MATH 113: PRACTICE FINAL SOLUTIONS

Note: The final is in Room T175 of Herrin Hall at 7pm on Wednesday, December 12th.

There are 9 problems; attempt all of them. Problem 9 (i) is a regular problem, but 9(ii)-(iii) are bonus problems, and they are not part of your regular score. So do them only if you have completed all other problems, as well as 9(i).

This is a closed book, closed notes exam, with no calculators allowed (they shouldn't be useful anyway). You may use any theorem, proposition, etc., proved in class or in the book, provided that you quote it precisely, and *provided you are not asked explicitly to prove it.* Make sure that you justify your answer to each question, including the verification that all assumptions of any theorem you quote hold. Try to be brief though.

If on a problem you cannot do part (i) or (ii), you may assume its result for the subsequent parts.

All unspecified vector spaces can be assumed finite dimensional unless stated otherwise.

Allotted time: 3 hours. Total regular score: 160 points.

Problem 1. *(20 points)*

(i) Suppose V is finite dimensional, $S, T \in \mathcal{L}(V)$. Show that $ST = I$ if and only if $TS = I$.

(ii) Suppose V, W are finite dimensional, $S \in \mathcal{L}(W, V)$, $T \in \mathcal{L}(V, W)$, $ST = I$. Does this imply that $TS = I$? Prove this, or give a counterexample.

(iii) Suppose that $S, T \in \mathcal{L}(V)$ and ST is nilpotent. Show that TS is nilpotent.

Solution 1. (i) We prove $ST = I$ implies $TS = I$; the converse follows by reversing the role of S and T. So suppose $ST = I$. Then S is surjective (since for $v \in V$, $v = S(Tv)$) and T is injective (for $Tv = 0$ implies $0 = S(Tv) = (ST)v = v$), so as V is finite dimensional, they are both invertible. Thus, S has an inverse R, so $RS = I$ in particular. But $T = IT = (RS)T = R(ST) = RI = R$, so $TS = I$ as claimed.

(ii) Suppose $V = \{0\}$, $W = \mathbb{F}$, $T0 = 0$ (V has only one element!), $Sw = 0$ for all $w \in W$. Then $ST0 = 0$, so $ST = I$, but $TSw = 0$ for all $w \in W$, and as W is not the trivial vector space, $TS \neq I$. (While this is the lowest dimensional possibility, with $\dim V = 0$, $\dim W = 1$, it is easy to add dimensions: e.g. $V = \mathbb{F}$, $W = \mathbb{F}^2$, $Tv = (v, 0)$, $S(w_1, w_2) = w_1$, so $STv = v$ for $v \in \mathbb{F}$, $TS(w_1, w_2) = (w_1, 0)$, which is not the identity map.)

(iii) Suppose $(ST)^k = 0$. Then $(TS)^{k+1} = T(ST)^k S = T0S = 0$.

Problem 2. *(15 points)* Suppose that $T \in \mathcal{L}(V, W)$ is injective and $(v_1, \ldots, v_n)$ is linearly independent in V. Show that $(Tv_1, \ldots, Tv_n)$ is linearly independent in W.

Solution 2. Suppose that $\sum_{j=1}^n a_j T v_j = 0$. Then $T(\sum_{j=1}^n a_j v_j) = \sum_{j=1}^n a_j T v_j = 0$. As T is injective, this implies $\sum_{j=1}^n a_j v_j = 0$. Since $(v_1, \ldots, v_n)$ is linearly independent in V, all a_j are 0. Thus, $(Tv_1, \ldots, Tv_n)$ is linearly independent in W.

Problem 3. *(20 points)* Consider the following map $(.,.) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$:

$$\langle x, y \rangle = (x_1 + x_2)(y_1 + y_2) + (2x_1 + x_2)(2y_1 + y_2);\ x = (x_1, x_2),\ y = (y_1, y_2).$$

(i) Show that $\langle .,. \rangle$ is an inner product on $\mathbb{R}^2$.
(ii) Find an orthonormal basis of $\mathbb{R}^2$ with respect to this inner product.
(iii) Let $\phi(x) = x_1$, $x = (x_1, x_2)$. Find $y \in \mathbb{R}^2$ such that $\phi(x) = \langle x, y \rangle$ for all $x \in \mathbb{R}^2$.

Solution 3. (i)

$$\langle x, x \rangle = (x_1 + x_2)^2 + (2x_1 + x_2)^2 \geq 0,$$

and is equal to 0 if and only if $x_1 = -x_2$ and $2x_1 = -x_2$, so $x_1 = x_2 = 0$. Thus, $\langle .,. \rangle$ is positive definite. As x, y are symmetric in the definition, it is clearly symmetric, and

$$\begin{aligned}\langle ax, y \rangle &= (ax_1 + ax_2)(y_1 + y_2) + (2ax_1 + ax_2)(2y_1 + y_2) \\ &= a((x_1 + x_2)(y_1 + y_2) + (2x_1 + x_2)(2y_1 + y_2)) = a\langle x, y \rangle,\end{aligned}$$

with a similar argument for $\langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle$.

(ii) One can apply Gram-Schmidt to the basis $((1,0),(0,1))$. As $\|(1,0)\|^2 = 1 + 2^2 = 5$, $e_1 = \frac{1}{\sqrt{5}}(1,0)$. Then $\langle (1,0),(0,1) \rangle = 1 \cdot 1 + 1 \cdot 2 = 3$, so $(0,1) - \langle (0,1), e_1 \rangle e_1 = (0,1) - \frac{3}{5}(1,0) = (\frac{-3}{5}, 1)$. Normalizing: $\|(-\frac{3}{5}, 1)\|^2 = (\frac{2}{5})^2 + (-\frac{1}{5})^2 = \frac{1}{5}$, so $e_2 = \sqrt{5}(-\frac{3}{5}, 1)$. Then (e_1, e_2) is an orthonormal basis.

(iii) If (e_1, e_2) is an orthonormal basis, then $y = \phi(e_1)e_1 + \phi(e_2)e_2$. Using the basis from the previous part gives

$$y = \frac{1}{\sqrt{5}} e_1 + \sqrt{5}(-\frac{3}{5}) e_2 = (2, -3).$$

We check that $\langle (x_1, x_2), (2, -3) \rangle = (x_1 + x_2)(-1) + (2x_1 + x_2) \cdot 1 = x_1$.

Problem 4. *(15 points)* Suppose V is a complex inner product space. Let $(e_1, e_2, \ldots, e_n)$ be an orthonormal basis of V. Show that

$$\operatorname{trace}(T^*T) = \sum_{j=1}^{n} \|Te_j\|^2.$$

Solution 4. If $A \in \mathcal{L}(V)$, then the matrix entries of A with respect to the basis $(e_1, \ldots, e_n)$ are $\langle Ae_j, e_i \rangle$, since

$$Ae_j = \langle Ae_j, e_1 \rangle e_1 + \ldots + \langle Ae_j, e_n \rangle e_n$$

as $(e_1, \ldots, e_n)$ is orthonormal. Thus,

$$\operatorname{trace} A = \operatorname{trace} \mathcal{M}(A, (e_1, \ldots, e_n)) = \sum_{j=1}^{n} \langle Ae_j, e_j \rangle.$$

Applying this with $A = T^*T$:

$$\operatorname{trace} T^* = \sum_{j=1}^{n} \langle T^*Te_j, e_j \rangle = \sum_{j=1}^{n} \langle Te_j, Te_j \rangle = \sum_{j=1}^{n} \|Te_j\|^2.$$

Problem 5. *(20 points)* Find $p \in \mathcal{P}_3(\mathbb{R})$ such that $p(0) = 0$, $p'(0) = 0$, and

$$\int_0^1 |1 - p(x)|^2 \, dx$$

is as small as possible.

Solution 5. Let U be the subspace of $\mathcal{P}_3(\mathbb{R})$ consisting of polynomials p with $p(0) = 0$ and $p'(0) = 0$. These are exactly the polynomials $ax^2 + bx^3$, $a, b \in \mathbb{R}$. Let $\langle p, q\rangle$ be the inner product $\langle p, q\rangle = \int_0^1 p(x)\, q(x)\, dx$ on $\mathcal{P}_3(\mathbb{R})$. As the constant function $1 \in \mathcal{P}_3(\mathbb{R})$, the $p \in U$ that minimizes $\|1-p\|^2$ (which is what we want) is the orthogonal projection $P_U 1$ of 1 onto U, so we just need to find this.

There are different ways one can proceed. One can find an orthonormal basis (e_1, e_2) of U using Gram-Schmidt applied to the basis (x^2, x^3), and then

$$P_U 1 = \langle 1, e_1\rangle e_1 + \langle 1, e_2\rangle e_2.$$

Explicitly, as $\int_0^1 (x^2)^2\, dx = \frac{1}{5}$, $e_1 = \sqrt{5}x^2$, then as $\int_0^1 x^3 x^2\, dx = \frac{1}{6}$,

$$x^3 - \langle x^3, e_1\rangle e_1 = x^3 - \frac{5}{6}x^2,$$

and finally

$$\int_0^1 (x^3 - \frac{5}{6}x^2)^2\, dx = \frac{1}{7 \cdot 36},$$

one has $e_2 = \sqrt{7} \cdot 6 \cdot (x^3 - \frac{5}{6}x^2)$. Thus, using

$$\int_0^1 1 \cdot x^2\, dx = \frac{1}{3},\ \int_0^1 1 \cdot (x^3 - \frac{5}{5}x^2)\, dx = \frac{1}{4} - \frac{5}{18} = -\frac{1}{36},$$

we deduce that

$$P_U 1 = \frac{5}{3}x^2 - 7(x^3 - \frac{5}{6}x^2).$$

A different way of proceeding is to work without orthonormal bases, rather use the characterization of $P_U 1$ as the unique element of U with $1 - P_U 1$ orthogonal to all vectors in U. So suppose $P_U 1 = ax^2 + bx^3$. Then we must have $\langle 1 - P_U 1, x^2\rangle = 0$ and $\langle 1 - P_U 1, x^3\rangle = 0$ which give

$$0 = \int_0^1 (1 - ax^2 - bx^3)x^2\, dx = \frac{1}{3} - \frac{a}{5} - \frac{b}{6},$$
$$0 = \int_0^1 (1 - ax^2 - bx^3)x^3\, dx = \frac{1}{4} - \frac{a}{6} - \frac{b}{7},$$

so

$$10 = 6a + 5b,\ \frac{21}{2} = 7a + 6b,$$

taking 7 times the first minus 6 times the second gives

$$70 - 63 = -b,$$

so $b = -7$, while taking 6 times the first minus 5 times the second gives

$$60 - \frac{5 \cdot 21}{2} = a,$$

so $a = \frac{120-105}{2} = \frac{15}{2}$.

Problem 6. *(20 points)* Suppose that $S, T \in \mathcal{L}(V)$, S invertible.

(i) Prove that if $p \in \mathcal{P}(\mathbb{F})$ is a polynomial then $p(STS^{-1}) = Sp(T)S^{-1}$.

(ii) Show that the generalized eigenspaces of STS^{-1} are given by

$$\operatorname{null}(STS^{-1} - \lambda I)^{\dim V} = \{Sv :\ v \in \operatorname{null}(T - \lambda I)^{\dim V}\}.$$

(iii) Show that $\operatorname{trace}(STS^{-1}) = \operatorname{trace}(T)$, where trace is the operator trace (sum of eigenvalues, with multiplicity). (Assume that $\mathbb{F} = \mathbb{C}$ in this part.)

Solution 6. (i) Observe that $(STS^{-1})^k = ST^kS^{-1}$ for all non-negative integers k. Indeed, this is immediate for $k = 0, 1$. In general, it is easy to prove this by induction: if it holds for some k, then

$$(STS^{-1})^{k+1} = (STS^{-1})^k STS^{-1} = ST^kS^{-1}STS^{-1} = ST^{k+1}S^{-1},$$

providing the inductive step and proving the claim.

Now, if $p \in \mathcal{P}(\mathbb{F})$ then $p = \sum_{j=0}^m a_j z^j$, $a_j \in \mathbb{F}$, and

$$p(STS^{-1}) = \sum_{j=0}^m a_j (STS^{-1})^j = \sum_{j=0}^m a_j ST^jS^{-1} = S(\sum_{j=0}^m a_j T^j)S^{-1} = Sp(T)S^{-1}$$

as claimed.

(ii) We first show

$$\{Sv : \ v \in \operatorname{null}(T - \lambda I)^{\dim V}\} \subset \operatorname{null}(STS^{-1} - \lambda I)^{\dim V}.$$

Indeed, let $p(z) = (z - \lambda)^{\dim V}$. If $v \in \operatorname{null}(T - \lambda I)^{\dim V}$, i.e. $p(T)v = 0$ then $0 = Sp(T)v = Sp(T)S^{-1}Sv = p(STS^{-1})Sv$, so $Sv \in \operatorname{null}(STS^{-1} - \lambda I)^{\dim V}$ as claimed.

As $T = S^{-1}(STS^{-1})S$, this also gives

$$\{S^{-1}w : \ w \in \operatorname{null}(STS^{-1} - \lambda I)^{\dim V}\} \subset \operatorname{null}(T - \lambda I)^{\dim V}.$$

As $S(S^{-1}w) = w$, we deduce that for $w \in \operatorname{null}(STS^{-1} - \lambda I)^{\dim V}$ there is in fact $v \in \operatorname{null}(T - \lambda I)^{\dim V}$ with $Sv = w$, namely $v = S^{-1}w$, so in summary

$$\{Sv : \ v \in \operatorname{null}(T - \lambda I)^{\dim V}\} = \operatorname{null}(STS^{-1} - \lambda I)^{\dim V}.$$

(iii) Let λ_j, $j = 1, \dots, k$ be the distinct eigenvalues of T with multiplicity $m_1, \dots, m_k$, respectively. That is, $\dim \operatorname{null}(T - \lambda_j I)^{\dim V} = m_j$. By what we have shown, the distinct eigenvalues of STS^{-1} are also exactly the λ_j, the generalized eigenspaces are $\{Sv : v \in \operatorname{null}(T - \lambda_j I)^{\dim V}\}$, and as

$$S|_{\operatorname{null}(T-\lambda_j I)^{\dim V}} : \operatorname{null}(T - \lambda_j I)^{\dim V} \to \operatorname{null}(STS^{-1} - \lambda_j I)^{\dim V}$$

is a bijection, hence an isomorphism, we deduce that these generalized eigenspaces also have dimension m_j. Thus,

$$\operatorname{trace} T = \sum_{j=1}^k m_j \lambda_j = \operatorname{trace}(STS^{-1}).$$

Problem 7. *(20 points)* Suppose that V is a complex inner product space.

(i) Suppose $T \in \mathcal{L}(V)$ and there is a constant $C \geq 0$ such that $\|Tv\| \leq C\|v\|$ for all $v \in V$. Show that all eigenvalues λ of T satisfy $|\lambda| \leq C$.

(ii) Suppose now that $T \in \mathcal{L}(V)$ in normal, and all eigenvalues satisfy $|\lambda| \leq C$. Show that $\|Tv\| \leq C\|v\|$ for all $v \in V$.

(iii) Give an example of an operator $T \in \mathcal{L}(\mathbb{C}^2)$ for which there is a constant $C \geq 0$ such that $|\lambda| \leq C$ for all eigenvalues λ of T, but there is a vector $v \in \mathbb{C}^2$ with $\|Tv\| > C\|v\|$.

Solution 7. (i) Suppose that λ is an eigenvalue of T. Then there is $v \in V$, $v \neq 0$, such that $Tv = \lambda v$. Thus, taking norms,

$$C\|v\| \geq \|Tv\| = \|\lambda v\| = |\lambda| \, \|v\|.$$

Dividing by $\|v\| \neq 0$ gives the conclusion.

(ii) Since T is normal, V has an orthonormal basis $(e_1, \dots, e_n)$ consisting of eigenvectors of T, with eigenvalue $\lambda_1, \dots, \lambda_n$. Recall that for any $v \in V$, $v = \sum_{j=1}^n \langle v, e_j \rangle e_j$, and that for any $a_j \in \mathbb{C}$, $\| \sum_{j=1}^n a_j e_j \|^2 = \sum_{j=1}^n |a_j|^2$.

Now we compute for $v \in V$:

$$\|Tv\|^2 = \|T(\sum_{j=1}^n \langle v, e_j \rangle e_j)\|^2 = \| \sum_{j=1}^n \langle v, e_j \rangle T e_j \|^2 = \| \sum_{j=1}^n \langle v, e_j \rangle \lambda_j e_j \|^2$$
$$= \sum_{j=1}^n |\langle v, e_j \rangle|^2 \, |\lambda_j|^2 \leq C^2 \sum_{j=1}^n |\langle v, e_j \rangle|^2 = C^2 \|v\|^2.$$

Taking positive square roots gives the conclusion.

(iii) Take any non-zero nilpotent operator: the eigenvalues are all 0, so one can take $C = 0$, but the operator is not the zero-operator, so there is some $v \neq 0$ such that $Tv \neq 0$, hence $\|Tv\| > 0 = 0\|v\| = C\|v\|$. For example, take $T \in \mathcal{L}(\mathbb{C}^2)$ given by $T(z_1, z_2) = (z_2, 0)$, so $T^2 = 0$, hence T is nilpotent, but $T(0,1) = (1,0) \neq 0$.

Problem 8. *(20 points)*

(i) Suppose that V, W are inner product spaces. Show that for all $T \in \mathcal{L}(V, W)$,

$$\dim \operatorname{range} T = \dim \operatorname{range} T^*, \ \dim \operatorname{null} T^* = \dim \operatorname{null} T + \dim W - \dim V.$$

(ii) Consider the map $T : \mathbb{R}^n \to \mathbb{R}^{n+1}$, $T(x_1, \dots, x_n) = (x_1, \dots, x_n, 0)$. Find T^* explicitly. What is $\operatorname{null} T^*$?

Solution 8. (i) As $\operatorname{range} T$ is the orthocomplement of $\operatorname{null} T^*$ in W,

$$\dim \operatorname{range} T + \dim \operatorname{null} T^* = \dim W.$$

As $\operatorname{range} T^*$ is the orthocomplement of $\operatorname{null} T$ in V,

$$\dim \operatorname{range} T^* + \dim \operatorname{null} T = \dim V.$$

By the rank-nullity theorem applied to T, one has

$$\dim \operatorname{range} T + \dim \operatorname{null} T = \dim V.$$

Subtracting the first of the three displayed equations from the third, we get

$$\dim \operatorname{null} T - \dim \operatorname{null} T^* = \dim V - \dim W,$$

which proves the claim regarding nullspaces, while subtracting the second displayed equation from the third gives

$$\dim \operatorname{range} T - \dim \operatorname{range} T^* = 0,$$

which proves the conclusion regarding ranges.

(ii) Recall that unless otherwise specified, we have the standard inner product on $\mathbb{R}^p$, $p = n, n+1$ here. For $y = (y_1, \dots, y_{n+1}) \in \mathbb{R}^{n+1}$, $T^* y \in \mathbb{R}^n$ is defined by the property that for all $x \in \mathbb{R}^n$

$$\langle x, T^* y \rangle_{\mathbb{R}^n} = \langle Tx, y \rangle_{\mathbb{R}^{n+1}} = \sum_{j=1}^{n+1} (Tx)_j y_j = \sum_{j=1}^n x_j y_j = \langle x, (y_1, \dots, y_n) \rangle.$$

Thus, $T^* y = (y_1, \dots, y_n)$. Correspondingly

$$\operatorname{null} T^* = \{(0, \dots, 0, y_{n+1}) : \ y_{n+1} \in \mathbb{R}\}.$$

Problem 9. (Bonus problem: do it only if you have solved all other problems.) If V, W, Z are vector spaces over a field $\mathbb{F}$, and $F : V \times W \to Z$, we say that F is bilinear if

$$F(v_1 + v_2, w) = F(v_1, w) + F(v_2, w),\ \ F(cv, w) = cF(v, w)$$
$$F(v, w_1 + w_2) = F(v, w_1) + F(v, w_2),\ \ F(v, cw) = cF(v, w)$$

for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, $c \in \mathbb{F}$, i.e. if F is linear in both slots.

(i) *(10 points)* Show that the set $\mathcal{B}(V, W; Z)$ of bilinear maps from $V \times W$ to Z is a vector space over $\mathbb{F}$ under the usual addition and multiplication by scalars:

$$(F + G)(v, w) = F(v, w) + G(v, w),\ \ (\lambda F)(v, w) = \lambda F(v, w),$$

$(v \in V,\ w \in W,\ c \in \mathbb{F})$ – make sure to check that $F + G$ and cF are indeed bilinear.

(ii) *(Bonus problem: do it only if you have solved all other problems.)* Recall that $V^* = \mathcal{L}(V, \mathbb{F})$ is the dual of V. For $f \in V^*$, $g \in W^*$, let $f \otimes g : V \times W \to \mathbb{F}$ be the map $(f \otimes g)(v, w) = f(v)g(w) \in \mathbb{F}$. Show that $f \otimes g \in \mathcal{B}(V, W; \mathbb{F})$, and the map $j : V^* \times W^* \to \mathcal{B}(V, W; \mathbb{F})$, $j(f, g) = f \otimes g$, is bilinear.

(iii) *(Bonus problem: do it only if you have solved all other problems.)* If V, W are finite dimensional with bases $(v_1, \dots, v_m)$, $(w_1, \dots, w_n)$, $(f_1, \dots, f_m)$ is the basis of V^* dual to $(v_1, \dots, v_m)$, and $(g_1, \dots, g_n)$ is the basis of W^* dual to $(w_1, \dots, w_n)$, let $e_{ij} = f_i \otimes g_j$, so $e_{ij}(v_r, w_s) = 0$ if $r \neq i$ or $s \neq j$, and $e_{ij}(v_r, w_s) = 1$ if $r = i$ and $s = j$. Show that

$$\{e_{ij} :\ i = 1, \dots, m,\ j = 1, \dots, n\}$$

is a basis for $\mathcal{B}(V, W; \mathbb{F})$. In particular, conclude that

$$\dim \mathcal{B}(V, W; \mathbb{F}) = (\dim V)(\dim W).$$

Remark: One could now define the *tensor product* of V^* and W^* as $V^* \otimes W^* = \mathcal{B}(V, W; \mathbb{F})$, or the tensor product of V and W, using $(V^*)^* = V$, as

$$V \otimes W = \mathcal{B}(V^*, W^*, \mathbb{F}).$$

Solution 9. (i) First, if F is bilinear and $\lambda \in \mathbb{F}$, then

$$\begin{aligned}(\lambda F)(cv, w) &= \lambda F(cv, w) = \lambda(cF(v, w)) = c(\lambda F(v, w)) = c(\lambda F)(v, w),\\ (\lambda F)(v_1 + v_2, w) &= \lambda F(v_1 + v_2, w) = \lambda(F(v_1, w) + F(v_2, w))\\ &= \lambda F(v_1, w) + \lambda F(v_2, w) = (\lambda F)(v_1, w) + (\lambda F)(v_2, w),\end{aligned}$$

so λF is linear in the first slot. Here the first and last equalities in both chains are the definition of λF, the second is that F is linear in the first slot, while the third is using the vector space structure in Z, namely that $\lambda(cz) = c(\lambda z)$ for all $z \in Z$, respectively the distributive law. The calculation that λF is linear in the second slot is identical, with the role of v's and w's interchanged.

A different way of phrasing this is the following: for each fixed $w \in W$, let $F^\sharp_w(v) = F(v, w)$, and for each fixed $v \in V$ let $F^\flat_v(w) = F(v, w)$. Then the statement that F is bilinear is equivalent to $F^\sharp_w : V \to Z$ being linear for each $w \in W$ and $F^\flat_v : W \to Z$ being linear for each $v \in V$. Since

$$(\lambda F)^\sharp_w(v) = (\lambda F)(v, w) = \lambda F(v, w) = \lambda F^\sharp_w(v),$$

$(\lambda F)^\sharp_w = \lambda F^\sharp_w$, so the fact that $(\lambda F)^\sharp_w$ is linear follows from the fact that $\lambda F^\sharp_w$ is – and the latter we have already shown (a scalar multiple of a linear map is linear). A similar calculation for $F^\flat_v$ shows that λF is bilinear as claimed.

Similarly, $F+G$ is bilinear if F and G are because $(F+G)^\sharp_w = F^\sharp_w + G^\sharp_w$, $F^\sharp_w$ and $G^\sharp_w$ are linear, with analogous formulae for $(F+G)^\flat_v$.

Now, if U is any set (and Z is a vector space as above), the set $\mathcal{M}(U;Z)$ of all maps from U to Z is a vector space with the usual addition and multiplication by scalars: $(\lambda F)(u) = \lambda F(u)$, $(F+G)(u) = F(u)+G(u)$. Now, $\mathcal{B}(V,W;Z)$ is a subset of $\mathcal{M}(V\times W;Z)$, and we have shown that it is closed under addition and multiplication by scalars. As the 0-map is bilinear, we deduce that $\mathcal{B}(V,W;Z)$ is a subspace of $\mathcal{M}(V\times W;Z)$, so in particular is a vector space itself.

(ii) $(f\otimes g)^\sharp_w(v) = (f\otimes g)(v,w) = f(v)g(w) = g(w)f(v)$, so $(f\otimes g)^\sharp_w = g(w)f$. As f is linear, $g(w)\in\mathbb{F}$, we deduce that $(f\otimes g)^\sharp_w$ is linear. The proof that $(f\otimes g)^\flat_v$ is linear is similar, and we deduce that $f\otimes g$ is bilinear, so $f\otimes g\in\mathcal{B}(V,W;\mathbb{F})$.

Now consider the map $j: V^*\times W^* \to \mathcal{B}(V,W;\mathbb{F})$, $j(f,g) = f\otimes g$. Then

$$\begin{aligned} j(\lambda f,g)(v,w) &= ((\lambda f)\otimes g)(v,w) = (\lambda f)(v)g(w) = \lambda f(v)g(w) \\ &= \lambda(f\otimes g)(w) = \lambda j(f,g)(v,w), \end{aligned}$$

with a similar calculation for homogeneity in the second slot. Additivity is also similar:

$$\begin{aligned} j(f_1+f_2,g)(v,w) &= ((f_1+f_2)\otimes g)(v,w) = (f_1+f_2)(v)g(w) = (f_1(v)+f_2(v))g(w) \\ &= f_1(v)g(w) + f_2(v)g(w) = (f_1\otimes g)(v,w) + (f_2\otimes g)(v,w) \\ &= (j(f_1,g)+j(f_2,g))(v,w), \end{aligned}$$

with a similar calculation in the second slot. Thus, j is indeed bilinear.

(iii) First, $\sum_{i,j} a_{ij}e_{ij} = 0$ means that $\sum_{i,j} a_{ij}e_{ij}(v,w) = 0$ for all $v\in V$, $w\in W$. Letting $v=v_r$, $w=w_s$ we deduce that $a_{rs}=0$ for all r,s, so $\{e_{ij}:\ i=1,\dots,m,\ j=1,\dots,n\}$ is linearly independent.

To see that it spans, suppose that F is a bilinear map. For $v\in V$, $w\in W$, write $v=\sum_i b_iv_i$, $w=\sum_j c_jw_j$. Then

$$\begin{aligned} F(v,w) &= F(\sum_i b_iv_i, \sum_j c_jw_j) = \sum_{i,j} b_ic_jF(v_i,w_j) \\ &= \sum_{i,j} F(v_i,w_j)e_{ij}(\sum_r b_rv_r, \sum_s c_sw_s) = \sum_{i,j} F(v_i,w_j)e_{ij}(v,w). \end{aligned}$$

Thus $F=\sum F(v_i,w_j)e_{ij}$, proving that $\{e_{ij}:\ i=1,\dots,m,\ j=1,\dots,n\}$ spans. Combined with the already shown linear independence, we deduce that it is a basis, so $\dim\mathcal{B}(V,W;\mathbb{F}) = mn = (\dim V)(\dim W)$, as claimed.