LA Final

2023/12/18

HW 9-5

5. "Suppose $Ax=\lambda x$. If $\lambda=0$, then the eigenvector x is in the nullspace. If $\lambda\neq 0$, then the eigenvector x is in the column space of A. The eigenvectors in the column space has r (rank of A) linearly independent vectors and the eigenvectors in the nullspace has $n-r$ linearly independent vectors. Since $n+(n-r)=n$, any $n\times n$ matrix A must have n linearly independent eigenvectors." What is wrong in the statement to lead to the incorrect conclusion? Find a 2×2 example (from the internet) that shows the statement is incorrect. Is the statement correct when A is a projection matrix?

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \lambda = 2(repeated) \rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, eigenvector = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

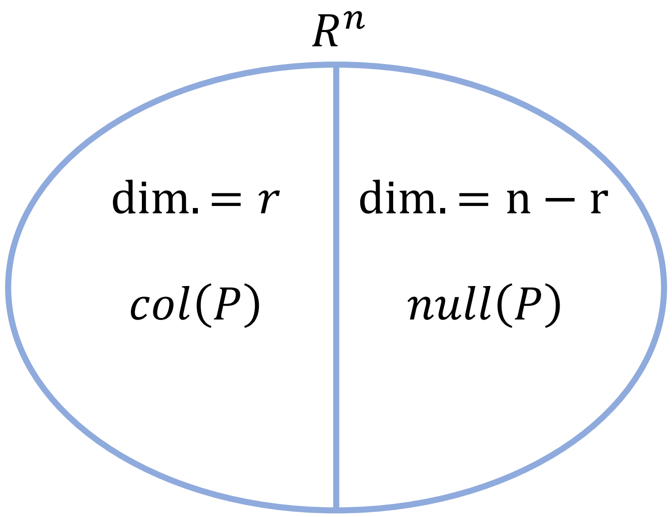
HW 9-5

5. "Suppose $Ax=\lambda x$. If $\lambda=0$, then the eigenvector x is in the nullspace. If $\lambda\neq 0$, then the eigenvector x is in the column space of A. The eigenvectors in the column space has r (rank of A) linearly independent vectors and the eigenvectors in the nullspace has $n-r$ linearly independent vectors. Since $n+(n-r)=n$, any $n\times n$ matrix A must have n linearly independent eigenvectors." What is wrong in the statement to lead to the incorrect conclusion? Find a 2×2 example (from the internet) that shows the statement is incorrect. Is the statement correct when A is a projection matrix?

P: projection matrix, vector x $\in R^n$

$P\ project\ x\ into\ a\ subspace(dim. = r)$
$Px \in col(P)$

$for\ P$:
$col(P) = row(P), null(P) = left\ null(P)$



HW 9-6

6. Show that the eigenvalues of $\boldsymbol{A}$ equal the eigenvalues of $\boldsymbol{A^T}$. Show by an example that the eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{A^T}$ are not the same.

$$\det(A-\lambda I)=\det(A-\lambda I)^T=\det(A^T-\lambda I^T)=\det(A^T-\lambda I)=0$$

$$A=\begin{bmatrix}1&1\\0&2\end{bmatrix}，\lambda=1\rightarrow eigenvector=\begin{bmatrix}1\\0\end{bmatrix}，\lambda=2\rightarrow eigenvector=\begin{bmatrix}1\\1\end{bmatrix}$$

$$A^T=\begin{bmatrix}1&0\\1&2\end{bmatrix}，\lambda=1\rightarrow eigenvector=\begin{bmatrix}1\\-1\end{bmatrix}，\lambda=2\rightarrow eigenvector=\begin{bmatrix}0\\1\end{bmatrix}$$

HW 9-7

7. A 3 by 3 matrix B is known to have eigenvalues 0, 1, 2, Is this information enough to find: (a) the rank of B, (b) the determinant of $B^{T}B$, (c) the eigenvalues of $B^{T}B$? How?

(a) $Dim.of\ Null(B) = 1 \rightarrow Rank(B) = 3 - 1 = 2$

(b) $\det\left(B^{T}B\right) = \det B^{T} \det B = (\det B)^{2} = (0 \cdot 1 \cdot 2)^{2} = 0$

(c) $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\lambda = 0, 1, 2$ or $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\lambda = 0, 1, 2$

$B^{T}B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $\lambda = 0, 1, 2$ or $B^{T}B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $\lambda = 0, 2, 4$

- We can't find all λ of $B^{T}B$. But , one λ of $B^{T}B$ is 0.

HW 10-2

2. If each number is the average of the two previous numbers, $G_{k+2} = \frac{1}{2}(G_{k+1} + G_k)$,

set up the matrix A and diagonalize it. Starting from $G_0 = 0$ and $G_1 = \frac{1}{2}$, find a

formula for G_k and compute its limit as $k \to \infty$.

$$Let\ u_k = \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}, \mathrm{k} = 0,1,2 \ldots \qquad \left(u_0 = \begin{bmatrix} G_1 \\ G_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}\right)$$

$$\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} G_{k+1} + \frac{1}{2} G_k \\ G_{k+1} \end{bmatrix} \to u_{k+1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} u_k$$

注意維度!

HW 10-2

2. If each number is the average of the two previous numbers, $G_{k+2} = \frac{1}{2}(G_{k+1} + G_k)$,

set up the matrix A and diagonalize it. Starting from $G_0 = 0$ and $G_1 = \frac{1}{2}$, find a

formula for G_k and compute its limit as $k \to \infty$.

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}, \det(A - \lambda I) = 0, \lambda = 1, -\frac{1}{2}$$

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

HW 10-2

2. If each number is the average of the two previous numbers, $G_{k+2} = \frac{1}{2}(G_{k+1} + G_k)$, set up the matrix A and diagonalize it. Starting from $G_0 = 0$ and $G_1 = \frac{1}{2}$, find a formula for G_k and compute its limit as $k \to \infty$.

$$\begin{bmatrix} G_k \\ G_{k-1} \end{bmatrix} = u_{k-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} u_{k-2} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}^{k-1} u_0 = A^{k-1}\, u_0 = (S\Lambda S^{-1})^{k-1} u_0$$

$$= S\Lambda^{k-1} \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} = S \begin{bmatrix} 1 & 0 \\ 0 & \left(-\frac{1}{2}\right)^{k-1} \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} = S \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3}\left(-\frac{1}{2}\right)^{k-1} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{3}\left(-\frac{1}{2}\right)^{k-1} \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \to G_k = \frac{1}{3} + \frac{1}{6}\left(-\frac{1}{2}\right)^{\mathrm{k}-1}, \text{if } \mathrm{k} \to \infty,\ G_k = \frac{1}{3}$$

HW 11-5

5. Rewrite the following matrices in the form $\lambda_1 x_1 x_1^H + \lambda_2 x_2 x_2^H$.

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \quad Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad R = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

$$P = 0 \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}^T + 1 \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T = 0 \cdot \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + 1 \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$Q = 1 \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}^T - 1 \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}^T = 1 \cdot \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} - 1 \cdot \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$R = 5 \cdot \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}^T - 5 \cdot \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}^T = 5 \cdot \begin{bmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{bmatrix} - 5 \cdot \begin{bmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

HW 11-6

6. Write one significant fact about the eigenvalues of each of the following

(a) A real symmetric matrix : $\boxed{\forall\ \lambda_i \in R}$

(b) A stable matrix (solutions of du/dt=Au approach zero) : $\boxed{\text{eigenvalues: a+b}i\text{, a<0, b} \in R}$

No ”=(Neutrally stable)”

(c) A Markov matrix : $\boxed{\lambda_{max} = 1,\ |\lambda_i| \leq 1}$

(d) A continuous Markov matrix : $\boxed{\lambda_{max} = 0,\ \lambda_i \leq 0}$

(e) A defective (nondiagonalizable) matrix : $\boxed{if\ A\ is\ n \times n, the\ number\ of\ distinct\ \lambda < \text{n}}$

(f) A singular matrix : $\boxed{Dim. of\ nullspace \geq 1 \rightarrow one\ eigenvalue = 0}$

HW 12-3

3. Show that an upper triangular and normal matrix must be diagonal.

Let $A = \{a_{ij}\}, a_{ij} = 0, for\ i > j$

For i $= 1, \because \mathrm{A^H}A = AA^H \ \rightarrow |a_{11}|^2 = |a_{11}|^2 + \sum_{j=2}^{n}\left|a_{1j}\right|^2 \rightarrow \sum_{j=2}^{n}\left|a_{1j}\right|^2 = 0$

$\rightarrow a_{1j} = 0, for\ j > 1$

For i $= 2, \because \mathrm{A^H}A = AA^H \ \rightarrow |a_{12}|^2 + |a_{22}|^2 = |a_{22}|^2 + \sum_{j=3}^{n}\left|a_{2j}\right|^2 \rightarrow \sum_{j=3}^{n}\left|a_{2j}\right|^2 = 0$

$\rightarrow a_{2j} = 0, for\ j >$2

For i $= k, \because \mathrm{A^H}A = AA^H \ \rightarrow \sum_{i=1}^{k-1}|a_{ik}|^2 + |a_{kk}|^2 = |a_{kk}|^2 + \sum_{j>k}^{n}\left|a_{kj}\right|^2 \rightarrow \sum_{j>k}^{n}\left|a_{kj}\right|^2 = 0$

$\rightarrow a_{kj} = 0, for\ j >$k

HW 12-3

3. Show that an upper triangular and normal matrix must be diagonal.

For i $= k, \because \mathrm{A}^{\mathrm{H}}A = AA^H \rightarrow \sum_{i=1}^{k-1}|a_{ik}|^2 + |a_{kk}|^2 = |a_{kk}|^2 + \sum_{j>k}^{n}\left|a_{kj}\right|^2 \rightarrow \sum_{j>k}^{n}\left|a_{kj}\right|^2 = 0$

$\rightarrow a_{kj} = 0, for\ j >$k

By mathematical induction, $a_{ij} = 0, for\ i < j$
$\therefore A\ must\ be\ diagonal.$

HW 12-4

4. Show that all permutation matrices are normal

$P: permutation\ matrix$
$\because\ P\ is\ orthogonal\ matrix\ \therefore\ P^{-1} = P^T$
$P^H = P^T$
$\rightarrow PP^H = PP^T = P^TP = P^HP = I$
$\therefore\ P\ is$ a normal matrix.

HW 12-PCA.3

3. Find the minimum values of

$$R(x)=\frac{x_1^2-x_1x_2+x_2^2}{x_1^2+x_2^2} \quad \text{and} \quad R(x)=\frac{x_1^2-x_1x_2+x_2^2}{2x_1^2+x_2^2} \text{(hint: let } y=\sqrt{2}x_1\text{)}$$

$$By\ Rayleigh's\ qoutient: R(x)=\frac{x^TAx}{x^Tx}$$

$$\lambda_{min} \leq R(x) \leq \lambda_{max}$$

$$R(x)=\frac{x_1^2-x_1x_2+x_2^2}{x_1^2+x_2^2} \rightarrow x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, A=\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

$$\rightarrow \lambda=\frac{1}{2},\frac{3}{2} \rightarrow minimum\ of\ R(x)=\frac{1}{2}$$

HW 12-PCA.3

3. Find the minimum values of

$$R(x)=\frac{x_1^2-x_1x_2+x_2^2}{x_1^2+x_2^2} \quad \text{and} \quad R(x)=\frac{x_1^2-x_1x_2+x_2^2}{2x_1^2+x_2^2} \text{(hint: let } y=\sqrt{2}x_1\text{)}$$

$$By\ Rayleigh's\ qoutient: R(x)=\frac{x^TAx}{x^Tx}$$

$$\lambda_{min} \le R(x) \le \lambda_{max}$$

$$R(x)=\frac{x_1^2-x_1x_2+x_2^2}{2x_1^2+x_2^2} \rightarrow x=\begin{bmatrix} \sqrt{2}x_1 \\ x_2 \end{bmatrix}, A=\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & 1 \end{bmatrix}$$

$$\rightarrow \lambda=\frac{3\pm\sqrt{3}}{4} \rightarrow minimum\ of\ R(x)=\frac{3-\sqrt{3}}{4}$$

HW 12-PCA.4

4. The ten largest U.S. industrial corporations yield the following data.

(a) Calculate the covariance and correlation matrices for the Sales and Profit

- Excel： COVARIANCE.S and CORREL

Covariance	Sales	Profit
Sales	1000509114	25575599.63
Profit	25575599.63	1430020.011

Correlation	Sales	Profit
Sales	1	0.67615
Profit	0.67615	1

HW 12-PCA.4

4. The ten largest U.S. industrial corporations yield the following data.

(b) Use the first eigenvector of the covariance matrix to find a weighted index of the sales and the profit so that the companies' performance can be best distinguished.

(c) Use the first eigenvector of the correlation matrix to find a weighted index so that the companies' performance can be distinguished.

Covariance	no.1	no.2		Correlation	no.1	no.2
eigenvalue	1001163400	775734.3		eigenvalue	1.67615	0.32385
eigenvector	0.99967293	0.02557405		eigenvector	0.70711	-0.7071
(unit vector)	0.02557405	-0.99967293		(unit vector)	0.70711	0.70711

Relative weights not weighted index!!

$$weighted\ index: z_k = e_i^T y_k = e_{i1}y_{k1} + e_{i2}y_{k2} + \cdots + e_{ip}y_{kp}$$

HW 12-PCA.4

4. The ten largest U.S. industrial corporations yield the following data.

(d) Compare and discuss the difference between the two indices found in (b) and (c).

- (c): normalized

(e) Use the second eigenvector of the correlation matrix to find a second weighted index. Show that this index is uncorrelated to the index in (c) and compare the two indexes.

$Cov(e_2^T y, e_1^T y) = e_2^T B^T B e_1 = 0 \ (\because e_1 \perp e_2)$

e_1: $First\ Principal\ component, \lambda_1$: the first largest variance
e_2: $Second\ Principal\ component, \lambda_2$: the second largest variance