LINEAR ALGEBRA FALL 2019

SOLUTIONS TO SELECTED PROBLEMS FROM HOMEWORK 5–8

Homwork 6. Question 1. (a) We prove the statement by induction on l. It clearly holds for l=1, if $p(x) = (x-\lambda)q(x)$, then $p(\lambda) = 0$. Now assume the statement holds for l, so for every polynomial $p(x)$ and for every λ, if

$$p(x) = (x-\lambda)^l q(x)$$

then

$$p(\lambda) = p'(\lambda) = \cdots = p^{(l-1)}(\lambda) = 0.$$

We show it holds for $l+1$. Suppose $p(x) = (x-\lambda)^{l+1}q(x)$. Then $p'(x) = (x-\lambda)^l(\lambda q(x) + (x-\lambda)q'(x))$. If we apply the induction hypothesis to $p'(x)$, we see that

$$p'(\lambda) = p''(\lambda) = \cdots = (p')^{(l-1)}(\lambda) = 0.$$

Clearly $p(\lambda) = 0$, so the above gives

$$p(\lambda) = p'(\lambda) = \cdots = p^{(l)}(\lambda) = 0.$$

(b) We apply the division algorithm to divide $p(x)$ by $(x-\lambda)^l$: we can write

$$p(x) = (x-\lambda)^l q(x) + r(x)$$

where $r(x)$ is the zero polynomial or $\deg r(x) < l$. If $r(x)$ is the zero polynomial, then we are done, so assume to the contrary that this is not the case. Set $k = \deg r(x) \leq l-1$, and write

$$r(x) = b_k x^k + \cdots + b_1 x + b_0, \qquad b_k \neq 0.$$

Then since by part (a), all the derivatives of $(x-\lambda)^k q(x)$ up to oder $l-1$ are zero at λ, and since the same holds for $p(x)$ by our assumption, all the derivatives of $r(x)$ up to order $l-1$ are equal to zero at λ as well. But

$$r^{(k)}(x) = k!\, b_k \neq 0,$$

a contradiction.

(c) If λ is a root of multiplicity l, then $p(\lambda) = \cdots = p^{(l-1)}(\lambda) = 0$ by part (a), and $p^{(l)}(\lambda) \neq 0$ since otherwise λ will be a root of multiplicity $\geq l+1$ by part (b). Conversely, if $p(\lambda) = \cdots = p^{(l-1)}(\lambda) = 0$, then by part (b) λ is a root of multiplicity $\geq l$, and it can't be a root of multiplicity $\geq l+1$ by part (a), so it is a root of multiplicity l.

4. 5. Fix $z_1, \ldots, z_{m+1}$ in $\mathbf{F}$ and define a function $L : \mathcal{P}_m(\mathbf{F}) \to \mathbf{F}^{m+1}$ by

$$L(p(x)) = (p(z_1), \ldots, p(z_{m+1})).$$

Clearly L is a linear map since for polynomials $p_1(x), p_2(x)$, we have $(p_1 + p_2)(z_i) = p_1(z_i) + p_2(z_i)$ for each i, and similarly $(\lambda p)(z_i) = \lambda p(z_i)$ for every $\lambda \in \mathbf{F}$ and for every i.

If we show that L is an isomorphism, then we are done: this will show that L is surjective and for every $(w_1, \ldots, w_{m+1})$ in $\mathbf{F}^{m+1}$, there is a unique polynomial $p(x)$ in $\mathcal{P}_m(\mathbf{F})$ such that $L(p(x)) = (w_1, \ldots, w_{m+1})$ which is the statement of the problem.

Note that the dimension of both $\mathcal{P}_m(\mathbf{F})$ and $\mathbf{F}^{m+1}$ is $m + 1$, so if we show L is injective, then we can conclude taht L is an isomorphism. To show L is injective, we show that the null space of L is zero. If $p(x)$ is in the null space of L, then $L(p(x)) = (0, \ldots, 0)$, so $p(z_1) = \cdots = p(z_{m+1}) = 0$. But this is possible only if $p(x)$ is the zero polynomial since the z_i are distinct by our assumption and a non-zero polynomial of degree at most m has at most m distinct zeros.

4. 6. First assume all the roots of $p(x)$ are distinct, so

$$p(x) = (x - \lambda_1) \ldots (x - \lambda_m)$$

where $\lambda_1, \ldots, \lambda_m$ are distinct. Then the zeros of $p(x)$ are the λ_i. For each i,

$$p'(x) = (x - \lambda_1) \ldots (x - \lambda_{i-1})(x - \lambda_{i+1}) \ldots (x - \lambda_m) + (x - \lambda_i)q(x)$$

where $q(x)$ is the derivative of $(x - \lambda_1) \ldots (x - \lambda_{i-1})(x - \lambda_{i+1}) \ldots (x - \lambda_m)$. Then $p'(\lambda_1) = (\lambda_1 - \lambda_2) \ldots (\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1}) \ldots (\lambda_1 - \lambda_m) \neq 0$. So none of the λ_i is a root of $p'(x)$.

Now assume $p(x)$ and $p'(x)$ have no common zeros. Then if the roots of $p(x)$ are not distinct, we can write

$$p(x) = (x - \lambda)^2 q(x)$$

for some polynomial $q(x)$ and some $\lambda \in \mathbf{C}$. But then

$$p'(x) = 2(x - \lambda)q(x) + (x - \lambda)^2 q'(x)$$

so $p'(\lambda) = 0$, so λ is a common zero of $p(x)$ and $p'(x)$, a contradiction.

5.B. 3. Let $v \in V$, and set $w = Tv - v$. Then $Tw = T(Tv - v) = T^2 v - Tv = v - Tv = -w$. Hence if $w \neq 0$, then since $Tw = -w$, -1 is an eigenvalue for T. By our assumption this is not the case, so $w = 0$, so $Tv = v$ for every $v \in V$.

5.B. 4. We first show that null $P \cap$ range $P = \{0\}$. Suppose that $v \in$ null $P \cap$ range P. Then $Pv = 0$ and $v = Pw$ for some $w \in V$. So

$$v = Pw = P^2w = P(Pw) = Pv = 0.$$

Next, we show that $V =$ null $P +$ range P. Let $v \in V$. Then

$$v = (v - Pv) + Pv.$$

We have $v - Pv \in$ null P since $P(v - Pv) = Pv - P^2v = Pv - Pv = 0$, and $Pv \in$ range P. So every vector in V is a sum of a vector in null P and a vector in range P.

5.C. 3. Clearly (a) implies (b), and (a) implies (c). To prove all the statements are equivalent, it is enough to show (b) implies (a) and (c) implies (a).

So we assume (b) holds. Then since $T : V \to V$ is a linear map,

$$\dim \operatorname{null} T + \dim \operatorname{range} T = \dim V.$$

On the other hand, by 2.43, we have

$$\begin{aligned}\dim(\operatorname{null} T + \operatorname{range} T) &= \dim \operatorname{null} T + \dim \operatorname{range} T - \dim(\operatorname{null} T \cap \operatorname{range} T)\\ &= \dim V - \dim(\operatorname{null} T \cap \operatorname{range} T)\end{aligned}$$

So $\dim(\operatorname{null}\ T \cap \operatorname{range}\ T) = 0$, so null $T \cap$ range $T = \{0\}$ and (a) holds. Similarly, we can use 2.43 to conclude that (c) implies (a).

5.C. 4. If V is the vector space $V = \{(a_1, a_2, \dots) \mid a_i \in \mathbf{R} \text{ for every } i\}$, and $T : V \to V$ is defined by $T(a_1, a_2, a_3, \dots) = (a_2, a_3, \dots)$, then null $T = \{(a_1, 0, 0, \dots)\}$, and T is surjective, so range $T = V$. So $V =$ null $V +$range T, but $(1, 0, 0, \dots) \in$ null $T \cap$ range T, so null $T +$ range T is not a direct sum.

6.A. 12. We use the Cauchy-Schwarz inequality (Example 6.17 of Section 6.A) and apply it to $y_1 = \cdots = y_n = 1$. We get

$$(x_1 + \cdots + x_n)^2 \leq n(x_1^2 + \cdots + x_n^2).$$

6.A. 31. This follows from the parallelogram Equality (6.22) if we look at the parallelogram formed by the sides with length a and b. The lengths of the diagonals of the parallelogram are $2d$ and c so by 6. 22, $(2b)^2 + c^2 = 2(a^2 + b^2)$.

6.B. 3. By Theorem 6.37, it is enough to apply the Gram-Schmidt method to the basis $v_1 = (1,0,0), v_2 = (1,1,1), v_3 = (1,1,2)$. To do this you can either directly use 6.31, or do it the way we did in class, namely first find an orthogonal basis using the formula $w_1 = v_1$ and

$$w_i = v_i - \frac{\langle v_i, w_1 \rangle}{\|w_1\|^2} w_1 - \cdots - \frac{\langle v_i, w_{i-1} \rangle}{\|w_{i-1}^2\|} w_{i-1}.$$

And then find an orthonormal basis by dividing each w_i by its norm. Applying this method, we get: $w_1 = (1,0,0)$,

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1 = (1,1,1) - \frac{1}{1}(1,0,0) = (0,1,1).$$

And

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2 \rangle}{\|w_2^2\|} w_2 = (1,1,2) - \frac{1}{1}(1,0,0) - \frac{3}{2}(0,1,1) = (0, -\frac{1}{2}, \frac{1}{2}).$$

Since $\|w_1\| = 1, \|w_2\| = \sqrt{2}, \|w_3\| = \frac{1}{\sqrt{2}}$, we have

$$u_1 = (1,0,0), u_2 = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), u_3 = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$

is an orthonormal bases.

6.B. 6. Note that if we set $v_1 = 1, v_2 = x, v_3 = x^2$, then $D(1) = 0, D(x) = 1, D(x^2) = 1 + 2x$. So the matrix of the linear transformation D with respect to v_1, v_2, v_3 is

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

which is upper triangular. Hence it is enough to apply Gram-Schmidt to the standard basis $1, x, x^2$ to obtain an orthonormal basis with respect to which the matrix of D is upper-triangular. We did the calculation in class, and got the orthonormal basis $1, \sqrt{12}(x - \frac{1}{2}), \sqrt{180}(x^2 - x + \frac{1}{6})$.