Linear Algebra and its Applications

HW#10

1. Find the eigenvalues of A, B, AB, and BA:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Are eigenvalues of AB equal to eigenvalues of A times eigenvalues of B? Are eigenvalues of AB equal to eigenvalues of BA?

2. If each number is the average of the two previous numbers, $G_{k+2} = \frac{1}{2}(G_{k+1} + G_k)$, set up the matrix $\boldsymbol{A}$ and diagonalize it. Starting from $G_0 = 0$ and $G_1 = \frac{1}{2}$, find a formula for G_k and compute its limit as $k \to \infty$.

3. Multinational companies in the U.S., Japan, and Europe have assets of \$4 trillion. At the start, \$2 trillion are in the U.S. and \$2 trillion in Europe. Each year 1/2 the U.S. money stays home, 1/4 goes to both Japan and Europe. For Japan and Europe, 1/2 stays home and 1/2 is sent to the U.S.

(a) Find the matrix that gives

$$\begin{bmatrix} US \\ J \\ E \end{bmatrix}_{yeark+1} = A \begin{bmatrix} US \\ J \\ E \end{bmatrix}_{yeark+1}$$

(b) Find the eigenvalues and eigenvectors of A.

(c) Find the limiting distribution of the \$4 trillion as the world ends.

(d) Find the distribution at year k.

4. A diagonal matrix Λ satisfies the usual rule $e^{\Lambda(t+s)} = e^{\Lambda t} e^{\Lambda s}$, because the rule holds for each diagonal entry.

(a) Explain why $e^{A(t+s)} = e^{At} e^{As}$, using the formula $e^{At} = Se^{\Lambda t}S^{-1}$.

(b) Show that $e^{A+B} = e^A e^B$ is *not true* for matrices, from the example

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$ (use the series for e^A and e^B)

5. (a) Use the series formula for e^{At} to show that $\frac{d}{dt}\left(e^{At}\right) = Ae^{At}$.

(b) Given (a), show that $u(t) = e^{At}u_0$ is the solution of $\frac{du}{dt} = Au, u = u_0$ at $t = 0$.

6. A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \text{ and } \frac{dw}{dt} = v - w$$

The total $v + w$ is constant (40 people).

(a) Find the matrix in $\frac{du}{dt} = Au$, and its eigenvalues and eigenvectors.

(b) What are v and w at $t = 1$?

(c) what are v and w as t approaches infinity?

(d) Reverse the diffusion of people to $du/dt = -Au$:

$$\frac{dv}{dt} = v - w \text{ and } \frac{dw}{dt} = w - v$$

The total $v + w$ still remains constant. How are the λ's changed now that A is changed to $-A$? What is v as t approaches infinity?