Monte Carlo Simulations

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Let us consider generic continuous functions of the form $f(\mathbf{x}, \mathbf{p})$, where $\mathbf{x}$ are the design variables and $\mathbf{p}$ are the design parameters. For a given setup of $\{\mathbf{x}, \mathbf{p}\}$, f usually presents a certain performance measure.

In most cases the engineering analysis of f assumes both $\mathbf{x}$ and $\mathbf{p}$ to be deterministic, namely we have full control and full knowledge of the values of these quantities without doubts or ambiguity. However, in reality these values might not be precisely known. Without lost of generality, let us consider only $\mathbf{x}$ in f, as $f(\mathbf{x})$. When the values of $\mathbf{x}$ are not deterministic, we remodel them into $\mathbf{X}$, stochastic or random quantity. At this point we only assume that X_i is a random quantity following certain probability distributions. Let $f_{X_i}(x)$ be the probability density function (PDF) of X_i. Our attentions are now focused on how to estimate the outcomes of f when $f_{\mathbf{X}}(\mathbf{x})$ are known.

Consider the special case when $\mathbf{X} \sim N(\mu_{\mathbf{X}}, \sigma^2_{\mathbf{X}})$, Gaussian distributions with mean $\mu_{\mathbf{X}}$ and standard deviations $\mu_{\mathbf{X}}$. Random realizations of $\mathbf{X}$ constitutes the values of $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ with n samples. If we evaluate $f_1 = f(\mathbf{x}_1), f_2 = f(\mathbf{x}_2), \cdots, f_n = f(\mathbf{x}_n)$, we then have n function outcomes. Statistical analysis of $f(\mathbf{X})$ with random variables using n values of f is generally called the Monte Carlo Simulation.

A standard Monte Carlo Simulation should include the following steps

1. Random generation : generate n realizations of the input vector $\mathbf{X}$ based on the PDF $f_{\mathbf{X}}$
2. Performance analysis : for each realization $\mathbf{x}_i$, calculate $y_i = f(\mathbf{x}_i)$
3. Results interpretation : with all n outcomes are obtained, extract the statistical information from $\mathbf{y}$.

Example: Let $X_1 \sim (2, 0.3^2)$ and $X_2 \sim (2, 0.3^2)$, please use Monte Carlo simulation to estimate the mean and variance of the function

$$f(X_1, X_2) = X_1^2 + 8X_2 - 75 \tag{1}$$