LINEAR ALGEBRA FALL 2019

SOLUTIONS TO SELECTED PROBLEMS FROM HOMEWORK 9–11

6.C. 14. (a) Let g be a continuous real-valued function on $[-1.1]$. We need to show if $< f, g >= 0$ for every $f \in U$, then $g = 0$. We have

$$\int_{-1}^{1} f(x)g(x) = 0$$

for every $f \in U$. Construct $f \in U$ as follows: $f(x) = xg(x)$ if $0 \leq x \leq 1$ and $f(x) = -xg(x)$ if $-1 \leq x \leq 0$. Since the limit of $f(x)$ as x approaches 0 from both right and left is 0, and since g is a continuous function, f is continuous. Clearly $f(0) = 0$, so $f \in U$. Now observe that

$$0 =< f, g >= \int_{-1}^{0} -xg(x)^2 \, dx + \int_{0}^{1} xg(x)^2 \, dx.$$

Since the integral of a non-negative function over an interval is non-negative, both integrals should be equal to 0. The integral of a non-negative function over an interval $[a, b], a < b$ is 0 exactly when the function is 0 on $[a, b]$. So we conclude g should be the 0 function on $[-1, 1]$.

(b) Since $U^\perp = \{0\}$ by part (a), $U + U^\perp = U \neq V$, so 6.47 does not hold. Similarly, $U^\perp = \{0\}$, and clearly $\{0\}^\perp = V$, so $(U^\perp)^\perp = V \neq U$, so 6.51 does not hold.

7.A. 11. First suppose that $P = P_U$. For $v, w \in V$, write $v = v_1 + v_2$ and $w = w_1 + w_2$ where $v_1, w_1 \in U$ and $v_2, w_2 \in U^\perp$. Then $P_U v = v_1$ and $P_U w = w_1$ so

$$\langle P_U v, w\rangle = \langle v_1, w_1 + w_2\rangle = \langle v_1, w_1\rangle$$

and

$$\langle v, P_U w\rangle = \langle v_1 + v_2, w_1\rangle = \langle v_1, w_1\rangle,$$

so P_u is self adjoint.

Conversely, suppose that P is self adjoint and let

$$U = \{v \in V \mid Pv = v\}.$$

Clearly U is a subspace of V. Note that for any $w \in V$, $P(Pw) = P^2 w = Pw$, so $Pw \in U$ by definition. We claim that $P = P_U$. First we show if $w \in U^\perp$, then $Pw = 0$. To show this note that since $Pw \in U$ we have,

$$0 = \langle Pw, w\rangle = \langle P(Pw), w\rangle = \langle Pw, Pw\rangle.$$

where the second inequality comes from the fact that $P^2 = P$ and the third inequality comes from the fact that P is self-adjoint. So $Pw = 0$.

Now for anny $v \in V$, we can write $v = v_1 + v_2$ where $v_1 \in U$ and $v_2 \in U^\perp$. Then $P_U v = v_1$ and $Pv = Pv_1 + Pv_2 = v_1$, so $P = P_U$.

8.A. 5. Suppose to the contrary that there is a combination

$$a_0 v + a_1 v + \cdots + a_{m-1} T^{m-1} v = 0.$$

Let r be the smallest index such that $a_r \neq 0$. We have

$$a_r T^r v + \cdots + a_{m-1} T^{m-1} v = 0.$$

Applying T^{m-r-1} to both side of the above equation, since $T^m v = 0$, we get

$$a_r T^{m-1} v = 0$$

Since $T^{m-1}v \neq 0$ this implies that $a_r = 0$, a contradiction.

8.A. 10. Since

$$\dim \operatorname{null} T^{n-1} + \dim \operatorname{range} T^{n-1} = n,$$

it is enough to show null $T^{n-1} \cap$ range $T^{n-1} = \{0\}$. Suppose $v \in T^{n-1} \cap$ range T^{n-1}. So $T^{n-1}v = 0$, and $v = T^{n-1}w$ for some $w \in V$. So $T^{2n-2}w = T^{n-1}v = 0$.

We have

$$\{0\} = \operatorname{null} I \subseteq \operatorname{null} T \subseteq \operatorname{null} T^2 \subseteq \cdots \subseteq \operatorname{null} T^n.$$

Since we assume T is not nilpotent, T^n is not the zero operator, so

$$\dim \operatorname{null} T^n \leq n - 1.$$

This implies that in the above sequence, there is $i \leq n-1$ such that $\dim \operatorname{null} T^i = \dim \operatorname{null} T^{i+1}$, and therefore null $T^{i-1} =$ null T^i, so by Theorem 8.3,

$$\operatorname{null} T^{i-1} = \operatorname{null} T^i = \operatorname{null} T^{i+1} \ldots.$$

Since $i \leq n-1$, we conclude

$$\operatorname{null} T^{n-1} = \operatorname{null} T^n = \cdots = \operatorname{null} T^{2n-2} = \ldots.$$

So $T^{2n-2}w = 0$ implies $T^{n-1}w = 0$, so $v = 0$.

8.B. 2. Let $T \in \mathcal{L}(\mathbf{R}^3)$ be the operator whose matrix in the standard basis is given by

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So $T(x, y, z) = (-y, x, 0)$. Clearly 0 is an eigenvalue since $T(0, 0, 1) = (0, 0, 0)$. If $\lambda \neq 0$ is an eigenvalue and $v = (x, y, z)$ is an eigenvector corresponding to λ, then $(-y, x, 0) = \lambda(x, y, z)$, so $-y = \lambda x$ and $x = \lambda y = \lambda(-\lambda x) = -\lambda^2 x$, so either $\lambda^2 = -1$ or $x = y = z = 0$. So 0 is the only eigenvalue.

We have $T^2(x, y, z) = (-x, -y, 0)$ and $T^3(x, y, z) = T(-y, -x, 0) = (x, y, 0)$, so T^3 is not the zero operator so T is not nilpotent by Theorem 8.18.

8.B. 3. Note that

$$(S^{-1}TS - \lambda I)^n = (S^{-1}(T - \lambda I)S)^n = S^{-1}(T - \lambda I)^n S.$$

Suppose that $v_1, \ldots, v_k$ is a basis for $G(\lambda_i, T) = \text{null } (T - \lambda_i I)^n$. Then since S is invertible, it is surjective, so there are vectors $u_1, \ldots, u_k$ such that $Su_1 = v_1, \ldots, Su_k = v_k$. Then clearly $u_1, \ldots, u_k$ are linearly independent: if $a_1u_1 + \cdots + a_ku_k = 0$, then S of the left hand side is 0, so $a_1v_1 + \cdots + a_kv_k = 0$. Also, for each j

$$(S^{-1}TS - \lambda_i I)^n u_j = S^{-1}(T - \lambda_i I)^n S u_j = S^{-1}(T - \lambda_i I)^n v_j = 0,$$

so $u_1, \ldots, u_k \in G(\lambda_i, S^{-1}TS) = \text{null } (S^{-1}TS - \lambda_i I)^n$. This shows that if λ_i is an eigenvalue for T, then it is an eigenvalue for $S^{-1}TS$ and $\dim G(\lambda_i, S^{-1}TS) \geq \dim G(\lambda_i, T)$. Now recall that the sum of the dimensions of generalized eigenspaces corresponding to all the eigenvalues is the dimension of V. If we apply this to T and $S^{-1}TS$, we see that if $\lambda_1, \ldots, \lambda_m$ are the distinct eigenvalues of T, then

$$n = \sum_i \dim G(\lambda_i, T) \leq \sum_i \dim G(\lambda_i, S^{-1}TS) \leq n$$

so we conclude that $\dim G(\lambda_i, T) = \dim G(\lambda_i, S^{-1}TS)$ for each i, and the λ_i are all the eigenvalues of $S^{-1}TS$.

8.C. 8. Let $p(x)$ be the minimal polynomial of T. We know T is invertible if and only if 0 is not an eigenvalue for T. We also know by Theorem 8.49 that the roots of $p(x)$ are exactly the eigenvalues of T, so we conclude that T is invertible if and only if 0 is not a root of $p(x)$.

So if T is invertible, 0 is not a root of $p(x)$, so $p(0) \neq 0$. Hence the constant term of $p(x)$ is non-zero. Conversely, if the constant term of $p(x)$ is non-zero, $p(0) \neq 0$, so 0 is not a root of $p(x)$ and therefore T is invertible.

8.C. 9. Suppose $p(x) = x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0$ is the minimal polynomial of an operator T and that T is invertible. By the previous problem, we know $a_0 \neq 0$. We have

$$T^m + a_{m-1}T^{m-1} \cdots + a_1 T + a_0 I = 0$$

and multiplying both sides by $(T^m)^{-1} = (T^{-1})^m = T^{-m}$, we get

$$I + a_{m-1}T^{-1} + \cdots + a_1(T^{-1})^{m-1} + a_0(T^{-1})^m = 0$$

So if we set $q(x) = 1 + a_{m-1}x + \cdots + a_1 x^{m-1} + a_0 x^m$, we have $q(T^{-1}) = 0$. So the degree of the minimal polynomial of T^{-1} is smaller than or equal to degree of the minimal polynomial of T, and similarly the degree of the minimal polynomial of $T = (T^{-1})^{-1}$ is smaller than or equal to the degree of the minimal polynomial of T^{-1}, so the degree of the minimal polynomials of T and T^{-1} are equal. Since $\frac{1}{a_0}q(x)$ is a monic polynomial it should be the minimal polynomial of T^{-1}.

8.C. 19. Suppose the matrix of T with respect to a basis $v_1, \ldots, v_n$ is an upper triangular matrix A. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T, and let d_i be the multiplicity of λ_i, so $d_i = \dim G(\lambda_i, T) = \dim \operatorname{null}(T - \lambda_i I)^n$. Let k_i be the number of times λ_i appears on the diagonal. Since

$$d_1 + \cdots + d_m = n = k_1 + \cdots + k_m,$$

if we have $d_i \leq k_i$ for each i, the only way the above equality holds is when $d_i = k_i$ for each i. So it is enough to show $d_i \leq k_i$ for each i.

The matrix of $T - \lambda_i I$ with respect to the basis $v_1, \ldots, v_n$ is $A - \lambda_i I$ which has exactly k_i zero entries on the diagonal. Since if A and B are two upper triangular matrices $(AB)_{ii} = A_{ii}B_{ii}$, the matrix of $(T - \lambda_i I)^n$ is $(A - \lambda_i I)^n$ and has also exactly k_i zero entries on the diagonal.

So we need to show if an operator (the operator is $(T - \lambda_i I)^n$ here.) has an upper triangular matrix M with k zero entries on the diagonal, then the dimension of its null space is at most k, or equivalently the dimension of its range is at least $n - k$. Suppose $j_1 < \cdots < j_{n-k}$ are the indices of the non-zero entries on the diagonal of M. Then we claim $Tv_{j_1}, \ldots, Tv_{j_{n-k}}$ are linearly independent. This shows the dimension of the range of T is at least $n - k$. To show the claim suppose to the contrary there is a linear combination

$$c_1 T v_{j_1} + \cdots + c_{n-k} T v_{j_{n-k}} = 0.$$

Let j_r be the largest index with a non-zero coefficient, so

$$c_1 T_{v_{j_1}} + \cdots + c_r T_{v_{j_r}} = 0 \tag{1}$$

and $c_r \neq 0$. Since M is upper triangular, for every j, Tv_j is a linear combination of $v_1, \ldots, v_j$:

$$Tv_j = M_{1j}v_1 + \cdots + M_{jj}v_j.$$

So the left hand side of Equation (1) is a combination of vectors $v_1, \ldots, v_{j_r}$ and since the j_r-th entry of the diagonal is non-zero, v_{j_2} appears with a non-zero coefficient in that combination. This contradict the fact that the v_j are linearly independent.