Linear Algebra and its Applications

HW#4

1. Choose three independent columns of U. Then make two other choices. Do the same for A. You have found bases for which spaces?

$$U = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 0 & 6 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 4 & 6 & 8 & 2 \end{bmatrix}.$$

2. Find a basis for each of these subspaces of $\mathbf{R}^4$:
   (a) All vectors whose components are equal.
   (b) All vectors whose components add to zero.
   (c) All vectors that are perpendicular to (1,1,0,0) and (1,0,1,1).
   (d) The column space (in $\mathbf{R}^2$) and nullspace (in $\mathbf{R}^5$) of

$$U = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

3. The nullspace of a 4 by 3 matrix A is the line through $(2,3,0)^T$.
   (a) What is the rank of A and the complete solution to Ax=0?
   (b) What is the exact row reduced echelon form U of A?

4. Prove that if either d=0 or f=0 (2 cases), the columns of U are dependent:

$$U = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

5. By performing the elimination to A and b so that A is reduced to a echelon form:

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}$$

   (a) What is the basis for the null space?
   (b) What is the basis for the left-null space?
   (c) What is the basis for the row space?
   (d) What is the basis for the column space?
   (e) Show that the inner product between any vector in the left-null space and

any vector in the column space is zero.

6. Find a 1 by 3 matrix whose nullspace consists of all vectors in R^3 such that $x_1+3x_3=0$. Find a 3 by 3 matrix with that same nullspace.

7. A is an m by n matrix of rank r. Suppose there are right-hand sides b for which $Ax = b$ has no solution.
   (a) What inequalities ($<$ or $\leq$) must be true between m, n, and r?
   (b) How do you know that $A^T y = 0$ has a nonzero solution?

8. If A is 2 by 3 and C is 3 by 2, show from its rank that there exit no C such that $CA=I$.

9. Calculate $(A^TA)^{-1}A^T$ or $A^T(AA^T)^{-1}$ and find a left-inverse and/or a right-inverse (when they exist) for

$$A=\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix} \quad \text{and} \quad M=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix} \quad \text{and} \quad T=\begin{bmatrix}a&b\\0&a\end{bmatrix}$$

10. Find the rank of the following A. If the matrix is written as $A=uv^T$ what are u and v?

$$A=\begin{bmatrix}1&0&0&3\\0&0&0&0\\2&0&0&6\end{bmatrix} \text{ and } A=\begin{bmatrix}2&-2\\6&-6\end{bmatrix}$$