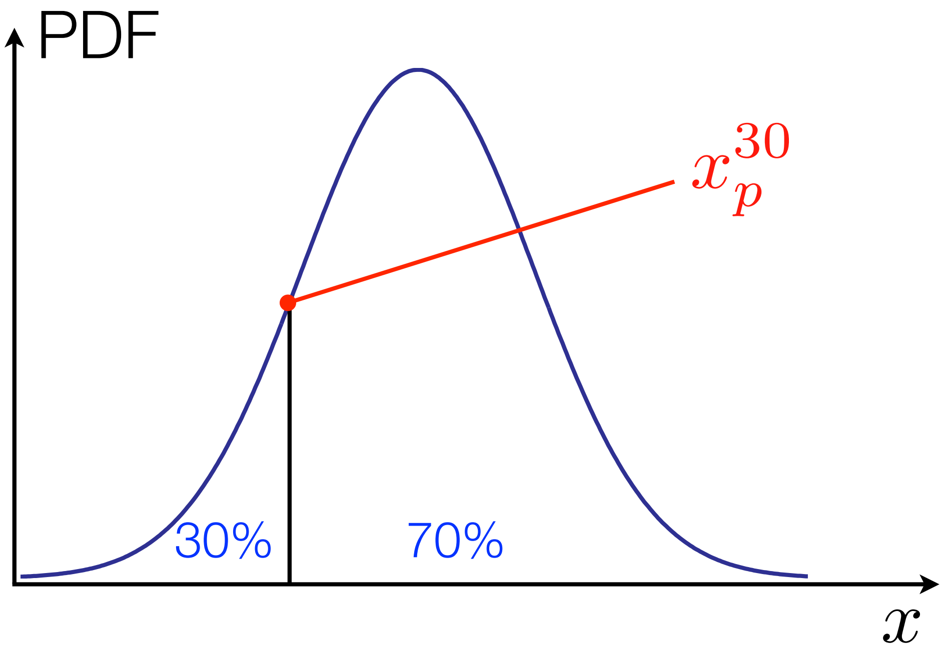


has higher kurtosis value,
more peaked at the center
and has flatter tails

Percentile

- The probability that the actual value will be less than the designed value is expressed as a percentage

$$x_p^n : \{F_X(x_p^n) = n\%\}$$



$$x_p^n = \boxed{F_X^{-1}}(n\%)$$

inverse CDF

Median (50th percentile) :

$$x_p^{50} : \{F_X(x_p^{50}) = 50\%\}$$

Compare Median and Mean

- Housing cost in Taipei were recorded by a realtor. The corresponding market prices are listed as below.

總價(萬)	單價(萬/坪)
7,500	33.9
6,996	98.1
6,068	94
5,700	88.4
4,850	106.1
4,800	94.8
4,780	131
3,960	76
3,730	97.8
Mean=5376	Mean=91
Median=4850	Median=94.8

Let's add an outlier

總價(萬)	單價(萬/坪)
100,000	500
Mean=14,838	Mean=132
Median=5,275	Median=96

Median values are less affected by heavy tails

Fitting Data to a Distribution