Homework 2

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Question 1.
discrete:

$$
\begin{aligned}
E[(X-a)^2] &= \sum_{x\in D}(x-a)^2\cdot p(x) = \sum_{x\in D}[(x-\mu)+(\mu-a)]^2\cdot p(x)\\
&= \sum_{x\in D}(x-\mu)^2 p(x) + \sum_{x\in D}2(x-\mu)(\mu-a)p(x) + \sum_{x\in D}(\mu-a)^2 p(x)\\
&\Rightarrow \sum_{x\in D}2(x-\mu)(\mu-a) = 2(\mu-a)\sum_{x\in D}(x-\mu)p(x) = 0\\
&\Rightarrow \sum_{x\in D}(\mu-a)^2 p(x) = (\mu-a)^2\cdot 1\\
\Rightarrow E[(X-a)^2] &= [\sum_{x\in D}(x-\mu)^2 p(x)] + (\mu-a)^2 = Var(X) + (\mu-a)^2
\end{aligned}
$$

continuous:

$$
\begin{aligned}
E[(X-a)^2] &= \int_D (x-a)^2\cdot f(x)dx = \int_D [(x-\mu)+(\mu-a)]^2\cdot f(x)dx\\
&= \int_D (x-\mu)^2 f(x)dx + \int_D 2(x-\mu)(\mu-a)f(x)dx + \int_D (\mu-a)^2 f(x)dx\\
&\Rightarrow \int_D 2(x-\mu)(\mu-a)f(x)dx = 2(\mu-a)\int_D [xf(x)-\mu f(x)]dx = 0\\
&\Rightarrow \int_D (\mu-a)^2 f(x)dx = (\mu-a)^2\\
\Rightarrow E[(X-a)^2] &= \int_D (x-\mu)^2 f(x)dx + (\mu-a)^2 = Var(X) + (\mu-a)^2
\end{aligned}
$$

Question 2.

$$
P(X=x;\lambda) = \frac{\lambda^{2x}}{(2x)!cosh(\lambda)}\ , for\ x = 1,2,3\cdots
$$

Part 1.

$$\frac{\lambda^{2x}}{(2x)!cosh(\lambda)} = \frac{\lambda^{2x}}{(2x)!\dfrac{e^{\lambda}+e^{-\lambda}}{2}} = \frac{2\lambda^{2x}}{(2x)!(e^{\lambda}+e^{-\lambda})}$$

$$\Rightarrow e^{\lambda}+e^{-\lambda} = \frac{\lambda^0+\lambda^0}{0!} + \frac{\lambda-\lambda}{1!} + \frac{2\lambda^2}{2!} \cdots = \sum_{n=0}^{\infty}\frac{2\lambda^{2n}}{2n!}$$

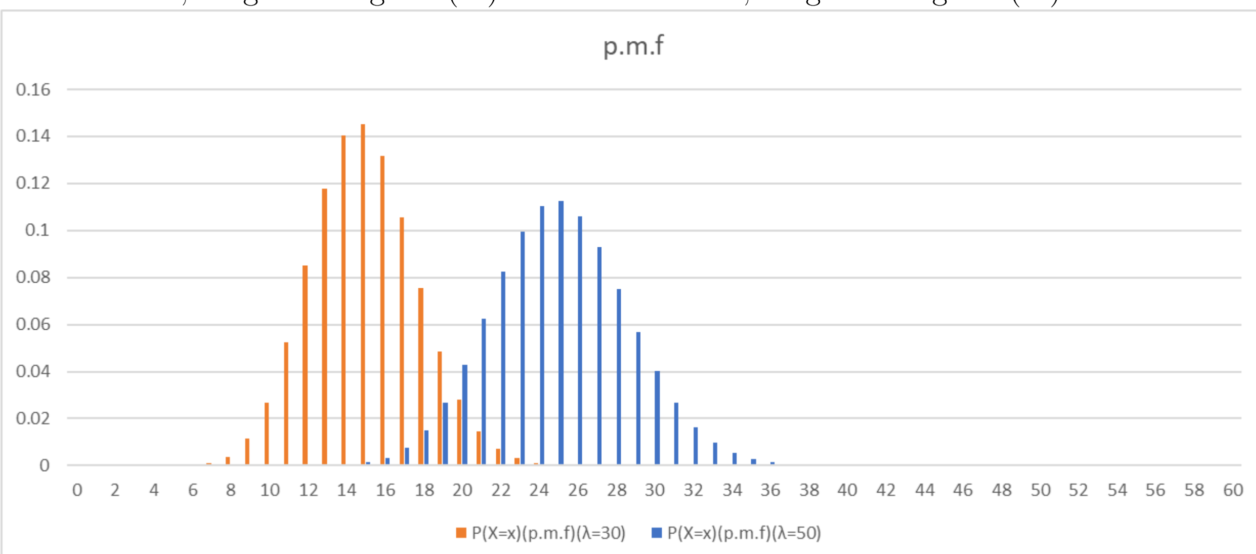
$$\Rightarrow \sum P(X) = \sum_{x=0}^{\infty}\left[\frac{2\lambda^{2x}}{(2x)!}\frac{1}{\sum_{n=0}^{\infty}\frac{2\lambda^{2n}}{(2n)!}}\right]$$

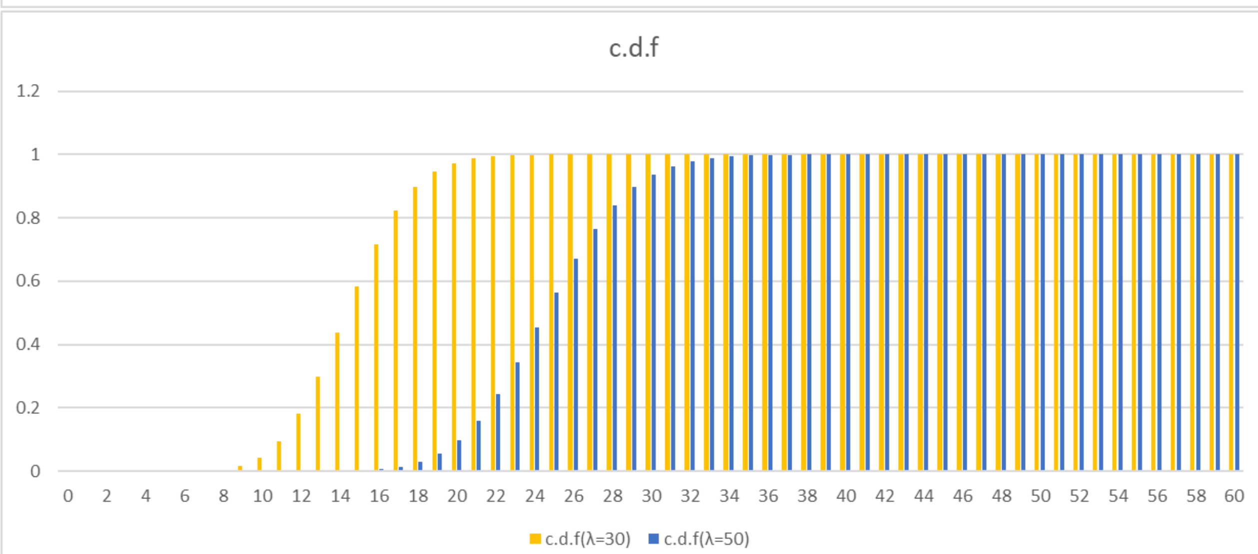
$$= \sum_{x=0}^{\infty}\frac{2\lambda^{2x}}{(2x)!}\cdot 1/\left[\sum_{n=0}^{\infty}\frac{2\lambda^{2n}}{(2n)!}\right]$$

$$= (e^{\lambda}+e^{-\lambda})\cdot\frac{1}{e^{\lambda}+e^{-\lambda}} = 1$$

Part 2.

For $\lambda = 30$, weight average: $E(X) = 15$. For $\lambda = 50$, weight average: $E(X) = 25$





The expected value for $\lambda = m$ is: $E(X) = m/2$

Question 3.

For Binomial distribution: $Var(X) = np(1-p)$

Note1:

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x} = \sum_{x=0}^{n} \frac{n!}{x!(n-x!)} a^x b^{n-x}$$

Note2:

$$\begin{aligned} \because E[X(X-1)] &= E(X^2 - X) = \sum_{x \in D} (x^2 - x) \cdot p(x) \\ &= \sum_{x \in D} x^2 \cdot p(x) - \sum_{x \in D} x \cdot p(x) \\ &= E(X^2) - E(X) \\ \therefore Var(X) &= E(X^2) - [E(X)]^2 = E[X(X-1)] + E(X) - [E(X)]^2 \end{aligned}$$

for Binomial Distribution $b(x; n, p)$:

$$\begin{aligned} E[X(X-1)] &= \sum_{x=0}^{n} x(x-1) \cdot \binom{n}{x} p^x (1-P)^{n-x} \\ &= \sum_{x=0}^{n} \frac{x(x-1)n!}{x!(n-x)!} p^x (1-p)^{n-x} \\ &= \sum_{x=2}^{n} \frac{x(x-1)n!}{x!(n-x)!} p^x (1-p)^{n-x} \\ &= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!} p^x (1-p)^{n-x} \\ &= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} (1-p)^{n-x} \\ &\Rightarrow (n-x) = (n-2) - (x-2) \\ E[X(X-1)] &= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)![(n-2)-(x-2)]!} p^{x-2} (1-p)^{(n-2)(x-2)} \\ &\Rightarrow m = n-2, y = x-2 \\ E[X(X-1)] &= n(n-1)p^2 \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^y (1-p)^{m-y} \\ &= n(n-1)p^2 [p + (1-p)]^m = n(n-1)p^2 \\ \Rightarrow Var(X) &= E(X^2) - [E(X)]^2 = E[X(X-1)] + E(X) - [E(X)]^2 \\ &= n(n-1)p^2 + np - (np)^2 = np(1-p) \end{aligned}$$

Question 4.

$$b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x} \ , \ E(X) = np \ , \ Var(X) = np(1-p)$$

$$p(x; \lambda) = \frac{e^{-\lambda} \lambda^x}{x!} \ , \ E(X) = \lambda \ , \ Var(X) = \lambda$$

Note1:

$$\lim_{n \to \infty} \lim_{p \to 0} \ b(x; n, p) = p(x; \lambda)$$

$$\Rightarrow \lambda = np \ , \ p = \frac{\lambda}{n}$$

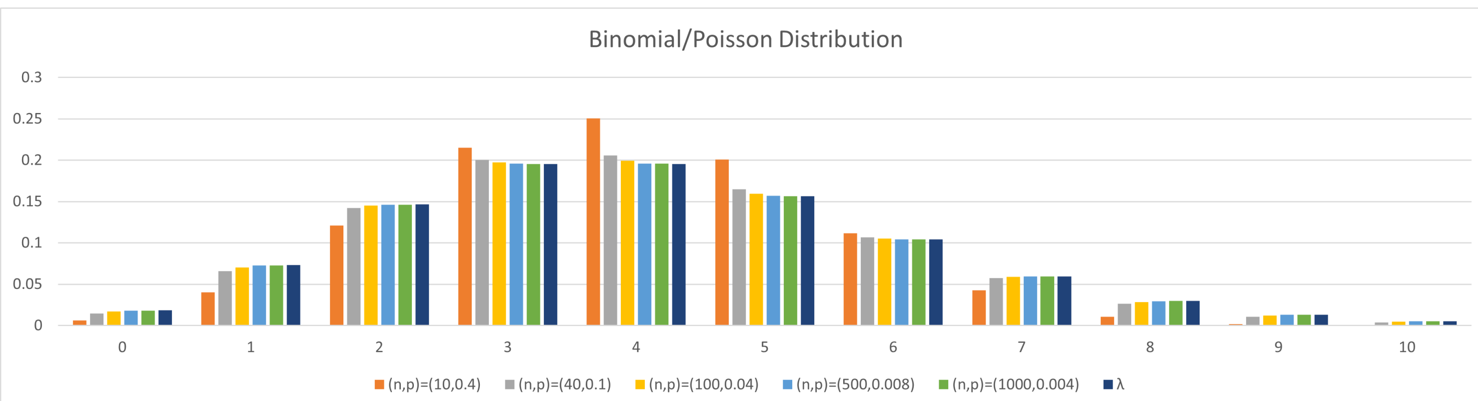
Note2:

$$e^x = \lim_{n \to \infty} (1 + \frac{x}{n})^n$$

for $lim_{n\to\infty}\ lim_{p\to 0}\ b(x;n,p)$:

$$\begin{aligned}
\underset{n\to\infty}{lim}\ \underset{p\to 0}{lim}\ b(x;n,p) &= \underset{n\to\infty}{lim} \binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} \\
&= \underset{n\to\infty}{lim}\left(\frac{\lambda^x}{x!}\right)\left(\frac{n!}{(n-x)!n^x}\right)\left(1-\frac{\lambda}{n}\right)^{n-x} \\
\Rightarrow \frac{n!}{(n-x)!n^x} &= \frac{n\cdot(n-1)(n-2)\dots(n-x+1)(n-x)!}{(n-x)!n^x} \\
&= \frac{n}{n}\cdot\frac{n-1}{n}\cdot\frac{n-1}{n}\cdots\frac{n-x+1}{n} \\
&= 1\cdot(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{x-1}{n}) \\
\Rightarrow \underset{n\to\infty}{lim}\binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x} &= \underset{n\to\infty}{lim}(\frac{\lambda^x}{x!})\cdot 1\cdot(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{x-1}{n})\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-x} \\
&= \frac{\lambda^x}{x!}\cdot 1\cdot e^{-\lambda}\cdot 1 = p(x;\lambda)
\end{aligned}$$

Question 5.



Observation: When Binomial's n getting bigger and p getting smaller, the distribution of probability approaches a Poisson distribution.

Question 6.

The probability for a number to appear in a darw of Power Lotto 638's first-set winning numbers:

$$p = \frac{6}{38} = 0.157894737$$

	$n=50$	$n=100$	$n=500$
Binomial Distribution $E(X)$	7.89473684210526	15.7894736842105	78.9473684210526
Statistic data average	7.89473684210527	15.7894736842105	78.9473684210526
Binomial Distribution $Var(X)$	6.64819944598341	13.2963988919667	66.4819944598394
Statistic data sample variance	6.85348506401132	13.7923186344238	55.8890469416785

Table 1. Winnig number appearance model/data

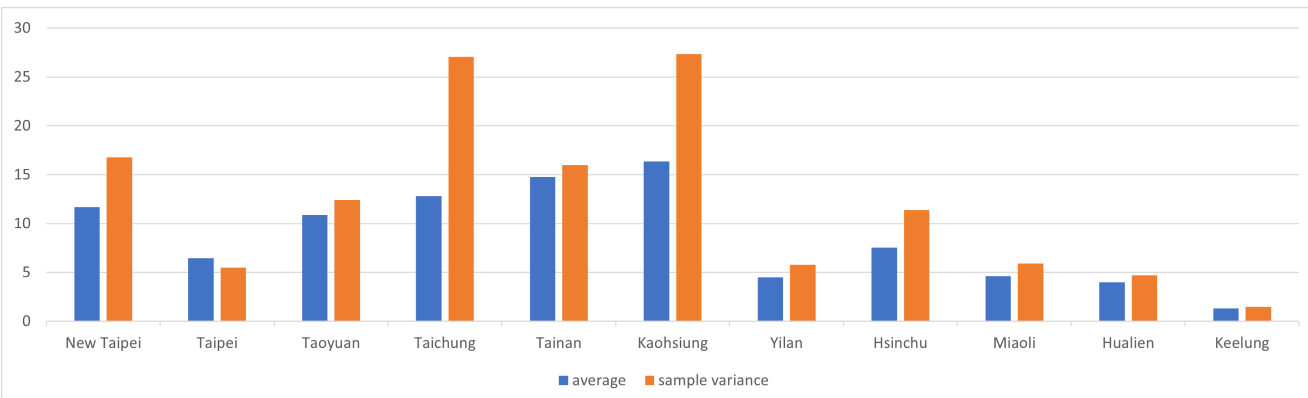
Question 7.

The Malayan Night Heron(黑冠麻鷺), once a rare sight in Taiwan, has become increasingly common in urban green spaces over the past decade. This shift in distribution has led to sightings of these elusive birds in unexpected places, including within the grounds of the National Taiwan University. Suppose we wish to observe the presence of Malayan Night Herons by randomly selecting a 100 square meter area within the university campus. Because the presence probability of a Malayan Night Heron in such area is relatively small (p is small), and the total campus of National Taiwan University is about 110,762,900 square meters (n is big). In such a scenario, we can model the occurrence of these birds using a Poisson Distribution.

Question 8.

City	New Taipei	Taipei	Taoyuan	Taichung	Tainan	Kaohsiung
average	12	6	11	13	15	16
sample variance	16.79104478	5.471752349	12.44278607	27.0671089	15.98817026	27.33311222
City	Yilan	Hsinchu	Miaoli	Hualien	Keelung	
average	4	8	5	4	1	
sample variance	5.803758983	11.38308458	5.90657822	4.708899945	1.47761194	

Table 2. Accident Death



While considering the modeling of this scenario, we can explore both the Binomial and Poisson distributions. However, upon closer examination, utilizing the Binomial distribution reveals that when sample variances are divided by the averages in the data, most results exceed 1. This outcome indicates a negative probability (p), which is inherently impossible. Additionally, certain sample variances exhibit significant disparities from the average, rendering them incompatible with a Poisson distribution. This discrepancy may stem from the sample size (n) not being sufficiently large to conform to the assumptions of these distributions