Eigenvalues and Eigenvectors - Introduction

$\boxed{Ax=\lambda x}$ $\boxed{A=S\Lambda S^{-1}}$

- **Applications**
  - **Applications of Matrix Diagonalization: Powers A^k and Exponential e^{At}**
    - ✓ **Markov processes (application of A^k)**
    - ✓ **Ordinary Differential Equation (the most often mentioned but is just one of many e^{At} applications)**
  - **Minimum and Maximum Principles**
    - ✓ **Gradient Search**
    - ✓ **Principal Component/Discriminant Analysis**
  - **Change of Basis = Similarity Transformation**
- $Ax=\lambda x \Rightarrow$ **this is an attempt to simplify the transformation by a matrix A to a simple multiplication by a number λ.**
- **Problem:**

  <u>Find certain x such that transformation A can be simplified into multiplication of a number λ</u>, i.e. $\boxed{Ax=\lambda x}$

  This is only possible for certain vectors, x, called *<u>eigenvectors</u>* in certain subspaces called *<u>eigenspaces</u>* and λ is called "*eigenvalue*" of A

Solutions of $Ax=\lambda x$

When λ and x are both unknown, $Ax=\lambda x$ is a nonlinear equation.

If λ is known $\Rightarrow$ $(A-\lambda I)x=0$ **a linear problem.**

(i) ***<u>The vector x is in the nullspace of $A-\lambda I$</u>***

(ii) ***<u>The number λ is chosen so that $A-\lambda I$ has a nullspace</u>***

- **The nullspace of $A-\lambda I$ is then called "*eigenspace*" of A. All x's in the *eigenspace* are *eigenvectors* corresponding to the same λ.**
- **The nullspace of A is an eigenspace of A with $\lambda=0$.**
- **We are interested in nonzero vector $x \Rightarrow A-\lambda I$ *must be singular* $\Rightarrow$** $\det(A-\lambda I)=0$. **(Characteristic equation)**
- **Conventional solution: find λ first by** $\det(A-\lambda I)=0$**. Then, the corresponding x can be found by the nullspace** $(A-\lambda I)x=0$.
- **Example:** $A=\begin{bmatrix}4 & -5\\ 2 & -3\end{bmatrix}$ **Find x and λ such that $Ax=\lambda x$**

$$\det(A-\lambda I)=\det\begin{bmatrix}4-\lambda & -5\\ 2 & -3-\lambda\end{bmatrix}=\mathbf{0} \Rightarrow (4-\lambda)(-3-\lambda)+10=0 \quad \Rightarrow$$

$$\lambda^2-\lambda-2=0 \Rightarrow \quad \lambda=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{1\pm\sqrt{9}}{2}=-1 \text{ or } 2$$

$$\lambda_1=-1:\ (A-\lambda_1 I)x=\begin{bmatrix}5 & -5\\ 2 & -2\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}.\quad x_1=c_1\begin{bmatrix}1\\ 1\end{bmatrix}.$$

$$\lambda_2=2:\ (A-\lambda_2 I)x=\begin{bmatrix}2 & -5\\ 2 & -5\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}.\quad x_2=c_2\begin{bmatrix}5\\ 2\end{bmatrix}.$$

More on $Ax=\lambda x$ solutions

- **Ax: Transformation of a vector x to Ax**
- **λx: a multiple of the vector $x \Rightarrow$ a vector in the same direction**
- **$Ax=\lambda x$: A transformation of "*certain*" vector x by A becomes a multiple of the vector x itself.**

Ex: $P=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$ **projects any vector onto** $\begin{bmatrix}1\\1\end{bmatrix}$ **Eigenvectors? Eigenvalues?**

$Px=1x=x \Rightarrow$ column space of P is the eigenspace with $\lambda_1=1$

$Px=0x=0 \Rightarrow$ nullspace is an eigenspace too with $\lambda_2=0$

If a projection matrix P has $\dim(\Re)=r$ and $\dim(\aleph)=n-r$, $\lambda=1$ has a r-dimensional eigenspace (repeats r times) and $\lambda=0$ has a $(n-r)$-dimensional eigenspace (repeats $n-r$ times).

- **Only for "*certain*" vector x, transformation A can be simplified? Useless for other vectors?**

Ex: $A=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}$ has $\lambda_1=3$ with $x_1=\begin{bmatrix}1\\0\end{bmatrix}$, $\lambda_2=2$ with $x_2=\begin{bmatrix}0\\1\end{bmatrix}$.

How about $x=\begin{bmatrix}1\\5\end{bmatrix}$**? Not an eigenvector!**

Let $x=x_1+5x_2 \Rightarrow$ $Ax=\lambda_1 x_1+5\lambda_2 x_2=\begin{bmatrix}3\\10\end{bmatrix}$. **The action of A can still be determined by eigenvalues and eigenvectors!**

Trace and Eigenvalues

Ex: $P=\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}$ has $\lambda=1,1,0,0.$

Ex: a triangular A

$$\det(A-\lambda I)=\begin{vmatrix}1-\lambda & 4 & 5\\ 0 & \frac{3}{4}-\lambda & 6\\ 0 & 0 & \frac{1}{2}-\lambda\end{vmatrix}=(1-\lambda)(\tfrac{3}{4}-\lambda)(\tfrac{1}{2}-\lambda).$$

λ = 1 or 3/4 or 1/2 $\Rightarrow$ eigenvalues are diagonal entries

- **How to transform matrix A into a diagonal or triangular matrix *without changing its eigenvalues*? Elimination doesn't work any more!**
- **To find eigenvalues is already a headache (unlike solving $Ax=b$ where elimination always works)**
- **Some checks on the eigenvalues (proof?)**

1. $\lambda_1+\cdots+\lambda_n=a_{11}+\cdots+a_{nn}=trace$

2. $\det A=\lambda_1\lambda_2\cdots\lambda_n$

$\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ has trace $a+d$, determinant $ad-bc$

$$\det\begin{bmatrix}a-\lambda & b\\ c & d-\lambda\end{bmatrix}=\lambda^2-(\text{trace})\lambda+\text{ determinant};\ \lambda=\frac{\text{trace}\pm\left[(\text{trace})^2-4\det\right]^{1/2}}{2}.$$

Diagonal Form of Matrix

- ***n×n* matrix *A* with <u>*n* linearly independent eigenvectors from</u> <u>eigenspaces</u>:** $S^{-1}AS = \Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$ **where *S'*s columns are formed by the eigenvectors and $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ may not be all distinct.**

Proof

$$AS = A\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix}.$$

$$\begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \cdots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}.$$

$$\Rightarrow \quad AS = S\Lambda, \quad \text{or} \quad S^{-1}AS = \Lambda, \quad \text{or} \quad A = S\Lambda S^{-1}.$$

- **<u>Not all matrixes are diagonalizable.</u> When diagonalizable, only *S* with eigenvector columns can diagonalize *A* into Λ with eigenvalues as its diagonal entries.**
- **If there are *n* distinct eigenvalues, all eigenvectors are independent ⇒ the matrix can be diagonalized.**
- ***S* is not unique. Example: *A*=*I*: eigenvalue=1; the eigenspace filling up the entire *n*-dimensional space.**

Independent Eigenvectors

Ex: defective matrix

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. **⇒ $\lambda_1=\lambda_2=0$ ⇒ $S^{-1}AS=0$ ⇒ $SS^{-1}ASS^{-1}=0$ ⇒ $A=0$**

But *A* is not zero!⇒ Contradictory! ⇒ *A* is not diagonalizable!

***A* is defective not because the eigenvalues are repeated zeros but because not enough independent eigenvectors can be found.**

Ex: $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ **and *A*=*I***

- ***Diagonalizability is concerned with the eigenvectors***

  Invertibility is concerned with the eigenvalues
- **If the eigenvectors correspond to *distinct* eigenvalues then those eigenvectors are linearly independent.**

  Eigenvalues Different "⇒" Eigenvectors Independent

***Reason*: For 2 by 2 matrix, let the eigenvectors be x_1 and x_2 corresponding to λ_1 and λ_2. Let** $c_1x_1+c_2x_2=0$ **(1) then** $A(c_1x_1+c_2x_2)=0$ **⇒**

$c_1\lambda_1x_1+c_2\lambda_2x_2=0$ **(2) ⇒ (2)−λ_2(1) ⇒**

$c_1(\lambda_1-\lambda_2)x_1 = 0.$ **But** $\lambda_1 \neq \lambda_2$ **and** $x_1 \neq 0$ **⇒** $c_1=0$

same for c_2 and for any number of eigenvectors.

Examples of Diagonalization

Ex: projection matrix $A=\begin{bmatrix}1/2 & 1/2\\ 1/2 & 1/2\end{bmatrix}$ $\Lambda=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$

$\Rightarrow\ S=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ and $AS=S\Lambda=\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$.

Ex: rotation $K=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$

K rotates any vector through 90^0. Are there vectors rotated without changing its direction? **Does *K* have eigenvalues?** **Yes!**

$\det(K-\lambda I)=\lambda^2+1=0$ $\Rightarrow$ **$\lambda_1=i$ and $\lambda_2=-i$**

$(K-\lambda_1 I)x_1=\begin{bmatrix}-i & -1\\ 1 & -i\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ and $x_1=\begin{bmatrix}1\\ -i\end{bmatrix}$

$(K-\lambda_2 I)x_2=\begin{bmatrix}i & -1\\ 1 & i\end{bmatrix}\begin{bmatrix}y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ and $x_2=\begin{bmatrix}1\\ i\end{bmatrix}$.

$\Rightarrow\ S=\begin{bmatrix}1 & 1\\ -i & i\end{bmatrix}$ and $S^{-1}KS=\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}$.

- **Complex numbers are needed even for real matrices**

Ex: $A^2x=A\lambda x=\lambda Ax=\lambda^2 x$. **eigenvalues for A^2 are the square of eigenvalues of *A*.**

Ex: $(S^{-1}AS)(S^{-1}AS)=\Lambda^2$ or $S^{-1}A^2S=\Lambda^2$.

More on Eigenvalues and Eigenvectors

Example 1: $\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{bmatrix}$

Example 2: $\begin{bmatrix}0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\end{bmatrix}$

Powers: $A^k = S\Lambda^k S^{-1}$

- $A^k = (S\Lambda S^{-1})(S\Lambda S^{-1})\dots(S\Lambda S^{-1}) = S\Lambda^k S^{-1}$
- **For invertible A,**

$$if \quad Ax = \lambda x \quad then \quad x = \lambda A^{-1}x \quad and \quad \frac{1}{\lambda}x = A^{-1}x.$$

The eigenvalues of A^{-1} *is* $1/\lambda$.

Example: rotation through 90°

$$K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad K^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad K^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

eigenvalues for square are –1 and –1;

eigenvalues for inverse are $1/i = -i$ and $1/(-i) = i$.

$$K^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and also} \quad \Lambda^4 = \begin{bmatrix} i^4 & 0 \\ 0 & (-i)^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Why? Physically. Rotate 90° 4 times = rotate 360°

Product of two matrices

- ***$ABx = A\mu x = \mu Ax = \mu\lambda x$* where μ is the eigenvalue of B**

THIS IS FALSE!! ***Eigenvectors are different for A and B!***

Example: $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$

- **A and B are diagonalizable:**

They share the same eigenvector matrix S if and only if $AB=BA$

Proof: **If** $A = S\Lambda_1 S^{-1}$ **and** $B = S\Lambda_2 S^{-1}$ **then**

$AB = S\Lambda_1 S^{-1} S\Lambda_2 S^{-1} = S\Lambda_1\Lambda_2 S^{-1}$ **and**

$BA = S\Lambda_2 S^{-1} S\Lambda_1 S^{-1} = S\Lambda_2\Lambda_1 S^{-1}$ **but** $\Lambda_2\Lambda_1 = \Lambda_1\Lambda_2$

$\Rightarrow AB=BA$

Opposite direction: Let $AB=BA \Rightarrow$

$ABx = BAx = B\lambda x = \lambda Bx.$

Bx and x are both eigenvectors of A corresponding to the same λ:

assuming distinct eigenvalues, Bx must be then multiple of $x \Rightarrow Bx = \mu x$

(for repeated eigenvalue corresponding to an eigenspace, the proof will be longer)

- $A = S\Lambda S^{-1}$ **is extremely useful when taking powers of A:**

$$\boxed{A^k = S\Lambda^k S^{-1}}$$

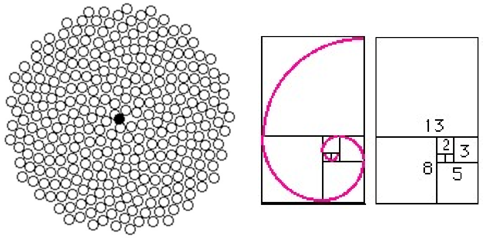
and $A=LU$ does not help at all in this aspect.

Fibonacci Sequence and Difference Equations

- **Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13,…**

$$F_{k+2} = F_{k+1} + F_k.$$

This is a form of difference equation. Numbers in the sequence turn up in fantastic natural patterns (e.g. sunflower seeds).



- **Question: 1000th Fibonacci number?**

Reduce it to a $u_{k+1} = Au_k$ **problem (like the compounded interest problem) Let** $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$$\begin{aligned} F_{k+2} &= F_{k+1} + F_k \\ F_{k+1} &= F_{k+1} \end{aligned} \quad \text{becomes} \quad u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k.$$

The second equation is trivial and is a standard trick for second order difference equation. For higher order equations, say order *s*, we need *s*-1 trivial equations.

- **The solution to $u_{k+1}=Au_k$:**

$$\boxed{u_k = A^k u_0}$$

Power of Matrix and Difference Equations

- **If** $A = S\Lambda S^{-1}$ **then**

$$u_k = A^k u_0 = \left(S\Lambda S^{-1}\right)\left(S\Lambda S^{-1}\right)\cdots\left(S\Lambda S^{-1}\right)u_0 = S\Lambda^k S^{-1} u_0.$$

By setting $S^{-1}u_0 = c \Rightarrow Sc = u_0$ **(i.e. expressing u_0 as linear combination of eigenvectors), the solution becomes**

$$u_k = S\Lambda^k c = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c_1\lambda_1^k x_1 + \cdots + c_n \lambda_n^k x_n.$$

The solution is a combination of the "pure solutions" $\lambda_i^k x_i$

- **Another way of looking at the solution:**

If $u_0 = c_1 x_1 + \cdots + c_n x_n$**, i.e. $u_0=Sc$ then**

$$\begin{aligned} u_k &= A^k u_0 = A^k (c_1 x_1 + \cdots + c_n x_n) \\ &= c_1 A^k x_1 + \cdots + c_n A^k x_n \\ &= c_1 \lambda_1^k x_1 + \cdots + c_n \lambda_n^k x_n. \end{aligned}$$

***c*'s are determined by the initial conditions:**

$$u_0 = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = Sc \quad \text{and} \quad c = S^{-1} u_0.$$

More on Fibonacci and Difference Equations

$A=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix},\quad \det(A-\lambda I)=\lambda^2-\lambda-1,\ \lambda_1=\frac{1+\sqrt{5}}{2},\quad \lambda_2=\frac{1-\sqrt{5}}{2}.$

and initial conditions: $c=S^{-1}u_0=\begin{bmatrix}\lambda_1 & \lambda_2\\ 1 & 1\end{bmatrix}^{-1}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}1/(\lambda_1-\lambda_2)\\ -1/(\lambda_1-\lambda_2)\end{bmatrix}.$

We have $u_k=c_1\lambda_1^k x_1+c_2\lambda_2^k x_2 \Rightarrow$

$$F_k=\frac{\lambda_1^k}{\lambda_1-\lambda_2}-\frac{\lambda_2^k}{\lambda_1-\lambda_2}=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k-\left(\frac{1-\sqrt{5}}{2}\right)^k\right].$$

Since the second term $[(1-\sqrt{5})/2]^k/\sqrt{5}$ **is less than 1/2 and is becoming insignificant when k is large.**

$\frac{F_{k+1}}{F_k}\approx\frac{1+\sqrt{5}}{2}\approx 1.618 \Rightarrow$ **Golden ratio!!**

- **Simplicity of A^k computation with diagonalization:**

$$A=\begin{bmatrix}-4 & -5\\ 10 & 11\end{bmatrix} \text{ has } \lambda_1=1,\ x_1=\begin{bmatrix}1\\ -1\end{bmatrix},\ \lambda_2=6,\ x_2=\begin{bmatrix}-1\\ 2\end{bmatrix}$$

$$A^k=S\Lambda^k S^{-1}=\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}1^k & 0\\ 0 & 6^k\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}2-6^k & 1-6^k\\ -2+2\cdot 6^k & -1+2\cdot 6^k\end{bmatrix}.$$

Markov Processes and Difference Equations

- **Example: each year 1/10 of the people outside California move in, and 2/10 of the people inside California move out. What is the population of California after 5 years, 10 years, or 100 years? (y: population outside, z: inside)**

$$\begin{aligned}y_1&=.9y_0+.2z_0\\ z_1&=.1y_0+.8z_0\end{aligned}\quad \text{or}\quad \begin{bmatrix}y_1\\ z_1\end{bmatrix}=\begin{bmatrix}.9 & .2\\ .1 & .8\end{bmatrix}\begin{bmatrix}y_0\\ z_0\end{bmatrix}.$$

- **Markov matrix: 1. Each column adds up to 1**

  2. All entries are nonnegative

- **Solving Markov process:**

$$A=\begin{bmatrix}.9 & .2\\ .1 & .8\end{bmatrix},\quad \det(A-\lambda I)=\lambda^2-1.7\lambda+.7,$$

$$\lambda_1=1 \quad\text{and}\quad \lambda_2=.7$$

$$A=S\Lambda S^{-1}=\begin{bmatrix}\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & -\frac{1}{3}\end{bmatrix}\begin{bmatrix}1 & \\ & .7\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -2\end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix}y_k\\ z_k\end{bmatrix}=A^k\begin{bmatrix}y_0\\ z_0\end{bmatrix}=\begin{bmatrix}\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & -\frac{1}{3}\end{bmatrix}\begin{bmatrix}1^k & \\ & .7^k\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix}y_0\\ z_0\end{bmatrix}$$
$$=(y_0+z_0)(1)^k\begin{bmatrix}\frac{2}{3}\\ \frac{1}{3}\end{bmatrix}+(y_0-2z_0)(.7)^k\begin{bmatrix}\frac{1}{3}\\ -\frac{1}{3}\end{bmatrix}.$$

- **Solution when time approaches infinity:** $\begin{bmatrix}y_\infty\\ z_\infty\end{bmatrix}=(y_0+z_0)\begin{bmatrix}\frac{2}{3}\\ \frac{1}{3}\end{bmatrix}$. **i.e.**

$$\begin{bmatrix}.9 & .2\\ .1 & .8\end{bmatrix}\begin{bmatrix}\frac{2}{3}\\ \frac{1}{3}\end{bmatrix}=\begin{bmatrix}\frac{2}{3}\\ \frac{1}{3}\end{bmatrix},\quad \text{or}\quad Au_\infty=u_\infty.$$

Stability of Markov Process

In the example of Markov process: ***The steady state is the eigenvector of A corresponding to*** $\lambda = 1$

- **For a Markov process:**

  (a) $\lambda_1 = 1$ **is an eigenvalue (each column adds up to 1)**

  (b) Its eigenvector x_1 is nonnegative and is the steady state since

  $Ax_1 = x_1 \Rightarrow A^{\infty}x_1 = x_1$ **(i.e. *A* can no longer change x_1)**

  (c) Other eigenvalues $|\lambda_i| \le 1$

  (d) If any power of *A* has all positive entries, then these other $|\lambda_i| < 1$. **Solution** $A^k u_0$ **then approaches a multiple of** $x_1 = u_\infty$

  (e.g. California's population approaches $(y_0 + z_0)\frac{1}{3}$ **)**

Reason **(a): each column of $A-1I$ adds up to 1−1=0 ⇒ rows of $A-I$ adds up to 0 ⇒ $A-I$ is singular ⇒** $\lambda_1 = 1$

(b): the steady state should maintain positive proportions

(c)(d): otherwise $u_k = c_1\lambda_1^k x_1 + \cdots + c_n\lambda_n^k x_n$ **will blow up. If** λ_i^k **(other than** λ_1 **=1) goes to zero when *k* becomes very large.** $u_k \to c_1 x_1 = u_\infty$

System Stability and Eigenvalues

Example:

Fibonacci number and compounded interest become larger and larger

⇒ unstable

Markov Process converges to constants ⇒ neutrally stable

- **Given:** $u_k = S\Lambda^k S^{-1}u_0 = c_1\lambda_1^k x_1 + \cdots + c_n\lambda_n^k x_n$.

  The difference equation $u_{k+1} = Au_k$ **is**

  stable **if all eigenvalues satisfy** $|\lambda_i| < 1$ **(** $A^k \to zero$ **)**

  neutrally stable **if some** $|\lambda_i| = 1$ **and other** $|\lambda_i| < 1$

  unstable **if at least one eigenvalue has** $|\lambda_i| > 1$

Example: stable matrix $A = \begin{bmatrix} 0 & 4 \\ 0 & \frac{1}{2} \end{bmatrix}$ **has eigenvalues 0 and 1/2**

$$u_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ u_1 = \begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix},\ u_2 = \begin{bmatrix} 2 \\ \frac{1}{4} \end{bmatrix},\ u_3 = \begin{bmatrix} 1 \\ \frac{1}{8} \end{bmatrix},\ u_4 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{16} \end{bmatrix},\cdots$$

The first step is to split u_0 into two eigenvectors:

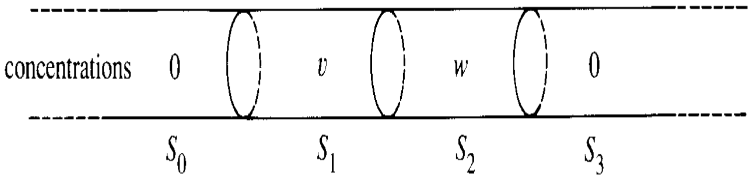
$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

The first vector is multiplied by its eigenvalue 0 and is thus annihilated.

The second vector is cut to half at every step.

Differential Equations Example: Diffusion Model

- **v and w are concentrations.**



- **At each time t the *diffusion rate* between two adjacent segments equals the *difference in concentrations*.**
- **Concentrations in S_0 and S_3 are forever zero because they have infinite ends.**
- **Differential equations:**

$$\begin{aligned}\frac{dv}{dt} &= (w-v)+(0-v)\\ \frac{dw}{dt} &= (0-w)+(v-w)\end{aligned} \Rightarrow u=\begin{bmatrix} v \\ w \end{bmatrix}, \frac{du}{dt}=\begin{bmatrix} -2v+w \\ v-2w \end{bmatrix}=\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}u.$$

- **Then the system becomes:**

$$\frac{du}{dt}=Au, \quad u=u_0 \text{ at } t=0.$$

Solution to Ordinal Differential Equation (ODE)

- **Solution? Look at** $\frac{du}{dt}=au, \quad u=u_0$ at $t=0$ **first**

Let $u(t)=ke^{at}$**. Given the initial values u=u_0 *at* t=0** $\Rightarrow u(t)=e^{at}u_0$.

- **a>0 ⇒ unstable; a=0 neutrally stable; a<0 stable.**
- **If $a=\alpha+i\beta$, $e^{at}=e^{\alpha t}e^{i\beta t}=e^{\alpha t}(\cos \beta t + i \sin \beta t)$**

⇒ α: stabiltiy β: oscillations

- **For systems**

$$\frac{du}{dt}=\begin{bmatrix} -2v+w \\ v-2w \end{bmatrix}=\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}u \text{ where } u=\begin{bmatrix} v \\ w \end{bmatrix} \text{ and } u=u_0 \text{ at } t=0$$

$$\frac{du}{dt}=Au, \quad u=u_0 \text{ at } t=0.$$

Solution:

$$\boxed{u(t)=e^{At}u_0=Se^{\Lambda t}S^{-1}u_0}$$

- **Difference equation: $u_k=A^ku_0$ depending on power of A**
- **Differential equation:** $u(t)=e^{At}u_0$ **depending on *exponential* of A**
- **Problem: what is *exponential* of A: e^{At}?**

Exponential of a Matrix

- **Imitate** $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots.$

$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$. **has the following properties:**

1. $\left(e^{As}\right)\left(e^{At}\right) = e^{A(s+t)}$ **2.** $\left(e^{At}\right)\left(e^{-At}\right) = I$ **3.** $\frac{d}{dt}\left(e^{At}\right) = Ae^{At}$

- **Property 3 is how we solve the differential equations**

- $e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots.$

$$e^{At} = I + S\Lambda S^{-1}t + \frac{S\Lambda^2 S^{-1}t^2}{2!} + \frac{S\Lambda^3 S^{-1}t^3}{3!} + \cdots$$

$$= S\left(I + \Lambda t + \frac{(\Lambda t)^2}{2!} + \frac{(\Lambda t)^3}{3!} + \cdots\right)S^{-1} = Se^{\Lambda t}S^{-1}.$$

- $e^{\Lambda t} = I + \Lambda t + \frac{(\Lambda t)^2}{2!} + \frac{(\Lambda t)^3}{3!} + \cdots$

$$= \begin{bmatrix} 1 + \lambda_1 t + \frac{(\lambda_1 t)^2}{2!} + \frac{(\lambda_1 t)^3}{3!} + \cdots & & \\ & \ddots & \\ & & 1 + \lambda_n t + \frac{(\lambda_n t)^2}{2!} + \frac{(\lambda_n t)^3}{3!} + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix}$$

More on Exponential: $e^{At}=Se^{\Lambda t}S^{-1}$

- **Example: exponential of** $A=\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

$$e^{At} = Se^{\Lambda t}S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{-t} & \\ & e^{-3t} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$$

$$= \frac{1}{2}\begin{bmatrix} e^{-t} + e^{-3t} & e^{-t} - e^{-3t} \\ e^{-t} - e^{-3t} & e^{-t} + e^{-3t} \end{bmatrix}.$$

- **If x_1 is the eigenvector of A corresponding to eigenvalue λ_1, then x_1 is the eigenvector of e^{At} corresponding to eigenvalue** $e^{\lambda_1 t}$**, i.e.**

$$\boxed{e^{At}x_1 = e^{\lambda_1 t}x_1}$$

Proof:

$$e^{At}x_1 = Ix_1 + Atx_1 + \frac{(At)^2}{2!}x_1 + \frac{(At)^3}{3!}x_1 + \cdots$$

$$= Ix_1 + Ax_1 t + \frac{t^2}{2!}A^2 x_1 + \frac{t^3}{3!}A^3 x_1 + \cdots$$

$$= Ix_1 + \lambda_1 x_1 t + \frac{t^2}{2!}\lambda_1^2 x_1 + \frac{t^3}{3!}\lambda_1^3 x_1 + \cdots$$

$$= Ix_1 + \lambda_1 t x_1 + \frac{(\lambda_1 t)^2}{2!}x_1 + \frac{(\lambda_1 t)^3}{3!}x_1 + \cdots$$

$$= (I + \lambda_1 t + \frac{(\lambda_1 t)^2}{2!} + \frac{(\lambda_1 t)^3}{3!} + \cdots)x_1 = e^{\lambda_1 t}x_1$$

- **<u>Matrix e^{At} is never singular</u>: its eigenvalue $e^{\lambda t}$ is never zero and**

det $e^{At} = e^{\lambda_1 t}e^{\lambda_2 t}\cdots e^{\lambda_n t} = e^{\text{trace}(At)} \Rightarrow$

Inverse of $e^{At} = e^{-At}$ which always exists

Matrix Exponential and ODE Solution

- **If A can be diagonalized, $A=S\Lambda S^{-1}$ then the differential equation *du/dt=Au* has the solution:**

$$u(t)=e^{At}u_0=Se^{\Lambda t}S^{-1}u_0.$$

The column of S are the eigenvectors of A, so that

$$u(t)=\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{bmatrix}S^{-1}u_0$$

By setting $S^{-1}u_0=c\Rightarrow Sc=u_0$ (i.e. expressing u_0 as linear combination of eigenvectors), the solution becomes

$$u(t)=\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_n\end{bmatrix}=c_1e^{\lambda_1 t}x_1+\cdots+c_ne^{\lambda_n t}x_n.$$

- **Another way of looking at the solution:**

  If $u_0=c_1x_1+\cdots+c_nx_n$, i.e. $u_0=Sc$ then

$$u(t)=e^{At}u_0=e^{At}(c_1x_1+\cdots+c_nx_n)$$
$$=c_1e^{At}x_1+\cdots+c_ne^{At}x_n$$
$$=c_1e^{\lambda_1 t}x_1+\cdots+c_ne^{\lambda_n t}x_n.$$

Back to ODE for Diffusion Model

- **For systems**

$$\frac{du}{dt}=\begin{bmatrix} -2v+w \\ v-2w\end{bmatrix}=\begin{bmatrix} -2 & 1 \\ 1 & -2\end{bmatrix}u \text{ where } u=\begin{bmatrix} v \\ w\end{bmatrix} \text{ and } u=u_0 \text{ at } t=0$$

$$\frac{du}{dt}=Au,\quad u=u_0 \text{ at } t=0.$$

- **First step: Diagonalize A**

$$\begin{bmatrix} -2 & 1 \\ 1 & -2\end{bmatrix}\begin{bmatrix} x_1 \\ x_2\end{bmatrix}=\lambda\begin{bmatrix} x_1 \\ x_2\end{bmatrix}\Rightarrow$$

$$A\begin{bmatrix} 1 \\ 1\end{bmatrix}=(-1)\begin{bmatrix} 1 \\ 1\end{bmatrix},\quad A\begin{bmatrix} 1 \\ -1\end{bmatrix}=(-3)\begin{bmatrix} 1 \\ -1\end{bmatrix}.$$

- **General solution:**

$$u(t)=c_1e^{\lambda_1 t}x_1+c_2e^{\lambda_2 t}x_2 \quad \text{or} \quad u=\begin{bmatrix} 1 & 1 \\ 1 & -1\end{bmatrix}\begin{bmatrix} e^{-t} & \\ & e^{-3t}\end{bmatrix}\begin{bmatrix} c_1 \\ c_2\end{bmatrix}.$$

At time zero (initial condition) ($e^0=1$):

$$u_0=c_1x_1+c_2x_2 \text{ or } u_0=\begin{bmatrix} 1 & 1 \\ 1 & -1\end{bmatrix}\begin{bmatrix} c_1 \\ c_2\end{bmatrix}=Sc \quad\Rightarrow\quad \boldsymbol{c=S^{-1}u_0}$$

Solution:

$$u(t)=\begin{bmatrix} 1 & 1 \\ 1 & -1\end{bmatrix}\begin{bmatrix} e^{-t} & \\ & e^{-3t}\end{bmatrix}\begin{bmatrix} c_1 \\ c_2\end{bmatrix}=S\begin{bmatrix} e^{-t} & \\ & e^{-3t}\end{bmatrix}S^{-1}u_0.=\boldsymbol{Se^{\Lambda t}S^{-1}u}$$

Stability of Differential Equation

$$u(t) = Se^{\Lambda t}S^{-1}u_0 = c_1e^{\lambda_1 t}x_1 + \cdots + c_n e^{\lambda_n t}x_n$$

- **Eigenvalues decide how u(t) behaves as** $t \to \infty$
- **Stability is governed by** $e^{\lambda_i t} \Rightarrow$ **by real part of** λ_i
- **If** $\lambda = a + ib$,

$$e^{\lambda t} = e^{at}e^{ibt} = e^{at}(\cos bt + i \sin bt) \quad \text{and} \quad \left|e^{\lambda t}\right| = e^{at}$$

Decays for *a*<0; becomes constant for *a*=0; and explodes for *a*>0.

- **The *du*/*dt*=*Au* system is**

Stable **and** $e^{At} \to 0$ **whenever all Re**λ_i**<0**

Neutrally stable **when all Re**$\lambda_i \le 0$ **and some Re**λ_i**=0**

Unstable **and** e^{At} **is unbounded if any eigenvalue has Re**λ_i**>0**

- **All solutions approaches zero *if and only if* all eigenvalues have a negative real part** $\Rightarrow$ ***asymptotic*** **stability**

Stability for a 2 by 2 System

$$\frac{du}{dt} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} u$$

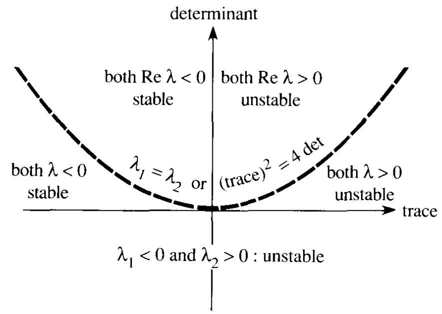
$$\det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = \lambda^2 - (\text{trace})\lambda + (\det) = 0. \Rightarrow$$

$$\lambda = \tfrac{1}{2}\left[\text{trace} \pm \sqrt{(\text{trace})^2 - 4(\det)}\right]$$

- **Stability test: 1. The trace *a*+*d* must be negative**

2. The determinant *ad*-*bc* must be positive

- **When the eigenvalues are real, the two tests guarantee them to be negative.**
- **When the eigen values are complex pair** $x \pm iy$, **the trace=2*x* and the determinant=**x^2+y^2



- **If *b*=*c* then** $(\text{trace})^2 - 4(\det) = (a+d)^2 - 4(ad - b^2) = (a-d)^2 + 4b^2 \ge 0$

$\Rightarrow$ **symmetric matrix is on or below parabola.**

- **Neutrally stable: boundaries of 2**$^{\text{nd}}$ **quadrant**

Example of 2 by 2 System

Example 2: diffusion equation $du/dt = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} u$ **is stable since eigenvalues are –1 and -3**

Example 3: diffusion model with two ends closed off.

$$\frac{du}{dt} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} u \quad \text{or} \quad \begin{aligned} dv/dt &= w - v \\ dw/dt &= v - w \end{aligned}$$

- **This is a *continuous Markov process.***

  Markov matrix: each column adds up to 1=$\lambda_{\max}$

  Cont. Markov matrix: each column adds up to 0=$\lambda_{\max}$

- ***A* is a Markov matrix if and only if *B=A-I* is a continuous Markov matrix.**

- **The steady state (*v*=*w*) is the eigenvector (=**$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$**) corresponding to** $\lambda_{\max}$

Skew-Symmetric Matrices

Example 1: $du/dt = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} u_0$

trace=0; det=1 $\Rightarrow \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \text{ so } \lambda = +i \text{ and } -i.$

eigenvectors: (1, -*i*) and (1, *i*) $u(t) = c_1 e^{it} \begin{bmatrix} 1 \\ -i \end{bmatrix} + c_2 e^{-it} \begin{bmatrix} 1 \\ i \end{bmatrix}.$

substituting $e^{it} = \cos t + i \sin t$**:** $u(t) = \begin{bmatrix} (c_1 + c_2)\cos t + i(c_1 - c_2)\sin t \\ -i(c_1 - c_2)\cos t + (c_1 + c_2)\sin t \end{bmatrix}$

At *t*=0, $u_0=(a, b) \Rightarrow Sc=u_0 \Rightarrow (c_1+c_2)=a$ **and** $-i(c_1-c_2)=b \Rightarrow$

$u(t) = \begin{bmatrix} a\cos t - b\sin t \\ b\cos t + a\sin t \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} u_0$ **sends** u_0 **around a circle.**

- **Also, since** $A^2=-I$

$$e^{At} = I + At + \frac{(At)^2}{2} + \cdots = \begin{bmatrix} \left(1 - \frac{t^2}{2} + \cdots\right) & \left(-t + \frac{t^3}{6} - \cdots\right) \\ \left(t - \frac{t^3}{6} + \cdots\right) & \left(1 - \frac{t^2}{2} + \cdots\right) \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

$\Rightarrow$ e^{At} **is an orthogonal matrix!**

FACT: $\boxed{\textbf{\textit{If A is skew-symmetric then } } e^{At} \textbf{\textit{ is an orthogonal matrix}}}$

$A^T = -A \quad \text{and} \quad \left(e^{At}\right)^T = e^{-At} \Rightarrow e^{At}\left(e^{At}\right)^T = \mathrm{I} \Rightarrow e^{At} \text{ is orthogoanl}$

$\Rightarrow \left\|e^{At}u_0\right\| = \|u_0\| \Rightarrow \text{Conservative systems : no energy is lost}$