Iterative Formula for solving the equality constrains in GRG

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First let us consider the simple case of solving (all in scalar)

$$h(d, s) = 0. \tag{1}$$

We have the perturbation at first order to be

$$\partial h = \frac{\partial h}{\partial d}\partial d + \frac{\partial h}{\partial s}\partial s \tag{2}$$

Let j be the iteration counter. For a given iteration, the design variable d is considered known and can be treated as a constant. Therefore $\partial d = 0$. In order to find the corresponding state variable, we need to find $h_{j+1} = 0$. (notice that we can not assume $\partial h = 0$ since at the current iteration we have not solved $h = 0$ yet)

From Eq.(2), for a known d we have $\partial d = 0$ and $\partial h \neq 0$, unless we are at the solution. Therefore Eq.(2) implies

$$\partial h = \frac{\partial h}{\partial s}\partial s$$

or in iterative equation

$$h_{j+1} - h_j = \left(\frac{\partial h}{\partial s}\right)_j (s_{j+1} - s_j). \tag{3}$$

In Eq.(3), we may land in an infeasible point ($h_j \neq 0$) but we want the next iteration to be feasible ($h_{j+1} = 0$). So we get

$$h_{j+1} = h_j + \left(\frac{\partial h}{\partial s}\right)_j (s_{j+1} - s_j) = 0 \tag{4}$$

and solving for s_{j+1} we get

$$s_{j+1} = s_j - \left(\frac{\partial h}{\partial s}\right)_j^{-1} h_j. \tag{5}$$

Let the inner iteration be the process of finding state variables with counter j and the outer iteration be the optimization process with counter k. Now we generalize for vector functions at point $\mathbf{x}_{k+1} = (\mathbf{d}_{k+1}, \mathbf{s}_{k+1})^T$ of the optimization iteration k (the outer iteration).

$$[\mathbf{s}_{k+1}]_{j+1} = \left[\mathbf{s}_{k+1} - \left(\frac{\partial \mathbf{h}}{\partial \mathbf{s}}\right)_{k+1}^{-1} \mathbf{h}(\mathbf{d}_{k+1}, \mathbf{s}_{k+1})\right]_j \tag{6}$$

Equation (6) is the updating equation (5.41) in the course textbook on page 187.

The remaining question is how to initialize the iteration formula (6), i.e., what is the first guess for $\mathbf{s}_{k+1}$ at $j = 0$. We can get a good guess for $\mathbf{s}_{k+1}$ (at $j = 0$) by picking the value that would satisfy the linear approximation of the constraints for a given change $\partial \mathbf{d}$ in two consecutive *outer* iterations, i.e., $\partial \mathbf{d} = \mathbf{d}_{k+1} - \mathbf{d}_k$ with corresponding vector $\partial \mathbf{s} = \mathbf{s}'_{k+1} - \mathbf{s}_k$, where the prime is used to distinguish $\mathbf{s}'_{k+1}$ that solves the linear approximation of $\mathbf{h} = 0$, from $\mathbf{s}_{k+1}$ that solves the actual $\mathbf{h} = 0$ at the conclusion of the j iteration.

Let us now set $\partial \mathbf{h} = 0$, we have

$$\partial \mathbf{s} = -\left(\frac{\partial \mathbf{h}}{\partial \mathbf{s}}\right)^{-1} \left(\frac{\partial \mathbf{h}}{\partial \mathbf{d}}\right) \partial \mathbf{d} \tag{7}$$

or

$$\mathbf{s}'_{k+1} = \mathbf{s}_k - \left(\frac{\partial \mathbf{h}}{\partial \mathbf{s}}\right)_k^{-1} \left(\frac{\partial \mathbf{h}}{\partial \mathbf{d}}\right)_k (\mathbf{d}_{k+1} - \mathbf{d}_k) \tag{8}$$

If we choose the gradient method to iterate in the reduced space of $\mathbf{d}$ in order to find the minimum, we will have

$$\partial \mathbf{d} = \mathbf{d}_{k+1} - \mathbf{d}_k = -\alpha_k \left(\frac{\partial z}{\partial \mathbf{d}}\right)_k^T \tag{9}$$

where $\frac{\partial z}{\partial \mathbf{d}}$ is the reduced gradient. Substituting Eq.(9) into (8) we get the formula that computes the initial guess of $\mathbf{s}'_{k+1}$ to be used for $j = 0$ in iteration (6). This is equation (5.39) in the course textbook on page 187.