Linear Algebra Term Project —The Fast Fourier Transform and its applications

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1 Rationale

The Fast Fourier Transform (FFT) stands as a pivotal advancement in contemporary science and technology, presenting an algorithm that efficiently computes the Discrete Fourier Transform (DFT). By substantially reducing computational complexity from O(N²) to O(N log), it becomes a cornerstone for diverse applications.

In the realm of mechanics, the FFT plays a pivotal role in solving Partial Differential Equations (PDEs) governing dynamic system behavior, offering a numerical approach. Its significance extends to signal analysis, seamlessly transforming signals from the time domain to the frequency domain. This transformation facilitates the identification of both noises and main frequencies, enhancing analytical precision.

Beyond its fundamental applications, the FFT proves indispensable in various fields, including Audio Compression, Image Compression, and Communication. These applications underscore its critical role in contemporary technological landscapes.

Engaging in a project involving the FFT presents a unique opportunity to grasp its inner workings through a linear algebra perspective. This endeavor promises to deepen understanding and proficiency in leveraging FFT for advanced applications in diverse domains.

2 Problem background

2.1 Fourier Series

2.1.1 Hilbert Space

One perspective for comprehending Fourier Series involves commencing with Hilbert Space. Hilbert Space is an infinite-dimensional space with vectors of finite length, exemplified by $\mathbf{v} = [1, 1/2, 1/3, \ldots]$. For vectors $\mathbf{v}$ and $\mathbf{w}$ within Hilbert Space:

$$|\langle \mathbf{v}, \mathbf{w} \rangle| \leq \|\mathbf{v}\|\|\mathbf{w}\|$$

$$|\langle \mathbf{v}, \mathbf{w} \rangle| = 0, \text{ if } \mathbf{v} \perp \mathbf{w}$$

In a finite domain ($-\pi \leq x \leq \pi$), where $f(x) = \sin x$ has continuous, infinite points, the square length of $f(x) = \sin x$ is given by:

$$\|\sin x\|^2 = \int_{-\pi}^{\pi} (\sin x)^2 \, dx = \pi$$

Thus, $\sin x$ $(-\pi \leq x \leq \pi)$ possesses infinite dimensions and finite length. The function space is Hilbert Space, as is the case with other functions $(\cos x, \sin 2x, \cos 2x, \ldots, \sin kx, \cos ekx, k \to \infty)$. Now, examine the inner product of two functions $f(x) = \sin x$ and $g(x) = \cos x$ in $(-\pi \leq x \leq \pi)$:

$$(\sin x, \cos x) = \int_{-\pi}^{\pi} \sin x \cdot \cos x \, dx = 0$$

This illustrates that the vectors of the two functions are perpendicular. This orthogonality extends to other sine and cosine functions $(\sin x, \cos x, \sin 2x, \cos 2x, \ldots, \sin kx, \cos kx, k \to \infty)$.

2.1.2 Fourier Series

Consider a function $y(x)$ defined on the domain $(-\pi \leq x \leq \pi)$ or a periodic function with a period of 2π. We can project this function $y(x)$ onto the coordinate space formed by the sine and cosine functions $(\sin x, \cos x, \sin 2x, \cos 2x, \ldots, \sin kx, \cos kx, k \to \infty)$ in an infinite-dimensional space. The projection formula is given by:

$$\hat{x} = \frac{\langle \mathbf{b}, \mathbf{y} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle} \mathbf{a}$$

where $\mathbf{y}(x) = a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \ldots + a_k \cos(kx) + b_k \sin(kx)$ with $k \to \infty$, and the coefficients are determined by:

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) \, dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \cos(nx) \, dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \sin(nx) \, dx \end{aligned}$$

2.2 Fourier Transform

2.2.1 promote the period 2π to arbitrary $p = 2L$

Consider a periodic function $f(x)$ with a period $p = 2L$. Introduce a new variable v, and let f be a function of v, such that $x = \frac{p}{2\pi} v$ and $v = \frac{2\pi}{p} x = \frac{\pi}{L} x$. The corresponding transformations are:

$$\begin{cases} v = \pi \Rightarrow x = L \\ v = -\pi \Rightarrow x = -L \end{cases}$$

This transformation can be expressed as:

$$f(x) = f\left(\frac{L}{\pi} v\right) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nv) + b_n \sin(nv)]$$

which can be further simplified to:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(\frac{n\pi}{L}x) + b_n \sin(\frac{n\pi}{L}x))$$

The coefficients are given by:

$$\begin{cases} a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\, dv \\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\cos(nv)\, dv \\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\sin(nv)\, dv \end{cases}$$

This is equivalent to:

$$\begin{cases} a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx \\ a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx \\ b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx \end{cases}$$

2.2.2 when $L \to \infty$

Define $w_n = \frac{n\pi}{L}$ and $\Delta w = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$. Let $f_L(x)$ with $p = 2L$ be expressed as:

$$f_L(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos w_n x + b_n \sin w_n x), \quad w_n = \frac{n\pi}{L}$$

When $f(x)$ is not a periodic function, we can represent f as $f_L(x)$ with $L \to \infty$ and $\Delta w = \frac{\pi}{L} \to 0$:

$$f(x) = \lim_{L\to\infty} f_L(x)$$

Then the sigma Σ becomes the integral $\int$:

$$f(x) = \int_0^{\infty} [A(w)\cos wx + B(w)\sin wx]\, dw$$

where

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\, dv$$

2.2.3 Fourier Transform and inverse Fourier Transform

Putting A and B into the above formula, we obtain:

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(v)[\cos(wv)\cos(wx) + \sin(wv)\sin(wx)]\, dv\, dw$$

Since $\int_{-\infty}^{\infty} f(v)[\cos(wv)\cos(wx)+\sin(wv)\sin(wx)]\,dv$ can be written as $\left[\int_{-\infty}^{\infty} f(v)\cos(wx-wv)\,dv\right]$, and the part in

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is an even function. $f(x)$ will change to:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\cos(wx-wv)\,dv\right]dw$$

If we change cos into sin in the above formula, we get:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\sin(wx-wv)\,dv\right]dw = 0$$

Multiplying the above formula by i and combining the two formulas, we get:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(v)\cos(wx-wv) + if(v)\sin(wx-wv)\,dv\right]dw$$

Using Euler's formula $e^{ix} = \cos x + i\sin x$, we form the complex Fourier integral:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-iwv}\,dv\right]e^{iwx}\,dw$$

The part inside the

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is a function of w, $f(w)$, called the Fourier Transform:

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx = \mathcal{F}(f)$$

The part outside $f(w)$ is the inverse Fourier Transform:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{iwx}\,dw = \mathcal{F}^{-1}(\hat{f})$$

2.3 Discrete Fourier Transform (DFT)

2.3.1 DFT

For all practical purposes, people seldom have a whole function $f(x)$. Most of the time, we only have finitely sampled points of $f(x)$. The Discrete Fourier Transform (DFT) helps people transform a bunch of discrete data from the time domain to the frequency domain. Without a continuous function, we can't perform the integral to obtain the coefficients of different frequencies. Instead, we get N samples from a function f and evaluate the summation from 0 to $N-1$:

$$\hat{f}_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n \cdot e^{-i\frac{2\pi kn}{N}}$$

Separate $e^{-i\frac{2\pi}{N}}$ from the above equation and define it as w. The entire part in the summation will become w^{kn}. So the equation will be like:

$$\hat{f}_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n \cdot w^{kn}, \quad w = e^{-i\frac{2\pi}{N}}$$

Turn this equation into the vector expression will be $\hat{f}_{N\times 1} = F_{N\times N} f_{N\times 1}$:

$$\begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{k=N} \end{bmatrix} = \begin{bmatrix} w^0 & w^0 & w^0 & \cdots & w^0 \\ w^0 & w^1 & w^2 & \cdots & w^{N-1} \\ w^0 & w^2 & w^4 & \cdots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w^0 & w^{N-1} & w^{2(N-1)} & \cdots & w^{(N-1)^2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n=N} \end{bmatrix}$$

The $F_{N\times N}$ is called the Fourier matrix F_N.

For example 1: If there is a function f representing a dynamic system, we take the 4 sample points f_n each second. Thus, each couple of sample $n \to n+1$ will have the time interval $\Delta t = 1$ (unit: second). And the $\hat{f}_k$ will represent the magnitude ($\|\hat{f}_k\|_N$) and phase of each frequency F_k, where $k\frac{N}{\Delta t} \Rightarrow F_k$.

For example 2: Set $N = 4$, the sample values are $f = [0, 1, 4, 9]^T$. The $w = e^{-i\frac{2\pi}{4}} = -i$, and the $\hat{f}$ will become:

$$\hat{f} = \begin{bmatrix} -i & -i & -i & -i \\ -i & 1 & i & -1 \\ -i & -1 & -i & 1 \\ -i & i & 1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ -4+8i \\ -6 \\ -4-8i \end{bmatrix}$$

In this example, we can check the orthogonality of each column in the Fourier matrix $F_{4\times 4}$ or F_4. Pick two arbitrary columns and calculate their inner product, just like below:

$$F_4^H F_4 = \begin{bmatrix} 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = 0$$

The result shows that each column of the Fourier matrix F_4 is orthogonal to each other. And the length of each column is $\sqrt{F_4^H F_4 n} = 2$. It is equal to $\sqrt{N}$, where $N = 4$. This feature can be expanded to any arbitrary Fourier matrix F_N. Thus, if we want to find the inverse of the Fourier matrix to do the inverse DFT, we will discover that $F_N^H F_N = NI$. The inverse of the Fourier matrix is just $\frac{1}{N}F_N^H = \frac{1}{N}\overline{F_N}$.

2.3.2 Difficulty

If the sampling number N is too small, the equation above may lead to the Aliasing Effect, which means signals at different frequencies have been stacked together. Thus, the sampling number N is typically chosen to be a large value in applications. However, this introduces a significant challenge. Each $\hat{f}_n$ in $\hat{f}$ requires $O(N)$ operations, and the entire $\hat{f}$ needs $O(N^2)$ operations. For example, when the sampling number N is

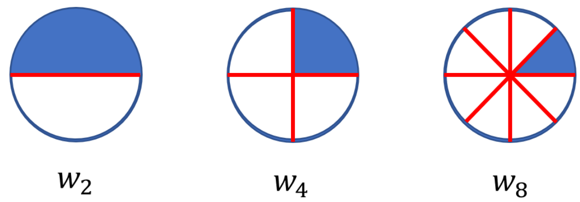

Figure 1: the rotation of different w

1000, the operations will scale up to 10^6. The larger the sampling number, the more computationally expensive the equation becomes.

2.4 Solutions with linear algebra theories and techniques

2.4.1 Thinking w_N as an angle rotation

We can conceptualize w_N as a rotation with an angle changing $2\pi/N$ in a unit circle. The power of w_N represents the number of times the angle changes. For instance, $w_{\frac{3}{4}}$ changes the angle 3 times, resulting in a rotation of $3 \cdot \frac{2\pi}{4}$. On the other hand, $w_{\frac{0}{4}}$ changes the angle 0 times, leading to zero rotation.

Upon examining the Fourier matrices with sampling numbers N and $M = \frac{N}{2}$, we observe connections between these two matrices. Specifically, $(w_N)^2 = w_M$ means the angle changing of w_M is twice that of w_N. Consequently, the Fourier matrix F_N can be expressed as a combination of w_N and w_M in each element. The elements that undergo angle changes $2\pi/N$ an even number of times can be written as powers of w_M. Conversely, the elements that undergo angle changes $2\pi/N$ an odd number of times will be expressed as powers of w_M multiplied by $c \cdot w_N$ to compensate for some angle $2\pi/N$.

2.4.2 FFT

What FFT is doing is called "Divide and Conquer." Suppose the sampling number N is even, and $M = N/2$. We divide f into two parts, fev and fod, and define:

$$f\hat{ev} = FM \cdot fev, \quad fev = \begin{bmatrix} f_0 & f_2 & f_4 & \dots & f_{N-4} & f_{N-2} \end{bmatrix}^T$$

$$f\hat{od} = FM \cdot fod, \quad fod = \begin{bmatrix} f_1 & f_3 & f_5 & \dots & f_{N-3} & f_{N-1} \end{bmatrix}^T$$

Assuming each $\hat{f}_k$ in $\hat{f}$ is a combination of $f\hat{ev}_k$ in $f\hat{ev}$ and $f\hat{od}_k$ in $f\hat{od}$, we start with the basic equation:

$$\hat{f} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \\ \dots \\ \hat{f}_{k=N-1} \end{bmatrix}, \quad \hat{f}_k = \frac{1}{N} \sum_{n=0}^{N-1} w_N^{nk} \cdot f_n$$

Dividing the summation into two parts:

$$\hat{f}_k = \left(\sum_{n=0}^{M-1} w_N^{2nk} \cdot f_{2n} \right) + \left(\sum_{n=0}^{M-1} w_N^{(2n+1)k} \cdot f_{2n+1} \right)$$

Since $w_M = w_N^{2n}$, we can rewrite the equation:

$$\hat{f}_k = \sum_{n=0}^{M-1} w_M^{nk} \cdot f_{2n} + w_N^k \cdot \sum_{n=0}^{M-1} w_N^{nk} \cdot f_{2n+1}$$

To express this in matrix form, let $k = 0, 1, 2, \ldots, M-1$, and consider the matrix multiplication:

$$\hat{f}_{k=0|M-1} = \begin{bmatrix} I & D \end{bmatrix} \begin{bmatrix} FM & 0 \\ 0 & FM \end{bmatrix} \begin{bmatrix} fev \\ fod \end{bmatrix}$$

Here, $[D]$ is a diagonal matrix:

$$[D] = \begin{bmatrix} w_N^0 & w_N^1 & w_N^2 & \ldots & w_N^{M-1} \end{bmatrix}$$

Next, consider $\hat{f}_{k=M|2M-1}$ using $\hat{f}'_k = \hat{f}_{k+M}$:

$$\hat{f}'_k = \begin{bmatrix} I & -D \end{bmatrix} \begin{bmatrix} FM & 0 \\ 0 & FM \end{bmatrix} \begin{bmatrix} fev \\ fod \end{bmatrix}$$

Now, combine $\hat{f}_{k=0|M-1}$ and $\hat{f}'_k$ to form $\hat{f}$:

$$\hat{f} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} FM & 0 \\ 0 & FM \end{bmatrix} \begin{bmatrix} fev \\ fod \end{bmatrix}$$

The sampling points vector, which has even-numbered elements above and odd-numbered elements below, can be expressed as a permutation matrix $[P]$ multiplying the vector f:

$$\begin{bmatrix} fev \\ fod \end{bmatrix} = [P]f = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix} f$$

Recalling the original DFT equation, we have the expansion of F_N:

$$\hat{f} = F_N f = \begin{bmatrix} I & DN \\ I & -DN \end{bmatrix} \begin{bmatrix} FM & 0 \\ 0 & FM \end{bmatrix} \begin{bmatrix} PN \end{bmatrix} f$$

By dividing F_N into FM with some multiplication, we can reduce the operations on a computer from $O(N^2)$ to $O^2(N/2)^2 + N/2$. Also, FM can be divided into $FM/2$ with some multiplication by using the same manipulation above. Finally, the operation times of DFT were reduced to $O(N \log^2 N)$. The entire algorithm, including dividing, calculating, and combining the Fourier matrix, is called the Fast Fourier Transform (FFT).

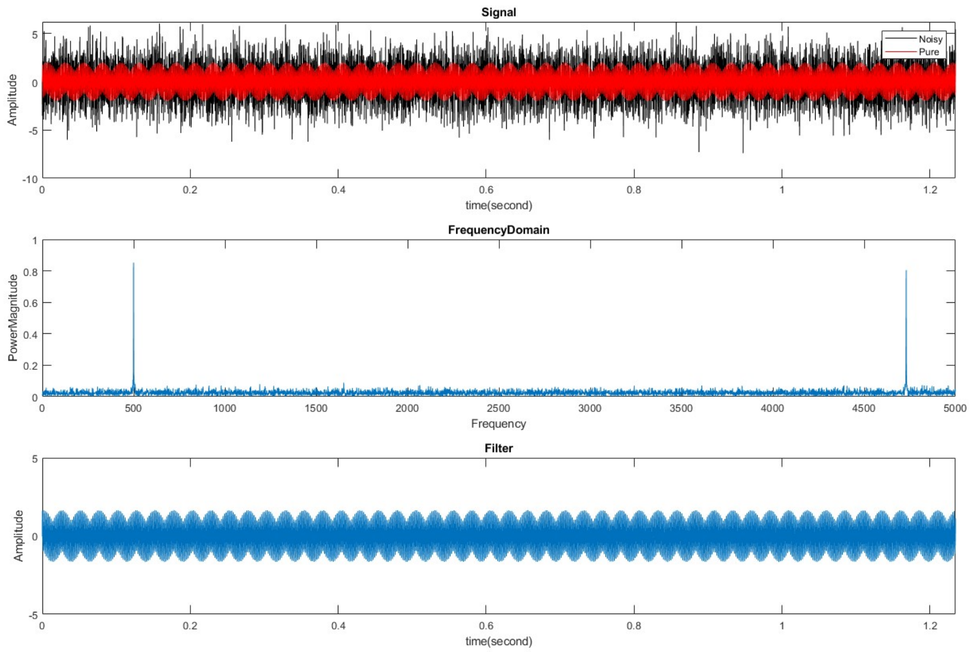

Figure 2: FFT denoising signal

2.5 Example (Denoise signal)

Suppose there is an input signal, a combination of a 500 Hz and 4730 Hz sinusoidal waveform with random noise. The original signal appears chaotic, making it challenging for the receiver to discern the underlying information. Leveraging the FFT algorithm, we can extract specific frequencies from the original signal in the frequency domain. Once we identify the primary frequencies within the signal, we can effectively filter out the noisy signal frequencies, retaining only the desired components. Finally, we can transform the filtered signal from the frequency domain back to the time domain using the inverse FFT.

code: FFT project Example.m(https://github.com/Yi-An-Wang/Linear-Algebra-project-example.git)
result: Figure 2: FFT denoising signal

2.6 Discussions

2.6.1 inverse DFT

Recall that the Fourier matrix is a symmetric matrix, with each column having a length of $\sqrt{N}$ and being orthogonal to one another. The inverse of the Fourier matrix with sampling numbers N, denoted as 1/N $F^{-}N$, serves as a key component in the transformation equation that converts ˆf back to f:

$$f = \frac{1}{N} F^{-} N$$

Given the similar characteristics of F¯N and FN, we can further decompose ˆf into two parts, namely odd and even, and perform operations analogous to those employed in the FFT algorithm:

$$f = \frac{1}{N} \begin{bmatrix} I & D^{-} \\ I & -D^{-} \end{bmatrix} \begin{bmatrix} FM[0] & 0 \\ 0 & FM \end{bmatrix} \begin{bmatrix} {}^{f}ev \\ {}^{f}od \end{bmatrix}$$

2.6.2 remaining and conclusion

When the sampling number is not a power of 2, practitioners often employ techniques such as Zero Padding or Truncation on the signal sets. Zero Padding involves adding zero terms to the signal set, effectively aligning the sampling points to a power of two. Conversely, Truncation trims the signal set to achieve the same goal. However, both methods introduce data distortion. Consequently, various variants of the FFT algorithm have been developed to address this issue. In summary, FFT stands as a powerful and efficient algorithm with wide-ranging applications. It plays a critical role in signal processing, image processing, communication, and various engineering computations.

2.7 References

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