Chapter 9 Kinematic Analysis of Spatial Mechanisms*

9.1 Introduction

It was shown that vector methods could be applied to a wide variety of problems in planar kinematic analysis. In this chapter, we will introduce matrix notation as a convenience in numerical calculations, particularly in those problems involving a spatial motion.

9.2 Position, Orientation and Location of a Rigid Body

In the study of the kinematics of robot manipulators, we are constantly dealing with the location of several bodies in space. The bodies of interest include the links of a manipulator, the tools, the work pieces, and so on. To identify the location of a body, a reference coordinate system is established. We called this reference coordinate system the fixed frame, although in reality, it may not necessarily be fixed to the ground. In what follows we employ a Cartesian coordinate system to describe the location of the a body, although other types of coordinate systems, such as the cylindrical coordinate system and spherical coordinate system, may also be used.

The location of a body with respect to a reference coordinate system is known if the positions of all the points of the body are known. If the body of interest is rigid, six independent parameters would be sufficient to describe its location in three-dimensional space. As shown in Fig. 9.1, we take the (x, y, z) coordinate system as the fixed frame. We also attach a (u, v, w) Cartesian coordinate system to the moving body and refer to it as the moving frame. Clearly, the positions of all the points of the rigid can be determined when the location of the moving frame with respect to the fixed frame is

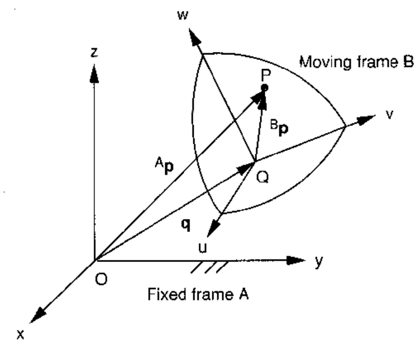

Figure 9.1 General spatial displacement

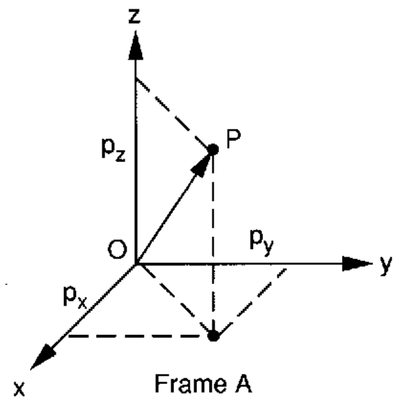

Figure 9.2 Position vector of a point P in space

known. This relative location can be considered as composed of the position of a point, say the origin Q, and the orientation of the moving frame with respect to the fixed frame initially, the location of the moving frame with respect to the fixed frame and the spatial displacement of a rigid body from the initial coincident position are equivalent.

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* Most of the material in this chapter is adopted from reference [1].

9.2.1 Description of a Position

The position of any point with respect to the reference frame can be described by a 3×1 position vector. For example, the position of a point P in the reference frame A as shown in Fig. 9.2 is written as

$$ {}^{A}\mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \qquad (9.1) $$

where the subscripts x, y and z represent the projections of the position vector onto the three coordinate axes of the reference frame. Since we are dealing with several coordinate systems, a leading superscript is used to indicate the coordinate system to which the vector is referred.

9.2.2 Description of an Orientation

The orientation of a rigid body with respect to the fixed frame can be described in several different ways. In what follows we first describe the direction cosine representation followed by the screw axis representation and then the Euler angle representation. To describe the orientation of a rigid body, we consider the motion of a moving frame B with respect to a fixed frame A with one point fixed. This is known as a rotation or a spherical motion. Without losing generality, we assume that the origin of the moving frame is fixed to that of the fixed frame, as shown Fig. 9.3.

(a) Direction Cosine Representation

One convenient way of describing the orientation of a rigid body is by means of the direction cosines of the coordinate axes of the moving frame with respect to the fixed frame. Let **i**, **j**, and **k** denote three unit vectors pointing along the coordinate axes of the fixed frame A, and **u**, **v**, and **w** denote three unit vectors pointing along the coordinate axes of the moving frame B, respectively, as shown in Fig. 9.3. The three unit vectors **u**, **v**, **w** can be expressed in the fixed frame A as follows:

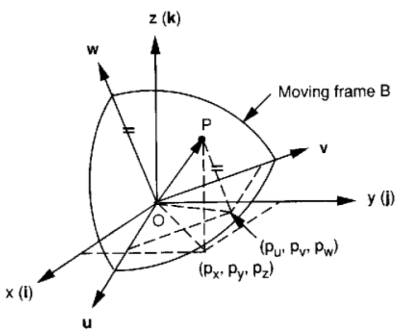


FIGURE 1.24. Spherical displacement.

Figure 9.3 Spherical displacement

$$ {}^{A}\mathbf{u} = u_x\,\mathbf{i} + u_y\,\mathbf{j} + u_z\,\mathbf{k} $$
$$ {}^{A}\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathrm{j} + v_z\,\mathbf{k} $$
$$ {}^{A}\mathbf{w} = w_x\,\mathbf{i} + w_y\,\mathbf{j} + w_z\,\mathbf{k} \qquad (9\text{-}2) $$

The position vector of a point P of the rigid body can be expressed either in the fixed frame A as

$$ {}^{A}\mathbf{p} = p_x\,\mathbf{i} + p_y\,\mathbf{j} + p_z\,\mathbf{k} \qquad (9\text{-}3) $$

Or in the rotated frame B as

$$ {}^{B}\mathbf{p} = p_u\,\mathbf{u} + p_v\,\mathbf{v} + p_w\,\mathbf{w} \qquad (9\text{-}4) $$

Since P is a point of the rigid body, ${}^{B}\mathbf{p}$ is constant. However, ${}^{A}\mathbf{p}$ depends on the orientation of B relative to A. Substituting Eq. (9-2) in (9-4), we obtain the vector **p** expressed in the fixed frame A as

$$ {}^{A}\mathbf{p} = (p_u u_x + p_v v_x + p_w w_x)\mathbf{i} + (p_u u_y + p_v v_y + p_w w_y)\mathbf{j} + (p_u u_z + p_v v_z + p_w w_z)\mathbf{k} \qquad (9\text{-}5) $$

Equating the x, y, and z components of ${}^{A}\mathbf{p}$ in Eq. (9-5) to the corresponding components in (9-3) yields

$$ \begin{aligned} p_x &= u_x p_u + v_x p_v + w_x p_w \\ p_y &= u_y p_u + v_y p_v + w_y p_w \\ p_z &= u_z p_u + v_z p_v + w_z p_w \end{aligned} \qquad (9\text{-}6) $$

Writing Eq. (9-6) in a matrix from we obtain

$$ {}^{A}\mathbf{p} = {}^{A}R_{B}\, {}^{B}\mathbf{p} \qquad (9\text{-}7) $$

Where

$$ {}^{A}R_{B} = \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad (9\text{-}8) $$

where the leading superscript A and the trailing subscript B indicate the order of transformation. In what follows we omit the leading superscript and trailing subscript from time to time whenever there are only two frames of reference and the order of transformation is clear.

We call the matrix ${}^{A}R_{B}$ the rotation matrix of the moving frame B with respect to the fixed frame A. The rotation matrix specifies the orientation of B completely with respect to A. It transforms the position vector of any point P from the moving frame B to the fixed frame A. From the definition above, we see that the columns of a rotation matrix represent three orthogonal unit vectors of the moving coordinate axes expressed in the fixed frame. Therefore, the rotation matrix is orthogonal. The orthogonal conditions can be stated as

$$ \begin{aligned} \mathbf{u}^2 &= 1 \\ \mathbf{v}^2 &= 1 \\ \mathbf{w}^2 &= 1 \end{aligned} \qquad (9\text{-}9) $$

and

$$ \begin{aligned} \mathbf{u}^T \mathbf{v} &= 0 \\ \mathbf{v}^T \mathbf{w} &= 0 \\ \mathbf{w}^T \mathbf{u} &= 0 \end{aligned} \qquad (9\text{-}10) $$

Because of the orthogonality conditions above, only three of the nine elements of ${}^{A}R_{B}$ are independent. Using the orthogonality conditions above, it can be shown that

$$ \begin{aligned} \mathbf{u} \times \mathbf{v} &= \mathbf{w} \\ \mathbf{v} \times \mathbf{w} &= \mathbf{u} \\ \mathbf{w} \times \mathbf{u} &= \mathbf{v} \end{aligned} \qquad (9\text{-}11) $$

It can also be shown that

$$ \det({}^{A}R_{B}) = 1 \qquad (9\text{-}12) $$

Furthermore, due to the orthogonality conditions, the inverse transformation of a rotation matrix is equal to its transpose：

$$^{B}R_{A} = {}^{A}R_{B}^{-1} = {}^{A}R_{B}^{T} \qquad (9\text{-}13)$$

Since the columns of $^{A}R_{B}$ represent three unit vectors of the coordinate axes of frame B expressed in frame A, it follows that the rows of $^{A}R_{B}$ represent the three unit vectors defined along the coordinate axes of frame A and expressed in frame B. Therefore, the rotation matrix can be interpreted as a set of three column vectors or a set of three row vectors：

$$^{A}R_{B} = [\ {}^{A}\mathbf{u} \quad {}^{A}\mathbf{v} \quad {}^{A}\mathbf{w}\ ] = \begin{bmatrix} {}^{B}i^{T} \\ {}^{B}j^{T} \\ {}^{B}k^{T} \end{bmatrix} \qquad (9\text{-}14)$$

Euler's Theorem

Euler's theorem states that the general displacement of a rigid body with one point fixed is a rotation about some axis. This unique axis of rotation is called the screw axis. In what follows, we show how the screw axis can be derived from a given rotation matrix.

Equation (9-7) provides an orthogonal transformation of the position of a point P in a moving frame B to a fixed frame A. Since the moving frame B coincides with the fixed frame A at the initial location, we may consider $^{B}\mathbf{p}$ as the first position and $^{A}\mathbf{p}$ as the second position of P of the rigid body B. Since the origin O is a stationary point, the screw axis passes through this point. Furthermore, if $\tilde{\mathbf{p}}$ lies on the axis of rotation, its position vector will not be affected by the rotation; that is

$$^{B}\tilde{\mathbf{p}} = {}^{A}\tilde{\mathbf{p}} \qquad (9\text{-}15)$$

The second position of P is governed by the orthogonal transformation given by Eq. (9-7) Substituting Eq.(9-15) into (9-7) and rearranging yields

$$({}^{A}R_{B} - I)\ {}^{A}\tilde{\mathbf{p}} = 0 \qquad (9\text{-}16)$$

where I denotes a 3×3 identity matrix.

We note that Eq. (9-16) is a special case of the following general eigenvalue problem：

$$({}^{A}R_{B} - \lambda I)\ {}^{A}\tilde{\mathbf{p}} = 0 \qquad (9\text{-}17)$$

where λ is called the eigenvalue value or characteristic value. Equation (9-17) consists of three linear homogeneous equations in three unknown, $\tilde{p}_x$, $\tilde{p}_y$, and $\tilde{p}_z$. The compatibility condition for the existence of nontrivial solutions is that the determinant of the coefficients must vanish; that is,

$$|{}^{A}R_{B} - \lambda I| = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix} = 0 \qquad (9\text{-}18)$$

where a_{ij} represents the (i, j) element of $^{A}R_{B}$.

Equation (9-18) is known as the characteristic equation and the values of λ for which the equation is satisfied are the eigenvalues. In general, the characteristic equation will have three roots with three corresponding eigenvectors. Expanding Eq. (9-18), say along the first column of ${}^{A}R_{B}$ and applying Eqs. (9-11) and (9-12), we obtain

$$\lambda^3 - \mathrm{tr}({}^{A}R_{B})\,\lambda^2 + \mathrm{tr}({}^{A}R_{B})\,\lambda - 1 = 0. \qquad (9\text{-}19)$$

where $\mathrm{tr}({}^{A}R_{B}) = a_{11}+a_{22}+a_{33}$.

We note that Eq.(9-19) contains λ-1 as a factor. Solving Eq.(9-19) yields

$$\lambda = 1,\ e^{i\theta},\ e^{-i\theta}$$

where

$$\theta = \cos^{-1}[(a_{11}+a_{22}+a_{33}-1)/2] \qquad (9\text{-}20)$$

defines the angle of rotation about the screw axis and where $\cos^{-1}x$ is the arccosine function. The eigenvector corresponding to the eigenvalue of λ=1 gives the direction of the screw axis. Hence the direction of the screw axis is obtained by solving Eq.(9-16) for the ratio $\tilde{p}_x / \tilde{p}_y / \tilde{p}_z$.

(b) Screw Axis Representation. In this section we seek a description of the orientation of a rigid body in terms of a rotation about a screw axis. As shown in Fig. 9.4, let the moving frame B be rotated an angle θ about an axis passing through the origin of the fixed frame A. The first position of a point P of the rigid body B is denoted by the vector $\mathbf{r}_1 = \overline{OP_1}$. The second position is denoted by $\mathbf{r}_2 = \overline{OP_2^r}$, and the direction of rotation is denoted by a unit vectors $\mathbf{s}(s_x, s_y, s_y)$. From the geometry of the figure, we obtain

$$\overline{S_pP_1} = \mathbf{r}_1 - (\mathbf{r}_1^T\mathbf{s})\mathbf{s}, \qquad (9\text{-}21)$$

$$\overline{S_pP_2^r} = \mathbf{r}_2 - (\mathbf{r}_2^T\mathbf{s})\mathbf{s}, \qquad (9\text{-}22)$$

Figure 9.5 shows a plane that contains both points P_1 and P_2^r and is normal to the axis of rotation. The point of intersection between the plane and the axis of rotation is taken as S_p.

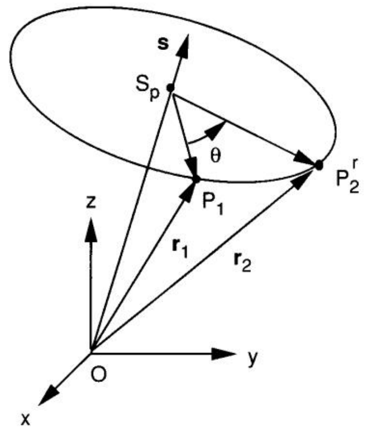


Figure 9.4 Spherical displacement

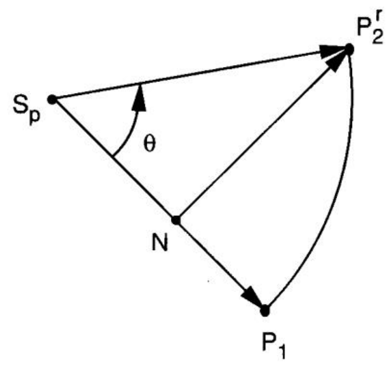


Figure 9.5 Plane normal to the axis of rotation

Let ($\overline{NP_2^r} \perp \overline{S_pP_1}$). Then, using of the fact that $|S_pP_1| = |S_pP_2^r|$ and $\mathbf{s}\times\overline{S_pP_1} = \mathbf{s}\times\mathbf{r}_1$, we have

$$\overline{S_pN} = \overline{S_pP_1}\, c\theta, \tag{9-23}$$

$$\overline{NP_2^r} = \mathbf{s}\times\mathbf{r}_1\, s\theta, \tag{9-24}$$

where $c\theta$ is a shorthand notation for $\cos\theta$ and $s\theta$ for $\sin\theta$. To derive a relation between $\mathbf{r}_2$ and $\mathbf{r}_1$, we express $\overline{S_pP_2^r}$ as a sum of two vectors:

$$\overline{S_pP_2^r} = \overline{S_pN} + \overline{NP_2^r}. \tag{9-25}$$

Substituting Eqs. (9-21) through (9-24) into (9.25), we obtain

$$\mathbf{r}_2 - (\mathbf{r}_2^T\mathbf{s})\mathbf{s} = [\mathbf{r}_1 - (\mathbf{r}_1^T\mathbf{s})\mathbf{s}]\, c\theta + \mathbf{s}\times\mathbf{r}_1 s\theta. \tag{9-26}$$

Substituting $\mathbf{r}_1^T\mathbf{s}=\mathbf{r}_2^T\mathbf{s}$ into Eq.(9-26) and rearranging yields

$$\mathbf{r}_2 = \mathbf{r}_1 c\theta + \mathbf{s}\times\mathbf{r}_1 s\theta + \mathbf{s}(\mathbf{r}_1^T\mathbf{s})(1-c\theta). \tag{9-27}$$

Equation (9-27) is known as *Rodrigue's formula* for a spherical displacement of a rigid body. By considering $\mathbf{r}_1$ as ${}^B\mathbf{p}$ and $\mathbf{r}_2$ as ${}^A\mathbf{p}$, Eq. (9-27) can be written in matrix form as

$${}^A\mathbf{p} = {}^AR_B\, {}^B\mathbf{p}, \tag{9-28}$$

where the elements of the rotation matrix are given by:

$$
\begin{aligned}
a_{11}&=(s_x^2-1)(1-\mathrm{c}\theta)+1,\\
a_{12}&=s_x s_y(1-\mathrm{c}\theta)-s_z\,\mathrm{s}\theta,\\
a_{13}&=s_x s_z(1-\mathrm{c}\theta)+s_y\,\mathrm{s}\theta,\\
a_{21}&=s_y s_x(1-\mathrm{c}\theta)+s_z\,\mathrm{s}\theta,\\
a_{22}&=(s_y^2-1)(1-\mathrm{c}\theta)+1,\\
a_{23}&=s_y s_z(1-\mathrm{c}\theta)-s_x\,\mathrm{s}\theta,\\
a_{31}&=s_z s_x(1-\mathrm{c}\theta)-s_y\,\mathrm{s}\theta,\\
a_{32}&=s_z s_y(1-\mathrm{c}\theta)+s_x\,\mathrm{s}\theta,\\
a_{33}&=(s_z^2-1)(1-\mathrm{c}\theta)+1.
\end{aligned}
\qquad (9\text{-}29)
$$

Equation (9-29) is called the *screw axis representation* of the orientation of a rigid body. This representation uses four parameters: three associated with the direction of the screw axis and one associated with the angle of rotation. However, only two of the three parameters associated with the direction of the screw axis are independent since they must satisfy the condition of a unit vector, $\mathbf{s}^T\mathbf{s}=1$.

Given the screw axis and angle of rotation, we compute the elements of the rotation matrix from Eq.(9-29). On the other hand, given a rotation matrix, we can compute the screw axis and the angle of rotation. The angle of rotation is obtained by summing the diagonal elements of the rotation matrix given by Eq.(9-29):

$$\theta=\pm\cos^{-1}[(a_{11}+a_{22}+a_{33}-1)/2] \qquad (9\text{-}30)$$

The direction of the screw axis is obtained by taking the differences between each pair of two opposing off-diagonal elements:

$$
\begin{aligned}
s_x &= (a_{32}-a_{23})/(2\mathrm{s}\theta),\\
s_y &= (a_{13}-a_{31})/(2\mathrm{s}\theta),\\
s_z &= (a_{21}-a_{12})/(2\mathrm{s}\theta).
\end{aligned}
\qquad (9\text{-}31)
$$

From Eqs.(9-30) and (9-31) it appears that there are two solutions of the screw axis, one being the negative of the other. In reality, these two solutions represent the same screw, since a $-\theta$ rotation about the $-\mathbf{s}$ axis produces the same result as a $+\theta$ rotation about the $\mathbf{s}$ axis.

(c) Euler Angle Representation. The direction cosine representation of an orientation contains nine parameters, and the screw axis representation requires four. Since rotation is a motion with 3 degrees of freedom, a set of three independent parameters are sufficient to describe the orientation of a rigid body in space. Several sets of three-parameter representations have been reported in the literature.

Perhaps, the most commonly used sets are the Euler angle. In an Euler representation, three successive rotations about the coordinate axes of either a fixed coordinate system or a moving coordinate system are used to describe the orientation of a rigid body.

For convenience, we introduce three basic rotation matrices from which an Euler angle representation can be derived in terms of their product. When a rigid body performs a rotation of θ about the z-axis, s_x=s_y=0 and s_z=1.
Hence the rotation matrix, Eq. (9-29), reduces to

$$\mathrm{R}(z,\ \theta)=\begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9-32}$$

Similarly, when a rigid body performs a rotation of ψ about the x-axis, s_y=s_z=0 and s_x=1. Hence the rotation matrix, Eq. (9-29), reduce to

$$\mathrm{R}\ (x,\ \psi)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\psi & -s\psi \\ 0 & s\psi & c\psi \end{bmatrix}, \tag{9-33}$$

and when a rigid body performs a rotation of ϕ about the y-axis, s_x=s_z=0, and s_y=1. Hence the rotation matrix, Eq. (9-29), reduce to

$$\mathrm{R}(y,\ \phi)=\begin{bmatrix} c\varphi & 0 & s\varphi \\ 0 & 1 & 0 \\ -s\varphi & 0 & c\varphi \end{bmatrix} \tag{9-34}$$

Roll-Pitch-Yaw Angle. We first consider three successive rotations of the moving frame B about the coordinate axes of the fixed frame A. Starting with the moving frame B coinciding with the fixed frame A, we rotate B about the x-axis by an angle ψ, resulting in an (u', v', w')system; followed by a second rotation of θ about the y-axis, resulting in an (u'', v'', w'') system; and then a third rotation of ϕ about the z-axis, resulting in the final (u, v, w) system, as shown in Fig. 9.6.

Since all rotation take place about the coordinate axes of a fixed frame, the resulting rotation matrix is obtained by premultiplying three basic rotation matrices:

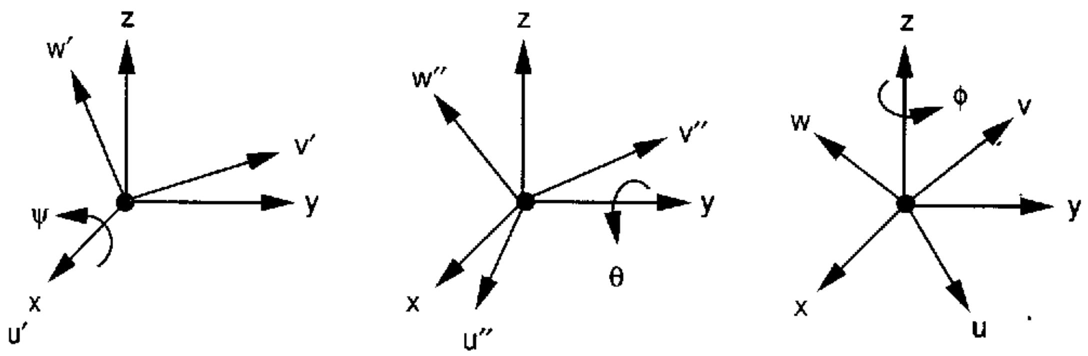


Figure 9.6 Successive rotations about the fixed coordinate axes

R(ψ, θ, ϕ)=R(z, ϕ) R(y, θ) R(x, ψ)

$$=\begin{bmatrix} c\varphi c\theta & c\varphi s\theta s\psi - s\varphi c\psi & c\varphi s\theta c\psi + s\varphi s\psi \\ s\varphi c\theta & s\varphi s\theta s\psi + c\varphi c\psi & s\varphi s\theta c\psi - c\varphi s\psi \\ -s\theta & c\theta s\psi & c\theta c\psi \end{bmatrix} \qquad (9\text{-}35)$$

The rotation about the x-axis is called a *roll*, the rotation about the y-axis is called a pitch, and the rotation about the z-axis is called a yaw. We call the convection of describing the orientation of a rigid body the roll-pitch-yaw angles representation. We note that successive rotations about the fixed coordinate axes result in a premultiplication of the matrices. Since finite rotations do not commute (Goldstein, 1980), the order of rotations cannot be exchanged arbitrarily.

Given the roll, pitch, and yaw angles, we can compute the overall rotation matrix from Eq. (9-35). On the other hand, given a rotation matrix, we can compute the equivalent roll-pitch-yaw angles as follows：

$$\theta = \sin^{-1}(-a_{31}),$$
$$\psi = \mathrm{Atan2}(a_{32}/c\theta,\ a_{33}/c\theta),$$
$$\phi = \mathrm{Atan2}(a_{21}/c\theta,\ a_{11}/c\theta), \qquad (9\text{-}36)$$

provided that $c\theta \neq 0$, where $\sin^{-1}x$ is the arcsine function and Atan2(y, x) is a two-argument arctangent function that yields one unique solution for the angle. Hence corresponding to a given rotation matrix, there are generally two possible solution of the roll-pitch-yaw angles, However, if $\theta = \pm\ 90^0$, the solution of Eq. (9-36) degenerate. For this special case, only the sum or the difference of ϕ and ψ can be computed.

w-u-w **Euler Angles**. Next, we consider three successive rotations of the rigid body about the coordinate axes of a moving frame B. Starting with the moving frame B coinciding with the fixed frame A, we rotate B about the body-attached w-axis by an angle ϕ, followed by a second rotation of θ about the rotated u'-axis, and then a third rotation of ψ about the rotated w''-axis as shown in Fig. 9.7. We note that each rotation occurs about an axis whose location depends on the preceding rotations. The first rotation causes the (u, v, w) frame to move into the (u', v', w') location .The second rotation causes the (u', v', w') frame to move into the (u'', v'', w'') location. The third location cause the (u'', v'', w'') frame to rotation into the final (u, v, w) location. Three such successive rotations are called *w-u-w* or *z-x-z* Euler angles.

The resulting rotation matrix can be derived by a kinematic inversion, that is, by considering the

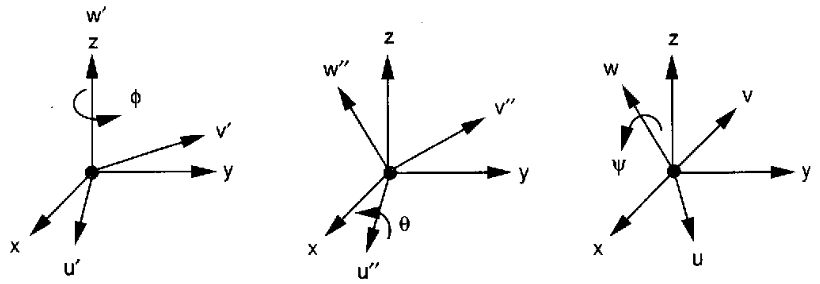


Figure 9.7 Successive rotations about the moving coordinate axes

orientation of frame A with respect to frame B. The inverse kinematics problem can be started as a rotation of frame A about the *w*-axis by angle $-\phi$, followed by a second rotation of -θ about the u-axis, and followed by a third rotation of $-\psi$ about the *w*-axis. Under the kinematic inversion, the coordinate axes of frame B are considered as fixed. Hence the overall rotation matrix, ${}^{B}R_{A}(-\phi, -\theta, -\psi)$, can be written as

$$ {}^{B}R_{A}(-\phi, -\theta, -\psi) = {}^{B}R_{A}(w, -\psi)\,{}^{B}R_{A}(u, -\theta)\,{}^{B}R_{A}(w, -\phi) \,. \qquad (9\text{-}37)$$

Since ${}^{A}R_{B} = [{}^{B}R_{A}]^{-1}$ and ${}^{B}R_{A}^{-1}(w, -\phi) = {}^{B}R_{A}(w, \phi)$, we can expand Eq.(9-37) as follows:

$$\begin{aligned}
{}^{A}R_{B}(\phi, \theta, \psi) &= [{}^{B}R_{A}(-\phi, -\theta, -\psi)]^{-1} \\
&= R(w, \phi)\, R(u, \theta)\, R(w, \psi) \\
&= \begin{bmatrix} c\varphi c\psi - s\varphi c\theta s\psi & -c\varphi s\psi - s\varphi c\theta c\psi & s\varphi s\theta \\ s\varphi c\psi + c\varphi c\theta s\psi & -s\varphi s\psi + c\varphi c\theta c\psi & -c\varphi s\theta \\ s\theta s\psi & s\theta c\psi & c\theta \end{bmatrix} \qquad (9\text{-}38)
\end{aligned}$$

We note that successive rotations about the rotated coordinate axes of a moving frame result in a *postmultiplication* by the matrices. Given the *w-u-w* Euler angles, we can compute the resulting rotation matrix from Eq.(9-38).On the other hand, given a rotation matrix, we can compute the *w-u-w* Euler angles as follows:

$$\theta = \cos^{-1} a_{33},$$
$$\phi = \text{Atan2}(a_{13}/s\theta,\ -a_{23}/s\theta), \qquad (9\text{-}39)$$
$$\psi = \text{Atan2}(a_{31}/s\theta,\ a_{32}/s\theta),$$

provided that $s\theta \neq 0$.When θ=0 or 180°, the solution of Eq.(9-39) degenerates. For this degenerated case, only the sum or the difference of ϕ and ψ can be computed.

w-v-w **Euler Angles**. Another type of Euler angle representation consists of a rotation of angle φ about the *w*-axis, followed by a second rotation of θ about the rotated *v*-axis, and followed by a third rotation of ψ about the rotated *w*-axis. The resulting rotation matrix is obtained by a postmultiplication of three basic rotation matrices as follows:

$${}^{A}R_{B}(\phi,\ \theta,\ \psi) = R(w,\ \phi)\ R(v,\ \theta)\ R(w,\ \psi)$$
$$= \begin{bmatrix} c\varphi c\theta c\psi - s\varphi s\psi & -c\varphi c\theta s\psi - s\varphi c\psi & c\varphi s\theta \\ s\varphi c\theta c\psi + c\varphi s\psi & -s\varphi c\theta s\psi + c\varphi c\psi & s\varphi s\theta \\ -s\theta c\psi & s\theta s\psi & c\theta \end{bmatrix} \qquad (9\text{-}40)$$

Again, given the *w-v-w* Euler angles, we can compute the resulting rotation matrix from Eq.(9-40).On the other hand, given a rotation matrix, we can compute the *w-v-w* Euler angles as follows:

$$\theta = \cos^{-1} a_{33},$$
$$\phi = \text{Atan2}(a_{23}/s\theta,\ a_{13}/s\theta), \qquad (9\text{-}41)$$
$$\psi = \text{Atan2}(a_{32}/s\theta,\ -a_{31}/s\theta),$$

provided that $s\theta \neq 0$.When θ=0 or 180°, the solution of Eq.(9-41) degenerates. In this case, only the sum or the difference of ϕ and ψ can be computed.

9.2.3 Description of a Location

As pointed out earlier, the location of the rigid body can be described by the position of the origin *Q* and the orientation of the moving frame with respect to the fixed frame. Figure 9.1 shows that the position of a point *P* of the rigid body can be expressed in the fixed frame A as ${}^{A}\mathbf{p} = \overline{OP}$. It can also be expressed in the moving frame B as ${}^{B}\mathbf{p} = \overline{QP}$. To derive a relation between ${}^{A}\mathbf{p}$ and ${}^{B}\mathbf{p}$, we construct the vector $\overline{OP}$ as sum of two vectors:

$$\overline{OP} = \overline{OQ} + \overline{QP} \qquad (9\text{-}42)$$

where $\overline{OQ} = {}^{A}\mathbf{q}$ denotes the position of Q with respect to the fixed frame A.

Let the orientation of the moving frame B with respect to the fixed frame A be defined by the rotation matrix ${}^{A}R_{B}$. Then Eq.(9-42) can be written as

$$ {}^{A}\mathbf{p} = {}^{A}R_{B}\, {}^{B}\mathbf{p} + {}^{A}\mathbf{q} \tag{9-43} $$

Equation (9-43) describes the position of a point in a rigid body in terms of the position of the origin Q and the orientation of the moving frame B with respected to the fixed frame A. The leading superscript of ${}^{B}\mathbf{p}$ cancels with the trailing subscript of ${}^{A}R_{B}$, leaving all quantities as vectors expressed in the fixed frame A.

Since we assume that initially, the moving frame coincides with the fixed frame, we may consider ${}^{B}\mathbf{p}$ as the first position of a point *P* and ${}^{A}\mathbf{p}$ as the second position of the same point expressed in the fixed frame A. The first term on the right hand side of Eq.(9-43) represent the contribution due to a rotation of the rigid body about some axis, and the second term represent the contribution due to a translation along the direction of ${}^{A}\mathbf{q}$. We observe that the general spatial displacement of a rigid body can be considered as a rotation plus a translation. This well known as Chasles' theorem.

9.3 Homogeneous Transformations

Equation (9-43) provides a general transformation of a position vector from the moving frame to the fixed frame. The first term on the right-hand side of the equation represents the contribution due to a rotation and the second term that due to a translation of the moving frame respect to fixed frame. The equation is not in a compact form, because the 3×3 rotation matrix does not provide for the translation. To write Eq.(9-43) in a better-appearing form, we introduce the concepts of homogeneous coordinates and homogeneous transformation matrix.

9.3.1 Homogeneous Coordinates

Let $\mathbf{p} = [p_x, p_y, p_z]^T$ be the position vector of a point with respect to a reference frame A in three-dimension space. We define the *homogeneous* coordinates of $\mathbf{p}$ as

$$ \hat{\mathbf{p}} \equiv [\rho p_x, \rho p_y, \rho p_z, \rho]^T \tag{9-44} $$

Thus the homogeneous coordinates of a point P in frame A are represented by a vector $\hat{\mathbf{p}}$ in a four-dimensional space. The fourth coordinate ρ is nonzero scaling factor. In general, an *N*-dimensional position vector becomes an (*N*+1)-dimensional vector in a coordinate system. The concept of homogeneous coordinates is useful in developing matrix transformations that include rotation, translation scaling, and perspective transformation (Ballard and Brown, 1982).

From the definition above, we see that a three-dimensional vector can be recovered from its four-dimensional homogeneous coordinates by dividing the first three homogeneous coordinates by the fourth coordinates; that is,

$$ p_x = \hat{p}_x/\rho,\ p_y = \hat{p}_y/\rho, \text{ and } p_z = \hat{p}_z/\rho \tag{9-45} $$

We note that the homogeneous coordinates $\hat{\mathbf{p}}$ are not unique, since any nonzero scaling factor ρ will yield the same three-dimension vector **p**. For example, $\hat{\mathbf{p}}_1 = [\rho_1 p_x, \rho_1 p_y, \rho_1 p_z, \rho_1]^T$ and $\hat{\mathbf{p}}_2 = [\rho_2 p_x, \rho_2 p_y, \rho_2 p_z, \rho_2]^T$ represent the same position vector $\mathbf{p}=[p_x, p_y, p_z]^T$ in a three-dimension space. For the kinematics of mechanisms and robot manipulators, we choose a scaling factor of ρ=1 for convenience. When the scaling factor is set to unity, the first three homogeneous coordinates represent the actual coordinates of a three-dimension vector. Hence the position vector of a point is given simply as

$$\hat{\mathbf{p}} = [p_x, p_y, p_z, 1]^T \qquad (9\text{-}46)$$

In subsequent chapter, we omit the *hats* from the equation for brevity.

9.3.2 Homogeneous Transformation Matrix

The *homogeneous transformation matrix* is a 4×4 matrix that is defined for the purpose of mapping a homogeneous position vector from one coordinate system into another. The matrix can be partitioned into four submatrices as follows:

$$^{A}T_{B} = \begin{bmatrix} ^{A}R_{B}(3\times3) & \vdots & ^{A}\mathbf{q}(3\times1) \\ \cdots\cdots\cdots & \vdots & \cdots\cdots \\ \gamma(1\times3) & \vdots & \rho(1\times1) \end{bmatrix} \qquad (9\text{-}47)$$

The upper left 3×3 submatrix $^{A}R_{B}$ denotes the orientation of a moving frame B with respect to a reference frame A, the upper right 3×1 submatrix $^{A}\mathbf{q}$ denotes the position of the origin of the moving frame relative to the fixed frame, the lower left 1×3 submatrix γ represents a perspective transformation, and the low right element ρ is a scaling factor. Using the homogeneous coordinates, the position and orientation of a vector can be written in a compact form:

$$^{A}\hat{\mathbf{p}} = {}^{A}T_{B}\,{}^{B}\hat{\mathbf{p}} \qquad (9\text{-}48)$$

where $^{A}T_{B} = \begin{bmatrix} ^{A}R_{B}(3\times3) & ^{A}\mathbf{q}(3\times1) \\ 0 \;\; 0 \;\; 0 & 1 \end{bmatrix}$, $^{A}\hat{\mathbf{p}} = [p_x, p_y, p_z, 1]^T$, and $^{B}\hat{\mathbf{p}} = [p_u, p_v, p_w, 1]^T$

Although the transformation matrix $^{A}T_{B}$ is not orthogonal, its inverse transformation does exist. Multiplying both sides of Eq.(9-43) by $^{A}R_{B}^{-1}$ and making use of the fact that $^{A}R_{B}^{-1} = {}^{A}R_{B}^{T}$, we obtain

$$^{B}\mathbf{p} = {}^{A}R_{B}^{T}\,{}^{A}\mathbf{p} - {}^{A}R_{B}^{T}\,{}^{A}\mathbf{q}$$

The inverse transformation of the above equation can be also obtained as:

$$^{B}\hat{\mathbf{p}} = {}^{A}T_{B}^{-1}\,{}^{A}\hat{\mathbf{p}} \qquad (9\text{-}49)$$

where $^{A}T_{B}^{-1} = {}^{B}T_{A} = \begin{bmatrix} {}^{A}R_{B}^{T} & -{}^{A}R_{B}^{T}\,{}^{A}\mathbf{q} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$.

Composite Homogeneous Transformation

The following rules are helpful for the determination of a composite transformation matrix:

1. At the initial position, the moving frame B and the fixed frame are coincident. Hence, T=I is an identity matrix.
2. Rotation and translation about the coordinate axes of a fixed frame results in a pre-multiplication of the two matrices.
3. Rotation and translation about the coordinate axes of a moving frame results in a post-multiplication of the two matrices.

Example 1: Find a homogeneous transformation matrix T that represents a rotation of α angle about the OX axis (fixed frame), followed by a translation of a units along the OX axis, followed by a translation of d units along the OZ axis, followed by a rotation of θ angle about the OZ axis.

Sol: $T = T_{z,\theta}\, T_{z,d}\, T_{x,a}\, T_{x,\alpha}$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 2: A T matrix is to be determined that represents a rotation of α angle about the OX axis (fixed frame), followed by a translation of b units along the rotated y axis.

Sol: $T = T_{x,\alpha}\, T_{v,b}$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

9.4 Hartenberg-Denavit Notation

Two coordinates system as shown in Fig. 9.8 are defined such that

1. The z_i-axis is aligned with the $(i+1)^{th}$ joint axis. The positive direction of rotation or translation can be chosen arbitrarily.

2. The x_i-axis is defined along the common normal between the i^{th} and $(i+1)^{th}$ joint axes and

points from the i^{th} to the $(i+1)^{th}$ joint axis. If the two axes are parallel, the x_i-axis can be chosen anywhere perpendicular to the two joint axes. In case of two intersecting joint axes, the x_i-axis can be defined either in the direction of the vector cross product $z_{i-1} \times z_i$ or in the opposite direction, and the origin is at the point of intersection.

3. The y_i-axis is determined by the right-hand rule.

Let H_{i-1} be the point of intersection of the x_i and z_{i-1} axes, and let O_i, the origin of the i^{th} coordinate system, be the point of intersection of x_i and z_i axes as shown in the Figure 9.8. The following parameters are uniquely determined by the geometry of the axes:

- a_i : offset distance between two adjacent joint axes, where $a_i = |H_{i-1}O_i|$.
- α_i : twist angle between two adjacent joint axes. It is the angle required to rotate the z_{i-1} axis into alignment with the z_i-axis about the positive x_i-axis according to the right-hand rule.
- θ_i : joint angle between two incident normals of a joint axis. It is the angle required to rotate the x_{i-1}-axis into alignment with the x_i-axis about the positive z_{i-1}-axis according to the right-hand rule.
- d_i : translational distance between two incident normals of a joint axis. $d_i = |O_{i-1}H_{i-1}|$ is positive if the vector $O_{i-1}H_{i-1}$ points in the positive z_{i-1}-direction; otherwise, it is negative.

Having established a coordinate system to each link of an object, a 4×4 transformation matrix relating two successive coordinate systems can be established. Observation of Fig 9.8 reveals that the i^{th} coordinate system can be thought of as being displaced from the $(i-1)^{th}$ coordinate system by the following successive rotations and translations. We may also think of the transformations above as four basic transformations about the moving coordinate axes. Therefore, the resulting transformation is given by

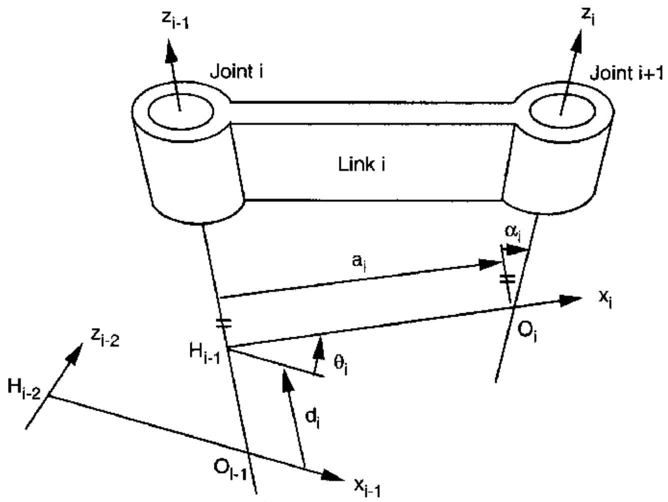


Figure 9.8 Definition of link parameters

$$\begin{pmatrix} {}^{i-1}x \\ {}^{i-1}y \\ {}^{i-1}z \\ 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}_{d_i\,/\,translate\text{ along Zi-1}} \begin{bmatrix} \cos\theta_i & -\sin(\theta_i) & 0 & 0 \\ \sin(\theta_i) & \cos(\theta_i) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{\theta_i\,/\,rotate\text{ about Zi-1}}$$

$$\begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{a_i\,/\,translate\text{ along Xi}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha_i) & -\sin(\alpha_i) & 0 \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{\alpha_i\,/\,rotate\text{ about Xi}} \begin{pmatrix} {}^{i}x \\ {}^{i}y \\ {}^{i}z \\ 1 \end{pmatrix} \quad (9\text{-}4.1a)$$

Expanding the overall transformation matrix, ${}^{i-1}T_i$ can be obtained by

$$\begin{bmatrix} \cos(x_{i-1}, x_i) & \cos(y_{i-1}, x_i) & \cos(z_{i-1}, x_i) & a_i \\ \cos(x_{i-1}, y_i) & \cos(y_{i-1}, y_i) & \cos(z_{i-1}, y_i) & b_i \\ \cos(x_{i-1}, z_i) & \cos(y_{i-1}, z_i) & \cos(z_{i-1}, z_i) & c_i \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (9\text{-}4.1b)$$

Equation (9-4.1b) is called the *Denavit-Hartenberg* (D-H) *transformation matrix*. The trailing subscript i and the leading superscript (i-1) denote that the transformation takes place from the i^{th} coordinate system to the $(i\text{-}1)^{th}$ coordinate system.

Example: A slider-crank mechanism is defined as shown in the diagram (Draw the example figure).

$X_1 = {}^1T_2\, X_2$ ${}^1T_2 = (a_2, 270°, 180°, d_2)$

$X_2 = {}^2T_3\, X_3$ ${}^2T_3 = (a_3, 0, \theta_3, 0)$

$X_3 = {}^3T_4\, X_4$ ${}^3T_4 = (a_4, 0, \theta_4, 0)$

$X_4 = {}^4T_1\, X_1$ ${}^4T_1 = (0, 270°, \theta_1, 0)$

$\Rightarrow X_1 = {}^1T_2{}^2T_3{}^3T_4{}^4T_1X_1$

$$\Rightarrow I = {}^1T_2\,{}^2T_3\,{}^3T_4\,{}^4T_1 = [T] \quad (9\text{-}4.2)$$

The matrix [T] contains elements that are functions of a_1, θ_1, a_2, θ_2, a_3, θ_3, and θ_4. Assuming that θ_4 is specified, we may solve for other unknown motion parameters θ_2, θ_3, and d_1 by equating corresponding elements of [T] and [I]. While solving Equation (9-4.2), it can be noted that the loop closure equation contains 16 scalar equations, four of which are trivial. Equating the upper right 3×1 sub-matrix results in three independent equations (two independent equations in two-dimensional case), representing the position equality. Equating the elements of the upper left 3×3 sub-matrix results in 9 equations. However, only three of the nine orientation equations (two of four in two-dimensional case) are independent because of the orthogonal conditions. One can rearrange the loop closure equation by redistributing the unknown variables on both sides of the equation as evenly as possible. Or, take the advantage of some special conditions, such as three consecutive intersecting joint axes or three consecutive parallel joint axes.

$$({}^{0}A_1)^{-1}\,({}^{0}A_n) = {}^{1}A_2\,{}^{2}A_3\ldots{}^{n-1}A_n \qquad (9\text{-}4.3a)$$

$$({}^{1}A_2)^{-1}\,({}^{0}A_1)^{-1}\,({}^{0}A_n) = {}^{2}A_3{}^{3}A_4\ldots{}^{n-1}A_n \qquad (9\text{-}4.3b)$$

$$({}^{2}A_3)^{-1}\,({}^{1}A_2)^{-1}\,({}^{0}A_1)^{-1}\,({}^{0}A_n) = {}^{2}A_3{}^{3}A_4\ldots{}^{n-1}A_n \qquad (9\text{-}4.3c)$$

9.5 Kinematics of Open-Loop Manipulators

Figure 9.9 shows the general structure of an open-loop (or serial) manipulator. There are two types of problems regarding the kinematics of robots:

1. Direct kinematics

   Given: link and joint parameters

   Find: the position and orientation of the end-effector

2. Inverse kinematics

   Given: position and orientation of the end-effector

   Find: joint angles

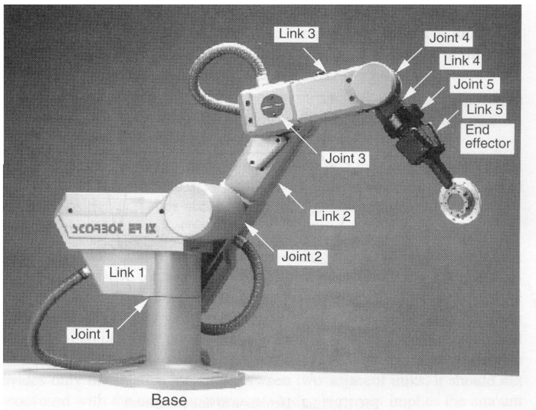


Fig. 9.9 A serial robot [1]

Methods of solution:

1. Matrix Method
2. Geometric and Vector approach
3. Screw algebra
4. Quaternion and Dual numbers

Example 1: A two-dof arm (Draw Fig. E3)

$${}^{0}T_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & a_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & a_1\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{1}T_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & a_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$ {}^0T_2 = {}^0T_1{}^1T_2 = \begin{bmatrix} \cos\theta_{12} & -\sin\theta_{12} & 0 & a_2\cos\theta_{12} + a_1\cos\theta_1 \\ \sin\theta_{12} & \cos\theta_{12} & 0 & a_2\sin\theta_{12} + a_1\cos\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ where } \theta_{12} = \theta_1+\theta_2 $$

Direct kinematics

Given θ_1, θ_2, find P(p_x, p_y) = f(θ_1, θ_2)

$$ \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}_0 = {}^0T_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_2\cos\theta_{12} + a_1\cos\theta_1 \\ a_2\sin\theta_{12} + a_1\sin\theta_1 \\ 0 \\ 1 \end{bmatrix} $$

an arbitrary point $[p_u, p_v]^T$ in the o_2-x_2-y_2 coordinate

$$ {}^0\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = {}^0T_2 \, {}^2\begin{bmatrix} p_u \\ p_v \\ 0 \\ 1 \end{bmatrix} $$

Inverse Kinematics

Given P(p_x, p_y) in the o-x-y coordinate system, find θ_1, θ_2.

(1) Specify the position of the origin of the 2nd coordinate system

$$ {}^0\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = [T] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_2\cos\theta_{12} + a_1\cos\theta_1 \\ a_2\sin\theta_{12} + a_1\sin\theta_1 \\ 0 \\ 1 \end{bmatrix} $$

$$ p_x^2+p_y^2 = a_2^2 + a_1^2 + 2a_1a_2(\cos\theta_{12}\cos\theta_1+\sin\theta_{12}\sin\theta_1) $$
$$ = a_1^2 + a_2^2 + 2a_1a_2\cos\theta_2 $$

$$ \Rightarrow \cos\theta_2 = (p_x^2+p_y^2 - a_1^2 - a_2^2)/(2a_1a_2) $$
$$ \Rightarrow \theta_2 = \pm \cos^{-1}[(p_x^2+p_y^2 - a_1^2 - a_2^2)/(2a_1a_2)] $$

Two solutions for θ_2

$p_x = (a_1+a_2\cos\theta_2)\cos\theta_1 - a_2\sin\theta_2\sin\theta_1$

$p_y = (a_1+a_2\cos\theta_2)\sin\theta_1 + a_2\sin\theta_2\cos\theta_1$

Thus, $\cos\theta_1 = \begin{vmatrix} p_x & -a_2\sin\theta_2 \\ p_y & (a_1+a_2\cos\theta_2) \end{vmatrix} / \Delta$, $\sin\theta_1 = \begin{vmatrix} (a_1+a_2\cos\theta_2) & p_x \\ a_2\sin\theta_2 & p_y \end{vmatrix} / \Delta$, and

$\Delta = (a_1+a_2\cos\theta_2)^2 + a_2^2(\sin\theta_2)^2$

$\theta_1 = \text{Atan2}(S_1, C_1)$ → one solution corresponding to one θ_2

Example 2: Scorbot Robot

In the Scorbot robot, the second, third, and fourth joint axes are parallel to one another and point into the paper at points A, B, and P, respectively. The first joint axis points up vertically, and the fifth joint axis intersects the fourth perpendicularly. Find the overall transformation matrix for the robot.

Table 9.1 D-H parameters of the Scorbot robot

	α_i	a_i	d_i	θ_i
1	$-\pi/2$	a_1	d_1	θ_1
2	0	a_2	0	θ_2
3	0	a_3	0	θ_3
4	$-\pi/2$	0	0	θ_4
5	0	0	d_5	θ_5

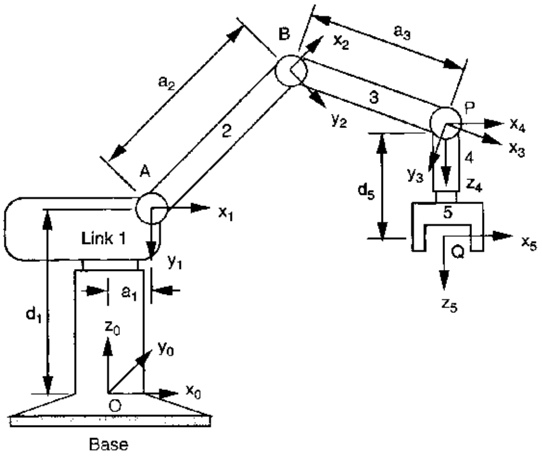


Fig. E4. Schematic diagram of the Scorbot robot

$$ {}^{0}A_1 = \begin{bmatrix} c\theta_1 & 0 & -s\theta_1 & a_1 c\theta_1 \\ s\theta_1 & 0 & c\theta_1 & a_1 s\theta_1 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(a)} $$

$$ {}^{1}A_2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2 c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2 s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(b)} $$

$$ {}^{2}A_3 = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_3 c\theta_3 \\ s\theta_3 & c\theta_3 & 0 & a_3 s\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(c)} $$

$$ {}^{3}A_4 = \begin{bmatrix} c\theta_4 & 0 & -s\theta_4 & 0 \\ s\theta_4 & 0 & c\theta_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(d)} $$

$$ {}^{4}A_5 = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 1 & d_5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(e)} $$

Multiplying (b), (c), (d) yields

$$
{}^{1}A_4 = \begin{bmatrix} c\theta_{234} & 0 & -s\theta_{234} & a_3 c\theta_{23} + a_2 c\theta_2 \\ s\theta_{234} & 0 & c\theta_{234} & a_3 s\theta_{23} + a_2 s\theta_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{(f)}
$$

Where $\theta_{ij}=\theta_i + \theta_j$, and $\theta_{ijk}=\theta_i + \theta_j + \theta_k$

Treat θ_2, θ_{23}, and θ_{234} as new variables. In this way, the rotation matrix contains only one variable, θ_{234}, while the position submatrix contains two variables, θ_2 and θ_{23}. Multiplying (a), (f), and (e) yields the overall transformation matrix ${}^{0}A_5$ as:

$$
\begin{aligned}
&u_x = c\theta_1 c\theta_{234} c\theta_5 + s\theta_1 s\theta_5 \quad , \quad u_y = s\theta_1 c\theta_{234} c\theta_5 - c\theta_1 s\theta_5 \quad , u_z = -s\theta_{234} c\theta_5 \\
&v_x = -c\theta_1 c\theta_{234} s\theta_5 + s\theta_1 c\theta_5 \ , v_y = -s\theta_1 c\theta_{234} s\theta_5 - c\theta_1 c\theta_5 \quad , \quad v_z = s\theta_{234} s\theta_5 \\
&w_x = -c\theta_1 s\theta_{234} \quad , \quad w_y = -s\theta_1 s\theta_{234} \quad , \quad w_z = -c\theta_{234} \\
&q_x = c\theta_1 (a_1 + a_2 c\theta_2 + a_3 c\theta_{23} - d_5 s\theta_{234}), \\
&q_y = s\theta_1 (a_1 + a_2 c\theta_2 + a_3 c\theta_{23} - d_5 s\theta_{234}), \\
&q_z = d_1 - a_2 s\theta_2 - a_3 s\theta_{23} - d_5 c\theta_{234}
\end{aligned} \qquad \text{(g)}
$$

Since this is a 5-dof manipulator, only five of the six parameters of the end-effector can be described. Very often, the desired position of a point and the direction of a line in the end-effector are specified.

(a) Direct Kinematics

For the direct kinematic problem, we simply substitute the given joint angles into Eq.(g) to obtain the end-effector position and the orientation in terms of **u**, **v**, and **w**.

(b) Inverse Kinematics

For the inverse kinematic problem, only 5 of the 12 parameters can be specified at will. This is because the manipulator has only 5-dof. It is obvious that the position vector $\vec{q}$ and the approach vector $\vec{w}$ cannot be specified simultaneously, due to the fact that **q** and **w** together depend only on 4 degrees of freedom of the manipulator. In this example, **q** and **u** are specified. A more straightforward approach by multiplying both sides of the loop-closure equation by $({}^{o}A_1)^{-1}$; that is

$$
({}^{o}A_1)^{-1}\, {}^{o}A_5 = {}^{1}A_2\, {}^{2}A_3\, {}^{3}A_4\, {}^{4}A_5 \qquad \text{(h)}
$$

Equating the first column of the (h),

$$
u_x c\theta_1 + u_y s\theta_1 = c\theta_{234} c\theta_5 \qquad \text{(i)}
$$

$$
-u_z = s\theta_{234} c\theta_5 \qquad \text{(j)}
$$

$$
-u_x s\theta_1 + u_y c\theta_1 = -s\theta_5 \qquad \text{(k)}
$$

Similarly, equating the fourth column of Eq.(h)

$$
q_x c\theta_1 + q_y s\theta_1 - a_1 = a_2 c\theta_2 + a_3 c\theta_{23} - d_5 s\theta_{234} \qquad \text{(l)}
$$

$$-q_z + d_1 = a_2 \mathrm{s}\theta_2 + a_3 \mathrm{s}\theta_{23} + d_5 \mathrm{c}\theta_{234} \qquad \text{(m)}$$

$$-q_x \mathrm{s}\theta_1 + q_y \mathrm{c}\theta_1 = 0 \qquad \text{(n)}$$

The first joint angle θ_1 is obtained by (n)：

$$\theta_1 = \tan^{-1} q_y / q_x$$

There are two solutions for $\theta_1 = \theta_1^*$ or $(\theta_1^* + \pi)$. Once θ_1 is found, two solutions for θ_5 are obtained from (k):

$$\theta_5 = \sin^{-1}(u_x \mathrm{s}\theta_1 - u_y \mathrm{c}\theta_1) = \theta_5^* \quad , \qquad \text{or} \quad \theta_5 = \pi - \theta_5^*.$$

Corresponding to each solution of (θ_1, θ_5), Eqs.(i) and (j) produce a unique solution of θ_{234}

$$\theta_{234} = \mathrm{ATAN2}[-u_z / \mathrm{c}\theta_5, (u_x \mathrm{c}\theta_1 + u_y \mathrm{s}\theta_1) / \mathrm{c}\theta_5]$$

Now, solving Eqs.(l) and (m) for θ_2 and θ_3

$$a_2 \mathrm{c}\theta_2 + a_3 \mathrm{c}\theta_{23} = k_1 \qquad \text{(o)}$$
$$a_2 \mathrm{s}\theta_2 + a_3 \mathrm{s}\theta_{23} = k_2 \qquad \text{(p)}$$

where $k_1 = q_x \mathrm{c}\theta_1 + q_y \mathrm{s}\theta_1 - a_1 + d_5 \mathrm{s}\theta_{234}$ and $k_2 = -q_z + d_1 - d_5 \mathrm{c}\theta_{234}$

Summing squares of the above equations yields

$$a_2^2 + a_3^3 + 2a_2 a_3 \mathrm{c}\theta_3 = k_1^2 + k_2^2$$

$$\Rightarrow \theta_3 = \cos^{-1}[(k_1^2 + k_2^2 - a_2^2 - a_3^2) / 2a_2 a_3]$$

and there are two solutions for θ_3. If $\theta_3 = \theta_3^*$ is also a solution, $\theta_3 = -\theta_3^*$ is also a solution. Once θ_3 is known, θ_2 can be solved by expanding Eqs.(o) and (p) as follows:

$$(a_2 + a_3 \mathrm{c}\theta_3)\mathrm{c}\theta_2 - (a_3 \mathrm{s}\theta_3)\mathrm{s}\theta_2 = k_1$$
$$(a_3 \mathrm{s}\theta_3)\mathrm{c}\theta_2 + (a_2 + a_3 \mathrm{c}\theta_3)\mathrm{s}\theta_2 = k_2$$

$$\Rightarrow \begin{cases} \mathrm{c}\theta_2 = A \\ \mathrm{s}\theta_2 = B \end{cases}$$

Hence, corresponding to each solution of $(\theta_1, \theta_3, \theta_5, \theta_{234})$, we obtain a unique solution of θ_2:

$$\theta_2 = \mathrm{Atan2}(B, A)$$

Finally, θ_4 is obtained by

$$\theta_4 = \theta_{234} - \theta_2 - \theta_3$$

We conclude that corresponding to each given end-effector location, there are at most eight inverse kinematic solutions.

9.6 Parallel Manipulators

Figure 9.10 shows a parallel manipulator (or platform manipulator) in which the moving platform is connected to a fixed base by several limbs or legs. Typically, the number of limbs is equal to the number of degrees of freedom such that every limb is controlled by one actuator and all the actuators can be mounted at or near the fixed base.

Advantages:

a) All actuators can be ground connected
   $\Rightarrow$ larger load carrying capacity

b) Reduction in mechanical transmission lines

c) Improvement of position accuracy

Disadvantages:

a) Reduced workspace

b) Mechanical complexity increased

c) Controllability

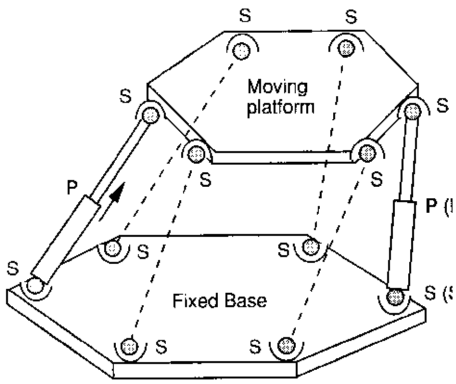


Fig. 9.10 Stewart-Gough platform

Structural Analysis of Parallel Manipulators

The following conditions are assumed:

- The number of limbs is equal to the number of degrees of freedom of the moving platform.
- The type and number of joints in all the limbs are arranged in an identical pattern.
- The number of and location of actuation joints in all limbs are the same.

We observe that in a symmetrical manipulator, the number of limbs, m, is equal to the number of degrees of freedom, F, which is also equal to the total number of loops, L+1; that is,

$$m=F=L+1 \tag{1}$$

Define the connectivity, C_k, of a limb as the degrees of freedom associated with all the joints in the limb. It follows that

$$\sum_{k=1}^{m} C_k = \sum_{i=1}^{j} f_i \tag{2}$$

where j is the number of joints in a mechanism.

Thus,

$$\sum_{k=1}^{m} C_k = (\lambda+1)F - \lambda \tag{3}$$

Furthermore, the connectivity of each limb should not be greater than the motion parameter and less than the degrees of freedom of the moving platform; i.e.

$$\lambda \ge C_k \ge F \tag{4}$$

Equations (3) and (4) are useful for enumeration and classification of parallel manipulators.

a. Planar Parallel Manipulators

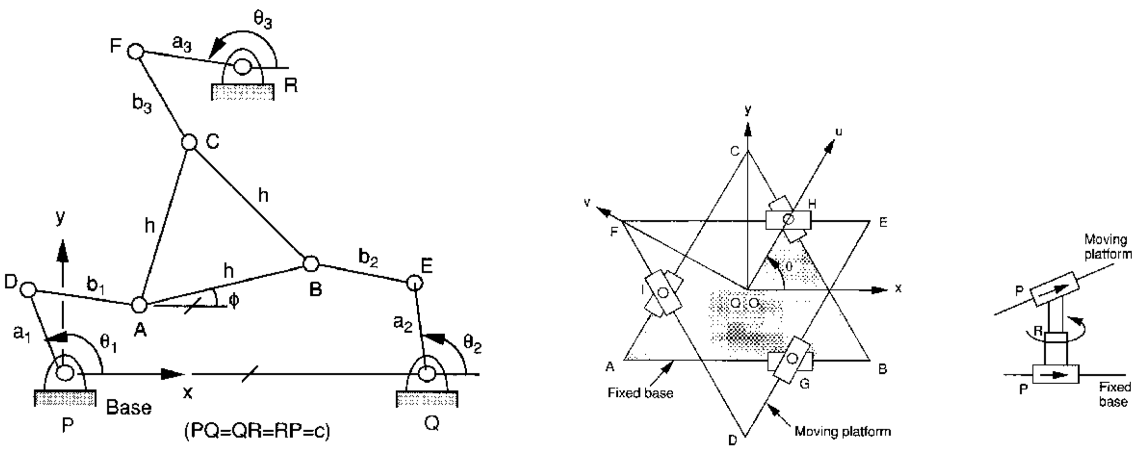
For planar 3-dof, three-limbed parallel manipulators, λ=3 and m=F=3. We obtain form Eq.(3)

$$C_1+C_2+C_3=9 \quad (5)$$

And Eq.(4) reduces to

$$3 \geq C_k \geq 3 \quad (6)$$

Hence the connectivity of each limb should be 3. Assuming that each limb consists of two links and three joints, each joint must be a 1-dof joint. Using only R and P joints as the kinematic pairs, we obtain seven possible limbs arrangements: RRR, RRP, RPR, PRR, RPP, PRP, and PPR. Figure 9.11 and – shows the parallel manipulator using 3RRRand 3PRP limb structure, respectively.



(a) (b)

Figure 9.11 Parallel manipulators (a)3 RRR (b)3 PRP

b. Spherical Parallel Manipulators

For spherical linkage manipulators, the only permissible joint type is the revolute joint, and all the joint axes must intersect at a common point, called the spherical center. Hence the only possible limb structure is the RRR configuration as shown in Figure 9.12. Note that one spherical joint can be installed at the center of a parallel manipulator. However, such a spherical joint can only be used as a passive joint.

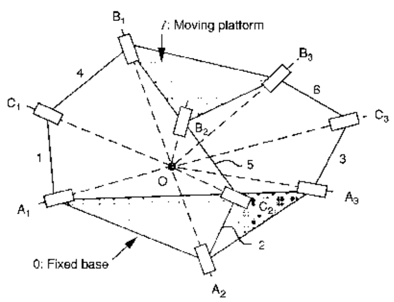


Figure 9.12 Spherical 3 RRR manipulator

c. Spatial Parallel Manipulators

Substituting λ=6 into Equations (3) and (4), we obtain

$$\sum C_k = 7F\text{-}6 \qquad (7)$$

and

$$6 \geq C_k \geq F \qquad (8)$$

Solving Eqs. (7) and (8) for positive integers of C_k, k=1,2,3,⋯, we can classify spatial parallel manipulators according to their degree-of-freedom and connectivity as given in Table 1.

Table 1. Classification of Spatial Parallel Manipulators

DOF F	No. of loops L	Sum of all joint freedoms $\sum_i f_i$	Connectivity C_k, k=1,2,⋯
2	1	8	(4,4),(5,3), (6,2)
3	2	15	(5,5,5), (6,5,4), (6,6,3)
4	3	22	(6,6,5,5), (6,6,6,4)
5	4	29	(6,6,6,6,5)
6	5	36	(6,6,6,6,6,6)

The number of links incorporated in each limb can be any as long as the sum of all joint freedoms is equal to the required connectivity. In practice, it is desirable to employ just two major links connecting the moving platform to the base by three joints. Figure 9.13 shows a few such limb configurations. Note that each of the limbs in Fig. 9.13 c, f, and h contains 1 passive dof, while the one shown in Fig. 9.13 g contains 2 passive degrees of freedom.

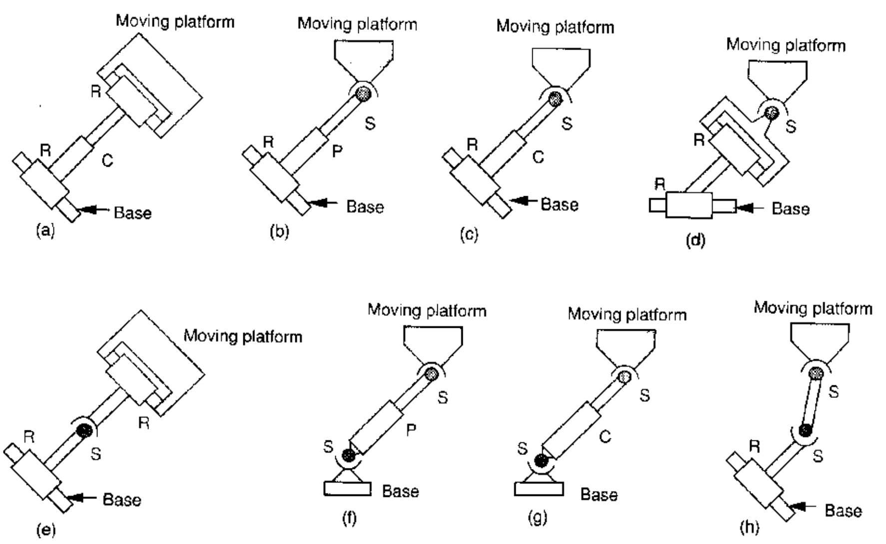


Figure 9.13 Some possible limb configurations

Example 1. A planar three DOF parallel manipulator (Fig. E5)

Degree of freedom:

$n = 8, j = 9$

$F = 3(8-1) - 2\times 9 = 21-18 = 3.$

Given: Location of the moving platform (x_A, y_A) and orientation ϕ

Solve: input angles θ_1 , θ_2, and θ_3

Expressing the position of A in the fixed coordinate system, yields

$$\begin{cases} x_A = x_p + a_1\, c\theta_1 + b_1 c(\theta_1+\varphi_1) \\ y_A = y_p + a_1\, s\theta_1 + b_1 s(\theta_1+\varphi_1) \end{cases} \qquad (1)$$

Since P is located at the origin, $x_p=y_p=0$. Eliminating φ_1 of Eq. (1), yields

$$x_A^2 + y_A^2 - 2x_A a_1 c\theta_1 - 2y_A a_1 s\theta_1 + a_1^2 - b_1^2 = 0 \qquad (2.a)$$

Similarly, two additional equations can be derived for limbs 2 and 3:

$$\begin{aligned} &x_A^2 + y_A^2 - 2x_A x_Q - 2y_A y_Q + x_Q^2 + y_Q^2 + h^2 + a_2^2 - b_2^2 + 2x_A h c\phi \\ &\quad + 2y_A h s\phi - 2x_A a_2 c\theta_2 - 2y_A a_2 s\theta_2 - 2a_2 h c\phi c\theta_2 - 2x_Q h c\phi - 2y_Q h s\phi \\ &\quad + 2x_Q a_2 c\theta_2 + 2y_Q a_2 s\theta_2 - 2a_2 h s\phi s\theta_2 = 0, \end{aligned} \qquad (2.b)$$

$$\begin{aligned} &x_A^2 + y_A^2 - 2x_A x_R - 2y_A y_R + x_R^2 + y_R^2 + h^2 + a_3^2 - b_3^2 \\ &\quad + 2x_A h c\left(\phi + \frac{\pi}{3}\right) + 2y_A h s\left(\phi + \frac{\pi}{3}\right) - 2x_A a_3 c\theta_3 \\ &\quad - 2y_A a_3 s\theta_3 - 2a_3 h c\left(\phi + \frac{\pi}{3}\right) c\theta_3 \\ &\quad - 2x_R h c\left(\phi + \frac{\pi}{3}\right) - 2y_R h s\left(\phi + \frac{\pi}{3}\right) + 2x_R a_3 c\theta_3 \\ &\quad + 2y_R a_3 s\theta_3 - 2a_3 h s\left(\phi + \frac{\pi}{3}\right) s\theta_3 = 0. \end{aligned} \qquad (2.c)$$

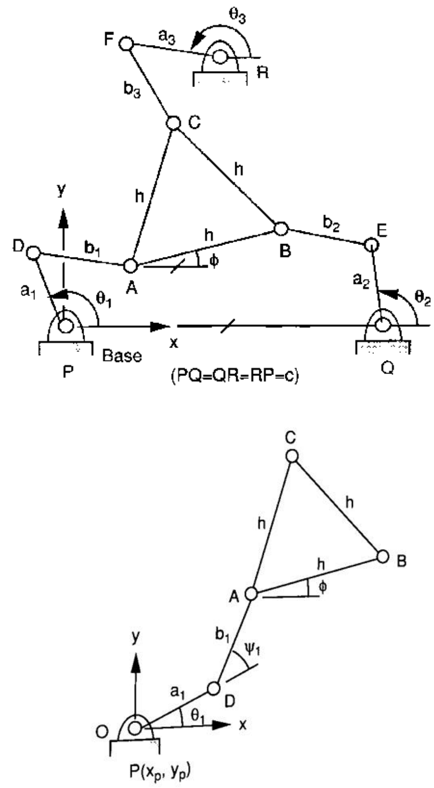


Fig. E5. Planar 3-dof, 3RRR parallel manipulator

1) <u>Inverse kinematics</u>

For the inverse kinematics, x_A, y_A, and ϕ are given, and the joint angles θ_1, θ_2, and θ_3 are to be found. This can be accomplished on a limb-by-limb basis. For limb 1, we arrange Eq. (2.a) in the following form:

$$e_1\, s\theta_1 + e_2\, c\theta_1 + e_3 = 0 \qquad (3)$$

where $e_1 = -2y_A a_1$, $e_2 = -2x_A a_1$, $e_3 = x_A^2 + y_A^2 + a_1^2 - b_1^2$.

Substituting the trigonometric identities

$s\theta_i = 2t_i/(1+t_i^2)$ and $c\theta_i = (1-t_i^2)/(1+t_i^2)$, where $t_i = \tan(\theta_i/2)$ into Equation (7), we obtain

$$(e_3 - e_2)\, t_1^2 + 2e_1 t_1 + (e_2 + e_3) = 0 \qquad (8)$$

Solving the equation for t_1 yields

$$\theta_1 = 2\tan^{-1}[(-e_1 \pm \sqrt{e_1^2 + e_2^2 - e_3^2}) / (e_3 - e_2)] \qquad (9)$$

Hence, corresponding to each given moving platform location, there are generally two solutions of θ_1 and therefore two configuration of limb 1. In general, there are a total of eight possible manipulator postures corresponding to a given end-effector location.

2) Direct kinematics

Given: θ_1, θ_2, and θ_3

To find: x_A, y_A, and ϕ

i. Eliminating x_A^2, y_A^2 by (2.a) and (2.b), and (2.a) and (2.c) will get two linear equations of (x_A, y_A) in terms of $\sin\phi$ and $\cos\phi$. We may then solve for (x_A, y_A) and then substitute back into Eq. (2.a). This will result in a fourth-degree polynomial in $\sin\phi$ and $\cos\phi$.

ii. The polynomial can be converted into an eighth-degree polynomial by using the half-tangent angle expressions. Hence, corresponding to each given set of input angles, there are at most eight possible manipulator configurations. (See Tsai L.W., "Robot Analysis" for more detail discussion.)

Example 2. Position analysis of a spatial 3RPS parallel manipulator (Figure E6)

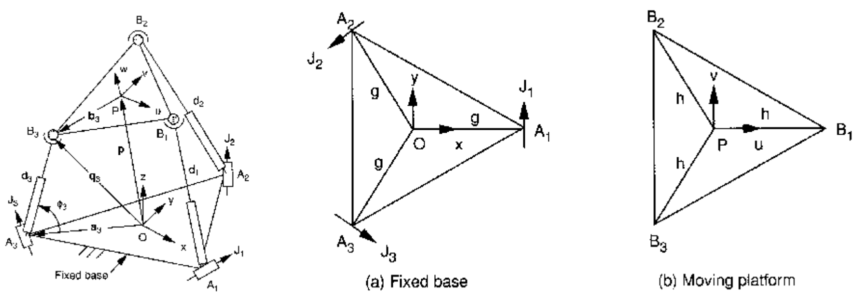


Figure E6. 3-RPS parallel manipulator

Let $\mathbf{a}_i$ and ${}^B\mathbf{b}_i$ be the position vectors of point A_i and B_i in the coordinate systems A and B. The coordinates of A_i and B_i are given by:

$\mathbf{a}_1 = [g, 0, 0]^T$ ${}^B\mathbf{b}_1 = [h, 0, 0]^T$

$\mathbf{a}_2 = [-g/2, \sqrt{3}g/2, 0]^T$ ${}^B\mathbf{b}_2 = [-h/2, \sqrt{3}h/2, 0]^T$

$\mathbf{a}_3 = [-g/2, -\sqrt{3}g/2, 0]^T$ ${}^B\mathbf{b}_3 = [-h/2, -\sqrt{3}h/2, 0]^T$

The position vector $\mathbf{q}_i$ of B_i with respect to the fixed coordinate system is obtained by the transformation:

$\mathbf{q}_i = \mathbf{p} + {}^{A}R_{B}\,{}^{B}\mathbf{b}_i$

$$\mathbf{q}_1 = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} + \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix} \cdot \begin{bmatrix} h \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p_x + hu_x \\ p_y + hu_y \\ p_z + hu_z \end{bmatrix} \tag{E.68}$$

$$\mathbf{q}_2 = \begin{bmatrix} p_x - \frac{h}{2}u_x + \frac{\sqrt{3}h}{2}v_x \\ p_y - \frac{h}{2}u_y + \frac{\sqrt{3}h}{2}v_y \\ p_z - \frac{h}{2}u_z + \frac{\sqrt{3}h}{2}v_z \end{bmatrix} \tag{E.69}$$

$$\mathbf{q}_3 = \begin{bmatrix} p_x - \frac{h}{2}u_x - \frac{\sqrt{3}h}{2}v_x \\ p_y - \frac{h}{2}u_y - \frac{\sqrt{3}h}{2}v_y \\ p_z - \frac{h}{2}u_z - \frac{\sqrt{3}h}{2}v_z \end{bmatrix} \tag{E.70}$$

Constraints imposed by the revolute joints are as

$$q_{1y} = 0 \tag{E.71}$$

$$q_{2y} = -\sqrt{3}\, q_{2x} \tag{E.72}$$

$$q_{3y} = \sqrt{3}\, q_{3x} \tag{E.73}$$

Substituting (71)–(73) into (68)~(70),

$$p_y + h\, u_y = 0 \tag{E.74}$$

$$p_y - h\, u_y/2 + \sqrt{3}\, h\, v_y/2 = -\sqrt{3}\,(p_x - h\, u_x/2 + \sqrt{3}\, h\, v_x/2) \tag{E.75}$$

$$p_y - h\, u_y/2 - \sqrt{3}\, h\, v_y/2 = \sqrt{3}\,(p_x - h\, u_x/2 - \sqrt{3}\, h\, v_x/2) \tag{E.76}$$

By 2×(74)-(75+76), yields

$$u_y = v_x \tag{E.77}$$

(76)-(75)

$$\Rightarrow \quad p_x = h(u_x - v_y)/2 \tag{E.78}$$

The length of a limb, d_i, is given by

$$d_i^2 = [q_i - a_i]^{T} \cdot [q_i - a_i]$$

$$\Rightarrow \quad d_1^2 = p_x^2 + p_y^2 + p_z^2 + 2h(p_x u_x + p_y u_y + p_z u_z) - 2gp_x - 2ghu_x + g^2 + h^2$$

$$d_2^2 = p_x^2 + p_y^2 + p_z^2 - h(p_x u_x + p_y u_y + p_z u_z) + \sqrt{3}\, h(p_x v_x + p_y v_y + p_z v_z)$$
$$+ g(p_x - \sqrt{3}\, p_y) - gh(u_x - \sqrt{3}\, u_y)/2 + gh(\sqrt{3}\, v_x - 3v_y)/2 + g^2 + h^2$$
$$d_3^2 = p_x^2 + p_y^2 + p_z^2 - h(p_x u_x + p_y u_y + p_z u_z) - \sqrt{3}\, h(p_x v_x + p_y v_y + p_z v_z)$$
$$+ g(p_x + \sqrt{3}\, p_y) - gh(u_x + \sqrt{3}\, u_y)/2 - gh(\sqrt{3}\, v_x + 3v_y)/2 + g^2 + h^2$$

<u>Inverse Kinematics</u>

Given the location of the moving platform, find the limb lengths d_1, d_2, and d_3.

Since the manipulator has only 3 degrees of freedom, the position and orientation must be specified in accordance with the constraints imposed by the revolute joints.

1. Equation (77) imposes the constraint on the orientation.
2. Equation (74) & (78) relate x & y components of **p** to the orientation of the moving platform.

Hence, only 3 of the12 parameters in ${}^{A}R_{B}$ and **p** can be specified arbitrarily.

In any case, the z-component of **p** must be specified since it does not appear in the constraint equations, while the other two parameters can be chosen from either the position vector **p** or the three unit vectors **u**, **v**, and **w**.

If the three components of **p** are chosen, there are eight corresponding platform orientations.

If the roll and pitch angles of the Euler roll-pitch-yaw angles and p_z are chosen, there are two corresponding platform locations.

Once the position vector and the rotation matrix of the moving platform are known, the limb length d_i's can then be computed.

<u>Direct Kinematics</u>

Given the limbs lengths d_1, d_2, and d_3, find the location of the moving platform.

See pp. 147 of Reference [1]

9.7 Jacobian Analysis of Serial Manipulators

9.7.1 Introduction

In previous chapters we have studied the kinematic relations between the end-effector location and the joint variables of serial and parallel manipulators. Both the direct and inverse kinematics have been analyzed. This knowledge enable us to bring the end effector to some desired locations in space. In this chapter we extend our study from a position analysis problem to a velocity analysis problem of serial manipulators.

For some applications, such as spray painting (Fig. 9.14), it is necessary to move the end effector of a manipulator along some desired paths with a prescribed speed. To achieve this goal, the motion of the individual joints of a manipulator must be carefully coordinated. There are two types of velocity coordination problems, called direct velocity and inverse velocity problem. For the *direct velocity problem*, the input joint rates are given and objective is to find the velocity state of the end effector. For the *inverse velocity problem,* the velocity state of the end effector is given and the input joint rates required to produce the desired velocity are to be found. In this chapter, the fundamental knowledge needed to achieve such a coordinated motion is developed.

We call the vector space spanned by the joint variables the *joint space*, and the vector space spanned by the end-effector location, the *end-effector space*. For robot manipulators, the *Jacobian matrix*, or simply *Jacobian*, is defined as the matrix that transforms the joint rates in the actuator space to the velocity state in the end-effector space.

The Jacobian matrix is a critical component for generating trajectories of prescribed geometry in the end-effector space. Most coordination algorithms employed by industrial robots avoid numerical inversion of the Jacobian matrix by deriving analytical inverse solutions on an ad hoc basis. Therefore, it is important that efficient algorithms be developed. Since the velocity state of the end effector can be defined in variety of Jacobian matrices and consequently, different methods of formulation have appeared in the literature (Craig, 1986; Featherstone, 1983; Hollerbach and Saher, 1983; Hunt, 1986, 1987a,b; Orin and Schrader, 1984; Waldron et al., 1985; Whitney, 1972). In what follows, two different definitions of the Jacobian matrix are described. The first is a *conventional Jacobian* and the second is a *screw-based Jacobian*.

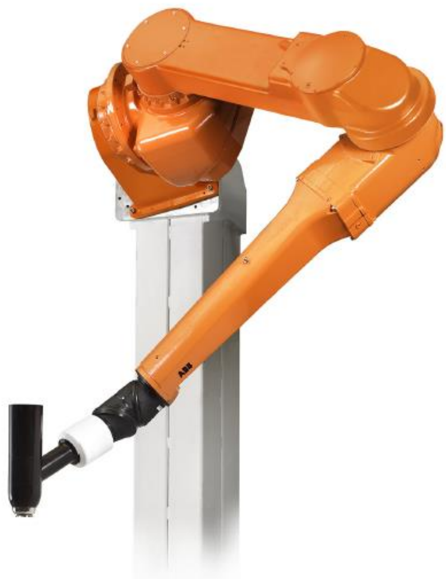


Figure 9.14 Spray painting robot. (ABB)

The Jacobian matrix is also useful in other applications. For some configurations of a manipulator, the Jacobian matrix may lose its full rank. Such conditions are called *singular conditions*. At a singular condition, a serial manipulator may lose one or more degrees of freedom while a parallel manipulator may gain one or more degrees of freedom. In this chapter, the singular conditions of serial manipulators are also studied.

9.7.2 Differential Kinematics of a Rigid Body

We first study the differential kinematics of a rigid body. Then these kinematic properties are applied for a derivation of the differential kinematics of the links in a manipulator and for a development of the Jacobian matrix. Since we will be dealing with many frames of reference, the following notations are made to identify the frame with respect to which a vector is defined. A vector **p** can be a function of time in one reference frame but constant in another reference frame. Thus, in general, we need two frames of references to describe the nature of a vector: one with respect to which the change of a vector is measured and another in which the vector is expressed. In this book we use an inner leading superscript to denote the frame with respect to which a vector is being measure, and an outer leading superscript to indicate the frame in which the vector is expressed.

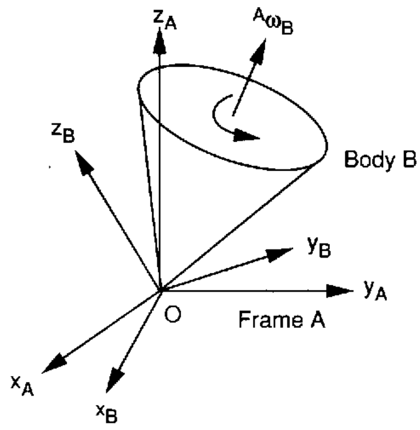


Figure 9.15 Instantaneous rotation of frame B with respect to A

For example, $^B\mathbf{p}$ denotes the position vector of a point *P* with respect to frame ***B***, and $^A(^B\mathbf{p})$ denotes $^B\mathbf{p}$ expressed in frame **A**. Similarly, the velocity of *P* is defined by taking the derivative of $^B\mathbf{p}$ with respect to time:

$$^B\mathbf{v}_P = \frac{d\,^B\mathbf{p}}{dt} \qquad (4.1)$$

However, once the differentiation is taken, the vector can be expressed in any other frame. Thus

$$^A(^B\mathbf{v}_P) = {}^A\left(\frac{d\,^B\mathbf{p}}{dt}\right) \qquad (4.2)$$

indicates that the differentiation is taken with respect to frame B and the resulting vector is expressed in frame A.

When the two leading superscripts are the same or when the frame with respect to which a vector quantity is being measured is clearly understood, the inner superscript will be omitted. For clarity, we often use the rotation matrix ^AR_B to transform a vector from one reference frame to another:

$$^A(^B\mathbf{p}) \equiv {}^AR_B\,{}^B\mathbf{p} \qquad (4.3)$$

Furthermore, when no specific reference frame is mentioned, either the base frame is implied or any reference frame can be used. Note that all vectors in one equation must be expressed in the same reference frame.

9.7.2-1 Angular Velocity of a Rigid Body

While the linear velocity describes the rate of change of the position of a point in space, the angular velocity vector describes the rate of change of the orientation of a rigid body. Figure 9.15 shows that frame *B* is rotating with respect to frame *A* with a fixed point *O*. The orientation of frame *B* with respect to *A* can be described by a rotation matrix, ^AR_B. Since the rotation matrix ^AR_B is orthogonal, the inverse transformation of ${}^{A}R_{B}$ is identical to the transpose. Hence

$${}^{A}R_{B}\,{}^{A}R_{B}^{T} = I \qquad (4.4)$$

where I is a 3 × 3 identity matrix.

Taking the derivative of Eq. (4.4) with respect to time, we obtain

$${}^{A}\dot{R}_{B}\,{}^{A}R_{B}^{T} + {}^{A}R_{B}\,{}^{A}\dot{R}_{B}^{T} = 0 \qquad (4.5)$$

Substituting ${}^{A}R_{B}^{T} = {}^{A}R_{B}^{-1}$ and ${}^{A}R_{B} = ({}^{A}R_{B}^{-1})^{T}$ into Eq. (4.5), we obtain

$$({}^{A}\dot{R}_{B}\,{}^{A}R_{B}^{-1}) + ({}^{A}\dot{R}_{B}\,{}^{A}R_{B}^{-1})^{T} = 0 \qquad (4.6)$$

Without losing generality, we may define the skew-symmetric matrix as

$$\Omega \equiv {}^{A}\dot{R}_{B}\,{}^{A}R_{B}^{-1} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \qquad (4.7)$$

Here ω_x, ω_y, and ω_z are to be identified as three independent parameters specifying the angular velocity of a rigid body. In what follows it will be shown that these three quantities from the components of a vector called the angular velocity vector of *B* in *A*.

The position vector of a point *P* that is embedded in frame *B* and measured with respect to frame A is given by

$${}^{A}\mathbf{p} = {}^{A}R_{B}\,{}^{B}\mathbf{p} \qquad (4.8)$$

Note that ${}^{B}\mathbf{p}$ is a constant vector in frame *B* since *P* is embedded in *B*. The velocity of *P* with respect to frame A is obtained by taking the derivative of Eq.(4.8) with respect to time:

$${}^{A}\mathbf{v}_{P} = \frac{d}{dt}\left({}^{A}R_{B}\,{}^{B}\mathbf{p}\right) = {}^{A}\dot{R}_{B}\,{}^{B}\mathbf{p} \qquad (4.9)$$

Solving ${}^{B}\mathbf{p}$ from Eq. (4.8) and substituting the resulting expression into Eq. (4.9) yields

$${}^{A}\mathbf{v}_{P} = {}^{A}\dot{R}_{B}\,{}^{A}R_{B}^{-1}\,{}^{A}\mathbf{p} \qquad (4.10)$$

Substituting Eq. (4.7) into (4.10) produces

$${}^{A}\mathbf{v}_{P} = \Omega\,{}^{A}\mathbf{p} \qquad (4.11)$$

We may ask ourselves the following question: Is there any point in B that has zero velocity at that instant? Assuming that $\tilde{P}$ is such a point,

$${}^{A}\mathbf{v}_{\tilde{p}} = \Omega\,{}^{A}\tilde{\mathbf{p}} = 0 \qquad (4.12)$$

Equation (4.12) consists of three homogeneous linear equations in three unknowns, $\tilde{p}_x$, $\tilde{p}_y$, and $\tilde{p}_z$. The compatibility condition for the existence of nontrivial solutions is that the determinant of the coefficient matrix must vanish (i.e., $|\Omega| = 0$). Since Ω is a 3 × 3 skew-symmetric matrix, this condition is satisfied automatically:

$$\begin{vmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{vmatrix} = \omega_x\omega_y\omega_z - \omega_x\omega_y\omega_z = 0$$

Hence only two of the three equations in Eq. (4.12) are independent. Solving Eq. (4.12) for the ratio

$\tilde{p}_x : \tilde{p}_y : \tilde{p}_z$, we obtain

$$\tilde{p}_x : \tilde{p}_y : \tilde{p}_z = \omega_x : \omega_y : \omega_z \qquad (4.13)$$

We conclude that there exist infinitely many stationary points, and these points lie on a line that passes through the origin and is parallel to the vector
${}^A\boldsymbol{\omega}_B = [\omega_x, \omega_y, \omega_z]^T$. We call the vector ${}^A\boldsymbol{\omega}_B$ the angular velocity vector and line the *instantaneous screw axis*. Using the vector notation, Eq.(4.11) can be written as

$${}^A\mathbf{v}_P = {}^A\boldsymbol{\omega}_B \times {}^A\mathbf{p} \qquad (4.14)$$

9.7.2-2 Linear Velocity of a Point

Figure 4.3 shows a rigid body B that is making an instantaneous rotation as well as translation with respect to a reference frame A. The position vector of a point P, which is not necessarily fixed in frame B, relative to frame A can be written as

$${}^A\mathbf{p} = {}^A\mathbf{q} + {}^AR_B\,{}^B\mathbf{p}, \qquad (4.15)$$

Where ${}^A\mathbf{q} = \overline{OQ}$ denotes the position vector of the origin Q of frame B with respect to frame A.

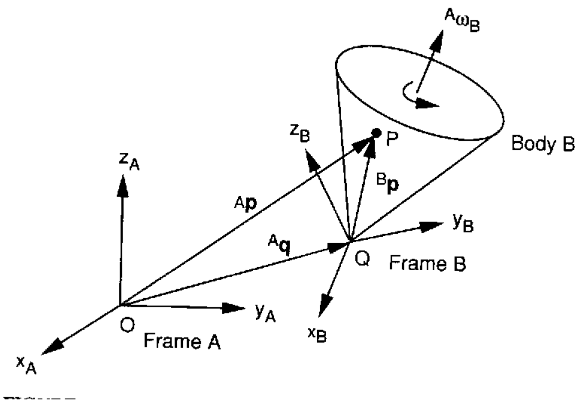


Figure 9.16 Instantaneous motion of a rigid body B with respect to frame A.

To derive the velocity of P, we first consider the rate of change of the second term in Eq.(4.15). This is essentially the case when frame B is rotating with respect to frame A with the origin Q fixed in A. Differentiating the second term of Eq.(4.15) with respect to time yields

$$\frac{d}{dt}({}^AR_B\,{}^B\mathbf{p}) = {}^AR_B\,{}^B\mathbf{v}_p + {}^A\dot{R}_B\,{}^B\mathbf{p} \qquad (4.16)$$

where ${}^B\mathbf{v}_p = \frac{d}{dt}{}^B\mathbf{p}$ denotes the velocity of P with respect frame B.

Postmultiplying both sides of Eq.(4.7) by ${}^AR_B$, we obtain

$${}^A\dot{R}_B = \Omega\,{}^AR_B \qquad (4.17)$$

Substituting Eq.(4.17) into (4.16) yields

$$\frac{d}{dt}({}^AR_B\,{}^B\mathbf{p}) = {}^AR_B\,{}^B\mathbf{v}_p + \Omega\,{}^AR_B\,{}^B\mathbf{p} \qquad (4.18)$$

Equation (4.18) can be written in vector from as

$$\frac{d}{dt}({}^AR_B\,{}^B\mathbf{p}) = {}^AR_B\,{}^B\mathbf{v}_p + {}^A\boldsymbol{\omega}_B \times ({}^AR_B\,{}^B\mathbf{p}) \qquad (4.19)$$

When the origin Q of frame B is moving with respect to frame A, we simply add a component representing the linear velocity of Q in A to Eq.(4.19). Hence a general equation of motion can be written as

$$ {}^{A}\mathbf{v}_{p}={}^{A}\mathbf{v}_{q}+{}^{A}R_{B}\,{}^{B}\mathbf{v}_{p}+{}^{A}\boldsymbol{\omega}_{B}\times({}^{A}R_{B}\,{}^{B}\mathbf{p}) \qquad (4.20)$$

where ${}^{A}\mathbf{v}_{q}={}^{A}\dot{\mathbf{q}}$ denotes the velocity of Q relative to frame A. The first term Eq.(4.20) is contributed by linear velocity of Q with respect to frame A, the second term is contributed by the relative motion of P with respect to frame B, and the third term is contributed by the rotation of frame B with respect to A.

Special Case. If point P is embedded in the moving frame B, ${}^{B}\mathbf{v}_{p}=0$ identically. Equation (4.20) reduces to

$$ {}^{A}\mathbf{v}_{p}={}^{A}\mathbf{v}_{q}+{}^{A}\boldsymbol{\omega}_{B}\times({}^{A}R_{B}\,{}^{B}\mathbf{p}) \qquad (4.21)$$

Although Eq. (4.21) is derived for the case in which Q is the origin of a moving frame, it is equally applicable to any two points fixed on the moving frame. In general, if P and Q are two points embedded in a rigid B, their velocities are related by the equation

$$ {}^{A}\mathbf{v}_{p}={}^{A}\mathbf{v}_{q}+{}^{A}\boldsymbol{\omega}_{B}\times({}^{A}\mathbf{p}-{}^{A}\mathbf{q}) \qquad (4.22)$$

9.7.2-3 Instantaneous Screw Axis

In this section we show that a general instantaneous motion of a rigid body can be described by a differential rotation about a unique axis and a differential translation along the same axis. This concept will be applied to the Jacobian analysis of serial manipulators.

For a general spatial motion of a rigid body B, are there any stationary points in B? If ${}^{B}\tilde{p}$ is a stationary point, ${}^{A}\mathbf{v}_{\tilde{p}}=0$ identically, and Eq.(4.21) reduces to

$$ {}^{A}\omega_{B}\times({}^{A}R_{B}\,{}^{B}\tilde{p})=-{}^{A}v_{q} \qquad (4.23)$$

Since the angular velocity ${}^{A}\boldsymbol{\omega}_{B}$ is derived from a 3×3 skew-symmetric matrix Ω, the coefficients matrix of Eq. (4.23) is singular. It follows that, in general, there are no solutions to Eq. (4.21). However, we may seek for those points whose linear velocity vectors point along the direction of the angular velocity. That is,

$$ {}^{A}\mathbf{v}_{\tilde{p}}=\lambda\,{}^{A}\boldsymbol{\omega}_{B} \qquad (4.24)$$

where λ is called a *pitch.*

Substituting Eq.(4.24) into (4.21) yields

$$ {}^{A}\mathbf{v}_{q}+{}^{A}\boldsymbol{\omega}_{B}?({}^{A}R_{B}\,{}^{B}\tilde{\mathbf{p}})=\lambda\,{}^{A}\boldsymbol{\omega}_{B} \qquad (4.25)$$

Dot-multiplying both sides of Eq.(4.25) by ${}^{A}\omega_{B}$, we obtain

$$ \lambda=\frac{{}^{A}\omega_{B}\cdot{}^{A}\mathbf{v}_{q}}{{}^{A}\omega_{B}^{2}} \qquad (4.26)$$

Equation (4.25) can be written in the form

$$ {}^{A}\omega_{B} \times ({}^{A}R_{B}\, {}^{B}\tilde{p}) = -{}^{A}v'_{q} \tag{4.27} $$

Where ${}^{A}v'_{q} = {}^{A}v_{q} - \lambda\, {}^{A}\omega_{B}$ is orthogonal to ${}^{A}\omega_{B}$; that is,

$$ {}^{A}\omega_{B} \cdot {}^{A}\mathbf{v}'_{q} = {}^{A}\omega_{B} \cdot ({}^{A}\mathbf{v}_{q} - \lambda\, {}^{A}\omega_{B}) \tag{4.28} $$

We now make use of the following result derived from vector algebra. Let vectors **a**, **b**, and **c** in Fig. 4.4 satisfy the following two conditions:

$$ \mathbf{a} \times \mathbf{c} = \mathbf{b} $$

$$ \mathbf{a} \cdot \mathbf{b} = \mathbf{0} $$

Then c has infinite number of solutions lying on a line:

$$ c = -\frac{a \times b}{a^{2}} + \mu a $$

where μ is an arbitrary scalar constant.

From the vector algebra above, we conclude that all solutions to Eqs. (4.27) and (4.28) are given by

$$ {}^{A}R_{B}\, {}^{B}\tilde{p} = \frac{{}^{A}\omega_{B} \times {}^{A}v'_{q}}{{}^{A}\omega_{B}^{2}} + \mu\, {}^{A}\omega_{B} \tag{4.29} $$

Applying Eq.(4.15),Eq.(4.29) can be written as

$$ {}^{A}\tilde{\mathbf{p}} = {}^{A}\mathbf{q} + \frac{{}^{A}\omega_{B} \times {}^{A}\mathbf{v}'_{q}}{{}^{A}\omega_{B}^{2}} + \mu\, {}^{A}\omega_{B} \tag{4.30} $$

Equation (4.29) or (4.30) states that the locus of all points whose instantaneous linear velocities point along the direction of the angular velocity vector is a line. This line that is parallel to the angular velocity vector is called the *instantaneous screw axis*. We conclude that the general spatial motion of a rigid body consists a differential rotation about, and a differential translation along, some axis.

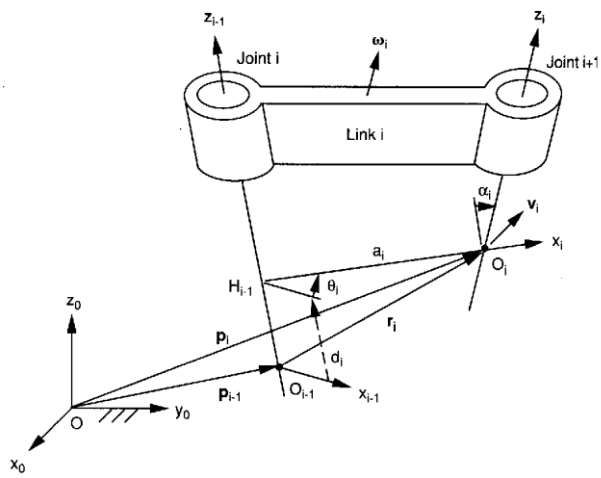


Figure 9.17 Geometry of link *i* and its motion state.

9.7.3 Differential Kinematics of Serial Manipulators

In this section we study the differential kinematics of a serial manipulator using the Denavit-Hartenberg transformation matrix. First, we study the differential motion of a link. Then we apply it to the differential motion of a serial manipulator.

9.7.3-1 Link Differential Transformation Matrix

Figure 9.17 shows a typical link, I, of a manipulator. According to the D-H convention, a Cartesian coordinate system (x_i, y_i, z_i) is attached to the distal end of link i, and the fixed coordinate system is denoted by frame (x_0, y_0, z_0). The location of link i can be described by a position vector $\mathbf{p}_i$ of O_i , and a rotation matrix ${}^0R_i$ of link i with respect to the reference frame 0. The velocity state of link i can be described by the linear velocity $\mathbf{v}_i$ of the origin O_i, and the angular velocity ω_i of link i relative to fixed reference frame.

The D-H transformation matrix is given by Eq.(9-4.1b) and the inverse transformation is given by Eq.(9-2.5). Taking the derivative of Eq.(9-4.1b) with respect to time, we obtain

$$ {}^{i-1}\dot{A}_i = \begin{bmatrix} -\dot{\theta}_i s\theta_i & -\dot{\theta}_i c\alpha_i c\theta_i & \dot{\theta}_i s\alpha_i c\theta_i & -\dot{\theta}_i a_i s\theta_i \\ \dot{\theta}_i c\theta_i & -\dot{\theta}_i c\alpha_i s\theta_i & \dot{\theta}_i s\alpha_i s\theta_i & \dot{\theta}_i a_i c\theta_i \\ 0 & 0 & 0 & \dot{d}_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.31} $$

In Eq. (4.31), both θ_i and d_i are treated as variables. For a revolute joint, $\dot{d}_i = 0$, and for a prismatic joint, $\dot{\theta}_i = 0$. Postmultiplying both sides of Eq. (4.31) by $({}^{i-1}A_i)^{-1}$, we obtain

$$ ({}^{i-1}\dot{A}_i)({}^{i-1}A_i)^{-1} = \begin{bmatrix} \dot{\theta}_i\, {}^{i-1}Z_{i-1} & \dot{d}_i\, {}^{i-1}\mathbf{z}_{i-1} \\ 0 & 0 \end{bmatrix} \tag{4.32} $$

where

$$ {}^{i-1}\mathbf{z}_{i-1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{4.33} $$

$$ {}^{i-1}\mathbf{Z}_{i-1} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.34} $$

Equation (4.33) represents a unit vector pointing along the z_{i-1} axis. Similarly, Eq. (4.34) represents a 3×3 skew-symmetric matrix whose nonzero elements denote a unit angular velocity of link i with respect to link i-1. Both ${}^{i-1}\mathbf{z}_{i-1}$ and ${}^{i-1}Z_{i-1}$ are expressed in the $(i\text{-}1)^{th}$ link frame. We conclude that the upper left 3×3 submatrix of $({}^{i-1}\dot{A}_i)({}^{i-1}A_i)^{-1}$ represents the angular velocity of link i relative to link i-1, and the fourth column represents the linear velocity of a point, which is embedded in link i and instantaneously coincident with O_{i-1}, relative to link i-1.

9.7.3-2 Overall Differential Transformation Matrix

We now apply the results derived earlier to the kinematic analysis of serial manipulators. Figure

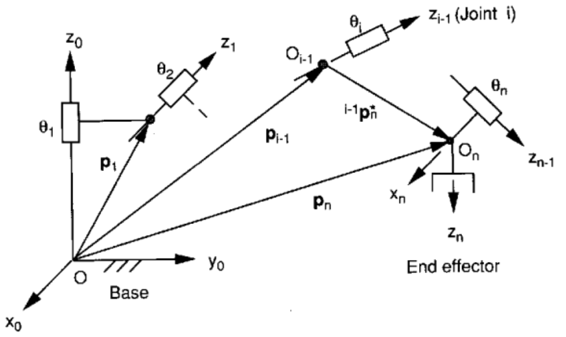


Figure 9.18 Link parameters of a serial manipulator

9.18 shows a schematic of a typical serial manipulator, where $\mathbf{p}_n$ denotes the position vector of the origin of the end-effector frame, and $\mathbf{p}_{i-1}$ denotes the position vector of the origin of the $(i\text{-}1)^{th}$ frame relative to the fixed frame. Further, ${}^{i-1}\mathbf{p}^*_n$ denotes the vector pointing from O_{i-1} to O_n and expressed in the fixed frame. The loop-closure equation for such an n-dof serial manipulator can be written as

$${}^0A_n={}^0A_1{}^1A_2{}^2A_3\ldots{}^{n-1}A_n. \qquad (4.35)$$

Taking the derivative of Eq. (4.35) with respect to time, we obtain

$${}^0\dot{A}_n=({}^0\dot{A}_1{}^1A_2\cdots{}^{n-1}A_n)+({}^0A_1{}^1\dot{A}_2\cdots{}^{n-1}A_n)+\cdots+({}^0A_1{}^1A_2\cdots{}^{n-1}\dot{A}_n) \qquad (4.36)$$

Equation (4.36) contains 12 nontrivial scalar equations that can be reduced to a system of six independent equations as follows. Postmultiplying Eq. (4.36) by ${}^0A_n^{-1}$, we obtain

$${}^0\dot{A}_n{}^0A_n^{-1}={}^0\dot{A}_1{}^0A_1^{-1}+{}^0A_1({}^1\dot{A}_2{}^1A_2^{-1}){}^0A_1^{-1}+({}^0A_1{}^1A_2)({}^2\dot{A}_3{}^2A_3^{-1})({}^0A_1{}^1A_2)^{-1}+\cdots \qquad (4.37)$$

The matrix ${}^0\dot{A}_n$ can be decomposed into two submatrices:

$${}^0\dot{A}_n=\begin{bmatrix}\dot{R}_n & \mathbf{v}_n\\ 0 & 0\end{bmatrix} \qquad (4.38)$$

where $\dot{R}_n$ denotes the rate of change of end-effector rotation matrix and $\mathbf{v}_n=\dot{\mathbf{p}}_n$ denotes the linear velocity of the origin of the hand coordinate system.

Similar to Eq.(4.32), we express the matrix products in Eq.(4.37) as

$${}^0\dot{A}_n({}^0A_n)^{-1}=\begin{bmatrix}\Omega_n & \mathbf{v}_0\\ 0 & 0\end{bmatrix} \qquad (4.39)$$

$${}^{i-1}\dot{A}_i({}^{i-1}A_i)^{-1}=\begin{bmatrix}\dot{\theta}_i{}^{i-1}Z_{i-1} & \dot{d}_i{}^{i-1}\mathbf{z}_{i-1}\\ 0 & 0\end{bmatrix} \qquad (4.40)$$

Note that $\Omega_n=\dot{R}_nR_n^T$ is a 3×3 skew-symmetric matrix whose elements represent the angular velocity of the end effector, and $\mathbf{v}_o$ represents the linear velocity of a point in the end effector that is instantaneously coincident with the origin of the fixed reference frame. For convenience, we define

$$ {}^{0}A_1 {}^{1}A_2 \cdots {}^{i\text{-}2}A_{i\text{-}1} = \begin{bmatrix} R_{i\text{-}1} & \mathbf{p}_{i-1} \\ 0 & 0 \end{bmatrix} \tag{4.41} $$

where $R_{i\text{-}1}$ and $\mathbf{p}_{i\text{-}1}$ denote the rotation matrix and the position of the origin of the (i-1)th frame with respect to the fixed reference frame. Substituting Eqs. (4.39) through (4.41) into (4.37) yields

$$ \begin{bmatrix} \Omega_n & v_0 \\ 0 & 0 \end{bmatrix} $$
$$ = \sum_{i=1}^{n} \begin{bmatrix} \dot{\theta}_i (R_{i-1} {}^{i-1}Z_{i-1} R_{i-1}^T) & -\dot{\theta}_i (R_{i-1} {}^{i-1}Z_{i-1} R_{i-1}^T) p_{i-1} + \dot{d}_i R_{i-1} {}^{i-1}z_{i-1} \\ 0 & 0 \end{bmatrix} \tag{4.42} $$
$$ = \sum_{i=1}^{n} \begin{bmatrix} \dot{\theta}_i Z_{i-1} & -\dot{\theta}_i Z_{i-1} p_{i-1} + \dot{d}_i z_{i-1} \\ 0 & 0 \end{bmatrix} $$

where

$$ Z_{i\text{-}1} = R_{i\text{-}1} {}^{i\text{-}1}Z_{i\text{-}1} R_{i\text{-}1}^T \tag{4.43} $$
$$ \mathbf{z}_{i\text{-}1} = R_{i\text{-}1} {}^{i\text{-}1}\mathbf{z}_{i\text{-}1} \tag{4.44} $$

Equation (4.42) contains only six independent equations. The (3,2), (1,3), and (2,1) elements form the angular velocity vector $\boldsymbol{\omega}_n$ of a point in the end effector that is instantaneously coincident of the origin of the fixed frame. Writing Eq. (4.42) in vector form, we obtain

$$ \boldsymbol{\omega}_n = \sum_{i=1}^{n} \dot{\theta}_i \mathbf{z}_{i-1} \tag{4.45} $$
$$ \mathbf{v}_o = \sum_{i=1}^{n} -\dot{\theta}_i \mathbf{z}_{i-1} \times \mathbf{p}_{i-1} + \dot{d}_i \mathbf{z}_{i-1} \tag{4.46} $$

Equations (4.45) and (4.46) imply that the angular velocities of the links are additive. We may think of the end effector as rotating instantaneous about and translating along all the joint axis, and the effect of the instantaneous motion about each joint axis can be added linearly. We note that the velocity, $\boldsymbol{v}_n$ of a point located at the origin of the hand coordinate system is related to $\boldsymbol{v}_o$ by the following transformation.

$$ \mathbf{v}_n = \mathbf{v}_0 + \boldsymbol{\omega}_n \times \mathbf{p}_n \tag{4.47} $$

9.7.4 Manipulator Jacobian Matrix

Let $x_i = f_i(q_1, q_2, q_3, \cdots, q_n)$ for i=1, 2, 3, . . ., m be a set of m equations, each a function of n independent variables. Then the time derivatives of x_i can be written as a function of $\dot{q}_i$ as follows：

$$ \dot{x}_i = \frac{\partial f_i}{\partial q_1}\dot{q}_1 + \frac{\partial f_i}{\partial q_2}\dot{q}_2 + \frac{\partial f_i}{\partial q_3}\dot{q}_3 + \cdots + \frac{\partial f_i}{\partial q_n}\dot{q}_n, \qquad i=1, 2, 3, \cdots, m . \tag{4.54} $$

Writing Eq. (4.54) in matrix form, we obtain

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \vdots & \frac{\partial f_1}{\partial q_n} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \vdots & \frac{\partial f_2}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial q_1} & \frac{\partial f_m}{\partial q_2} & \cdots & \frac{\partial f_m}{\partial q_n} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}, \qquad (4.55)
$$

or simply

$$
\dot{\mathbf{x}} = \mathrm{J}\,\dot{\mathbf{q}} \qquad (4.56)
$$

where $\dot{\mathbf{x}}=[\dot{x}_1, \dot{x}_2, \cdots, \dot{x}_m]^{\mathrm{T}}$ denotes an m-dimensional vector, $\dot{\mathbf{q}} = [\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n]^{\mathrm{T}}$ denotes an n-dimensional vector, and J denotes the $m\times n$ matrix of the partial derivatives in Eq.(4.55).

We call J the *Jacobian matrix,* or simply *Jacobian*. The Jacobian matrix is a linear transformation matrix that maps an n-dimensional velocity vector $\dot{\mathbf{q}}$ into an m-dimensional velocity vector $\dot{\mathbf{x}}$. We may think of the elements of J as the influence coefficients of the vector function x. The (i, j) element of J describes how a differential change in q_j affects the differential change in x_i. In general, the vector **x** is a nonlinear function of **q**. Hence the Jacobian matrix is also a function of **q**. Thus, the Jacobian matrix is configuration dependent. For robot manipulators, the Jacobian matrix is defined as the coefficient matrix of any set of equations that relates the velocity state of the end effector to the actuated joint rates. The joint rates are defined as

$$
\dot{q}_i = \begin{cases} \dot{\theta}_i & \text{for a revolute joint} \\ \dot{d}_i & \text{for a prismatic joint} \end{cases}
$$

The velocity state of the end effector, x, can be expressed in several different ways. Perhaps the most commonly used definitions are the *conventional Jacobian and screw-based Jacobian.*

1. *Conventional Jacobian*. In a conventional Jacobian, the end-effector velocity state is expressed in terms of the linear velocity of the origin of the end-effector coordinate frame, $\mathbf{v}_n$, and the angular velocity of the end effector, $\boldsymbol{\omega}_n$.

$$
\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{v_n} \\ \boldsymbol{\omega}_\mathbf{n} \end{bmatrix}. \qquad (4.58)
$$

2. *Screw-based Jacobian*. The screw-based Jacobian is defined in terms of the angular velocity of the end effector, $\boldsymbol{\omega}_n$, and the linear velocity of a reference point, $\boldsymbol{v}_o$, in the end effector that is instantaneously coincident with the origin of a reference frame in which the screws are expressed:

$$
\dot{\boldsymbol{x}} = \begin{bmatrix} \boldsymbol{w}_n \\ \boldsymbol{v}_o \end{bmatrix}. \qquad (4.59)
$$

We note that the end-effector velocity, $\dot{\mathbf{x}}$, for the screw-based Jacobian is defined with its angular and linear velocity vectors arranged in reverse order from the conventional Jacobian.

In general, the Jacobian matrix is an $m\times n$ matrix, where m denotes the degrees of freedom of the end-effector space and n denotes the number of actuated joint variables. For a 6-dof spatial manipulator, $m = n = 6$, the Jacobian matrix is a 6×6 square matrix. For a manipulator with less than 6 degrees of freedom, the end-effector velocity state may contain just the linear velocity vector, or the angular velocity vector, or a combination of some linear and angular velocity components. For example, the working space of a planar manipulator is confined to a two-dimensional space. A three-component vector $\dot{\mathbf{x}} = [v_x, v_y, v_z]^T$ is sufficient to describe the velocity state of the end effector. Hence the Jacobian reduces to a 3×3 matrix. Similarly, for a point positioning device $\dot{\mathbf{x}} = [v_x, v_y, v_z]^T$ and for a body-orienting mechanism $\dot{\mathbf{x}} = [\omega_x, \omega_y, \omega_z]^T$. On the other hand, for a manipulator with redundant degrees of freedom, we may have $n > 6$. In this book, we concentrate on nonredundant robot manipulators.

9.7.5 Conventional Jacobian

As mentioned earlier, any point in the end effector can be chosen as the reference point to describe the velocity state of the end effector. A logical choice is the origin. O_n, of the end-effector frame (Whitney, 1972). Using this definition, the end-effector velocity state can be expressed in terms of the joint rates as follows:

$$\mathbf{v}_n = \sum_{i=1}^{n} \left[\dot{\theta}_i \left(\mathbf{z}_{i-1} \times {}^{i-1}\mathbf{p}_n^*\right) + \mathbf{z}_{i-1}\dot{d}_i\right] \tag{4.60}$$

$$\boldsymbol{\omega}_n = \sum_{i=1}^{n} \dot{\theta}_i \mathbf{z}_{i-1} \tag{4.61}$$

where $\dot{\theta}_i$ and $\dot{d}_i$ are the rate of rotation about and translation along the i^{th} joint axis, $\mathbf{z}_{i-1}$ is a unit vector along the i^{th} joint axis, and ${}^{i-1}\mathbf{p}_n^*$ is a vector defined from the origin of the $(i - 1)^{th}$ link frame, O_{i-1}, to the origin of the end effector frame, as shown in Fig. 9.18. Note that all vectors in Eqs. (4.60) and (4.61) are expressed in the fixed coordinate frame, (x_0, y_0, z_0). Writing Eqs. (4.60) and (4.61) in matrix form, we obtain

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{v}_n \\ \boldsymbol{\omega}_n \end{bmatrix} = \mathbf{J}\dot{\mathbf{q}} \ , \tag{4.62}$$

where

$$J = [J_1,\ J_2, \cdots, J_n],$$

$$J_i = \begin{bmatrix} \mathbf{z}_{i-1} \times {}^{i-1}\mathbf{p}_n^* \\ \mathbf{z}_{i-1} \end{bmatrix} \quad \text{for a revolute joint,}$$

$$J_i = \begin{bmatrix} \mathbf{z}_{i-1} \\ \mathbf{O} \end{bmatrix} \quad \text{for a prismatic joint.}$$

The left-hand side of Eq. (4.62) is a 6×1 vector composed of the elements of $\mathbf{v}_n$ and $\boldsymbol{\omega}_n$, while the right-hand side is a product of the Jacobian matrix and the vector of joint rates. The vector of joint rates consists of all the actuated joint rates, $\dot{\mathbf{q}}_i$, for $i = 1, 2, \ldots, n$. The i^{th} column of the Jacobian matrix, J_i, represents the effect of the i^{th} joint rate on the velocity state of the end effector.

Equation (4.62) implies that to compute the Jacobian matrix, the direction and location of each joint axis should be determined first. This can be accomplished by the following matrix operations:

$$\mathbf{z}_{i-1} = {}^{0}\mathbf{R}_{i-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad (4.63)$$

$${}^{i-1}\mathbf{p}_n^* = {}^{0}\mathbf{R}_{i-1}\,{}^{i-1}\mathbf{r}_i + {}^{i}\mathbf{p}_n^* \qquad (4.64)$$

where

$${}^{i-1}\mathbf{r}_i = \begin{bmatrix} a_i c\theta_i \\ a_i s\theta_i \\ d_i \end{bmatrix}$$

denotes the vector $\overline{O_{i-1}O_i}$, expressed in the $(i - 1)$th link frame, while ${}^{i-1}\mathbf{p}_n^*$ denotes the vector $\overline{O_{i-1}O_n}$ expressed in the fixed frame. Once the Jacobian is known, the end-effector velocity can be computed directly from Eq. (4.62) for any given joint rates. On the other hand, given a desired end-effector velocity, the inverse transformation of Eq. (4.62) can be solved for the joint rates.

9.7.5-1 Jacobian of a Planar 2-DOF Manipulator

In this example we derive the Jacobian of a planar 2-dof manipulator shown in Fig. 9.19. The manipulator is made up of two revolute joints, with both axes pointing out of the paper. A coordinate system is attached to each link according to the D-H convention for the purpose of analysis. The (x_0, y_0) coordinate system is attached to the base with its origin located at the fixed pivot O. The x_0-axis points to the right. We first compute the vectors $\mathbf{z}_{i-1}$ and ${}^{i-1}\mathbf{p}_2^*$ by applying Eqs. (4.63) and (4.64):

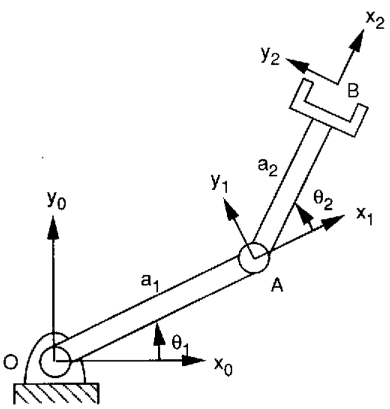


Fig. 9.19 Planar 2-dof, 2R manipulator.

$$\mathbf{z}_0 = \mathbf{z}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$ {}^{1}\mathbf{p}_2^* = \begin{bmatrix} a_2 \mathrm{c}\theta_{12} \\ a_2 \mathrm{s}\theta_{12} \\ 0 \end{bmatrix}, $$

$$ {}^{0}\mathbf{p}_2^* = \begin{bmatrix} a_1 \mathrm{c}\theta_1 + a_2 \mathrm{c}\theta_{12} \\ a_1 \mathrm{s}\theta_1 + a_2 \mathrm{s}\theta_{12} \\ 0 \end{bmatrix}, $$

where $\theta_{12} = \theta_1 + \theta_2$. We note the expressions above can be obtained directly from the geometry of the links without using Denavit-Hartenberg transformation matrices. Substituting the expressions above into Eq. (4.62), we obtain

$$ \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} -a_1 \mathrm{s}\theta_1 - a_2 \mathrm{s}\theta_{12} & -a_2 \mathrm{s}\theta_{12} \\ a_1 \mathrm{c}\theta_1 + a_2 \mathrm{c}\theta_{12} & a_2 \mathrm{c}\theta_{12} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}. \tag{4.65} $$

Hence the Jacobian matrix is given by

$$ \mathrm{J} = \begin{bmatrix} -a_1 s\theta_1 - a_2 s\theta_{12} & -a_2 s\theta_{12} \\ a_1 c\theta_1 + a_2 c\theta_{12} & a_2 c\theta_{12} \end{bmatrix}. \tag{4.66} $$

9.7.5-2 Jacobian of a Planar 3-DOF Manipulator

As a second example, we study the conventional Jacobian of the 3-dof manipulator shown in Fig. 2.3. We first compute the vectors $\mathbf{z}_{i-1}$ and ${}^{i-1}\mathbf{p}_3^*$ from Eqs. (4.63) and (4.64), for i = 1, 2, and 3 as follows:

$$ \mathbf{z}_0 = \mathbf{z}_1 = \mathbf{z}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

$$ {}^{2}\mathbf{p}_3^* = \begin{bmatrix} a_3 \mathrm{c}\theta_{123} \\ a_3 \mathrm{s}\theta_{123} \\ 0 \end{bmatrix}, $$

$$ {}^{1}\mathbf{p}_3^* = \begin{bmatrix} a_2 \mathrm{c}\theta_{12} + a_3 \mathrm{c}\theta_{123} \\ a_2 \mathrm{s}\theta_{12} + a_3 \mathrm{s}\theta_{123} \\ 0 \end{bmatrix}, $$

$$ {}^{0}\mathbf{p}_3^* = \begin{bmatrix} a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123} \\ a_1 s\theta_1 + a_2 s\theta_{12} + a_3 s\theta_{123} \\ 0 \end{bmatrix}, $$

where $\theta_{12} = \theta_1 + \theta_2$ and $\theta_{123} = \theta_1 + \theta_2 + \theta_3$ .Substituting the expressions above into Eq. (4.62), we obtain

$$\begin{bmatrix} v_x \\ v_y \\ \omega_z \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix},$$

where

$$\mathrm{J} = \begin{bmatrix} -\left(a_1\mathrm{s}\theta_1 + a_2\mathrm{s}\theta_{12} + a_3\mathrm{s}\theta_{123}\right) & -\left(a_2\mathrm{s}\theta_{12} + a_3\mathrm{s}\theta_{123}\right) & -a_3\mathrm{s}\theta_{123} \\ \left(a_1\mathrm{c}\theta_1 + a_2\mathrm{c}\theta_{12} + a_3\mathrm{c}\theta_{123}\right) & \left(a_2\mathrm{c}\theta_{12} + a_3\mathrm{c}\theta_{123}\right) & -a_3\mathrm{c}\theta_{123} \\ 1 & 1 & 1 \end{bmatrix} \tag{4.67}$$

We note that if the reference point is chosen at origin of the (x_2, y_2) frame, the Jacobian matrix reduces to

$$\mathrm{J} = \begin{bmatrix} -\left(a_1\mathrm{s}\theta_1 + a_2\mathrm{s}\theta_{12}\right) & -\left(a_2\mathrm{s}\theta_{12}\right) & 0 \\ \left(a_1\mathrm{c}\theta_1 + a_2\mathrm{c}\theta_{12}\right) & \left(a_2\mathrm{c}\theta_{12}\right) & 0 \\ 1 & 1 & 1 \end{bmatrix}. \tag{4.68}$$

9.7.5-3 Jacobian of the Stanford Manipulator

Figure 4.8 shows a schematic diagram of the Stanford arm described in Chapter 2. To simplify the analysis, the origin of the fixed coordinate frame is located at the point of intersection of the first two joint axes, and the origin of the (x_6, y_6, z_6) frame is located at the point of intersection of the last three joint axes.

The D-H link parameters are listed in Table 4.1, from which the D-H transformation matrices are derived as follows:

$${}^0A_1 = \begin{bmatrix} c\theta_1 & 0 & -s\theta_1 & 0 \\ s\theta_1 & 0 & c\theta_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^1A_2 = \begin{bmatrix} c\theta_2 & 0 & s\theta_2 & 0 \\ s\theta_2 & 0 & -c\theta_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$
{}^{2}A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^{3}A_4 = \begin{bmatrix} c\theta_4 & 0 & -s\theta_4 & 0 \\ s\theta_4 & 0 & c\theta_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^{4}A_5 = \begin{bmatrix} c\theta_5 & 0 & s\theta_5 & 0 \\ s\theta_5 & 0 & -c\theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^{5}A_6 = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ s\theta_6 & c\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

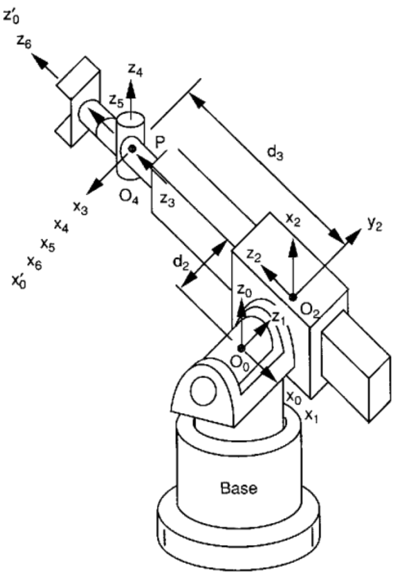


Figure 9.20 Stanford manipulator.

TABLE 4.1. D-H Link Parameters of the Stanford Arm

Joint i	α_i	a_i	d_i	θ_i
1	−90°	0	0	θ_1 (variable)
2	90°	0	d_2 (constant)	θ_2 (variable)
3	0°	0	d_3 (variable)	−90° (constant)
4	−90°	0	0	θ_4 (variable)
5	90°	0	0	θ_5 (variable)
6	0°	0	0	θ_6 (variable)

The directions of the joint axes, $\mathbf{z}_{i-1}$, are derived by applying Eq. (4.63):

$$
\mathbf{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{4.69}
$$

$$
\mathbf{z}_1 = {}^{0}R_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -s\theta_1 \\ c\theta_1 \\ 0 \end{bmatrix}, \tag{4.70}
$$

$$
\mathbf{z}_2 = \mathbf{z}_3 = {}^{0}\mathrm{R}_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathrm{c}\theta_1\mathrm{s}\theta_2 \\ \mathrm{s}\theta_1\mathrm{s}\theta_2 \\ \mathrm{c}\theta_2 \end{bmatrix}, \tag{4.71}
$$

$$
\mathbf{z}_4 = {}^{0}\mathrm{R}_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\mathrm{s}\theta_1\mathrm{s}\theta_4 + \mathrm{c}\theta_1\mathrm{c}\theta_2\mathrm{c}\theta_4 \\ \mathrm{c}\theta_1\mathrm{s}\theta_4 + \mathrm{s}\theta_1\mathrm{c}\theta_2\mathrm{c}\theta_4 \\ -\mathrm{s}\theta_2\mathrm{c}\theta_4 \end{bmatrix}, \tag{4.72}
$$

$$
\mathbf{z}_5 = {}^{0}\mathrm{R}_5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathrm{s}\theta_1\mathrm{c}\theta_4\mathrm{s}\theta_5 + \mathrm{c}\theta_1\mathrm{c}\theta_2\mathrm{s}\theta_4\mathrm{s}\theta_5 + \mathrm{c}\theta_1\mathrm{s}\theta_2\mathrm{c}\theta_5 \\ -\mathrm{c}\theta_1\mathrm{c}\theta_4\mathrm{s}\theta_5 + \mathrm{s}\theta_1\mathrm{c}\theta_2\mathrm{s}\theta_4\mathrm{s}\theta_5 + \mathrm{s}\theta_1\mathrm{s}\theta_2\mathrm{c}\theta_5 \\ -\mathrm{s}\theta_2\mathrm{s}\theta_4\mathrm{s}\theta_5 + \mathrm{c}\theta_2\mathrm{c}\theta_5 \end{bmatrix}. \tag{4.73}
$$

The position vectors, ${}^{i-1}\mathbf{p}_6^*$, are derived by applying Eq. (4.64)

$$
{}^{3}\mathbf{p}_6^* = {}^{4}\mathbf{p}_6^* = {}^{5}\mathbf{p}_6^* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{4.74}
$$

$$
{}^{2}\mathbf{p}_6^* = \begin{bmatrix} d_3\mathrm{c}\theta_1\mathrm{s}\theta_2 \\ d_3\mathrm{s}\theta_1\mathrm{s}\theta_2 \\ d_3\mathrm{c}\theta_2 \end{bmatrix}, \tag{4.75}
$$

$$
{}^{1}\mathbf{p}_6^{*} = \begin{bmatrix} d_3 \mathrm{c}\theta_1 \mathrm{s}\theta_2 - d_2 \mathrm{s}\theta_1 \\ d_3 \mathrm{s}\theta_1 \mathrm{s}\theta_2 + d_2 \mathrm{c}\theta_1 \\ d_3 \mathrm{c}\theta_2 \end{bmatrix}, \tag{4.76}
$$

$$
{}^{0}\mathbf{p}_6^{*} = \begin{bmatrix} d_3 \mathrm{c}\theta_1 \mathrm{s}\theta_2 - d_2 \mathrm{s}\theta_1 \\ d_3 \mathrm{s}\theta_1 \mathrm{s}\theta_2 + d_2 \mathrm{c}\theta_1 \\ d_3 \mathrm{c}\theta_2 \end{bmatrix}. \tag{4.77}
$$

The Jacobian matrix is derived by applying Eq. (4.62) column by column:

$$
\begin{bmatrix} v_{6x} \\ v_{6y} \\ v_{6z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \mathrm{J} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{d}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{bmatrix},
$$

where the Jacobian matrix is given by

$$
\mathrm{J} = \begin{bmatrix} -d_3 \mathrm{s}\theta_1 \mathrm{s}\theta_2 - d_2 \mathrm{c}\theta_1 & d_3 \mathrm{c}\theta_1 \mathrm{c}\theta_2 & \mathrm{c}\theta_1 \mathrm{s}\theta_2 & 0 & 0 & 0 \\ d_3 \mathrm{c}\theta_1 \mathrm{s}\theta_2 - d_2 \mathrm{s}\theta_1 & d_3 \mathrm{s}\theta_1 \mathrm{c}\theta_2 & \mathrm{s}\theta_1 \mathrm{s}\theta_2 & 0 & 0 & 0 \\ 0 & -d_3 \mathrm{s}\theta_2 & \mathrm{c}\theta_2 & 0 & 0 & 0 \\ 0 & -\mathrm{s}\theta_1 & 0 & \mathrm{c}\theta_1 \mathrm{s}\theta_2 & j_{45} & j_{46} \\ 0 & \mathrm{c}\theta_1 & 0 & \mathrm{s}\theta_1 \mathrm{s}\theta_2 & j_{55} & j_{56} \\ 1 & 0 & 0 & \mathrm{c}\theta_2 & j_{65} & j_{66} \end{bmatrix}, \tag{4.78}
$$

where j_{45}, j_{55}, and j_{65} represent the x, y, and z components of unit vector $\mathbf{z}_4$, and j_{46}, j_{56}, and j_{66} are the x, y, and z components of $\mathbf{z}_5$.

As a consequence of the concurrence of axes 4, 5 and 6, all the elements in the upper right 3×3 submatrix are equal to zero.

<u>Singular Analysis</u>

The Jacobian matrix, J, transforms joint rates of a manipulator into the end-effector velocity state. Thus, given the joint rates, we can compute the end-effector velocities directly. In a trajectory planning problem, however, the end-effector velocities are usually given along a desired path in the end- effector space, and these velocities must be converted into the joint rates in the joint space. This requires a computation of the inverse transformation of Jacobian

$$
\dot{\mathbf{q}} = \mathrm{J}^{-1} \dot{\mathbf{x}} \tag{9-7.3}
$$

Equation (9-7.3) provides a means of calculating the joint rates required to produce certain desired end-effector velocities. It is obvious that the joint rates depend on the condition of the Jacobian matrix. At certain manipulator configurations, the Jacobian matrix may lose its full rank.

Hence as the manipulator approaches these configurations, the Jacobian matrix becomes ill conditioned and may not be invertible. Under such a condition, numerical solution of Eq. (9-7.3) results in infinite joint rates. A manipulator is said to be at a singular configuration when the Jacobian matrix loses its full rank. Physically, it means that the instantaneous screws spanning the n-dimensional space of the Jacobian matrix becomes linearly dependent. Therefore, at a singular configuration, a serial manipulator may lose one or more degrees of freedom, and it won't be able to move in some directions in the end-effector space.

Singular configurations can be found by setting the determinant of the Jacobian matrix to zero. In general, this will result in a single algebraic equation. For serial manipulator, the singular condition is a function of the intermediate joint variables, not of the first and last joint variables. This is because the presence of a singularity depends solely on the relative locations of the joint axes. Rotation of the entire manipulator about the first axis does not change the relative locations of the joint axes. Similarly, rotation of the end-effector about the last joint axis does not affect the location of any joint axes. Therefore, the first and last joint variables do not appear in the determinant of the Jacobian matrix. There are two types of singularities for a serial manipulator: boundary singularity and interior singularity. A boundary singularity occurs when the end effector is on the surface of the workspace boundary, and it usually happens when the manipulator is either in a fully stretched-out or a folded-back configuration. Boundary singularity can also occur when one of its actuators reaches its mechanical limit. An interior singularity occurs inside the workspace boundary. Several conditions may lead to an interior singularity. For example, when two or more joint axes line up on a straight line, the effects of a rotation about one joint axis can be canceled by counter-rotation about another joint axis. Thus the end effector remains stationary even though the intermediate links of the manipulator may move in space. Another example of interior singularity occurs when four revolute joint axes are parallel to one another or intersect at a common point. For a manipulator of general geometry, the problem of identifying interior singularities becomes a much more complex problem. Basically, an interior singularity occurs whenever the screws of two or more joint axes become linearly dependent. Boundary singularities are not particularly serious, since they can always be avoided by arranging the tasks of manipulation far away from the workspace boundary. Interior singularity is more troublesome because it is more difficult to predict during the path planning process. The following examples illustrate the physical meaning of boundary and interior singularities.

Example: Planar 3-DOF manipulator (Fig. E7)

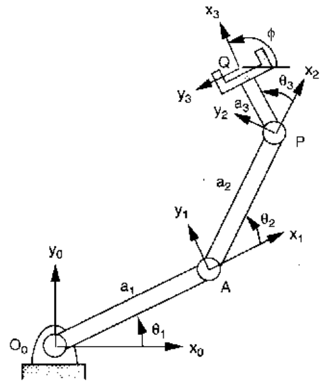


Fig. E7

$$\det(\mathrm{J}) = \begin{bmatrix} -(a_1 s\theta_1 + a_2 s\theta_{12} + a_3 s\theta_{123}) & -(a_2 s\theta_{12} + a_3 s\theta_{123}) & -a_3 s\theta_{123} \\ (a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123}) & (a_2 c\theta_{12} + a_3 c\theta_{123}) & a_3 c\theta_{123} \\ 1 & 1 & 1 \end{bmatrix}$$

$= a_1\, a_2\, s\theta_2 = 0$

Since θ_1 and θ_3 do not appear in the equation, the singularity occurs at $\theta_2 = 0$ or $\theta_2 = \pi$.

(i) $\theta_2 = 0 \Rightarrow$ a stretched-out configuration

(ii) $\theta_2 = \pi \Rightarrow$ a folded-back configuration

Both configurations lose 1 degree of freedom, the end effector can only move along the tangential direction of the arm; motion along the radial direction is not possible. This is a typical example of workspace boundary singularity.

Jacobian Matrices for Parallel manipulators

The kinematic constraints imposed by the limbs can be written in the general form

$$\mathrm{f}(\mathbf{x}, \mathbf{q}) = 0 \qquad (9\text{-}7.4)$$

where $\mathbf{q}$: actuated joint variables, $\mathbf{x}$: location of the moving platform. Differentiating the above equation, we obtain a relationship as

$$\mathrm{J_x}\,\dot{\mathbf{x}} = \mathrm{J_q}\,\dot{\mathbf{q}} \qquad (9\text{-}7.5)$$

where $\mathrm{J_x} = \partial\mathbf{f}/\partial\mathbf{x}$ and $\mathrm{J_q} = -\partial\mathbf{f}/\partial\mathbf{q}$. The overall Jacobian matrix, J, can be expressed as

$$\dot{\mathbf{q}} = \mathrm{J}\,\dot{\mathbf{x}} \qquad (9\text{-}7.6)$$

where $\mathrm{J} = (\mathrm{J_q}^{-1}\mathrm{J_x})$.

Singular conditions

(1) Inverse kinematic singularities — when the determinant of $\mathrm{J_q}$ goes to zero

$$\det(\mathrm{J_q}) = 0 \qquad (9\text{-}7.7)$$

It means there exist some nonzero $\dot{q}$ vectors that result in zero $\dot{x}$ vectors. Infinitesimal motion of the moving platform along certain directions cannot be accomplished. The manipulator loses one or more degrees of freedom. On the other side, at an inverse kinematic singular configuration the manipulator can resist forces or moments in some directions with zero actuator forces or torques.

(2) Direct Kinematic Singularities — when the determinant of $\mathrm{J_x}$ goes to zero

det $(J_x) = 0$ (9-7.8)

In this condition, there exist some nonzero $\dot{x}$ that result in zero $\dot{q}$ vector, that is, the moving platform can possess infinitesimal motion in some directions while all the actuators are completely locked. The moving platform gains 1 or more degrees of freedom. In other words, at a direct kinematic singular configuration, the manipulator can not resist forces or moments in some directions.

Example 1: Jacobian of a Planar 3RRR Parallel Manipulator (Figure E8).

A loop closure equation can be written for each limb. For example,

$$\overline{PG} + \overline{GA} = \overline{PD} + \overline{DA} \qquad (1)$$

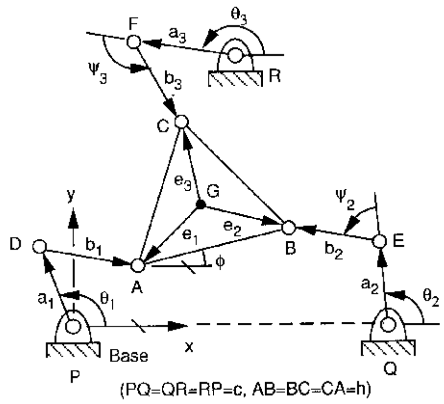


Fig. E8

A velocity vector-loop equation is obtained by taking the derivative of the above equation with respect to time

$$\boldsymbol{v_g} + \dot{\phi}(\boldsymbol{k} \times \boldsymbol{e_i}) = \dot{\theta}_i(\boldsymbol{k} \times \boldsymbol{a_i}) + (\dot{\theta}_i + \dot{\psi}_i)(\boldsymbol{k} \times \boldsymbol{b_i}) \quad (2)$$

To eliminate ψ_i, we dot-multiply both sides of Eq.(2) by $\mathbf{b_i}$,

$$\boldsymbol{b_i} \cdot \boldsymbol{v_g} + \dot{\phi}\, \boldsymbol{k} \cdot (\boldsymbol{e} \times \boldsymbol{b_i}) = \dot{\theta}_i \boldsymbol{k} \cdot (\boldsymbol{a_i} \times \boldsymbol{b_i}) \qquad (3)$$

Writing Eq. (3) three times, once for each i=1, 2, and 3, yields three scalar equations in matrix form:

$$J_x \dot{\mathbf{x}} = J_q \dot{\mathbf{q}} \qquad (4)$$

where $J_x = \begin{bmatrix} b_{1x} & b_{1y} & e_{1x}b_{1y} - e_{1y}b_{1x} \\ b_{2x} & b_{2y} & e_{2x}b_{2y} - e_{2y}b_{2x} \\ b_{3x} & b_{3y} & e_{3x}b_{3y} - e_{3y}b_{3x} \end{bmatrix}$, $J_q = \begin{bmatrix} a_{1x}b_{1y} - a_{1y}b_{1x} & 0 & 0 \\ 0 & a_{2x}b_{2y} - a_{2y}b_{2x} & 0 \\ 0 & 0 & a_{3x}b_{3y} - a_{3y}b_{3x} \end{bmatrix}$,

and where $\dot{\mathbf{x}} = \begin{bmatrix} V_{gx} & V_{gy} & \dot{\varphi} \end{bmatrix}^T$, $\dot{\mathbf{q}} = \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 \end{bmatrix}^T$.

(a) Inverse Kinematic Singularities

When one of diagonal elements of J_q vanishes,

$a_{ix}b_{iy} - a_{iy}b_{ix} = 0$ i=1, or 2, or 3

It represents the magnitude of $\boldsymbol{a_i} \times \boldsymbol{b_i} = 0$.

⇒Inverse Kinematic Singularities arises whenever any limb is in a fully stretched-out or folded-back configuration. The manipulator loses 1, 2, or 3 degrees of freedom depending on whether 1, 2, or 3 limbs are singular.

At an inverse kinematic singularity, infinitesimal rotation of the input link, which is in a stretched-out or folded-back configuration, results in no output motion of the moving platform.

(b) Direct Kinematic Singularities

det (J_x) =0

The last column represents the magnitudes of $\bar{e}_i \times \bar{b}_i$. These three elements will vanish when the vector $\mathbf{b}_i$ is in line with $\mathbf{e}_i$ for all limbs.

⇒ The Direct Kinematic Singularity arises whenever the three vectors, b_i, for i=1, 2, and 3 intersect at a common point. At this configuration, the moving platform can make an infinitesimal rotation about point G while the actuators are locked. The moving platform gains 1 degree-of-freedom and it cannot withstand any external moment about G.

Another direct kinematic singularities arises when the three vectors $\mathbf{b}_i$, for i=1, 2, and 3, are parallel to one another. At this configuration, the moving platform can make an infinitesimal translation along a direction that is perpendicular to the vector $\mathbf{b}_i$ while all the actuators are locked. Any force applied to the moving platform in that direction cannot be resisted by the actuators.

(c) Combined Singularities

A combined singularity occurs when both det (J_x) & det (J_q) are both equal to zero.

Architecture 1

PQ=QR=RP, AB=BC=CA, a_i =PQ/$\sqrt{3}$,

and $b_i = e_i$ for i=1, 2, and 3

⇒ Pivots D, E, and F meet at the centroid of the base triangle. Because of the special link length ratios, the centroid of the moving platform also coincides with the centroid of the base triangle. Therefore, the moving platform can rotate about the centroid while the actuators are locked. On the other hand, the moving platform can be held stationary at ϕ=0 while the input links make some infinitesimal rotations about their corresponding pivots. This is due to the fact that the elements of J_q and the last column of J_x are all equal to zero.

Architecture 2

PQ=QR=RP= AB=BA=CA,

and $a_i = b_i$ for i=1, 2, and 3

⇒ three moving pivots A, B, and C are coincident with the three fixed pivots P, Q, and R, respectively. Hence the three input links can make arbitrary rotations while the moving platform is locked. Furthermore, with the actuators locked at $\theta_1 = -150°$, $\theta_2 = -30°$, and θ_3=90°, the moving platform can perform an infinitesimal rotation about its centroid.

[**End of this Chapter]**

References:

1. Tsai, L.W., *Robot Analysis-The Mechanics of Serial and Parallel Manipulators,* John Wiley &

Sons, Inc., 1999.

Appendix I Direct kinematics of the planar 3 RRR parallel manipulator.

Referring to Fig. E5, Equations (2.a), (2.b) and (2.c) can be written in the following forms

$$x_A^2 + y_A^2 + e_{11}x_A + e_{12}y_A + e_{13} = 0, \qquad (3.21)$$

$$x_A^2 + y_A^2 + e_{21}x_A + e_{22}y_A + e_{23} = 0, \qquad (3.22)$$

$$x_A^2 + y_A^2 + e_{31}x_A + e_{32}y_A + e_{33} = 0, \qquad (3.23)$$

where

$$\begin{aligned}
e_{11} &= -2a_1 c\theta_1,\\
e_{12} &= -2a_1 s\theta_1,\\
e_{13} &= a_1^2 - b_1^2,\\
e_{21} &= -2x_Q + 2hc\phi - 2a_2c\theta_2,\\
e_{22} &= -2y_Q + 2hs\phi - 2a_2s\theta_2,\\
e_{23} &= x_Q^2 + y_Q^2 + h^2 + a_2^2 - b_2^2 - 2a_2hc\phi c\theta_2 - 2a_2hs\phi s\theta_2\\
&\quad - 2x_Qhc\phi - 2y_Qhs\phi + 2x_Qa_2c\theta_2 + 2y_Qa_2s\theta_2,\\
e_{31} &= -2x_R + 2hc\left(\phi + \frac{\pi}{3}\right) - 2a_3c\theta_3,\\
e_{32} &= -2y_R + 2hs\left(\phi + \frac{\pi}{3}\right) - 2a_3s\theta_3,\\
e_{33} &= x_R^2 + y_R^2 + h^2 + a_3^2 - b_3^2 - 2a_3hc\left(\phi + \frac{\pi}{3}\right)c\theta_3 - 2a_3hs\left(\phi + \frac{\pi}{3}\right)s\theta_3\\
&\quad - 2x_Rhc\left(\phi + \frac{\pi}{3}\right) - 2y_Rhs\left(\phi + \frac{\pi}{3}\right) + 2x_Ra_3c\theta_3 + 2y_Ra_3s\theta_3.
\end{aligned}$$

Note that e_{11}, e_{12}, and e_{13} are constants, while e_{21}, e_{22}, e_{23}, e_{31}, e_{32} and e_{33} are linear functions of $\sin\phi$ and $\cos\phi$.

Equations (3.21), (3.22), and (3.23) constitute three nonlinear equations in three unknowns, x_A, y_A, and ϕ. This system of equations can be simplified by performing the following operations. Subtracting Eq. (3.22) from (3.21) yields

$$e'_{11}x_A + e'_{12}y_A + e'_{13} = 0. \qquad (3.24)$$

Subtracting Eq. (3.23) from (3.21) yields

$$e'_{21}x_A + e'_{22}y_A + e'_{23} = 0. \qquad (3.25)$$

Here $e'_{11} = e_{11} - e_{21}$, $e'_{12} = e_{12} - e_{22}$, $e'_{13} = e_{13} - e_{23}$, $e'_{21} = e_{11} - e_{31}$, $e'_{22} = e_{12} - e_{32}$, and $e'_{23} = e_{13} - e_{33}$ are linear functions of $\sin\phi$ and $\cos\phi$.

Equation (3.21) together with (3.24) and (3.25) fform a new system of equations. We may solve Eqs. (3.24) and (3.25) for x_A and y_A and then substitute the resulting expressions into Eq. (3.21). This results in a fourth-degree polynomial in $s\phi$ and $c\phi$:

$$\delta_1^2 + \delta_2^2 + e_{11}\delta\delta_1 + e_{12}\delta\delta_2 + e_{13}\delta^2 = 0, \tag{3.26}$$

where

$$\delta = e'_{11}e'_{22} - e'_{12}e'_{21},$$
$$\delta_1 = e'_{12}e'_{23} - e'_{13}e'_{22},$$
$$\delta_2 = e'_{13}e'_{21} - e'_{11}e'_{23}.$$

Equation (3.26) can be converted into an eighth-degree polynomial by using the half-tangent angle expressions. Hence, corresponding to each given set of input joint angles, there are at most eight possible manipulator configurations. A numerical method of solution can be found in Gosselin and Sefrioui (1991).