

Linear Algebra Term Project —The Fast Fourier Transform and its applications

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1 Rationale

The Fast Fourier Transform (FFT) is one of the most important breakthrough in contemporary science and technology. It is an algorithm that computes the Discrete Fourier Transform (DFT) really efficiently, by reducing the computational complexity from $O(N^2)$ to $O(N)\log_2$.

In mechanical field, FFT help solving the Partial differential equation (PDE) of such a dynamic system behavior in numerical way. When it comes to signal analysis, the Fast Fourier Transform can powerfully change the signal from the time domain into the frequency domain. This help us easier to identify the noises and the main frequencies. Other applications of FFT such as Audio Compression, Image Compression and Communication are also critical in nowadays.

I consider the project is a great opportunity for me to realize how FFT work in the linear algebra thinking.

2 Problem background

2.1 Fourier Series

2.1.1 Hilbert Space

One of the perspective to understand the Fourier Series is starting with the **Hilbert Space**. Hilbert Space is a infinite Dimensions space with finite length vectors \mathbf{v} . example: $\mathbf{v} = [1, 1/2, 1/3, \dots]$ If \mathbf{v}, \mathbf{w} are in Hilbert Space:

$$\begin{aligned} |\mathbf{v}^T \mathbf{w}| &\leq \|\mathbf{v}\| \|\mathbf{w}\| \\ |\mathbf{v}^T \mathbf{w}| &= 0, \quad \text{if } \mathbf{v} \perp \mathbf{w} \end{aligned}$$

In a finite domain $(-\pi \leq x \leq \pi)$, $f(x) = \sin x$ has continuous, infinite point. The length square of $f(x) = \sin x$ is:

$$\|f(x)\|^2 = \int_{-\pi}^{\pi} (\sin x)^2 dx = \pi$$

Thus, $\sin x$ $(-\pi \leq x \leq \pi)$ has infinite dimension and finite length. The function space is **Hilbert Space**, and so do other function $(\cos x, \sin 2x, \cos 2x, \dots, \sin kx, \cos kx, \quad k \rightarrow$

∞). Now, check the inner product of two functions $f(x) = \sin x$ and $g(x) = \cos x$ in $(-\pi \leq x \leq \pi)$:

$$(f, g) = \int_{-\pi}^{\pi} \sin x * \cos x dx = 0$$

This shows that the vectors of the two functions are perpendicular to each other. And this orthogonality can be promoted to other sin,cos functions:

$(\sin x, \cos x, \sin 2x, \cos 2x, \dots, \sin kx, \cos kx, \quad k \rightarrow \infty)$.

2.1.2 Fourier Series

Consider a function $y(x)$ has domain $(-\pi \leq x \leq \pi)$, or a periodic function whose period is 2π . We can project this function $y(x)$ to the coordinate formed by the sin,cos function $(\sin x, \cos x, \sin 2x, \cos 2x, \dots, \sin kx, \cos kx, \quad k \rightarrow \infty)$ in infinity dimensions space. Use the projection formula $\hat{\mathbf{x}} = \frac{\mathbf{b}^T \mathbf{a}}{\mathbf{a}^T \mathbf{a}}$:

$$\begin{aligned} y(x) &= a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \dots + a_k \cos(kx) + b_k \sin(x) \\ &, \quad k \rightarrow \infty \\ &= a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \end{aligned}$$

where

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \cos(nx) dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \sin(nx) dx \end{aligned}$$

2.2 Fourier Transform

2.2.1 promote the period 2π to arbitrary $p = 2L$

Consider a periodic function $f(x)$ has a period $p = 2L$. Introduce a new variable v , and let f be a function of v .

$$x = \frac{p}{2\pi} v, \quad v = \frac{2\pi}{p} x = \frac{\pi}{L} x \quad \Rightarrow \quad \begin{cases} v = \pi \longrightarrow x = L \\ v = -\pi \longrightarrow x = -L \end{cases}$$

the changing can be written as:

$$\begin{aligned} f(x) &= f\left(\frac{L}{\pi} v\right) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nv) + b_n \sin(nv)] \\ &\Downarrow \\ f(x) &= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \end{aligned}$$

the coefficients are:

$$\begin{cases} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)dv \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\cos(nv)dv \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\sin(nv)dv \end{cases} \Rightarrow \begin{cases} a_0 &= \frac{1}{2L} \int_{-L}^L f(x)dx \\ a_n &= \frac{1}{L} \int_{-L}^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ b_n &= \frac{1}{L} \int_{-L}^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{cases}$$

2.2.2 when $L \rightarrow \infty$

Define $w_n = \frac{n\pi}{L}$, and $\Delta w = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$ with $p = 2L$ become:

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x), \quad w_n = \frac{n\pi}{L}$$

When $f(x)$ is not a periodic function, we can write f as $f_L(x)$ with $L \rightarrow \infty$ and $\Delta w = \frac{\pi}{L} \rightarrow 0$:

$$f(x) = \lim_{L \rightarrow \infty} f_L(x)$$

Then the sigma Σ will become the integral \int :

$$\begin{aligned} f(x) &= \int_0^{\infty} [A(w) \cos wx + B(w) \sin wx] dw \\ A(w) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv dv, \\ B(w) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv dv \end{aligned}$$

2.2.3 Fourier Transform and inverse Fourier Transform

Put A and B into the above formula, we can get:

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(v) [\cos wv \cos wx + \sin wv \sin wx] dv dw$$

Since $\int_{-\infty}^{\infty} f(v) [\cos wv \cos wx + \sin wv \sin wx] dv$ can be write as $\left[\int_{-\infty}^{\infty} f(v) \cos(wx - wv) dv \right]$. and the part in [...] is an even function. The $f(x)$ will change into:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) \cos(wx - wv) dv \right] dw$$

If we change \cos into \sin in the above formula, we will get:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) \sin(wx - wv) dv \right] dw = 0$$

Multiply the above formula i and combine the two formula:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(v) \cos(wx - wv) + i f(v) \sin(wx - wv) dv \right] dw$$

Use the Euler formula $e^{ix} = \cos x + i \sin x$, form the complex Fourier integral:

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v) e^{iw(x-v)} dv dw \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-i w v} dv \right] e^{i w x} dw \end{aligned}$$

The part inside the [...] is a function of w , $f(w)$ called the **Fourier Transform**:

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i w x} dx = \mathcal{F}(f)$$

The part outside $f(w)$ is the **inverse Fourier Transform**:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{i w x} dw = \mathcal{F}^{-1}(\hat{f})$$

2.3 Discrete Fourier Transform (DFT)

2.3.1 DFT

For all practical purposes, people seldom get a whole function $f(x)$. Most of the time, we only have finite sampled points of $f(x)$. The Discrete Fourier Transform (DFT) help people transform a bounch of Discrete data from time domain to frequency domain. Without a continious function, we can't do the integral to get the coefficients of different frequencies. Instead, we get N samples from a funtion f , and evaluate the summation from 0 to $N - 1$:

$$\hat{f}_k = \sum_{n=0}^{N-1} f_n \cdot e^{-\frac{i 2\pi k n}{N}}$$

Separate the $e^{-\frac{i 2\pi k n}{N}}$ from the above equation and defined it w . The antire part in the summation will become w^{kn} . So the equation will be like :

$$\hat{f}_k = \sum_{n=0}^{N-1} f_n \cdot w^{kn}, \quad w = e^{-\frac{i 2\pi}{N}}$$

Turn this equation into the vector expression will be $\hat{\mathbf{f}}_{N \times 1} = \mathbf{F}_{N \times N} \mathbf{f}_{N \times 1}$:

$$\begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{k=N} \end{bmatrix}_{N \times 1} = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 & \dots & w^0 \\ w^0 & w^1 & w^2 & w^3 & \dots & w^{N-1} \\ w^0 & w^2 & w^4 & w^6 & \dots & w^{2(N-1)} \\ w^0 & w^3 & w^6 & w^9 & \dots & w^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ w^0 & w^{N-1} & w^{2(N-1)} & w^{3(N-1)} & \dots & w^{(N-1)^2} \end{bmatrix}_{N \times N} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n=N} \end{bmatrix}_{N \times 1}$$

The $\mathbf{F}_{N \times N}$ is called the Fourier matrix \mathbf{F}_N

For example 1: If there is function f representing a dynamic system. We take the

samples points f_n each seconds. Thus, each couple of sample $n \rightarrow n + 1$ will have the time interval $\Delta t = 1$ (unit: second). And the \hat{f}_k will represent the magnitude ($\frac{\|\hat{f}_k\|}{N}$) and phase of each frequency F_k , where $\frac{k}{N \cdot \Delta t} \Rightarrow F_k$.

For example 2: Set $N = 4$, the sample values are $\mathbf{f} = [0 \ 1 \ 4 \ 9]^T$. The $w = e^{-i2\pi/4} = -i$ and the \hat{f} will become :

$$\hat{\mathbf{f}} = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} \mathbf{f} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ -4 + 8i \\ -6 \\ -4 - 8i \end{bmatrix}$$

In this example, we can check the orthogonality of each column in Fourier matrix $\mathbf{F}_{4 \times 4}$ or \mathbf{F}_4 . Pick two arbitrary columns and calculate their inner product, just like below:

$$\mathbf{F}_{4_2}^H \mathbf{F}_{4_4} = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}^H \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = [1 \ i \ -1 \ -i] \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = 0$$

The result shows that each columns of the Fourier matrix \mathbf{F}_4 are orthogonal to each other. And the length of each columns is $\sqrt{\mathbf{F}_{4_n}^H \mathbf{F}_{4_n}} = 2$. It is equal to \sqrt{N} , $N = 4$. This feature can be expanded to any arbitrary Fourier matrix \mathbf{F}_N . Thus, if we want to find the inverse of Fourier matrix to do the inverse DFT, we will discover that $\mathbf{F}_N^H \mathbf{F}_N = N\mathbf{I}$. The inverse of Fourier matrix is just $\frac{1}{N} \mathbf{F}_N^H = \frac{1}{N} \bar{\mathbf{F}}_N$.

2.3.2 Difficulty

If sampling number N is too small, the equation above will lead to the Aliasing Effect, which mean the signal in different frequencies have been stacked together. Thus, the sampling number N is usually a huge value in applications. This led to a difficult issue. Each \hat{f}_n in $\hat{\mathbf{f}}$ needs $O(N)$ operations. The hole $\hat{\mathbf{f}}$ need $O(N^2)$ operations. For instance, when sampling numbers N is 1000, the operations will up to 10^6 . The large number a sampling number is, the more expensive the equation will be.

2.4 Solutions with linear algebra theories and techniques

2.4.1 Thinking w_N as an angle rotation

we can think w_N as a rotation with angle changing $2\pi/N$ in a unit circle. The power of w_N represents the times of angle changing. For example, w_4^3 changes the angle 3 times, the rotation is $3 \cdot \frac{2\pi}{4}$. While w_4^0 changes angle 0 times, so the rotation is zero. If we check the Fourier matrices with N and $M = N/2$ sampling numbers, we will discover there are some connections between those two matrix. First, $(w_N)^2 = w_M$ means the angle changing of w_M is twice of the changing of w_N .

Thus, the Fourier matrix \mathbf{F}_N can be written as the combination of w_N and w_M in each

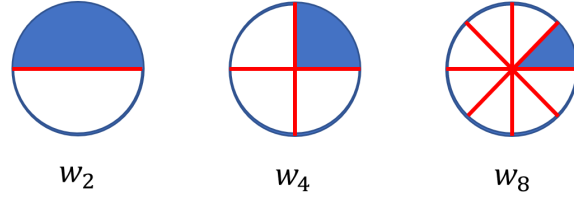


Figure 1: the rotation of different w

element. The elements which do the angle changing $2\pi/N$ even times can be written as the power of w_M . The elements which do the angle changing $2\pi/N$ odd times will be written as the power of w_M and multiply $c \cdot w_N$ to compensating some angle $2\pi/N$.

2.4.2 FFT

What FFT is doing is called "Divide and Conquer". Suppose the sampling number N is even and $M = N/2$. We divide \mathbf{f} into two parts, \mathbf{f}_{ev} and \mathbf{f}_{od} , and make:

$$\begin{aligned}\hat{\mathbf{f}}_{ev} &= \mathbf{F}_M \mathbf{f}_{ev}, \quad \mathbf{f}_{ev} = [f_0 \quad f_2 \quad f_4 \quad \cdots \quad f_{N-4} \quad f_{N-2}]^T \\ \hat{\mathbf{f}}_{od} &= \mathbf{F}_M \mathbf{f}_{od}, \quad \mathbf{f}_{od} = [f_1 \quad f_3 \quad f_5 \quad \cdots \quad f_{N-3} \quad f_{N-1}]^T\end{aligned}$$

Assume each \hat{f}_k in $\hat{\mathbf{f}}$ is a combination of $\hat{f}_{ev,k}$ in $\hat{\mathbf{f}}_{ev}$ and $\hat{f}_{od,k}$ in $\hat{\mathbf{f}}_{od}$, and try to find out the coefficients of those elements. We start from the basic equation:

$$\hat{\mathbf{f}} = [\hat{f}_1 \quad \hat{f}_2 \quad \hat{f}_3 \quad \cdots \quad \hat{f}_{k=N-1}]^T, \quad \hat{f}_k = \sum_{n=0}^{N-1} w_N^{nk} \cdot f_n$$

And divide the summation into two parts:

$$\begin{aligned}\hat{f}_k &= (w_N^{0k} \cdot f_0 + w_N^{2k} \cdot f_2 + \cdots + w_N^{N-2k} \cdot f_{N-2}) \\ &\quad + (w_N^{1k} \cdot f_1 + w_N^{3k} \cdot f_3 + \cdots + w_N^{N-1k} \cdot f_{N-1}) \\ &= \sum_{n=0}^{M-1} w_N^{2nk} \cdot f_{2n} + \sum_{n=0}^{M-1} w_N^{(2n+1)k} \cdot f_{2n+1}\end{aligned}$$

Since $w_m = w_n^2$, we can rewrite the equation:

$$\begin{aligned}\hat{f}_k &= \sum_{n=0}^{M-1} w_M^{nk} \cdot f_{2n} + \sum_{n=0}^{M-1} w_N^{nk+k} \cdot f_{2n+1} \\ &= \sum_{n=0}^{M-1} w_M^{nk} \cdot f_{2n} + w_N^k \cdot \sum_{n=0}^{M-1} w_N^{nk} \cdot f_{2n+1}\end{aligned}$$

First, we care about the \hat{f}_k from number 0 to $M - 1$. In order to realize the equation, we can write it in an expansion:

$$\begin{aligned}\hat{f}_k &= \begin{bmatrix} w_M^0 & w_M^{1k} & w_M^{2k} & \dots & w_M^{(M-1)k} \end{bmatrix} \begin{bmatrix} f_0 \\ f_2 \\ f_4 \\ \vdots \\ f_{M-2} \end{bmatrix} \\ &+ w_N^k \cdot \begin{bmatrix} w_M^0 & w_M^{1k} & w_M^{2k} & \dots & w_M^{(M-1)k} \end{bmatrix} \begin{bmatrix} f_1 \\ f_3 \\ f_5 \\ \vdots \\ f_{M-1} \end{bmatrix} \\ &, k = (0, 1, 2, \dots, M - 1)\end{aligned}$$

The equation can be written as three matrix multiplication:

$$\hat{\mathbf{f}}_{k=0|M-1} = \begin{bmatrix} [I] & [D] \end{bmatrix} \begin{bmatrix} \mathbf{F}_M & [0] \\ [0] & \mathbf{F}_M \end{bmatrix} \begin{bmatrix} \mathbf{f}_{ev} \\ \mathbf{f}_{od} \end{bmatrix}$$

where $[D]$ is a diagonal matrix:

$$[D] = \begin{bmatrix} w_N^0 & & & & \\ & w_N^1 & & & \\ & & w_N^2 & & \\ & & & \ddots & \\ & & & & w_N^{M-1} \end{bmatrix}$$

Next, we check the \hat{f}_k from number M to $2M - 1$. We use $\hat{f}_{k'}$ to express those sample points. Here, $k' = k + M$ and $k = (0, 1, 2, \dots, M - 1)$. The expansion can be written as:

$$\begin{aligned}\hat{f}_{k'} &= \begin{bmatrix} w_M^0 & w_M^{1k'} & w_M^{2k'} & \dots & w_M^{(M-1)k'} \end{bmatrix} \begin{bmatrix} f_0 \\ f_2 \\ f_4 \\ \vdots \\ f_{M-2} \end{bmatrix} \\ &+ w_N^{k'} \cdot \begin{bmatrix} w_M^0 & w_M^{1k'} & w_M^{2k'} & \dots & w_M^{(M-1)k'} \end{bmatrix} \begin{bmatrix} f_1 \\ f_3 \\ f_5 \\ \vdots \\ f_{M-1} \end{bmatrix} \\ &\text{where } k' = k + M, k = (0, 1, 2, \dots, M - 1)\end{aligned}$$

Since $w_N^{k'} = w_N^{k+M} = w_N^k \cdot w_N^{N/2}$. We can consider $w_N = e^{-i2\pi/N} \Rightarrow w_N^{N/2} = e^{-i\pi} = -1$. And rewrite the expansion of $\hat{f}_{k'}$ as:

$$\begin{aligned} \hat{f}_{k'} &= \begin{bmatrix} w_M^0 & w_M^{1k'} & w_M^{2k'} & \cdots & w_M^{(M-1)k'} \end{bmatrix} \begin{bmatrix} f_0 \\ f_2 \\ f_4 \\ \vdots \\ f_{M-2} \end{bmatrix} \\ &\quad - w_N^k \cdot \begin{bmatrix} w_M^0 & w_M^{1k'} & w_M^{2k'} & \cdots & w_M^{(M-1)k'} \end{bmatrix} \begin{bmatrix} f_1 \\ f_3 \\ f_5 \\ \vdots \\ f_{M-1} \end{bmatrix} \\ &\text{where } k' = k + M, \quad k = (0, 1, 2, \dots, M-1) \end{aligned}$$

The equation of $\hat{f}_{k'}$ can be written as three matrix multiplication:

$$\hat{\mathbf{f}}_{k'=M|2M-1} = \begin{bmatrix} [I] & [-D] \end{bmatrix} \begin{bmatrix} \mathbf{F}_M & [0] \\ [0] & \mathbf{F}_M \end{bmatrix} \begin{bmatrix} \mathbf{f}_{ev} \\ \mathbf{f}_{od} \end{bmatrix}$$

Now, we can combine $\hat{\mathbf{f}}_{k=0|M-1}$ and $\hat{\mathbf{f}}_{k'=M|2M-1}$ together to form $\hat{\mathbf{f}}$:

$$\hat{\mathbf{f}} = \begin{bmatrix} [I] & [D] \\ [I] & [-D] \end{bmatrix} \begin{bmatrix} \mathbf{F}_M & [0] \\ [0] & \mathbf{F}_M \end{bmatrix} \begin{bmatrix} \mathbf{f}_{ev} \\ \mathbf{f}_{od} \end{bmatrix}$$

The sampling points vector which has the even number elements above and odd number elements below can be express as a permutation matrix $[P]$ multiplying the vector \mathbf{f} :

$$\begin{bmatrix} \mathbf{f}_{ev} \\ \mathbf{f}_{od} \end{bmatrix} = [P] \mathbf{f} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \mathbf{f}$$

Recall the original DFT equation. We have the expansion of \mathbf{F}_N :

$$\begin{aligned} \hat{\mathbf{f}} &= \mathbf{F}_N \mathbf{f} \\ &= \begin{bmatrix} [I] & [D] \\ [I] & [-D] \end{bmatrix} \begin{bmatrix} \mathbf{F}_M & [0] \\ [0] & \mathbf{F}_M \end{bmatrix} [P] \mathbf{f} \\ \Rightarrow \mathbf{F}_N &= \begin{bmatrix} [I] & [D_N] \\ [I] & [-D_N] \end{bmatrix} \begin{bmatrix} \mathbf{F}_M & [0] \\ [0] & \mathbf{F}_M \end{bmatrix} [P_N] \end{aligned}$$

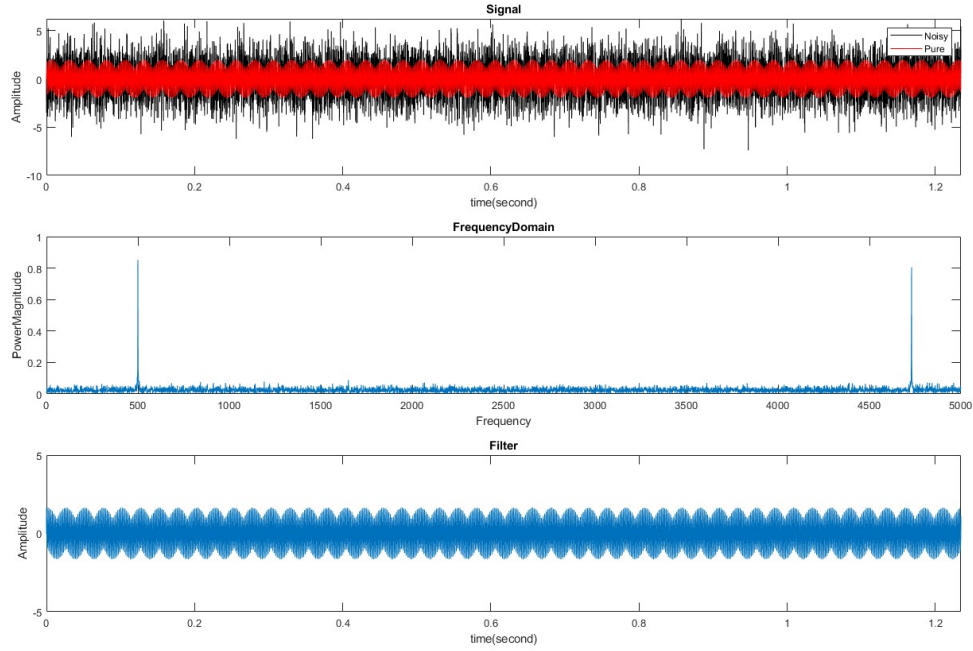


Figure 2: FFT denoising signal

By divide the \mathbf{F}_N into \mathbf{F}_M with some multiplication, we can reduce the operations in a computer from $O(N^2)$ to $O(2(N/2)^2 + N/2)$. Also, \mathbf{F}_M can be divided into $\mathbf{F}_{M/2}$ with some multiplication by using the same manipulation above. Finally, the operation times of DFT was reduced to $O(N)\log_2 N$. And the hole algorithm including dividing, calculating and combining the Fourier matrix is call the Fast Fourier Transform (FFT).

2.5 Example (Denoise signal)

Supposed there is a input signal which is a combination of $500Hz$ and $4730Hz$ Sinwave with random noize. The original signal is too messy for the receiver to understand the information. By FFT algorithm, we can extract the specific frequencies from the original signal in the frequency domain. After we know the main frequencies inside the signal, we can easily remove the noisy signal's frequencies and only reserve the frequencies we want. Finally, we can transform the remaining signal from the frequency domain back to the time domain by the inverse FFT.

code: [FFT project Example.m\(https://github.com/Yi-An-Wang/Linear-Algebra-project-example.git\)](https://github.com/Yi-An-Wang/Linear-Algebra-project-example.git)

result: Figure 2: FFT denoising signal

2.6 Discussions

2.6.1 inverse DFT

Recall the Fourier matrix is a symmetric matrix with each column has a length \sqrt{N} and orthogonal to each other. $\frac{1}{N}\bar{\mathbf{F}}_N$ will be the inverse of Fourier matrix with sampling numbers N . Thus, the transformation equation which changes the $\hat{\mathbf{f}}$ back to \mathbf{f} is:

$$\mathbf{f} = \frac{1}{N}\bar{\mathbf{F}}_N\hat{\mathbf{f}}$$

Because $\bar{\mathbf{F}}_N$ and \mathbf{F}_N almost have the same features, we can separate $\hat{\mathbf{f}}$ into two parts, odd and even. Do the same operation just like what FFT is doing.

$$\mathbf{f} = \frac{1}{N} \begin{bmatrix} [I] & [\bar{D}] \\ [I] & [-\bar{D}] \end{bmatrix} \begin{bmatrix} \mathbf{F}_M & [0] \\ [0] & \mathbf{F}_M \end{bmatrix} \begin{bmatrix} \hat{\mathbf{f}}_{ev} \\ \hat{\mathbf{f}}_{od} \end{bmatrix}$$

2.6.2 remaining and conclusion

While the sampling number is not the power of 2, people may do Zero padding or Truncatoin on the signal sets. The former add zero terms to the signal set, making the sampling points become the power of two. The latter, on the contrary, cut the signal set to make the sampling points be the power of two. Both of the methods will make the data distortion. Thus, there are some variants of FFT algorithm to deal with this problem. In summary, FFT is a powerful and efficient algorithm with broad applications. For example, in signal processing, image processing, communication and various of engineering computing, FFT all plays a critical role.

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Linear Algebra Term Project —The Fast Fourier Transform and its applications—optimized by ChatGPT

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1 Rationale

The Fast Fourier Transform (FFT) stands as a pivotal advancement in contemporary science and technology, presenting an algorithm that efficiently computes the Discrete Fourier Transform (DFT). By substantially reducing computational complexity from $O(N^2)$ to $O(N \log N)$, it becomes a cornerstone for diverse applications.

In the realm of mechanics, the FFT plays a pivotal role in solving Partial Differential Equations (PDEs) governing dynamic system behavior, offering a numerical approach. Its significance extends to signal analysis, seamlessly transforming signals from the time domain to the frequency domain. This transformation facilitates the identification of both noises and main frequencies, enhancing analytical precision.

Beyond its fundamental applications, the FFT proves indispensable in various fields, including Audio Compression, Image Compression, and Communication. These applications underscore its critical role in contemporary technological landscapes.

Engaging in a project involving the FFT presents a unique opportunity to grasp its inner workings through a linear algebra perspective. This endeavor promises to deepen understanding and proficiency in leveraging FFT for advanced applications in diverse domains.

2 Problem background

2.1 Fourier Series

2.1.1 Hilbert Space

One perspective for comprehending Fourier Series involves commencing with Hilbert Space. Hilbert Space is an infinite-dimensional space with vectors of finite length, exemplified by $\mathbf{v} = [1, 1/2, 1/3, \dots]$. For vectors \mathbf{v} and \mathbf{w} within Hilbert Space:

$$\begin{aligned} |\langle \mathbf{v}, \mathbf{w} \rangle| &\leq \|\mathbf{v}\| \|\mathbf{w}\| \\ |\langle \mathbf{v}, \mathbf{w} \rangle| &= 0, \text{ if } \mathbf{v} \perp \mathbf{w} \end{aligned}$$

In a finite domain $(-\pi \leq x \leq \pi)$, where $f(x) = \sin x$ has continuous, infinite points, the square length of $f(x) = \sin x$ is given by:

$$\|\sin x\|^2 = \int_{-\pi}^{\pi} (\sin x)^2 dx = \pi$$

Thus, $\sin x$ $(-\pi \leq x \leq \pi)$ possesses infinite dimensions and finite length. The function space is Hilbert Space, as is the case with other functions $(\cos x, \sin 2x, \cos 2x, \dots, \sin kx, \cos kx, k \rightarrow \infty)$. Now, examine the inner product of two functions $f(x) = \sin x$ and $g(x) = \cos x$ in $(-\pi \leq x \leq \pi)$:

$$(\sin x, \cos x) = \int_{-\pi}^{\pi} \sin x \cdot \cos x dx = 0$$

This illustrates that the vectors of the two functions are perpendicular. This orthogonality extends to other sine and cosine functions $(\sin x, \cos x, \sin 2x, \cos 2x, \dots, \sin kx, \cos kx, k \rightarrow \infty)$.

2.1.2 Fourier Series

Consider a function $y(x)$ defined on the domain $(-\pi \leq x \leq \pi)$ or a periodic function with a period of 2π . We can project this function $y(x)$ onto the coordinate space formed by the sine and cosine functions $(\sin x, \cos x, \sin 2x, \cos 2x, \dots, \sin kx, \cos kx, k \rightarrow \infty)$ in an infinite-dimensional space. The projection formula is given by:

$$\hat{x} = \frac{\langle \mathbf{b}, \mathbf{y} \rangle}{\langle \mathbf{a}, \mathbf{a} \rangle} \mathbf{a}$$

where $\mathbf{y}(x) = a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \dots + a_k \cos(kx) + b_k \sin(kx)$ with $k \rightarrow \infty$, and the coefficients are determined by:

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} y(x) dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \cos(nx) dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} y(x) \sin(nx) dx \end{aligned}$$

2.2 Fourier Transform

2.2.1 promote the period 2π to arbitrary $p = 2L$

Consider a periodic function $f(x)$ with a period $p = 2L$. Introduce a new variable v , and let f be a function of v , such that $x = \frac{p}{2\pi}v$ and $v = \frac{2\pi}{p}x = \frac{\pi}{L}x$. The corresponding transformations are:

$$\begin{cases} v = \pi \Rightarrow x = L \\ v = -\pi \Rightarrow x = -L \end{cases}$$

This transformation can be expressed as:

$$f(x) = f\left(\frac{L}{\pi}v\right) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(nv) + b_n \sin(nv)]$$

which can be further simplified to:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(\frac{n\pi}{L}x) + b_n \sin(\frac{n\pi}{L}x))$$

The coefficients are given by:

$$\begin{cases} a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right) dv \\ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right) \cos(nv) dv \\ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right) \sin(nv) dv \end{cases}$$

This is equivalent to:

$$\begin{cases} a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx \\ a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \\ b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \end{cases}$$

2.2.2 when $L \rightarrow \infty$

Define $w_n = \frac{n\pi}{L}$ and $\Delta w = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$. Let $f_L(x)$ with $p = 2L$ be expressed as:

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos w_n x + b_n \sin w_n x), \quad w_n = \frac{n\pi}{L}$$

When $f(x)$ is not a periodic function, we can represent f as $f_L(x)$ with $L \rightarrow \infty$ and $\Delta w = \frac{\pi}{L} \rightarrow 0$:

$$f(x) = \lim_{L \rightarrow \infty} f_L(x)$$

Then the sigma Σ becomes the integral \int :

$$f(x) = \int_0^{\infty} [A(w) \cos wx + B(w) \sin wx] dw$$

where

$$\begin{aligned} A(w) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos wv dv \\ B(w) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin wv dv \end{aligned}$$

2.2.3 Fourier Transform and inverse Fourier Transform

Putting A and B into the above formula, we obtain:

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(v) [\cos(wv) \cos(wx) + \sin(wv) \sin(wx)] dv dw$$

Since $\int_{-\infty}^\infty f(v) [\cos(wv) \cos(wx) + \sin(wv) \sin(wx)] dv$ can be written as $\left[\int_{-\infty}^\infty f(v) \cos(wx - wv) dv \right]$, and the part in

...

is an even function. $f(x)$ will change to:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(v) \cos(wx - wv) dv \right] dw$$

If we change \cos into \sin in the above formula, we get:

$$\frac{1}{2\pi} \int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(v) \sin(wx - wv) dv \right] dw = 0$$

Multiplying the above formula by i and combining the two formulas, we get:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \left[\int_{-\infty}^\infty f(v) \cos(wx - wv) + i f(v) \sin(wx - wv) dv \right] dw$$

Using Euler's formula $e^{ix} = \cos x + i \sin x$, we form the complex Fourier integral:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(v) e^{-i w v} dv \right] e^{i w x} dw$$

The part inside the

...

is a function of w , $\hat{f}(w)$, called the Fourier Transform:

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(x) e^{-i w x} dx = \mathcal{F}(f)$$

The part outside $\hat{f}(w)$ is the inverse Fourier Transform:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \hat{f}(w) e^{i w x} dw = \mathcal{F}^{-1}(\hat{f})$$

2.3 Discrete Fourier Transform (DFT)

2.3.1 DFT

For all practical purposes, people seldom have a whole function $f(x)$. Most of the time, we only have finitely sampled points of $f(x)$. The Discrete Fourier Transform (DFT) helps people transform a bunch of discrete data from the time domain to the frequency domain. Without a continuous function, we can't perform the integral to obtain the

coefficients of different frequencies. Instead, we get N samples from a function f and evaluate the summation from 0 to $N - 1$:

$$\hat{f}_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n \cdot e^{-i \frac{2\pi kn}{N}}$$

Separate $e^{-i \frac{2\pi}{N}}$ from the above equation and define it as w . The entire part in the summation will become w^{kn} . So the equation will be like:

$$\hat{f}_k = \frac{1}{N} \sum_{n=0}^{N-1} f_n \cdot w^{kn}, \quad w = e^{-i \frac{2\pi}{N}}$$

Turn this equation into the vector expression will be $\hat{f}_{N \times 1} = F_{N \times N} f_{N \times 1}$:

$$\begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \vdots \\ \hat{f}_{k=N} \end{bmatrix} = \begin{bmatrix} w^0 & w^0 & w^0 & \cdots & w^0 \\ w^0 & w^1 & w^2 & \cdots & w^{N-1} \\ w^0 & w^2 & w^4 & \cdots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w^0 & w^{N-1} & w^{2(N-1)} & \cdots & w^{(N-1)^2} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{n=N} \end{bmatrix}$$

The $F_{N \times N}$ is called the Fourier matrix F_N .

For example 1: If there is a function f representing a dynamic system, we take the 4 sample points f_n each second. Thus, each couple of sample $n \rightarrow n + 1$ will have the time interval $\Delta t = 1$ (unit: second). And the \hat{f}_k will represent the magnitude ($\|\hat{f}_k\|_N$) and phase of each frequency F_k , where $k \frac{N}{\Delta t} \Rightarrow F_k$.

For example 2: Set $N = 4$, the sample values are $f = [0, 1, 4, 9]^T$. The $w = e^{-i \frac{2\pi}{4}} = -i$, and the \hat{f} will become:

$$\hat{f} = \begin{bmatrix} -i & -i & -i & -i \\ -i & 1 & i & -1 \\ -i & -1 & -i & 1 \\ -i & i & 1 & -i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 14 \\ -4 + 8i \\ -6 \\ -4 - 8i \end{bmatrix}$$

In this example, we can check the orthogonality of each column in the Fourier matrix $F_{4 \times 4}$ or F_4 . Pick two arbitrary columns and calculate their inner product, just like below:

$$F_4^H F_4 = \begin{bmatrix} 1 & -i & -1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} = 0$$

The result shows that each column of the Fourier matrix F_4 is orthogonal to each other. And the length of each column is $\sqrt{F_4^H F_4 n} = 2$. It is equal to \sqrt{N} , where $N = 4$. This feature can be expanded to any arbitrary Fourier matrix F_N . Thus, if we want to find the inverse of the Fourier matrix to do the inverse DFT, we will discover that $F_N^H F_N = NI$. The inverse of the Fourier matrix is just $\frac{1}{N} F_N^H = \frac{1}{N} \overline{F_N}$.

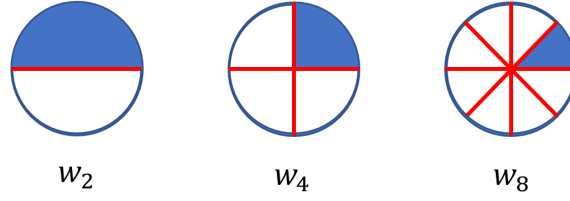


Figure 1: the rotation of different w

2.3.2 Difficulty

If the sampling number N is too small, the equation above may lead to the Aliasing Effect, which means signals at different frequencies have been stacked together. Thus, the sampling number N is typically chosen to be a large value in applications. However, this introduces a significant challenge. Each \hat{f}_n in \hat{f} requires $O(N)$ operations, and the entire \hat{f} needs $O(N^2)$ operations. For example, when the sampling number N is 1000, the operations will scale up to 10^6 . The larger the sampling number, the more computationally expensive the equation becomes.

2.4 Solutions with linear algebra theories and techniques

2.4.1 Thinking w_N as an angle rotation

We can conceptualize w_N as a rotation with an angle changing $2\pi/N$ in a unit circle. The power of w_N represents the number of times the angle changes. For instance, $w_{\frac{3}{4}}$ changes the angle 3 times, resulting in a rotation of $3 \cdot \frac{2\pi}{4}$. On the other hand, $w_{\frac{0}{4}}$ changes the angle 0 times, leading to zero rotation.

Upon examining the Fourier matrices with sampling numbers N and $M = \frac{N}{2}$, we observe connections between these two matrices. Specifically, $(w_N)^2 = w_M$ means the angle changing of w_M is twice that of w_N . Consequently, the Fourier matrix F_N can be expressed as a combination of w_N and w_M in each element. The elements that undergo angle changes $2\pi/N$ an even number of times can be written as powers of w_M . Conversely, the elements that undergo angle changes $2\pi/N$ an odd number of times will be expressed as powers of w_M multiplied by $c \cdot w_N$ to compensate for some angle $2\pi/N$.

2.4.2 FFT

What FFT is doing is called "Divide and Conquer." Suppose the sampling number N is even, and $M = N/2$. We divide f into two parts, fev and fod , and define:

$$\hat{fev} = FM \cdot fev, \quad fev = [f_0 \ f_2 \ f_4 \ \dots \ f_{N-4} \ f_{N-2}]^T$$

$$\hat{fod} = FM \cdot fod, \quad fod = [f_1 \ f_3 \ f_5 \ \dots \ f_{N-3} \ f_{N-1}]^T$$

Assuming each \hat{f}_k in \hat{f} is a combination of \hat{fev}_k in \hat{fev} and \hat{fod}_k in \hat{fod} , we start

with the basic equation:

$$\hat{f} = \begin{bmatrix} \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \\ \vdots \\ \hat{f}_{k=N-1} \end{bmatrix}, \quad \hat{f}_k = \frac{1}{N} \sum_{n=0}^{N-1} w_N^{nk} \cdot f_n$$

Dividing the summation into two parts:

$$\hat{f}_k = \left(\sum_{n=0}^{M-1} w_N^{2nk} \cdot f_{2n} \right) + \left(\sum_{n=0}^{M-1} w_N^{(2n+1)k} \cdot f_{2n+1} \right)$$

Since $w_M = w_N^{2n}$, we can rewrite the equation:

$$\hat{f}_k = \sum_{n=0}^{M-1} w_M^{nk} \cdot f_{2n} + w_N^k \cdot \sum_{n=0}^{M-1} w_N^{nk} \cdot f_{2n+1}$$

To express this in matrix form, let $k = 0, 1, 2, \dots, M-1$, and consider the matrix multiplication:

$$\hat{f}_{k=0|M-1} = \begin{bmatrix} I & D \end{bmatrix} \begin{bmatrix} FM & 0 \\ 0 & FM \end{bmatrix} \begin{bmatrix} fev \\ fod \end{bmatrix}$$

Here, $[D]$ is a diagonal matrix:

$$[D] = \begin{bmatrix} w_N^0 & w_N^1 & w_N^2 & \dots & w_N^{M-1} \end{bmatrix}$$

Next, consider $\hat{f}_{k=M|2M-1}$ using $\hat{f}'_k = \hat{f}_{k+M}$:

$$\hat{f}'_k = \begin{bmatrix} I & -D \end{bmatrix} \begin{bmatrix} FM & 0 \\ 0 & FM \end{bmatrix} \begin{bmatrix} fev \\ fod \end{bmatrix}$$

Now, combine $\hat{f}_{k=0|M-1}$ and \hat{f}'_k to form \hat{f} :

$$\hat{f} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} FM & 0 \\ 0 & FM \end{bmatrix} \begin{bmatrix} fev \\ fod \end{bmatrix}$$

The sampling points vector, which has even-numbered elements above and odd-numbered elements below, can be expressed as a permutation matrix $[P]$ multiplying the vector f :

$$\begin{bmatrix} fev \\ fod \end{bmatrix} = [P]f = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} f$$

Recalling the original DFT equation, we have the expansion of F_N :

$$\hat{f} = F_N f = \begin{bmatrix} I & DN \\ I & -DN \end{bmatrix} \begin{bmatrix} FM & 0 \\ 0 & FM \end{bmatrix} [PN] f$$

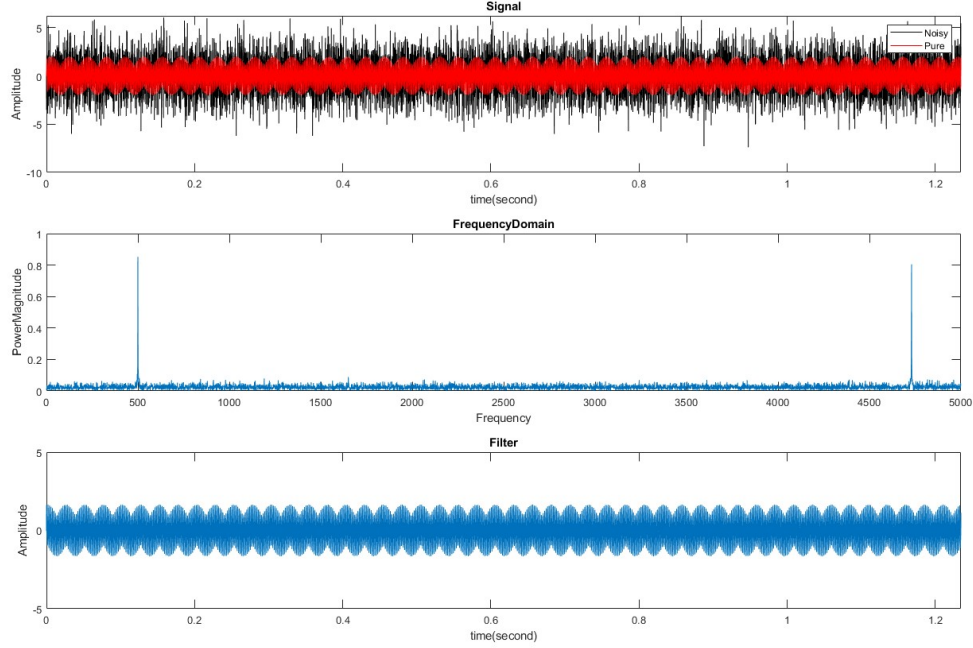


Figure 2: FFT denoising signal

By dividing F_N into FM with some multiplication, we can reduce the operations on a computer from $O(N^2)$ to $O^2(N/2)^2 + N/2$. Also, FM can be divided into $FM/2$ with some multiplication by using the same manipulation above. Finally, the operation times of DFT were reduced to $O(N \log^2 N)$. The entire algorithm, including dividing, calculating, and combining the Fourier matrix, is called the Fast Fourier Transform (FFT).

2.5 Example (Denoise signal)

Suppose there is an input signal, a combination of a 500 Hz and 4730 Hz sinusoidal waveform with random noise. The original signal appears chaotic, making it challenging for the receiver to discern the underlying information. Leveraging the FFT algorithm, we can extract specific frequencies from the original signal in the frequency domain. Once we identify the primary frequencies within the signal, we can effectively filter out the noisy signal frequencies, retaining only the desired components. Finally, we can transform the filtered signal from the frequency domain back to the time domain using the inverse FFT.

code: [FFT project Example.m\(https://github.com/Yi-An-Wang/Linear-Algebra-project-example.git\)](https://github.com/Yi-An-Wang/Linear-Algebra-project-example.git)

result: Figure 2: FFT denoising signal

2.6 Discussions

2.6.1 inverse DFT

Recall that the Fourier matrix is a symmetric matrix, with each column having a length of \sqrt{N} and being orthogonal to one another. The inverse of the Fourier matrix with sampling numbers N , denoted as $1/N F^{-N}$, serves as a key component in the transformation equation that converts \hat{f} back to f :

$$f = \frac{1}{N} F^{-N}$$

Given the similar characteristics of F^{-N} and FN , we can further decompose \hat{f} into two parts, namely odd and even, and perform operations analogous to those employed in the FFT algorithm:

$$f = \frac{1}{N} \begin{bmatrix} I & D^- \\ I & -D^- \end{bmatrix} \begin{bmatrix} FM[0] & 0 \\ 0 & FM \end{bmatrix} \begin{bmatrix} f_{ev} \\ f_{od} \end{bmatrix}$$

2.6.2 remaining and conclusion

When the sampling number is not a power of 2, practitioners often employ techniques such as Zero Padding or Truncation on the signal sets. Zero Padding involves adding zero terms to the signal set, effectively aligning the sampling points to a power of two. Conversely, Truncation trims the signal set to achieve the same goal. However, both methods introduce data distortion. Consequently, various variants of the FFT algorithm have been developed to address this issue. In summary, FFT stands as a powerful and efficient algorithm with wide-ranging applications. It plays a critical role in signal processing, image processing, communication, and various engineering computations.

2.7 References

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