MATH 113: MIDTERM

Each problem is 20 points. Attempt all problems. Problem 5, part (5) is extra credit only, so work on it *only if* you have completed all other problems.

This is a closed book, closed notes exam, with no calculators allowed (they shouldn't be useful anyway). You may use any theorem, proposition, etc., proved in class or in the book *provided that you quote it precisely.* Make sure that you justify your answer to each question, including the verification that all assumptions of any theorem you quote hold. Try to be brief though.

If on a problem you cannot do part (1), (2) or (3), you may assume its result for the subsequent parts.

Allotted time: 60 minutes.

Problem 1. Let $\mathcal{P}^m(\mathbb{R})$ denote the set of polynomials of degree $\leq m$ on $\mathbb{R}$ with real coefficients. Suppose that $m \geq 1$ and $(p_0, p_1, \dots, p_m)$ is a list of elements of $\mathcal{P}^m(\mathbb{R})$ satisfying $p_j(0) = p_j(2)$, $j = 0, 1, \dots, m$. Show that $(p_0, p_1, \dots, p_m)$ is linearly dependent. Is this true for $m = 0$?

Problem 2. Show that a vector space V is finite dimensional if and only if there exists a natural number n such that every linearly independent list $(v_1, \dots, v_k)$, ($v_j \in V$, $j = 1, \dots, k$) has length $k \leq n$.

Problem 3. Consider the linear map

$$T : \mathbb{F}^4 \to \mathbb{F}^2,\ T(x_1, x_2, x_3, x_4) = (x_4 + 2x_3, 3x_1 - x_2).$$

(1) Show that T is surjective.
(2) What is $\dim \operatorname{null} T$?
(3) Find a basis for $\operatorname{null} T$.

Problem 4. Suppose that V, W are finite dimensional vector spaces, U is a subspace of V and Z is a subspace of W. Show that there is a linear map $S \in \mathcal{L}(V, W)$ such that $\operatorname{null} S = U$ and $\operatorname{range} S = Z$ if and only if $\dim U + \dim Z = \dim V$.

Problem 5. Suppose that V is a vector space over $\mathbb{F}$, and W is a subspace of V. For $v \in V$, let

$$[v] = \{v' \in V :\ v' - v \in W\} = \{v + w :\ w \in W\} \subset V.$$

(1) Show that for $v_1, v_2 \in V$, we have $[v_1] = [v_2]$ if and only if $v_1 - v_2 \in W$.
(2) Show that $[v]$ is a subspace of V if and only if $v \in W$.
(3) Show that if $[v_1] = [v_1']$ and $[v_2] = [v_2']$ ($v_1, v_1', v_2, v_2' \in V$) then $[v_1 + v_2] = [v_1' + v_2']$, and if $c \in \mathbb{F}$, $v_1, v_1' \in V$, $[v_1] = [v_1']$ then $[cv_1] = [cv_1']$.
(4) Let V/W denote the set $\{[v] :\ v \in V\}$, and we define addition and multiplication by scalars on V/W by $[v_1] + [v_2] = [v_1 + v_2]$ for $v_1, v_2 \in V$, and for $c \in \mathbb{F}$, $v \in V$, let $c[v] = [cv]$. (These make sense independent of the choice of v_1, etc., by the previous part.) List the properties you would need to check to show that V/W is a vector space over $\mathbb{F}$ with these operations, and show that addition is commutative: $[v_1] + [v_2] = [v_2] + [v_1]$ and the following distributive law holds $c([v_1] + [v_2]) = c[v_1] + c[v_2]$.
(5) (Extra credit only!) Show that if V is finite dimensional, then V/W is finite dimensional and $\dim(V/W) = \dim V - \dim W$. (You may assume that V/W is a vector space.)