ANOVA
Analysis of Variances

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Replicated 2^2 Factorial Design and Sum of Squares (SS's)

Test	x_1	x_2	y_{i1}	y_{i2}	y_{i3}	y_{i4}	y_{i5}	$\bar{y}_i$
1	−1	−1	11	7	10	15	7	10
2	+1	−1	48	43	52	55	47	49
3	−1	+1	31	24	27	23	20	25
4	+1	+1	37	33	34	37	34	35

$i = 1,\ldots,m \quad m = \#\text{ of Tests } = 4 \qquad j = 1,\ldots,n \quad n = \#\text{ of replicates} = 5$

$$\bar{y}_i = \frac{\sum_{j=1}^{n} y_{ij}}{n} \quad i = 1,\ldots,m \qquad \bar{\bar{y}} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}}{mn} = 29.75$$

$$y_{ij} = \bar{\bar{y}} + (\bar{y}_i - \bar{\bar{y}}) + (y_{ij} - \bar{y}_i) \qquad \text{Sum of Squares=SS} = \sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^2$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^2 = mn\bar{\bar{y}}^2 + n\sum_{i=1}^{m}(\bar{y}_i - \bar{\bar{y}})^2 + \sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij} - \bar{y}_i)^2$$

$$\text{SS} = \text{SS(mean)} + \text{SS(between tests)} + \text{SS(within tests)}$$

Sum of Squares

$$\sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^2 - mn\bar{\bar{y}}^2 = \sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij}-\bar{\bar{y}})^2 = n\sum_{i=1}^{m}(\bar{y}_i-\bar{\bar{y}})^2 + \sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij}-\bar{y}_i)^2$$

$$\text{SS} - \text{SS(mean)} = \text{SS(total)} = \text{SS(between tests)} + \text{SS(within tests)}$$

$$\text{SS(total)} = \sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij}-\bar{\bar{y}})^2 = \sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^2 - mn\bar{\bar{y}}^2$$

$$= 21969 - 17701.25$$

$$\text{SS(between tests)} = n\sum_{i=1}^{m}(\bar{y}_i-\bar{\bar{y}})^2 = 4053.75$$

$$\text{SS(within tests)} = \sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij}-\bar{y}_i)^2 = 214$$

$$= \text{Sum of Squared Errors} = \text{SSE}$$

3

Mean Squares

- Mean squares within tests: Mean Squared Error (MSE)

$$\boxed{\hat{\sigma}_\varepsilon^2 = s_p^2} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij}-\bar{y}_i)^2}{m\quad(n-1)} = \frac{\text{SS(within tests)}}{m(n-1)} = \frac{\text{SS(within tests)}}{\text{degrees of freedom}}$$

$$= \frac{214}{4(5-1)} = 13.375 = \text{Mean Square (within tests)=MS(within tests)}$$

$$= \text{Mean Squared Error} = \text{MSE}$$

- Mean squares between tests: MS(between tests)

$$s_{\bar{y}}^2 = \frac{\sum_{i=1}^{m}(\bar{y}_i-\bar{\bar{y}})^2}{m-1} \Rightarrow \boxed{s_{\bar{y}}^2 = \frac{s_y^2}{n} \Rightarrow s_y^2 = ns_{\bar{y}}^2} \quad n \text{ is the sample size of } \bar{y}_i$$

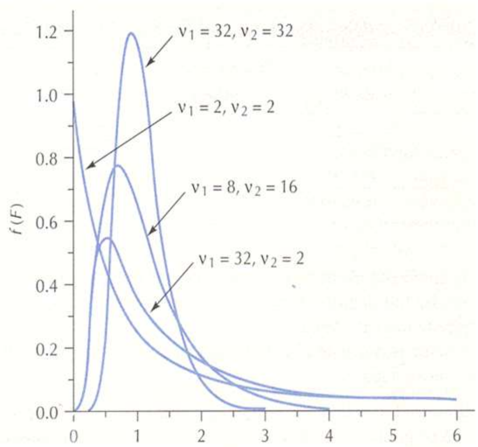
$$s_y^2 = ns_{\bar{y}}^2 = \frac{n\sum_{i=1}^{m}(\bar{y}_i-\bar{\bar{y}})^2}{m-1} = \frac{\text{SS(between tests)}}{\text{DOF(between tests)}}$$

$$= \frac{4053.75}{4-1} = 1351.25 = \text{MS(between tests)}$$

4

Recall *F* Statistic and Distributions

- Two ***normal*** populations have the ***same variance***
- s_1^2 : sample variances from population 1 with sample size n_1
- s_2^2 : sample variances from population 2 with sample size n_2
- Then, s_1^2 / s_2^2 follows $F_{v1,\ v2}$ distribution
  - v_1=n_1-1; v_2=n_2-1
- F table look-up



5

F-test

- H_0: *MS*(between tests)=*MS*(within tests)
  H_a: *MS*(between tests)>*MS*(within tests)
- Test statistic:
  F_{calc}=*MS*(between tests)/*MS*(within tests)
- Reject H_0 when $F_{calc} > F_{v1,v2,1-\alpha}$
- Reject H_0: **Difference between tests are not caused by noise (estimated by difference within tests)**
- Example:

$$F_{calc} = \frac{MS(\text{between tests})}{MS(\text{within tests})} = \frac{1351.25}{13.375} = 101.03 > F_{3,16,0.99} = 5.29$$

6

ANOVA Table for Replicated 2^2

Source of variation	Sum of Squares (SS)	Degrees of Freedom *(DOF)*	Mean square (MS)	*F Ratio*
Mean	$mn\bar{\bar{y}}^2$	1	SS(Mean)/ DoF(Mean)	MS(Mean)/MS(Pure error)
Between tests	$n\sum_{i=1}^{m}(\bar{y}_i - \bar{\bar{y}})^2$	*m*–1	SS(B/W Tests)/DoF(B/W Tests)	MS(B/w Tests)/MS(Pure error)
Pure error (within tests)	$\sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij} - \bar{y}_i)^2$	*m*(*n*–1)	SS(Pure error)/DoF(Pure error)	
SS	$\sum_{i=1}^{m}\sum_{j=1}^{n}y_{ij}^2$	*mn*		

Source of variation	Sum of Squares	*DOF*	Mean square	F_{calc}
Mean	17,701.25	1	17,701.25	1,323.46
Between tests	4,053.75	3	1,351.25	101.03
Pure error (within tests)	214.00	16	13.375	
SS	**21,969.00**	**20**		

ANOVA Table without "Mean"

$$SS(\text{total}) = \sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij} - \bar{\bar{y}})^2 = \sum_{i=1}^{m}\sum_{j=1}^{n}y_{ij}^2 - mn\bar{\bar{y}}^2$$

Source of variation	Sum of Squares (SS)	Degrees of Freedom *(DOF)*	Mean square (MS)	*F Ratio*
Between tests	$n\sum_{i=1}^{m}(\bar{y}_i - \bar{\bar{y}})^2$	*m*–1	SS(B/W Tests)/DoF(B/W Tests)	MS(B/w Tests)/MS(Pure error)
Pure error (within tests)	$\sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij} - \bar{y}_i)^2$	*m*(*n*–1)	SS(Pure error)/DoF(Pure error)	
SS(Total)= SS-mean	$\sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij} - \bar{\bar{y}})^2$	*mn*–1		

ANOVA Table without "Mean"

$$\text{SS(total)} = \sum_{i=1}^{m}\sum_{j=1}^{n}(y_{ij} - \overline{\overline{y}})^2 \quad = \sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^2 - mn\overline{\overline{y}}^2$$

$$= 21969 - 17701.25 = 4267.75$$

Source of variation	Sum of Squares	*DOF*	Mean square	F_{calc}
Between tests	4,053.75	3	1,351.25	101.03
Pure error (within tests)	214.00	16	13.375	
SS-Mean =SS(Total)	**4,267.75**	**19**		

Linear Regression Model

$$\hat{y} = 29.75 + 12.25x_1 + 0.25x_2 - 7.25x_1x_2$$

	X1	X2	X1X2	Y
Replicate 1	-1	-1	1	11
	1	-1	-1	48
	-1	1	-1	31
	1	1	1	37
Replicate 2	-1	-1	1	7
	1	-1	-1	43
	-1	1	-1	24
	1	1	1	33
Replicate 3	-1	-1	1	10
	1	-1	-1	52
	-1	1	-1	27
	1	1	1	34
Replicate 4	-1	-1	1	15
	1	-1	-1	55
	-1	1	-1	23
	1	1	1	37
Replicate 5	-1	-1	1	7
	1	-1	-1	47
	-1	1	-1	20
	1	1	1	34

迴歸統計	
R 的倍數	0.974606
R 平方	0.949856
調整的 R	0.940455
標準誤	3.657185
觀察值個	20

ANOVA

	自由度	SS	MS	F	顯著值
迴歸	3	4053.75	1351.25	101.028	1.3E-10
殘差	16	214	13.375		
總和	19	4267.75			

	係數	標準誤	t 統計	P-值	下限 95%	上限 95%
截距	29.75	0.817771	36.37936	8.18E-17	28.0164	31.4836
X1	12.25	0.817771	14.97974	7.8E-11	10.5164	13.9836
X2	0.25	0.817771	0.305709	0.763768	-1.4836	1.983597
X1X2	-7.25	0.817771	-8.86556	1.43E-07	-8.9836	-5.5164

R^2 to Assess the Model

Toal Variation in y SST= SSR + SSE

Overall variability in y

Explained in part by → The regression model

Remains, in part, unexplained → The error

- R^2 measures the proportion of the variation in y that is explained by the variation in x.

$$R^2 = \frac{\text{Variation explained by Model}}{\text{Total variation in y}} = \frac{\text{SSR}}{\text{SST}} = 1 - \frac{\text{SSE}}{\text{SST}}$$

Example of R^2

迴歸統計	
R 的倍數	0.974606
R 平方	0.949856
調整的 R	0.940455
標準誤	3.657185
觀察值個	20

$$R = \sqrt{R^2} = \sqrt{0.949856} = 0.974606$$

$$R^2 = \frac{\text{SSR}}{\text{SST}} = \frac{4053.75}{4267.75} = 0.949856$$

ANOVA

	自由度	SS	MS	F	顯著值
迴歸	3	4053.75	1351.25	101.028	1.3E-10
殘差	16	214	13.375		
總和	19	4267.75			

	係數	標準誤	t 統計	P-值	下限 95%	上限 95%
截距	29.75	0.817771	36.37936	8.18E-17	28.0164	31.4836
X1	12.25	0.817771	14.97974	7.8E-11	10.5164	13.9836
X2	0.25	0.817771	0.305709	0.763768	-1.4836	1.983597
X1X2	-7.25	0.817771	-8.86556	1.43E-07	-8.9836	-5.5164

Adjusted R²

$$R^2 = \frac{\text{SSR}}{\text{SST}} = 1 - \frac{\text{SSE}}{\text{SST}} \Rightarrow \text{Adjusted } R^2 = 1 - \frac{\text{SSE/DoF}}{\text{SST/DoF}}$$

迴歸統計	
R 的倍數	0.974606
R 平方	0.949856
調整的 R	0.940455
標準誤	3.657185
觀察值個	20

$$\text{Adjusted } R^2 = 1 - \frac{214/16}{4267.75/19} = 0.940455$$

ANOVA

	自由度	SS	MS	F	顯著值
迴歸	3	4053.75	1351.25	101.028	1.3E-10
殘差	16	214	13.375		
總和	19	4267.75			

	係數	標準誤	t 統計	P-值	下限 95%	上限 95%
截距	29.75	0.817771	36.37936	8.18E-17	28.0164	31.4836
X1	12.25	0.817771	14.97974	7.8E-11	10.5164	13.9836
X2	0.25	0.817771	0.305709	0.763768	-1.4836	1.983597
X1X2	-7.25	0.817771	-8.86556	1.43E-07	-8.9836	-5.5164

Two-Way ANOVA

- Example: agriculture experiments
  - Three tomato varieties (*i*): Harvester, Pusa Early Dwarf and Ife No. 1
  - Four plantation densities (*j*): 10, 20, 30, and 40 thousand plants per hectare)
  - Three replicates (*k*)
- Experimental results: x_{ijk}

Variety	Plantation Density 10k	20k	30k	40k	sum	$X_{i\bullet\bullet}$
H	10.5, 9.2, 7.9	12.8, 11.2, 13.3	12.1, 12.6, 14.0	10.8, 9.1, 12.5	136.0	11.3
Ife	8.1, 8.6, 10.1	12.7, 13.7, 11.5	14.4, 15.4, 13.7	11.3, 12.5, 14.5	146.5	12.21
P	16.1, 15.3, 17.5	16.6, 19.2, 18.5	20.8, 18.0, 21.0	18.4, 18.9, 17.2	217.5	18.13
sum	103.3	129.5	142.0	125.2	500.00	
$X_{\bullet j\bullet}$	11.48	14.39	15.78	13.91		13.89

Sum of Squares

$i = 1, \cdots, I$
$j = 1, \cdots, J$
$k = 1, \cdots, K$
$N = \#\ of\ total\ trials (= IJK)$
$X_{\bullet\bullet\bullet} = \sum_i \sum_j \sum_k X_{ijk} / N$
$X_{ij\bullet} = \sum_k X_{ijk} / K$
$X_{i\bullet\bullet} = \sum_j \sum_k X_{ijk} / (N / I)(= JK)$
$X_{\bullet j\bullet} = \sum_i \sum_k X_{ijk} / (N / J)(= IK)$

$$\text{SS(Total)} = \sum_i \sum_j \sum_k (X_{ijk} - X_{\bullet\bullet\bullet})^2$$

$$\text{SSE} = \sum_i \sum_j \sum_k (X_{ijk} - X_{ij\bullet})^2$$

$$\text{SS(Variety)} = \sum_i \sum_j \sum_k (X_{i\bullet\bullet} - X_{\bullet\bullet\bullet})^2 = \frac{N}{I} \sum_i (X_{i\bullet\bullet} - X_{\bullet\bullet\bullet})^2$$

$$\text{SS(Density)} = \sum_i \sum_j \sum_k (X_{\bullet j\bullet} - X_{\bullet\bullet\bullet})^2 = \frac{N}{J} \sum_j (X_{\bullet j\bullet} - X_{\bullet\bullet\bullet})^2$$

$$\text{SS(Interaction)} = \sum_i \sum_j \sum_k [(X_{ij\bullet} - X_{\bullet\bullet\bullet}) - (X_{i\bullet\bullet} - X_{\bullet\bullet\bullet}) - (X_{\bullet j\bullet} - X_{\bullet\bullet\bullet})]^2$$

$$= \sum_i \sum_j \sum_k (X_{ij\bullet} - X_{i\bullet\bullet} - X_{\bullet j\bullet} + X_{\bullet\bullet\bullet})^2 = \frac{N}{IJ} \sum_i \sum_j (X_{ij\bullet} - X_{i\bullet\bullet} - X_{\bullet j\bullet} + X_{\bullet\bullet\bullet})^2$$

$$\text{SS(Total)} = \text{SS(Variety)} + \text{SS(Density)} + \text{SS(Interaction)} + \text{SSE}$$

Mean Squares

$$\text{MSE} = \sum_i \sum_j \sum_k (X_{ijk} - X_{ij\bullet})^2 / (K-1)IJ$$

$$\text{MS(Variety)} = \sum_i \sum_j \sum_k (X_{i\bullet\bullet} - X_{\bullet\bullet\bullet})^2 / (I-1) = \frac{N}{I} \sum_i (X_{i\bullet\bullet} - X_{\bullet\bullet\bullet})^2 / (I-1)$$

$$\text{MS(Density)} = \sum_i \sum_j \sum_k (X_{\bullet j\bullet} - X_{\bullet\bullet\bullet})^2 / (J-1) = \frac{N}{J} \sum_j (X_{\bullet j\bullet} - X_{\bullet\bullet\bullet})^2 / (J-1)$$

$$\text{MS(Interaction)} = \sum_i \sum_j \sum_k (X_{ij\bullet} - X_{i\bullet\bullet} - X_{\bullet j\bullet} + X_{\bullet\bullet\bullet})^2 / (I-1)(J-1)$$

$$= \frac{N}{IJ} \sum_i \sum_j (X_{ij\bullet} - X_{i\bullet\bullet} - X_{\bullet j\bullet} + X_{\bullet\bullet\bullet})^2 / (I-1)(J-1)$$

ANOVA Table

Source of variation	Sum of Squares	*DOF*	Mean square	F_{calc}
Variety	327.60	2	163.8	163.8/1.59
Density	86.69	3	28.9	28.9/1.59
Interaction	8.03	6	1.34	1.34/1.59
Pure error (within tests)	38.04	24	1.59	
Total	460.36	35		

$F_{2,24,0.99} = 5.61 \quad F_{3,24,0.99} = 4.241 \quad F_{6,24,0.99} = 3.63$

17

Two-Way ANOVA in Excel

		Plantation Density			
		10K	20K	30K	40K
Variety	Harvester	10.5	12.8	12.1	10.8
		9.2	11.2	12.6	9.1
		7.9	13.3	14	12.5
	Ife No.1	8.1	12.7	14.4	11.3
		8.6	13.7	15.4	12.5
		10.1	11.5	13.7	14.5
	Pusa Early Dwarf	16.1	16.6	20.8	18.4
		15.3	19.2	18	18.9
		17.5	18.5	21	17.2

ANOVA

變源	SS	自由度	MS	F	P-值	臨界值
樣本	327.5972	2	163.7986	103.343	1.61E-12	3.402826
欄	86.68667	3	28.89556	18.23063	2.21E-06	3.008787
交互作用	8.031667	6	1.338611	0.84455	0.548361	2.508189
組內	38.04	24	1.585			
總和	460.3556	35				

18

Additive Model with Interactions

Let $\mu = \sum_i \sum_j \mu_{ij} / IJ \quad \mu_{i\bullet} = \sum_j \mu_{ij} / J \quad \mu_{\bullet j} = \sum_i \mu_{ij} / I$

$\alpha_i = \mu_{i\bullet} - \mu$ = effect of factor *A* at level *i*

$\beta_j = \mu_{\bullet j} - \mu$ = effect of factor *B* at level *j*

$\gamma_{ij} = (\mu_{ij} - \mu) - \alpha_i - \beta_j$ = interaction effect of *A* at level *i* and *B* at level *j*

Then, the additive linear model is:

$X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}$

19

ANOVA Table for All Terms (Back to Text-book Example)

Source of variation	Sum of Squares	*DOF*	Mean square	F_{calc}
E_1	3,001.25	1	3,001.25	**224.39***
E_2	1.25	1	1.25	0.09
E_{12}	1,051.25	1	1,051.25	**78.60***
Pure error (within tests)	214.00	16	13.375	
Total	**4,267.75**	**19**		

20

ANOVA Table for Partial Model

- Partial model: $\hat{y} = \hat{b}_0 + \hat{b}_1 x_1 + \hat{b}_{12} x_1 x_2$

Source of variation	Sum of Squares	*DOF*	Mean square	F_{calc}
E_1	3,001.25	1	3,001.25	**224.39***
E_{12}	1,051.25	1	1,051.25	**78.60***
Residual E_2	1.25	1	1.25	0.09
Pure error	214.00	16	13.375	
Total	**4,267.75**	**19**		

$$R^2 = \frac{\text{SSR}}{\text{SST}} = \frac{4052.5}{4267.75} = 0.949564$$

Linear Regression for the Partial Model

	X1	X1X2	Y
Replicate 1	-1	1	11
	1	-1	48
	-1	-1	31
	1	1	37
Replicate 2	-1	1	7
	1	-1	43
	-1	-1	24
	1	1	33
Replicate 3	-1	1	10
	1	-1	52
	-1	-1	27
	1	1	34
Replicate 4	-1	1	15
	1	-1	55
	-1	-1	23
	1	1	37
Replicate 5	-1	1	7
	1	-1	47
	-1	-1	20
	1	1	34

迴歸統計	
R 的倍數	0.974456
R 平方	0.949564
調整的 R	0.94363
標準誤	3.558337
觀察值個	20

$$R^2 = \frac{\text{SSR}}{\text{SST}} = \frac{4052.5}{4267.75} = 0.949564$$

ANOVA

	自由度	SS	MS	F	顯著值
迴歸	2	4052.5	2026.25	160.029	9.4E-12
殘差	17	215.25	12.66176		
總和	19	4267.75			

	係數	標準誤	t 統計	P-值	下限 95%	上限 95%	下限 95.0%
截距	29.75	0.795668	37.38995	9.12E-18	28.07128	31.42872	28.07128
X1	12.25	0.795668	15.39586	2.05E-11	10.57128	13.92872	10.57128
X1X2	-7.25	0.795668	-9.11184	5.94E-08	-8.92872	-5.57128	-8.92872

Sum of Squares (Interactions)

$i = 1, \cdots, I$ (Factor 1)
$j = 1, \cdots, J$ (Factor 2)
$l = 1, \cdots, L$ (Factor 3)
$k = 1, \cdots, K$ (Replicates)
$N = \#\ of\ total\ trials (=IJLK)$

$$X_{\bullet\bullet\bullet\bullet} = \sum_i \sum_j \sum_l \sum_k X_{ijlk} / N$$

$$X_{i\bullet\bullet\bullet} = \sum_j \sum_l \sum_k X_{ijlk} / (N/I)(= JLK)$$

$$X_{ij\bullet\bullet} = \sum_l \sum_k X_{ijlk} / (N/IJ)(= LK)$$

$$X_{ijl\bullet} = \sum_k X_{ijlk} / K$$

$$\text{SS}(ij \text{ interaction}) = \frac{N}{IJ} \sum_i \sum_j [X_{ij\bullet\bullet} - X_{i\bullet\bullet\bullet} - X_{\bullet j\bullet\bullet} + X_{\bullet\bullet\bullet\bullet}]^2$$

$$= \frac{N}{IJ} \sum_i \sum_j [(X_{ij\bullet\bullet} - X_{\bullet\bullet\bullet\bullet}) - (X_{i\bullet\bullet\bullet} - X_{\bullet\bullet\bullet\bullet}) - (X_{\bullet j\bullet\bullet} - X_{\bullet\bullet\bullet\bullet})]^2$$

$$\text{SS}(il \text{ interaction}) = \frac{N}{IL} \sum_i \sum_l [X_{i\bullet l\bullet} - X_{i\bullet\bullet\bullet} - X_{\bullet\bullet l\bullet} + X_{\bullet\bullet\bullet\bullet}]^2$$

$$= \frac{N}{IL} \sum_i \sum_l [(X_{i\bullet l\bullet} - X_{\bullet\bullet\bullet\bullet}) - (X_{i\bullet\bullet\bullet} - X_{\bullet\bullet\bullet\bullet}) - (X_{\bullet\bullet l\bullet} - X_{\bullet\bullet\bullet\bullet})]^2$$

$$\text{SS}(jl \text{ interaction}) = \frac{N}{JL} \sum_j \sum_l [X_{\bullet jl\bullet} - X_{\bullet j\bullet\bullet} - X_{\bullet\bullet l\bullet} + X_{\bullet\bullet\bullet\bullet}]$$

$$= \frac{N}{JL} \sum_j \sum_l [(X_{\bullet jl\bullet} - X_{\bullet\bullet\bullet\bullet}) - (X_{\bullet j\bullet\bullet} - X_{\bullet\bullet\bullet\bullet}) - (X_{\bullet\bullet l\bullet} - X_{\bullet\bullet\bullet\bullet})]^2$$

SS(ijl interaction)

$$\begin{aligned}
&= K \sum_i \sum_j \sum_l \{(X_{ijl\bullet} - X_{\bullet\bullet\bullet\bullet}) - (X_{i\bullet\bullet\bullet} - X_{\bullet\bullet\bullet\bullet}) - (X_{\bullet j\bullet\bullet} - X_{\bullet\bullet\bullet\bullet}) - (X_{\bullet\bullet l\bullet} - X_{\bullet\bullet\bullet\bullet}) \\
&\qquad - [(X_{ij\bullet\bullet} - X_{\bullet\bullet\bullet\bullet}) - (X_{i\bullet\bullet\bullet} - X_{\bullet\bullet\bullet\bullet}) - (X_{\bullet j\bullet\bullet} - X_{\bullet\bullet\bullet\bullet})] \\
&\qquad - [(X_{i\bullet l\bullet} - X_{\bullet\bullet\bullet\bullet}) - (X_{i\bullet\bullet\bullet} - X_{\bullet\bullet\bullet\bullet}) - (X_{\bullet\bullet l\bullet} - X_{\bullet\bullet\bullet\bullet})] \\
&\qquad - [(X_{\bullet jl\bullet} - X_{\bullet\bullet\bullet\bullet}) - (X_{\bullet j\bullet\bullet} - X_{\bullet\bullet\bullet\bullet}) - (X_{\bullet\bullet l\bullet} - X_{\bullet\bullet\bullet\bullet})]\}^2 \\
&= K \sum_i \sum_j \sum_l [X_{ijl\bullet} - X_{ij\bullet\bullet} - X_{i\bullet l\bullet} - X_{\bullet jl\bullet} + X_{i\bullet\bullet\bullet} + X_{\bullet j\bullet\bullet} + X_{\bullet\bullet l\bullet} - X_{\bullet\bullet\bullet\bullet}]^2
\end{aligned}$$

Analysis of Surface Defects Data

Factor	Average η by Factor Level (dB) 1	2	3	Degree of Freedom	Sum of Squares	Mean Square	F
A. Temperature	− 24.23	−50.10	− 61.76	2	4427	2214	27
B. Pressure	− 27.55	−47.44	− 61.10	2	3416	1708	21
C. Nitrogen	−39.03	− 55.99	− 41.07	2	1030	515	6.4
D. Silane	− 39.20	− 46.85	−50.04	2	372	186	2.3
E. Settling time	−51.52	− 40.54	− 44.03	2	378	189	2.3
F. Cleaning method	−45.56	− 41.58	− 48.95	2	164†	82	
Error				5	405†	81	
Total				17	10192		
(Error)				(7)	(569)	(81)	

* Overall mean η = −45.36 dB. Underscore indicates starting level.
† Indicates the sum of squares added together to form the pooled error sum of squares shown in parentheses.

Analysis of Thickness Data

Factor	Average η′ by Level (dB) 1	2	3	Degree of Freedom	Sum of Squares	Mean Square	F
A. Temperature	35.12	34.91	24.52	2	440	220	16
B. Pressure	31.61	30.70	32.24	2	7†	3.5	
C. Nitrogen	34.39	27.86	32.30	2	134	67	5.0
D. Silane	31.68	34.70	28.17	2	128	64	4.8
E. Settling time	30.52	32.87	31.16	2	18†	9	
F. Cleaning method	27.04	33.67	33.85	2	181	90.5	6.8
Error				5	96†	19.2	
Total				17	1004	59.1	
(Error)				(9)	(121)	(13.4)	

* Overall mean η′ = 31.52 dB. Underscore indicates starting level.
† Indicates the sum of squares added together to form the pooled error sum of squares shown in parentheses.

25

Analysis of Deposition Rate Data

Factor	Average η″ by Factor Level (dBam) 1	2	3	Degree of Freedom	Sum of Squares	Mean Square	F
A. Temperature	28.76	34.13	39.46	2	343.1	171.5	553
B. Pressure	32.03	34.78	35.54	2	41.0	20.5	66
C. Nitrogen	32.81	35.29	34.25	2	18.7	9.4	30
D. Silane	32.21	34.53	35.61	2	36.3	18.1	58
E. Settling time	34.06	33.99	34.30	2	0.3†	0.2	
F. Cleaning method	33.81	34.10	34.44	2	1.2†	0.6	
Error				5	1.3†	0.26	
Total				17	441.9	25.9	
(Error)				(9)	(2.8)	(0.31)	

* Overall mean η″ = 34.12 dBam. Underscore indicates starting level.
† Indicates the sum of squares added together to form the pooled error sum of squares shown in parentheses.

26

More on Linear Regression

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27

Multiple Regression?

Example: Fuel Consumption

Y=FUEL=1000×FUELC/POP
= motor fuel consumption, gallons per person

X_1=TAX, cents per gallon

X_2=DLIC=NLIC/POP
=proportion of population with driver's licenses.

Six columns of Table 2.1 list values, for each of the 48 contiguous states, of the following quantities:

POP = 1971 population (in thousands).
TAX = 1972 motor fuel tax in cents per gallon.
NLIC = 1971 number of licensed drivers (in thousands).
INC = 1972 per capita personal income in dollars.
ROAD = 1971 length of Federal-aid primary highways, in miles.

STATE	POP	X_1 TAX	NLIC	X_3 INC	X_4 ROAD	FUELC	X_2 DLIC	Y FUEL
1 ME	1029	9.00	540	3571	1976	557	0.525	541
2 NH	771	9.00	441	4092	1250	404	0.572	524
3 VT	462	9.00	268	3865	1586	259	0.580	561
4 MA	5787	7.50	3060	4870	2351	2396	0.529	414
5 RI	968	8.00	527	4399	431	397	0.544	410
6 CN	3082	10.00	1760	5342	1333	1408	0.571	457
7 NY	18366	8.00	8278	5319	11868	6312	0.451	344
8 NJ	7367	8.00	4074	5126	2138	3439	0.553	467
9 PA	11926	8.00	6312	4447	8577	5528	0.529	464
10 OH	10783	7.00	5948	4512	8507	5375	0.552	498
11 IN	5291	8.00	2804	4391	5939	3068	0.530	580
12 IL	11251	7.50	5903	5126	14186	5301	0.525	471
13 MI	9082	7.00	5213	4817	6930	4768	0.574	525
14 WI	4520	7.00	2465	4207	6580	2294	0.545	508
15 MN	3896	7.00	2368	4332	8159	2204	0.608	566
16 IA	2883	7.00	1689	4318	10340	1830	0.586	635
17 MO	4753	7.00	2719	4206	8508	2865	0.572	603
18 ND	632	7.00	341	3718	4725	451	0.540	714
19 SD	579	7.00	419	4716	5915	501	0.724	865
20 NE	1525	8.50	1033	4341	6010	976	0.677	640
21 KS	2258	7.00	1496	4593	7834	1466	0.663	649
22 DE	565	8.00	340	4983	602	305	0.602	540
23 MD	4056	9.00	2073	4897	2449	1883	0.511	464
24 VA	4764	9.00	2463	4258	4686	2604	0.517	547
25 WV	1781	8.50	982	4574	2619	819	0.551	460
26 NC	5214	9.00	2835	3721	4746	2953	0.544	566
27 SC	2665	3.00	1460	3448	5399	1537	0.548	577
28 GA	4720	7.50	2731	3846	9061	2979	0.579	631
29 FA	7259	8.00	4084	4188	5975	4169	0.563	574
30 KY	3299	9.00	1626	3601	4650	1761	0.493	534
31 TN	4031	7.00	2088	3640	6905	2301	0.518	571
32 AL	3510	7.00	1801	3333	6594	1946	0.513	554
33 MS	2263	8.00	1309	3063	6524	1306	0.578	577
34 AR	1978	7.50	1081	3357	4121	1242	0.547	628
35 LA	3720	8.00	1813	3528	3495	1812	0.487	487
36 OK	2634	6.58	1657	3802	7834	1695	0.629	644
37 TX	11649	5.00	6595	4045	17782	7451	0.566	640
38 MT	719	7.00	421	3897	6385	506	0.586	704

Statistical Meaning of Multiple Regression

?

$$\hat{FUEL} = \hat{\beta}_0 + \hat{\beta}_1 TAX + \hat{\beta}_2 DLIC$$

?

$$\hat{FUEL} = 984.0 - 53.11 TAX$$

Removing Effects of DLIC

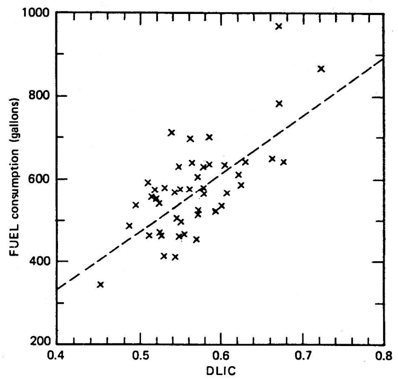


Figure 2.2

$$\hat{FUEL} = 22.73 + 1409.8 DLIC$$

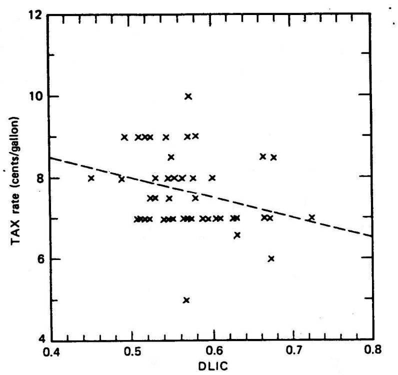


Figure 2.4

$$\hat{TAX} = 10.48 - 4.94 DLIC$$

True Effect of TAX

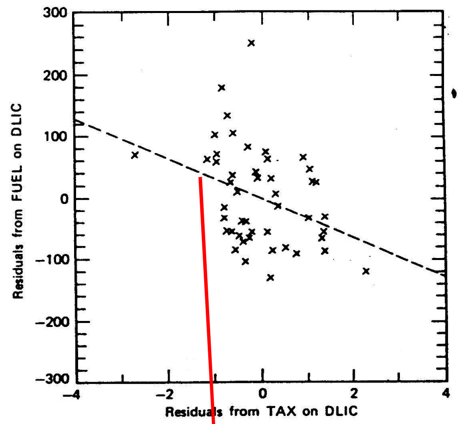


Figure 2.5

$$\hat{FUEL} = 109.0 - 32.08TAX + 1251.5DLIC$$

In Most Designs of Experiments

- X's are designed to be orthogonal (statistically uncorrelated) to one another
- L_9 (3^4)
- 2^3 Factorial

Expt. No.	Column 1	2	3	4
1	1	1	1	1
2	1	2	2	2
3	1	3	3	3
4	2	1	2	3
5	2	2	3	1
6	2	3	1	2
7	3	1	3	2
8	3	2	1	3
9	3	3	2	1

		Main Effects			Interactions			
Test	I	x_1	x_2	x_3	x_1x_2	x_1x_3	x_2x_3	$x_1x_2x_3$
1	+	-1	-1	-1	+1	+1	+1	-1
2	+	+1	-1	-1	-1	-1	+1	+1
3	+	-1	+1	-1	-1	+1	-1	+1
4	+	+1	+1	-1	+1	-1	-1	-1
5	+	-1	-1	+1	+1	-1	-1	+1
6	+	+1	-1	+1	-1	+1	-1	-1
7	+	-1	+1	+1	-1	-1	+1	-1
8	+	+1	+1	+1	+1	+1	+1	+1

32

With Uncorrelated Variables X's

- Back to the example of the Glove Box 2^4 factorial design of experiments

Multiple Regression b_i = Simple Regression b_i

	係數	標準誤	t 統計	P-值	下限 95%	上限 95%	下限 95.0%	上限 95.0%
截距	-0.08719	0.079537	-1.09619	0.28922	-0.2557976	0.08142257	-0.2557976	0.08142257
X1	-0.32719	0.079537	-4.11367	0.000813	-0.4957976	-0.1585774	-0.4957976	-0.15857743
X2	0.397188	0.079537	4.993769	0.000133	0.22857743	0.56579757	0.22857743	0.56579757
X3	0.319063	0.079537	4.011517	0.001007	0.15045243	0.48767257	0.15045243	0.48767257
X4	0.160938	0.079537	2.023439	0.060064	-0.0076726	0.32954757	-0.0076726	0.32954757
X1X2	0.073438	0.079537	0.923317	0.369557	-0.0951726	0.24204757	-0.0951726	0.24204757
X1X3	-0.05844	0.079537	-0.73472	0.473139	-0.2270476	0.11017257	-0.2270476	0.11017257
X1X4	-0.01531	0.079537	-0.19252	0.849756	-0.1839226	0.15329757	-0.1839226	0.15329757
X2X3	-0.09531	0.079537	-1.19835	0.248231	-0.2639226	0.07329757	-0.2639226	0.07329757
X2X4	-0.07719	0.079537	-0.97046	0.346258	-0.2457976	0.09142257	-0.2457976	0.09142257
X3X4	0.004688	0.079537	0.058935	0.953734	-0.1639226	0.17329757	-0.1639226	0.17329757
X1X2X3	0.085938	0.079537	1.080477	0.295948	-0.0826726	0.25454757	-0.0826726	0.25454757
X1X2X4	0.050313	0.079537	0.63257	0.535951	-0.1182976	0.21892257	-0.1182976	0.21892257
X1X3X4	-0.06906	0.079537	-0.86831	0.398062	-0.2376726	0.09954757	-0.2376726	0.09954757
X2X3X4	-0.05219	0.079537	-0.65614	0.521056	-0.2207976	0.11642257	-0.2207976	0.11642257
X1X2X3X	0.060313	0.079537	0.758298	0.459297	-0.1082976	0.22892257	-0.1082976	0.22892257

	係數	*標準誤*	*t 統計*	*P-值*	*下限 95%*	*上限 95%*	*下限 95.0%*	*上限 95.0%*
截距	-0.08719	0.112713	-0.77354	0.445257	-0.31738	0.143003	-0.31738	0.143003
X2	0.397188	0.112713	3.523892	0.001386	0.166997	0.627378	0.166997	0.627378

33

Multicollinearity in Linear Regression

- Example: House Price
  - A real estate agent believes that a house selling price can be predicted using the house size, number of bedrooms, and lot size.
  - A random sample of 100 houses was drawn and data recorded.

Price	Bedrooms	H Size	Lot Size
124100	3	1290	3900
218300	4	2080	6600
117800	3	1250	3750
.	.	.	.
.	.	.	.

  - Analyze the relationship among the four variables

- Solution
- The proposed model is
  PRICE = β_0 **+** β_1**BEDROOMS +** β_2**H-SIZE +**β_3**LOTSIZE +** ε
  - Excel solution

Regression Statistics	
Multiple R	0.74833
R Square	0.559998
Adjusted F	0.546248
Standard E	25022.71
Observatio	100

The model is valid, but no variable is significantly related to the selling price !!

ANOVA

	df	*SS*	*MS*	*F*	*ignificance F*
Regressior	3	7.65E+10	2.55E+10	40.7269	4.57E-17
Residual	96	6.01E+10	6.26E+08		
Total	99	1.37E+11			

	Coefficients	*tandard Err*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	37717.59	14176.74	2.660526	0.009145	9576.963	65858.23
Bedrooms	2306.081	6994.192	0.329714	0.742335	-11577.3	16189.45
H Size	74.29681	52.97858	1.402393	0.164023	-30.8649	179.4585
Lot Size	-4.36378	17.024	-0.25633	0.798244	-38.1562	29.42862

- However,
  - when regressing the price on each independent variable alone, it is found that each variable is strongly related to the selling price.
  - Multicollinearity is the source of this problem.

	Price	*Bedrooms*	*H Size*	*Lot Size*
Price	1			
Bedrooms	0.645411	1		
H Size	0.747762	0.846454	1	
Lot Size	0.740874	0.83743	0.993615	1

- Multicollinearity causes two kinds of difficulties:
  - The t statistics appear to be too small.
  - The β coefficients cannot be interpreted as “slopes”.

Nonlinearity in Linear Regression

- Regression analysis is considered powerful for several reasons:
  - It can cover variety of mathematical models
    - linear relationships.
    - non - linear relationships.
    - qualitative variables.
  - It provides efficient methods for model building, to select the best fitting set of variables.

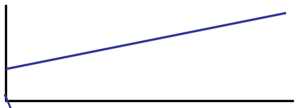
Polynomial Models

- The independent variables may appear as functions of a number of predictor variables.
  - Polynomial models of order p with one predictor variable: $y = \beta_0 + \beta_1 x + \beta_2 x^2 + \ldots + \beta_p x^p + \varepsilon$
  - Polynomial models with two predictor variables
    For example:
    $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$
    $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 + \varepsilon$

- Polynomial models with one predictor variable

  – First order model (p = 1)

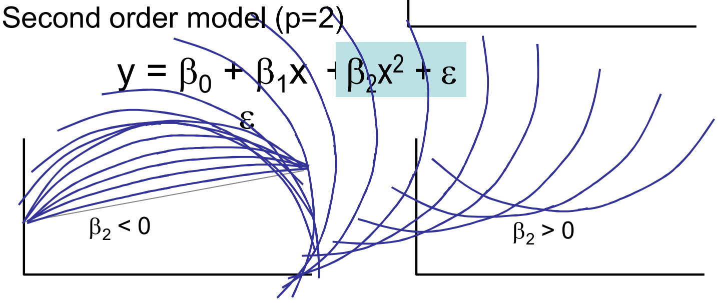
$$y = \beta_0 + \beta_1 x + \varepsilon$$



  – Second order model (p=2)

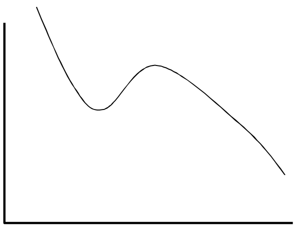
$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \varepsilon$$

ε

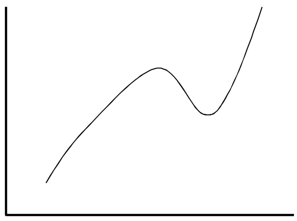


– Third order model (p=3)

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \varepsilon$$



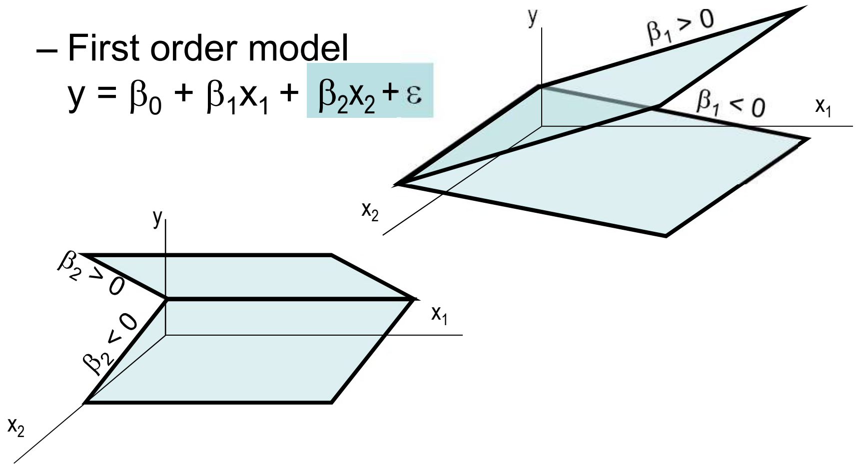
$\beta_3 < 0$



$\beta_3 > 0$

- Polynomial models with two predictor variables

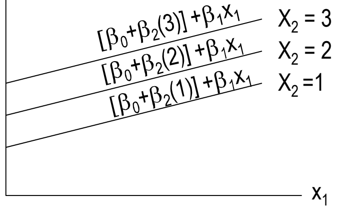
  – First order model
  $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$



- Polynomial models with two predictor variables

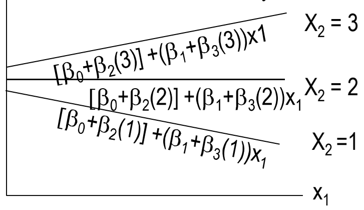
  – First order model
  $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon$

The effect of one predictor variable on y is independent of the effect of the other predictor variable on y.

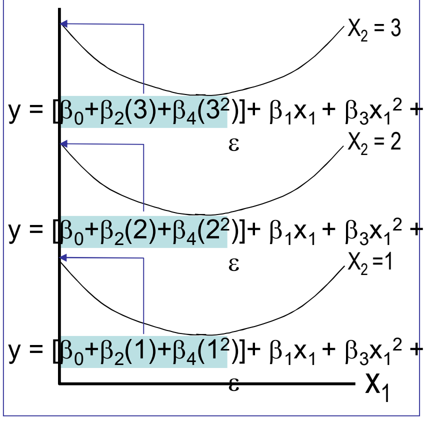


  – First order model with interaction
  $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 + \varepsilon$

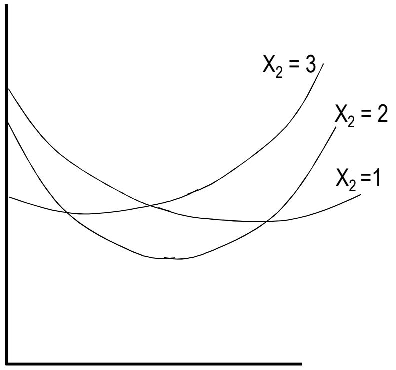
The two variables interact to affect the value of y.



– Second order model
$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1^2 + \beta_4 x_2^2 + \varepsilon$

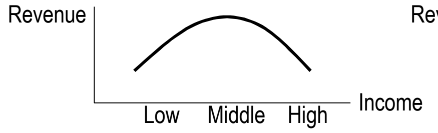


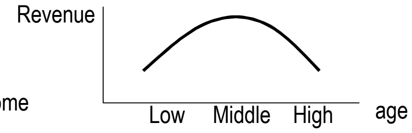
– Second order model with interaction
$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1^2 + \beta_4 x_2^2 + \beta_5 x_1 x_2 + \varepsilon$



- Example Location for a new restaurant
  - A fast food restaurant chain tries to identify new locations that are likely to be profitable.
  - The primary market for such restaurants is middle-income adults and their children (between the age 5 and 12).
  - Which regression model should be proposed to predict the profitability of new locations?

- Solution
  - The dependent variable will be **Gross Revenue**
  - There are quadratic relationships between Revenue and each predictor variable. Why?
    - Members of middle-class families are more likely to visit a fast food family than members of poor or wealthy families.

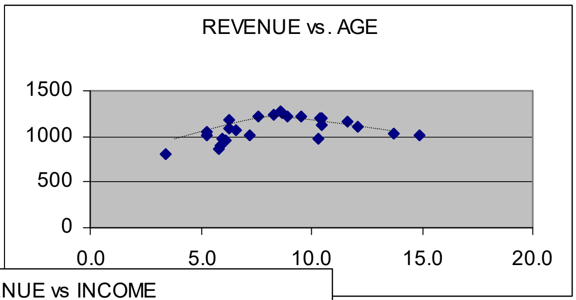


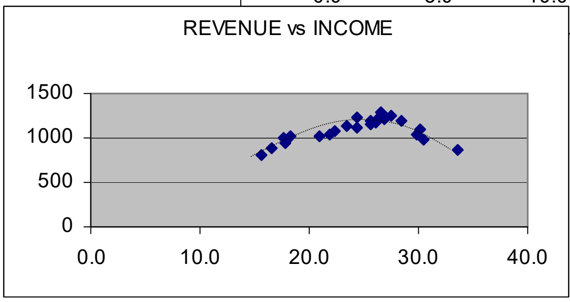


- Families with very young or older kids will not visit the restaurant as frequent as families with mid-range ages of kids.

$$\textbf{Revenue} = \beta_0 + \beta_1\textbf{Income} + \beta_2\textbf{Age} + \beta_3\textbf{Income}^2 + \beta_4\textbf{Age}^2 + \beta_5(\textbf{Income})(\textbf{Age}) + \varepsilon$$

- Example
  - To verify the validity of the model proposed in example, 25 areas with fast food restaurants were randomly selected.
  - Data collected included (see Xm19-02.xls):
    - Previous year's annual gross sales.
    - Mean annual household income.
    - Mean age of children





SUMMARY OUTPUT

Regression Statistics	
Multiple R	0.952121
R Square	0.906535
Adjusted F	0.881939
Standard E	44.69533
Observatio	25

The model provides a good fit

ANOVA

	df	*SS*	*MS*	*F*	*ignificance F*
Regressior	5	368140.4	73628.08	36.85692	3.86E-09
Residual	19	37955.78	1997.673		
Total	24	406096.2			

	Coefficients	*tandard Err*	*t Stat*	*P-value*	*Lower 95%*	*Upper 95%*
Intercept	-1133.98	320.0193	-3.54348	0.00217	-1803.79	-464.173
INCOME	173.2032	28.20399	6.141086	6.66E-06	114.1715	232.2348
AGE	23.54996	32.23447	0.730583	0.473947	-43.9176	91.01751
INC sq	-3.72613	0.542156	-6.8728	1.48E-06	-4.86087	-2.59138
AGE sq	-3.86871	1.179054	-3.28119	0.003928	-6.3365	-1.40092
INC X AGE	1.967268	0.944082	2.08379	0.050921	-0.00872	3.943255

The model can be used to make predictions.

However, do not interpret the coefficients or test them. Multicollinearity is a problem!!

In excel: Tools > Data Analysis > Correlation

	INCOME	AGE	INC sq	AGE sq	INC X AGE
INCOME	1				
AGE	0.0201	1			
INC sq	$\boxed{0.9945}$	-0.045	1		
AGE sq	-0.042	$\boxed{0.9845}$	-0.099	1	
INC X AGE	0.4596	$\boxed{0.8861}$	0.3968	$\boxed{0.8405}$	1

The multicollinearity can be reduced by modifying the original predictor variables

Income = **Income - average income**
Age = **Age - average Age**

Income = **Income - 24.2**

Age = **Age - 8.392**

Original data

REVENUE	INCOME	AGE	INC sq	AGE sq	INC X AGE
1128	23.5	10.5	552.25	110.25	246.75
1005	17.6	7.2	309.7	1.84	126.72
1212	26.3	7.6	691.69	57.76	199.88
893	1				
1073	2				
1179	2				
1109	2				
1019	2				
1228	2				
.	.	.	.	.	.
.	.	.	.	.	.

$(-.7)^2$ $(-.7)(2.11)$

REVENUE	Income	Age	Inc-sq	Age sq	Inc X Age
1128	-0.7	2.11	0.49	4.4437	-1.4756
1005	-6.6	-1.2	43.56	1.4209	7.8672
1212	2.1	-0.8	4.41	0.6273	-1.6632
.	.	.	.	.	.
.	.	.	.	.	.

Modified data

Regression results of the modified model

SUMMARY OUTPUT

Multicolinearity is not a problem anymore

Regression Statistics	
Multiple R	0.952121
R Square	0.906535
Adjusted R Sq	0.881939
Standard Error	44.69533
Observations	25

	Income	Age	Inc-sq	Age sq	Inc X Age
Income	1				
Age	0.020058	1			
Inc-sq	-0.17263	-0.61236	1		
Age sq	-0.31219	0.400929	0.096296	1	
Inc X Age	-0.61438	-0.37221	0.167909	-0.08263	1

ANOVA

	df	SS	MS	F	Significance F
Regression	5	368140.4	73628.08	36.85692	3.86E-09
Residual	19	37955.78	1997.673		
Total	24	406096.2			

	Coefficients	Standard Err	t Stat	P-value	Lower 95%	Upper 95%
Intercept	1200.066	15.08728	79.54156	1.91E-25	1168.488	1231.644
Income	9.367849	2.743887	3.41408	0.00291	3.624827	15.11087
Age	6.225473	5.472777	1.137534	0.269457	-5.229186	17.68013
Income sq	-3.72613	0.542156	-6.8728	1.48E-06	-4.860874	-2.59138
Age sq	-3.86871	1.179054	-3.28119	0.003928	-6.336497	-1.40092
Inc X Age	1.967268	0.944082	2.08379	0.050921	-0.008718	3.943255