MATH 113: PRACTICE FINAL

Note: The final is in Room T175 of Herrin Hall at 7pm on Wednesday, December 12th.

There are 9 problems; attempt all of them. Problem 9 (i) is a regular problem, but 9(ii)-(iii) are bonus problems, and they are not part of your regular score. So do them only if you have completed all other problems, as well as 9(i).

This is a closed book, closed notes exam, with no calculators allowed (they shouldn't be useful anyway). You may use any theorem, proposition, etc., proved in class or in the book, provided that you quote it precisely, and *provided you are not asked explicitly to prove it.* Make sure that you justify your answer to each question, including the verification that all assumptions of any theorem you quote hold. Try to be brief though.

If on a problem you cannot do part (i) or (ii), you may assume its result for the subsequent parts.

All unspecified vector spaces can be assumed finite dimensional unless stated otherwise.

Allotted time: 3 hours. Total regular score: 160 points.

Problem 1. *(20 points)*

(i) Suppose V is finite dimensional, $S, T \in \mathcal{L}(V)$. Show that $ST = I$ if and only if $TS = I$.

(ii) Suppose V, W are finite dimensional, $S \in \mathcal{L}(W, V)$, $T \in \mathcal{L}(V, W)$, $ST = I$. Does this imply that $TS = I$? Prove this, or give a counterexample.

(iii) Suppose that $S, T \in \mathcal{L}(V)$ and ST is nilpotent. Show that TS is nilpotent.

Problem 2. *(15 points)* Suppose that $T \in \mathcal{L}(V, W)$ is injective and $(v_1, \ldots, v_n)$ is linearly independent in V. Show that $(Tv_1, \ldots, Tv_n)$ is linearly independent in W.

Problem 3. *(20 points)* Consider the following map $(.,.) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$:

$$\langle x, y\rangle = (x_1 + x_2)(y_1 + y_2) + (2x_1 + x_2)(2y_1 + y_2);\ x = (x_1, x_2),\ y = (y_1, y_2).$$

(i) Show that $\langle ., .\rangle$ is an inner product on $\mathbb{R}^2$.

(ii) Find an orthonormal basis of $\mathbb{R}^2$ with respect to this inner product.

(iii) Let $\phi(x) = x_1$, $x = (x_1, x_2)$. Find $y \in \mathbb{R}^2$ such that $\phi(x) = \langle x, y\rangle$ for all $x \in \mathbb{R}^2$.

Problem 4. *(15 points)* Suppose V is a complex inner product space. Let $(e_1, e_2, \ldots, e_n)$ be an orthonormal basis of V. Show that

$$\operatorname{trace}(T^*T) = \sum_{j=1}^{n} \|Te_j\|^2.$$

Problem 5. *(20 points)* Find $p \in \mathcal{P}_3(\mathbb{R})$ such that $p(0) = 0$, $p'(0) = 0$, and

$$\int_0^1 |1 - p(x)|^2\, dx$$

is as small as possible.

Problem 6. *(20 points)* Suppose that $S, T \in \mathcal{L}(V)$, S invertible.

(i) Prove that if $p \in \mathcal{P}(\mathbb{F})$ is a polynomial then $p(STS^{-1}) = Sp(T)S^{-1}$.

(ii) Show that the generalized eigenspaces of STS^{-1} are given by

$$\operatorname{null}(STS^{-1} - \lambda I)^{\dim V} = \{Sv : \ v \in \operatorname{null}(T - \lambda I)^{\dim V}\}.$$

(iii) Show that $\operatorname{trace}(STS^{-1}) = \operatorname{trace}(T)$, where trace is the operator trace (sum of eigenvalues, with multiplicity). (Assume that $\mathbb{F} = \mathbb{C}$ in this part.)

Problem 7. *(20 points)* Suppose that V is a complex inner product space.

(i) Suppose $T \in \mathcal{L}(V)$ and there is a constant $C \geq 0$ such that $\|Tv\| \leq C\|v\|$ for all $v \in V$. Show that all eigenvalues λ of T satisfy $|\lambda| \leq C$.

(ii) Suppose now that $T \in \mathcal{L}(V)$ in normal, and all eigenvalues satisfy $|\lambda| \leq C$. Show that $\|Tv\| \leq C\|v\|$ for all $v \in V$.

(iii) Give an example of an operator $T \in \mathcal{L}(\mathbb{C}^2)$ for which there is a constant $C \geq 0$ such that $|\lambda| \leq C$ for all eigenvalues λ of T, but there is a vector $v \in \mathbb{C}^2$ with $\|Tv\| > C\|v\|$.

Problem 8. *(20 points)*

(i) Suppose that V, W are inner product spaces. Show that for all $T \in \mathcal{L}(V, W)$,

$$\dim \operatorname{range} T = \dim \operatorname{range} T^*, \ \dim \operatorname{null} T^* = \dim \operatorname{null} T + \dim W - \dim V.$$

(ii) Consider the map $T : \mathbb{R}^n \to \mathbb{R}^{n+1}$, $T(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0)$. Find T^* explicitly. What is $\operatorname{null} T^*$?

Problem 9. If V, W, Z are vector spaces over a field $\mathbb{F}$, and $F : V \times W \to Z$, we say that F is bilinear if

$$F(v_1 + v_2, w) = F(v_1, w) + F(v_2, w), \ F(cv, w) = cF(v, w)$$
$$F(v, w_1 + w_2) = F(v, w_1) + F(v, w_2), \ F(v, cw) = cF(v, w)$$

for all $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, $c \in \mathbb{F}$, i.e. if F is linear in both slots.

(i) *(10 points)* Show that the set $\mathcal{B}(V, W; Z)$ of bilinear maps from $V \times W$ to Z is a vector space over $\mathbb{F}$ under the usual addition and multiplication by scalars:

$$(F + G)(v, w) = F(v, w) + G(v, w), \ (\lambda F)(v, w) = \lambda F(v, w),$$

$(v \in V, \ w \in W, \ c \in \mathbb{F})$ – make sure to check that $F + G$ and cF are indeed bilinear (there are four properties to check to make sure that a map is bilinear; on this exam it suffices to choose one of these properties and check it carefully).

(ii) *(Bonus problem: do it only if you have solved all other problems.)* Recall that $V^* = \mathcal{L}(V, \mathbb{F})$ is the dual of V. For $f \in V^*$, $g \in W^*$, let $f \otimes g : V \times W \to \mathbb{F}$ be the map $(f \otimes g)(v, w) = f(v)g(w) \in \mathbb{F}$. Show that $f \otimes g \in \mathcal{B}(V, W; \mathbb{F})$, and the map $j : V^* \times W^* \to \mathcal{B}(V, W; \mathbb{F})$, $j(f, g) = f \otimes g$, is bilinear.

(iii) *(Bonus problem: do it only if you have solved all other problems.)* If V, W are finite dimensional with bases $(v_1, \ldots, v_m)$, $(w_1, \ldots, w_n)$, $(f_1, \ldots, f_m)$ is the basis of V^* dual to $(v_1, \ldots, v_m)$, and $(g_1, \ldots, g_n)$ is the basis of W^* dual to $(w_1, \ldots, w_n)$, let $e_{ij} = f_i \otimes g_j$, so $e_{ij}(v_r, w_s) = 0$ if $r \neq i$ or $s \neq j$, and $e_{ij}(v_r, w_s) = 1$ if $r = i$ and $s = j$. Show that

$$\{e_{ij} : \ i = 1, \ldots, m, \ j = 1, \ldots, n\}$$

is a basis for $\mathcal{B}(V, W; \mathbb{F})$. In particular, conclude that

$$\dim \mathcal{B}(V, W; \mathbb{F}) = (\dim V)(\dim W).$$

Remark: One could now define the *tensor product* of V^* and W^* as $V^* \otimes W^* = \mathcal{B}(V, W; \mathbb{F})$, or the tensor product of V and W, using $(V^*)^* = V$, as

$$V \otimes W = \mathcal{B}(V^*, W^*, \mathbb{F}).$$