LINEAR ALGEBRA FALL 2019

SOLUTIONS TO SELECTED PROBLEMS FROM HOMEWORK 1–4

1.C. 6. (a) Yes, because if a and b are real numbers, then $a^3 = b^3$ if and only if $a = b$. Clearly the subset $\{(a, b, c) \mid a = b\}$ is closed under addition and scalar multiplication and is therefore a subspace.

(b) No because $(1, 1, 0)$ and $(1, \omega, 0)$ where ω is a 3rd root of 1 other than 1 are in this subset. But $(1, 1, 0) + (1, \omega, 0) = (2, 1 + \omega, 0)$ is not in this subset: We have $\omega^3 - 1 = 0$. Since $\omega^3 - 1 = (\omega - 1)(\omega^2 + \omega + 1)$ and since $\omega \neq 1$, $\omega^2 + \omega + 1 = 0$. So $(1 + \omega)^3 = 1 + 3\omega + 3\omega^2 + 1 = -1 \neq 2^3$.

1.C. 8. The union of the x-axis and the y-axis.

2.A. 5: (a) If we think of $\mathbf{C}$ as a vector space over $\mathbf{R}$, and if there are scalars $c_1, c_2 \in \mathbf{R}$ such that $c_1(1 + i) + c_2(1 - i) = 0$, then $(c_1 + c_2) + i(c_1 - c_2) = 0$, so $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$ so both c_1 and c_2 are zero.

(b) If we think of $\mathbf{C}$ as a vector space over $\mathbf{C}$, then the scalars can be any complex numbers. And we have $i(1 + i) + (1 - i) = 0$, so $1 + i$ and $1 - i$ are not linearly independent.

2.C. 2. Let V be a subspace of $\mathbf{R}^2$. By 2.39, since $\dim \mathbf{R}^2 = 2$, any two linearly independent vectors in $\mathbf{R}^2$ span $\mathbf{R}^2$, So If V contains 2 linearly independent vectors it is equal to $\mathbf{R}^2$. If V does not contain 2 linearly independent vectors, it is either $\{0\}$ or is spanned by 1 non-zero vector (a, b). In the latter case, $V = \operatorname{span}((a, b)) = \{(\lambda a, \lambda b)\}$ which is the line through the origin and (a, b).

3.A. 12. Let $v_1, \ldots, v_n$ be a basis for V. To show that $\mathcal{L}(V, W)$ is infinite dimensional, it is enough to show that for any positive integer m, there are m linearly independent maps $T_1, \ldots, T_m$ in $\mathcal{L}(V, W)$. (because if a vector space is finite dimensional of dimension say r, then there are at most r linearly independent vectors in that space.) Since W is infinite dimensional, we can choose m linearly independent vectors $w_1, \ldots, w_m$ in W. Now by 3.5, for each $1 \leq i \leq m$, there is a unique linear map T_i which sends v_1 to w_i and $v_2, \ldots, v_n$

to zero. We claim $T_1, \dots, T_m$ are linearly independent. If $c_1T_1 + \cdots + c_mT_m$ is the zero map, then it sends every vector to 0. So in particular

$$c_1T_1 + \cdots + c_mT_m(v_1) = 0,$$

so $c_1w_1 + \cdots + c_mw_m = 0$. Since the w_i are linearly independent, this implies that all the c_i should be zero, so $T_1, \dots, T_m$ are linearly independent.

3.C. 3 Let $\dim V = n$ and $\dim \operatorname{range}(T) = k$. Then $\dim \operatorname{null}(T) = n - k$. Pick a basis $u_{k+1}, \dots, u_n$ for $\operatorname{null}(T)$ and extend it to a basis $u_1, \dots, u_n$ for V. Let $w_i = T(u_i)$ for $1 \leq i \leq k$. Then $w_1, \dots, w_k \in \operatorname{range}(T)$ and they are linearly independent: To see this assume $c_1w_1 + \cdots + c_kw_k = 0$. Then $T(c_1u_1 + \dots c_ku_k) = 0$, so $c_1u_1 + \dots c_ku_k \in \operatorname{null}(T)$, so

$$c_1u_1 + \dots c_ku_k = c_{k+1}u_{k+1} + \cdots + c_nu_n$$

for some $c_{k+1}, \dots, c_n$. Since the u_i are linearly independent, this implies that all the c_i should be 0, so $w_1, \dots, w_k$ are linearly independent.

We now extend $w_1, \dots, w_k$ to a basis $w_1, \dots, w_m$ for W. For $1 \leq i \leq k$, $T(u_i) = w_i$, so there is a 1 in the i-th entry of the i-th column, and the rest of the entries of the i-th column are zero. For $k + 1 \leq i \leq n$, $T(u_i) = 0$, so the i-th column is zero.

3.C. 4. The vector $T(v_1)$ is either 0 or not. If $T(v_1) = 0$, then for any basis $w_1, \dots, w_m$ of W, we have

$$T(v_1) = 0w_1 + \cdots + 0w_m$$

so the first column of the matrix is zero.

If $T(v_1) \neq 0$, then we let $w_1 = T(v_1)$ and extend w_1 to a basis $w_1, \dots, w_m$ for W. We have

$$T(v_1) = 1w_1 + 0w_2 + \cdots + 0w_m.$$

So the first column of the matrix with respect to this basis has 1 in the first row and 0 everywhere else.

3.D. 4. Asume $\operatorname{null}(T_1) = \operatorname{null}(T_2)$. Let $w_1, \dots, w_k$ be a basis for $\operatorname{range}(T_1)$. For each $1 \leq i \leq k$, choose v_i in V such that $T_1(v_i) = w_i$. Let $w'_i = T_2(v_i)$. Then $w'_1, \dots, w'_k$ are linearly independent in W since if a linear combination of them is zero

$$c_1w'_1 + \cdots + c_kw'_k = 0,$$

then $T_2(c_1v_1 + \cdots + c_kv_k) = 0$, and so $c_1v_1 + \cdots + c_kv_k \in \operatorname{null}(T_2) = \operatorname{null}(T_1)$. Therefore, $c_1v_1 + \cdots + c_kv_k \in \operatorname{null}(T_1)$ and so $0 = T_1(c_1v_1 + \cdots + c_kv_k) =$

$c_1w_1 + \cdots + c_kw_k$. So each c_i should be 0 since by our assumption the w_i are linaerly independent. This show the w_i' are linaerly independent, and so

$$\dim \operatorname{range}(T_2) \geq \dim \operatorname{range}(T_1) = k.$$

A similar argument shows $\dim \operatorname{range}(T_1) \geq \dim \operatorname{range}(T_2)$, and so

$$\dim \operatorname{range}(T_2) = \dim \operatorname{range}(T_1) = k.$$

We conclude that $w_1', \ldots, w_k'$ form a basis for range(T_2). Extend $w_1, \ldots, w_k$ to a basis $w_1, \ldots, w_m$ for W and extend $w_1', \ldots, w_k'$ to a basis $w_1', \ldots, w_m'$ for W. Define S to be the unique linear map such that $S(w_i') = w_i$ for each i, $1 \leq i \leq m$. (See Theorem 3.5 on Page 54.) Clearly S is injective and surjective and is therefore invertible.

To show that $T_1 = ST_2$ note that for every $v \in V$, if $T_1(v) = c_1w_1 + \cdots + c_kw_k$, then $T_1(v - c_1v_1 - \cdots - c_kv_k) = 0$, so $v - c_1v_1 - \cdots - c_kv_k \in \operatorname{null}(T_1) = \operatorname{null}(T_2)$. Therefore

$$T_2(v - c_1v_1 - \cdots - c_kv_k) = 0$$

so $T_2(v) = T_2(c_1v_1 + \cdots + c_kv_k) = c_1w_1' + \cdots + c_kw_k'$. Hence

$$ST_2(v) = S(c_1w_1' + \cdots + c_kw_k') = c_1w_1 + \cdots + c_kw_k = T_1(v).$$

3.D. 5. Suppose $\dim V = n$, and range(T_1) = range(T_2). Since V is finite dimensional, range(T_1) and range(T_2) are both finite dimensional. Let

$$k = \dim \operatorname{range}(T_1) = \dim \operatorname{range}(T_2).$$

Pick a basis $w_1, \ldots, w_k$ for range(T_1). (which is equal to range(T_2).) Let $v_1, \ldots, v_k$ be such that $T_1(v_i) = w_i$ for each $1 \leq i \leq k$. We know that

$$\dim \operatorname{null}(T_1) = \dim V - \dim \operatorname{range}(T_1) = n - k.$$

Pick a basis $v_{k+1}, \ldots, v_n$ for null(T_1). We claim that $v_1, \ldots v_n$ form a basis for V. It is enough to show that they are linearly independent. To show this, assume that

$$c_1v_1 + \ldots c_nv_n = 0.$$

Then

$$0 = T_1(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_kw_k.$$

Since the w_i are linearly independent, each $c_i = 0$ for $1 \leq i \leq k$, so $c_{k+1}v_{k+1} + \cdots + c_nv_n = 0$. Since $v_{k+1}, \ldots v_n$ form a basis for null(T_1), they are linearly independent, so all the c_i should be equal to 0, so $v_1, \ldots, v_n$ are linearly independent.

Similarly, pick $v_1', \ldots, v_k'$ in V such that $T_2(v_i') = w_i$ for $1 \leq i \leq k$. Pick a basis $v_{k+1}', \ldots, v_n'$ for null(T_2). Then $v_1', \ldots, v_n'$ is a basis for V.

By Theorem 3.5, there is a linear map S such that $S(v_i) = v_i'$ for each $1 \leq i \leq n$. We show $T_1 = T_2S$. We have $T_1(v_i) = 0$ and $T_2S(v_i) = T_2(v_i') = 0$

for each $k+1 \leq i \leq n$. And $T_1(v_i) = w_i$ and $T_2S(v_i) = T_2(v_i') = w_i$ for each $1 \leq i \leq k$. So $T_1 = T_2S$.

3.D: 16. We first show that for every v in V, Tv is a scalar multiple of v. To do this, we assume Tv and v are linearly independent and get a contradiction. If Tv and v are linearly independent, then we can extend Tv, v to a basis $Tv, v, v_2, \ldots, v_n$ for V. Then by Theorem 3.5 on Page 54, there is a linear map S which maps Tv to a non-zero vector, and maps $v, v_2, \ldots, v_n$ to the zero vector. For such a linear map $TS(v) = T(0) = 0$, but $ST(v) = S(Tv) \neq 0$, a contradiction.

Now suppose $u_1, \ldots, u_n$ is a basis for V. We know for each i, there is λ_i such that $Tu_i = \lambda_i u_i$. For any $i \neq j$, we also know by the above argument that $T(v_i + v_i)$ is a scalar multiple of $v_i + v_j$. Suppose that

$$T(v_i + v_j) = \lambda(v_i + v_j).$$

But we assumed $T(v_i) = \lambda_i v_i$ and $T(v_j) = \lambda_j v_j$, so

$$\lambda v_i + \lambda v_j = \lambda_i v_i + \lambda_j v_j.$$

Since v_i and v_j are linearly independent, this implies that $\lambda = \lambda_i = \lambda_j$, so all the λ_i are equal.

3.E. 12. This is easy if we assume V is finite dimensional since the dimension of V and $U \times V/U$ are equal, but here there is no assumption on the dimension of V. Let $v_1 + U, \ldots, v_n + U$ be a basis for V/U. Then every vector in V/U can be written uniquely as $c_1(v_1 + U) + \cdots + c_n(v_n + U)$. (which is equal to $(c_1v_1 + \cdots + c_nv_n) + U$.)

Define a map T from $U \times V/U$ to V as follows

$$T(u, v + U) = u + c_1v_1 + \cdots + c_nv_n$$

where $v + U = c_1v_1 + \cdots + c_nv_n + U$. Clearly T is linear. To show that T is injective , assume $T(u, v+U) = 0$. This means that $u + c_1v_1 + \cdots + c_nv_n = 0$, so $c_1v_1 + \cdots + c_nv_n \in U$. By 3.85 this is equivalent to say $c_1(v_1 + U) + \cdots + c_n(v_n + U) = 0$ in V/U. (the zero vector in V/U is $0 + U = U$.) Since the $v_i + U$ is a basis for V/U, this implies that each c_i is zero, and so u is also zero.

To show T is surjective, we note that if $v \in V$, then $v+U = c_1v_1+\cdots+c_nv_n+U$ for some $c_1, \ldots, c_n$. Again by 3.85 this means that $v - c_1v_1 - \cdots - c_nv_n \in U$. Call this vector u. Then $T(u, c_1v_1 + \cdots + c_nv_n + U) = v$, so T is surjective.

3.E. 16. Note that here the dimension of V might be infinite. For example, if $F = \mathbf{R}$, V is the vector space of all polynomials with real coefficients, and U

is the subspace of all polynomials f such that $f(0) = 0$, then V/U is spanned by $1 + U$, so V/U is 1-dimensional, but V and U are infinite dimensional.

Let $v + U$ be a basis for V/U. (For any vector v which is not in U, $v + U$ is linearly independent in V/U and is therefore a basis for V/U.) For every $v' \in V$, there is a unique c such that $v' + U = c(v + U)$. Define $T : U \to F$ such that $T(v') = c$. Clearly T is linear.

We show that $\mathrm{null}(T) = U$. First if $u \in U$, then $u + U = 0 + U = 0(v + U)$, so by our definition of T, $T(u) = 0$. Conversely, if $T(v') = 0$, then $v' + U = 0(v + U) = 0 + U$, so $v' \in U$ by 3.85.