Lecture notes on topological insulators

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Contents

APPENDIX A: Differential form

The natural language for the calculus in high dimension is **differential form**, which is developed by the great mathematician Cartan. In 3-dimensional space, we have,

$$\begin{aligned}
&0-\text{form} \quad f(x,y,z) && \text{(A1)}\\
&1-\text{form} \quad A = A_1dx + A_2dy + A_3dz && \text{(A2)}\\
&2-\text{form} \quad B = B_1dy\wedge dz + B_2dz\wedge dx + B_3dz\wedge dy && \text{(A3)}\\
&3-\text{form} \quad G = gdx\wedge dy\wedge dz && \text{(A4)}
\end{aligned}$$

The coefficients A_i, B_i, g are all functions of (x,y,z). A 0-form f is just the usual multi-variable function. A 1-form is like a line segment $\mathbf{A}\cdot d\ell$. A 2-form is like a surface element $\mathbf{B}\cdot d^2\mathbf{S}$. The 3-form is like a volume element gdV.

1. Wedge product

The wedge $\wedge$ between dx and dy is called a **wedge product**. One requires

$$dx\wedge dy = -dy\wedge dx. \tag{A5}$$

As a result, $dx\wedge dx = 0$. That is, the surface element $dx\wedge dy$ automatically carries a sign. The signs correspond to two possible normals of the surface element.
Example 1:

$$\begin{aligned}
A &= A_1dx + A_2dy + A_3dz && \text{(A6)}\\
B &= B_1dx + B_2dy + B_3dz, && \text{(A7)}
\end{aligned}$$

then

$$\begin{aligned}
& A\wedge B\\
=\ & (A_1dx + A_2dy + A_3dz)\wedge(B_1dx + B_2dy + B_3dz)\\
=\ & (A_2B_3 - A_3B_2)dy\wedge dz + (A_3B_1 - A_1B_3)dz\wedge dx\\
+\ & (A_1B_2 - A_2B_1)dx\wedge dy. && \text{(A8)}
\end{aligned}$$

That is, the wedge product of two 1-forms becomes a 2-form. Also, one can see that $A\wedge B$ is like $(\mathbf{A}\times\mathbf{B})\cdot d^2\mathbf{S}$.

If A is the same as above, but B is a 2-form,

$$B = B_1dy\wedge dz + B_2dz\wedge dx + B_3dz\wedge dy, \tag{A9}$$

then

$$\begin{aligned}
A\wedge B &= (A_1dx + A_2dy + A_3dz)\\
&\wedge\ (B_1dy\wedge dz + B_2dz\wedge dx + B_3dz\wedge dy)\\
&= (A_1B_1 + A_2B_2 + A_3B_3)dx\wedge dy\wedge dz. && \text{(A10)}
\end{aligned}$$

That is, their wedge product is a 3-form that is related to $\mathbf{A}\cdot\mathbf{B}dV$.

Finally, in general, if A is a p-form, B is a q-form, then

$$A\wedge B = (-1)^{pq}B\wedge A. \tag{A11}$$

2. Exterior derivative

An **exterior derivative** transforms a p-form to a $(p+1)$-form. It obeys the rules (A is a p-form),

$$\begin{aligned}
&1.\ d(A+B) = dA + dB && \text{(A12)}\\
&2.\ d(A\wedge B) = dA\wedge B + (-1)^pAd\wedge B, && \text{(A13)}\\
&3.\ d^2x = 0 && \text{(A14)}\\
&4.\ df = \frac{\partial f}{\partial x_i}dx_i. && \text{(A15)}
\end{aligned}$$

The last rule shows that df is just the usual total derivative of f, or $\nabla f\cdot d\boldsymbol{\ell}$.

With these rules, it follows that

$$d^2 f = 0, \text{ and } d^2 A = 0 \tag{A16}$$

for any p-form A.
Example 2:
If

$$A = A_1 dx + A_2 dy + A_3 dz, \tag{A17}$$

then

$$\begin{aligned} dA &= dA_1 \wedge dx + dA_2 \wedge dy + dA_3 \wedge dz \\ &= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) dy \wedge dz + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) dz \wedge dx \\ &+ \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) dx \wedge dy. \end{aligned} \tag{A18}$$

That is, dA is like $\nabla \times \mathbf{A} \cdot d^2\mathbf{S}$.
Example 3:
If

$$B = B_1 dy \wedge dz + B_2 dz \wedge dx + B_3 dx \wedge dy, \tag{A19}$$

then

$$\begin{aligned} dB &= dB_1 \wedge dy \wedge dz + dB_2 \wedge dz \wedge dx + dB_3 \wedge dx \wedge dy \\ &= \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z}\right) dx \wedge dy \wedge dz. \end{aligned} \tag{A20}$$

That is, dB is like $\nabla \cdot \mathbf{B} dV$.

In summary, the exterior derivative of 0-form f, 1-form A, and 2-form B correspond to ∇f, $\nabla \times \mathbf{A}$, and $\nabla \cdot \mathbf{B}$ respectively. That is, it naturally generates gradient, curl, and divergence – the 3 basic operations in vector calculus.

Furthermore, $d^2 f = 0$ and $d^2 A = 0$ correspond to

$$\nabla \times (\nabla f) = 0, \text{ and } \nabla \cdot (\nabla \times \mathbf{A}) = 0. \tag{A21}$$

That is, the curl of a gradient is zero, and the divergence of a curl is zero – the 2 basic theorems in vector calculus.

3. Stokes theorem

The **Stokes theorem** is

$$\int_S d^2\mathbf{S} \cdot \nabla \times \mathbf{A} = \int_C d\ell \cdot \mathbf{A}. \tag{A22}$$

In the language of the differential form, it becomes,

$$\int_S dA = \int_C A. \tag{A23}$$

The **Gauss theorem** is

$$\int_V d^3V \nabla \cdot \mathbf{B} = \int_S d^2\mathbf{S} \cdot \mathbf{B}. \tag{A24}$$

In the language of the differential form, it becomes,

$$\int_V dB = \int_S B. \tag{A25}$$

You can see that the theorems under two different names actually look the same when written in differential forms. The only difference is that whether A, B is a 1-form or a 2-form. The new language both simplifies and unifies the theorems in vector calculus.

We have used 3D space to illustrate the usage of differential forms. But the same formulation can be applied to higher dimensions. For example, in general, for a p-form ω, we have

$$\int_M d\omega = \int_{\partial M} \omega, \tag{A26}$$

where M is a $(p+1)$-dimensional manifold, and ∂M is the boundary of M. This is the **generalized Stokes theorem** that is valid in any dimension.

4. Old wine in new bottle

We now write everything in differential forms. For the Abelian case, the **connection 1-form** (for a particular band-n) is,

$$A = A_1 dk_1 + A_2 dk_2 + A_3 dk_3, \tag{A27}$$

where

$$A_\ell = i\langle u_n | \frac{\partial}{\partial k_\ell} | u_n \rangle. \tag{A28}$$

The **curvature 2-form** is

$$F = \frac{1}{2} F_{ij} dk_i \wedge dk_j, \tag{A29}$$

where

$$F_{ij} = \partial_i A_j - \partial_j A_i. \tag{A30}$$

It follows that,

$$F = dA. \tag{A31}$$

The first Chern number is,

$$C_1 = \frac{1}{2\pi} \int_M \text{tr } F. \tag{A32}$$

For the non-Abelian case, the connection 1-form is

$$A = \mathsf{A}_1 dk_1 + \mathsf{A}_2 dk_2 + \mathsf{A}_3 dk_3, \tag{A33}$$

where

$$(A_\ell)_{\alpha\beta} = i\langle u_{n\alpha} | \frac{\partial}{\partial k_\ell} | u_{n\beta} \rangle, \tag{A34}$$

where $\alpha, \beta = 1, 2, \cdots, N$, the number of energy levels considered. The curvature 2-form is

$$F = \frac{1}{2}\mathsf{F}_{ij} dk_i \wedge dk_j, \tag{A35}$$

where

$$\mathsf{F}_{ij} = (\partial_i \mathsf{A}_j - \partial_j \mathsf{A}_i) - i[\mathsf{A}_i, \mathsf{A}_j]. \tag{A36}$$

It follows that,

$$F = dA - iA \wedge A. \tag{A37}$$

Now,

$$\begin{aligned}
& \frac{1}{4}\epsilon_{ijkl} \operatorname{tr}(\mathsf{F}_{ij}\mathsf{F}_{kl})\, dk_1 \wedge dk_2 \wedge dk_3 \wedge dk_4 & \text{(A38)}\\
=\ & \frac{1}{4} \operatorname{tr}(\mathsf{F}_{ij}\mathsf{F}_{kl})\, dk_i \wedge dk_j \wedge dk_k \wedge dk_l & \text{(A39)}\\
=\ & \operatorname{tr}(F \wedge F). & \text{(A40)}
\end{aligned}$$

The **Chern-Simons identity** (see Eq. (**??**)) becomes,

$$\operatorname{tr}(F \wedge F) = d \operatorname{tr}\left(A \wedge dA - \frac{2}{3}iA \wedge A \wedge A\right). \tag{A41}$$

Or, $\operatorname{tr}(F \wedge F) = dK$, where K is called the **Chern-Simons form**,

$$K = \operatorname{tr}\left(A \wedge dA - \frac{2}{3}iA \wedge A \wedge A\right). \tag{A42}$$

In the new language, the derivation of the CS identity is much simpler. Finally, the second Chern number is,

$$\begin{aligned}
C_2 &= \frac{1}{8\pi^2}\int_M \operatorname{tr}(F \wedge F) & \text{(A43)}\\
&= \frac{1}{8\pi^2}\int_{\partial M} K. & \text{(A44)}
\end{aligned}$$

For higher Chern numbers, see App. C.

Connection form and curvature form can also be written in projection operators (Avron, 1995; Budich and Trauzettel, 2013). For example, for the Abelian case, given

$$P = |u_n\rangle\langle u_n|, \tag{A45}$$

and

$$\hat{A}_\ell \equiv |u_n\rangle A_\ell \langle u_n|, \tag{A46}$$

one has (Sec. 4 of Liu *et al.*, 2012),

$$\begin{aligned}
\hat{A} &= \hat{\mathsf{A}}_\ell dk_\ell & \text{(A47)}\\
&= -idPP = -i[dP, P], & \text{(A48)}
\end{aligned}$$

and

$$\begin{aligned}
\hat{F} &= \hat{\mathsf{F}}_{kl} dk_k \wedge dk_\ell & \text{(A49)}\\
&= iPdP \wedge dPP. & \text{(A50)}
\end{aligned}$$

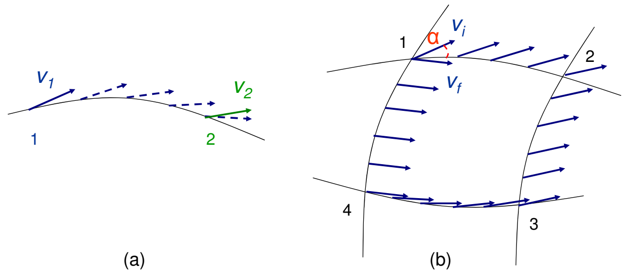

FIG. 1 (a) Parallel transport of a vector from 1 to 2. It offers a way to compare $\mathbf{v}_1$ and $\mathbf{v}_2$ on a curved surface. (b) A vector is parallel transported around a closed path. The final vector could point to a different direction from the initial vector.

The first Chern number is,

$$C_1 = \frac{1}{2\pi}\int \operatorname{tr}(iPdP \wedge dPP). \tag{A51}$$

For example, given

$$H = \mathbf{B} \cdot \boldsymbol{\sigma}, \tag{A52}$$

one has,

$$P_\pm = \frac{1 \pm \hat{\mathbf{B}} \cdot \boldsymbol{\sigma}}{2}, \tag{A53}$$

and

$$\operatorname{tr} \hat{F}(P_\pm) = \mp\frac{1}{2}\hat{\mathbf{B}} \cdot d\hat{\mathbf{B}} \times d\hat{\mathbf{B}}. \tag{A54}$$

APPENDIX B: Differential geometry

The discussion here is brief, and only a heuristic picture is provided. For more details, one can read, for example, Schutz, 1982, or Nakahara, 2003.

1. Parallel transport and holonomy angle

At each point p on a 2D surface, there is a vector space T_p formed by the tangent vectors at that point. For an ant living on a curved surface, how can it compare two vectors at different locations? One can follow the rule of **parallel transport** (Levi-Civita, 1917): Starting from point 1, the ant can carry the vector in such a way that it makes a fixed angle with the tangent vector along the path (see Fig. 1(a)), and compare it with the vector already at point 2.

For a closed loop on a curved surface, after the parallel transport, the final vector $\mathbf{v}_f$ could differ from the initial vector $\mathbf{v}_i$ (see Fig. 1b). The angle between this two vectors is called the **holonomy angle** (or **defect angle**). For example, for a sphere with radius r, the defect angle α_A for a vector transported around a **spherical triangle** with area A is equal to the solid angle of this triangle,

$$\alpha_A = \frac{A}{r^2}. \tag{B1}$$

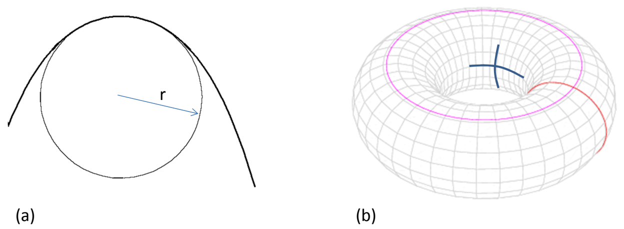

FIG. 2 (a) An osculating circle for a particular point on a curve. Similar concept can be extended to a 2D curved surface: a patch of the surface near a particular point can be approximated by a quadratic surface. (b) The inner side of a torus is similar to the surface of a saddle and has negative curvature. The outer side of the torus has positive curvature.

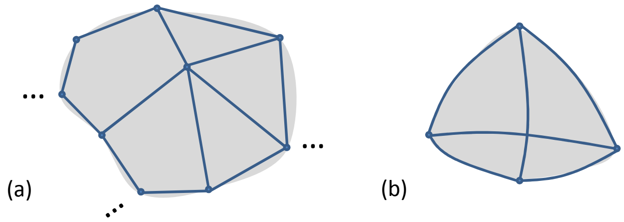

FIG. 3 (a) A surface is divided into cells. (b) Division of a sphere-like surface.

2. Intrinsic curvature and extrinsic curvature

One can define the **Gaussian curvature** at point p as the ratio between α_A and A for an infinitesimal loop around p,

$$K = \lim_{A\to 0} \frac{\alpha_A}{A}. \tag{B2}$$

According to this definition, the sphere has a constant Gaussian curvature $K = 1/r^2$ everywhere on the surface.

You can apply the same definition to find out the Gaussian curvature of a cylinder. The result would be zero. That is why we can cut it open and lay it down on top of a desk easily without stretching.

For a 3D creature, a cylindrical surface looks curved. This can be measured by the **mean curvature** (details below). That is, one needs to distinguish between bending and stretching/squeezing. The former could change the mean curvature (extrinsic), but only the latter could change the Gaussian curvature (intrinsic). A 2D creature could only feel the intrinsic curvature, but not the extrinsic curvature.

Consider a smooth 2D surface. Near a point p, the surface can be approximated by a quadratic surface. That is, there is no difference between this patch of surface and the quadratic surface up to the second order Taylor expansion of the coordinate (see Fig. 2(a)).

A quadratic surface must have two principal directions with maximum and minimum radii r_1, r_2. They correspond to two **principle curvatures** $k_1 = 1/r_1, k_2 = 1/r_2$ (up to a sign). The Gaussian curvature at p is the product of k_1, k_2,

$$K = \frac{1}{r_1}\frac{1}{r_2}. \tag{B3}$$

On the other hand, the mean curvature H is the sum of k_1, k_2,

$$H = \frac{1}{r_1} + \frac{1}{r_2}. \tag{B4}$$

For a cylinder, the radius r_1 for a straight line along a ridge is infinite, so $K = 0$, but $H \neq 0$.

On the inner side of a torus, the two principal directions curve toward opposite directions (like the surface of a saddle, see Fig. 2(b)). Therefore, the Gaussian curvature K is negative. On the outer side of the torus, K is positive.

3. Euler characteristic

First, consider a two-dimensional surface M (with or without boundary). Divide it into a patchwork of cells as in Fig. 3(a). Assume there are β_0 vertices (aka 0-simplexes), β_1 edges (1-simplexes), and β_2 faces (2-simplexes). Then the **Euler characteristic** of this patchwork is defined as,

$$\chi(M) = \beta_0 - \beta_1 + \beta_2. \tag{B5}$$

For example, for the sphere-like surface shown in Fig. 3(b), we have

$$\chi(S^2) = 4 - 6 + 4 = 2. \tag{B6}$$

This number does not depend on how the surface is being divided. Also, if M' is **homeomorphic** to (topologically the same as) M, then $\chi(M') = \chi(M)$ (**Poincaré-Alexander theorem**). Thus, $\chi(M)$ reveals the topology of M.

As an exercise, verify that the Euler characteristics of a disk and a torus are 1 and 0 respectively. F a general closed 2D surface M,

$$\chi(M) = 2(1-g), \tag{B7}$$

where g is the number of holes in the surface. For example, for a torus, $g = 1$ and $\chi = 0$; for a sphere, $g = 0$ and $\chi = 2$.

In general, for a surface M with dimension D, we can divide it into a patchwork of cells, and define

$$\chi(M) = \sum_{k=0}^{D} (-1)^k \beta_k, \tag{B8}$$

where β_k is the number of k-**simplexes**.

4. Gauss-Bonnet theorem

Gauss-Bonnet theorem tells us that the total Gaussian curvature of a *closed* 2D surface M is $2\pi\chi$ (see **??**),

$$\frac{1}{2\pi}\int_M d^2a\ K = \chi(M). \tag{B9}$$

That is, the total Gaussian curvature is a topological invariant.

As we have mentioned, if the radius of a sphere is r, then $K = 1/r^2$. In this case, the Gauss-Bonnet theorem is trivially satisfied. What is amazing is that no matter how you squeeze and stretch the sphere to redistribute the K's, the total curvature is always 4π.

There are many ways to calculate the Gaussian curvature. So far we have mentioned two in Eq. (B2) and Eq. (B3). The third way is as follows (Huang, 1978): Suppose there is a small area A covering a point p on the surface. The unit normal vector of A draws out another area G_A on the surface of a unit sphere S (known as the **Gauss map**). The Gaussian curvature can be defined as,

$$K = \lim_{A\to 0}\frac{G_A}{A}. \tag{B10}$$

From this definition, it is easy to see that the total curvature of a sphere-like object is 4π.

For reference, for a 2D surface with a boundary that is sectionally smooth (with corners), the Gauss-Bonnet theorem is generalized as,

$$\int_M d^2a\ K + \int_{\partial M} d\ell\ \kappa_g + \sum_i(\pi - \theta_i) = 2\pi\chi(M), \tag{B11}$$

in which κ_g is the **geodesic curvature** that measures the deviation from a geodesic curve, and the third term is a sum of exterior angles at the corners. The generalization of the Gauss-Bonnet theorem to higher dimension can be found in App. C.5.

5. Hopf-Poincaré theorem

Given a **zero** (such as a **source** or a **drain**) of a vector field, one can define an index according to the pattern of the surrounding flow. First consider a flow on a 2D surface: For an ant walking clockwise around the zero once, if the vectors of the flow on the ant's path rotate clockwise n-times, then the index is n. If they rotate counter-clockwise n-times, then the index is $-n$. For example, for the top-left figure in Fig. 4, the index is 1; for the top-right figure in Fig. 4, the index is -1.

On a D-dimensional manifold, the circular path surrounding the zero is replaced by a sphere S^{D-1}. The vectors on the sphere would trace out another sphere S'^{D-1}, and the index is given by the winding number of S'^{D-1} over S^{D-1}.

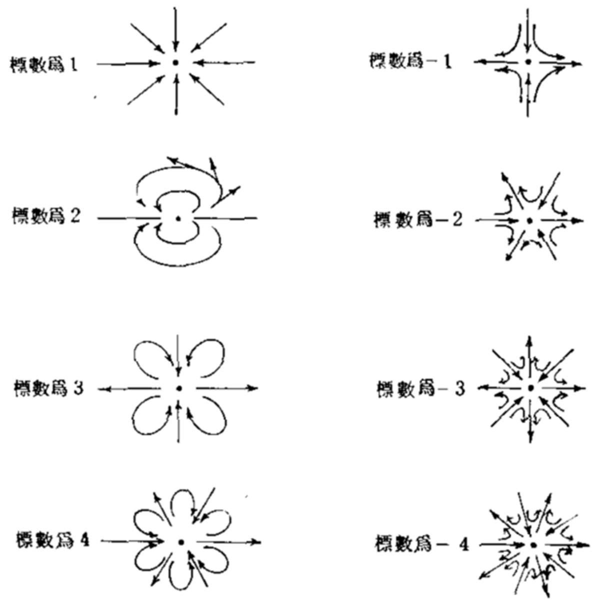

FIG. 4 Several examples of the index of a singularity in a 2D vector field. Note that if the arrows in the top-left figure are reversed, the index is still +1 (this is so only in even dimension). The figure is from Huang, 1978.

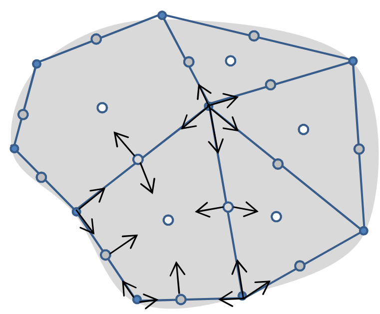

FIG. 5 Assign a flow to the surface divided by cells. Dark, grey, and white dots are sources, saddle points, and sinks.

Hopf-Poincaré theorem (1927) states that, for a flow distributed on a *closed* manifold M, the total index of the vector field is equal to the Euler characteristic of M. That is,

$$\sum_i \mathrm{ind}(\mathbf{v}_i) = \chi(M). \tag{B12}$$

For a 2D surface, this theorem can be proved as follows. This proof is based on Lect 12 of Tadashi's youtube lecture on Topology and Geometry:
First, divide the surface into cells with β_0 vertices, β_1 edges, and β_2 faces (see Fig. 5). Second, place a source at each vertex, a saddle point at the middle of each edge, and a sink in the middle of each face. We then have a continuous flow filling the surface. Since the indices of a source, a saddle point, and a sink are $+1, -1, +1$ respec-

tively. We have

$$\sum_i \mathrm{ind}(\mathbf{v}_i) = (+1)\beta_0 + (-1)\beta_1 + (+1)\beta_2, \tag{B13}$$

which is exactly the Euler characteristic defined in Eq. (B5). End of proof.

For example, the total index of a vector field on the surface of a S^2 is 2, according to this theorem. In Fig. 6, we show 3 possible flow patterns, and their total indices are indeed all equal to 2. You may try to see if it's possible to find a vector field that breaks this rule. In general,

$$\chi(S^n) = 1 + (-1)^n. \tag{B14}$$

Whenever $\chi(M) = 0$, the surface M is *parallelizable* (i.e., could be "combed"). Therefore, S^1 and S^3 are parallelizable.

The Hopf-Poincaré theorem on S^2 is sometimes called the **hairy ball theorem**: it's impossible to have a hairy ball free of any vortex (assuming the hair lies on the surface, of course). An alternative scenario is that, for the flow of wind on the surface of the earth, there must exist at least one location that has no wind at all.

Since the Euler characteristic of T^2 is zero, it's possible to have a smooth flow on T^2 without any vortex. For example, a flow with all vectors point to the azimuthal direction. On the other hand, if there is a vortex with index 1 somewhere on the surface of a torus, then there must be another vortex with index -1. In general, for any dimension n,

$$\chi(T^n) = 0. \tag{B15}$$

Such a fact is related to the **Nielsen-Ninomiya theorem**, aka **fermion-doubling theorem**: massless lattice fermions always have to come up in pairs. This is valid in any *odd* spatial dimension, and neither TR nor SI symmetry needs be presumed. The Weyl point of a fermion is a source or a drain of the Berry flux. It is the zero of the vector field of Berry connection. The Berry index (topological charge) of this nodal point can be identified as the index of the zero.

Finally, based on the Hopf-Poincaré theorem, one can deduce that the Euler characteristic $\chi(M)$ of any (closed) odd-dimensional manifold M is zero: After reversing the direction of the vector field, $\mathbf{v} \to -\mathbf{v}$, one has $\mathrm{ind}(\mathbf{v}_i) \to -\mathrm{ind}(\mathbf{v}_i)$ for each i (in *odd* dimension!). Thus $\sum_i \mathrm{ind}(\mathbf{v}_i) = 0$ (see p.39 of Milnor, 1965).

6. Curvature in higher dimension

The curvature of a surface in higher dimension can also defined from the holonomy of a vector transported around a closed loop. Before that, it helps to introduce the covariant derivative.

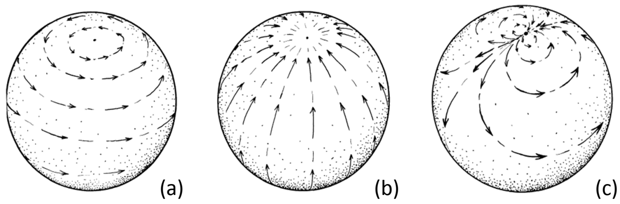

FIG. 6 Three examples of possible flows on S^2 surface. According to the Hopf-Poincaré theorem, the total index of the vector field must be 2.

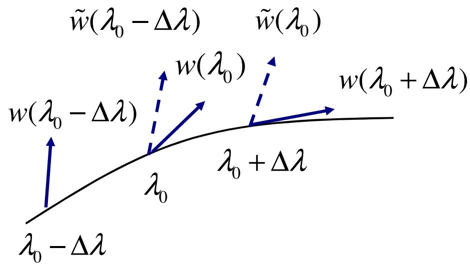

FIG. 7 The vector field along a path is indicated by solid-line vectors. Parallel transport of the vector at $\lambda_0 - \Delta\lambda$ are indicated by dotted-line vectors.

a. Covariant derivative

Suppose there is a vector field on a D-dimensional surface, $\mathbf{w}(\mathbf{r}) = w^i(\mathbf{r})\hat{e}_i(\mathbf{r})$, $\hat{e}_i$ are coordinate bases of tangent vectors ($i = 1, \cdots, D$). A path on the surface is parametrized by λ. The **covariant derivative** of a vector $\mathbf{w}$ along the tangent vector, $\mathbf{u} = u^i\hat{e}_i = (dx^i/d\lambda)\hat{e}_i$, of the path is defined as,

$$\nabla_{\mathbf{u}}\mathbf{w}(\lambda_0) = \lim_{\Delta\lambda\to 0} \frac{\mathbf{w}(\lambda_0) - \tilde{\mathbf{w}}(\lambda_0 - \Delta\lambda)}{\Delta\lambda}, \tag{B16}$$

in which $\tilde{\mathbf{w}}(\lambda_0 - \Delta\lambda)$ is $\mathbf{w}(\lambda_0 - \Delta\lambda)$ being parallel transported to the location λ_0 (see Fig. 7). The covariant derivative has the following properties,

$$\begin{aligned} \nabla_{\mathbf{u}}(f\mathbf{w}) &= f\nabla_{\mathbf{u}}\mathbf{w} + \mathbf{w}\nabla_{\mathbf{u}}f, && \text{(B17)} \\ \nabla_{f\mathbf{u}+g\mathbf{v}}\mathbf{w} &= f\nabla_{\mathbf{u}}\mathbf{w} + g\nabla_{\mathbf{v}}\mathbf{w}, && \text{(B18)} \end{aligned}$$

in which $\nabla_{\mathbf{u}}f = df/d\lambda$.

Based on these properties, suppose we know how the bases move,

$$\nabla_i\hat{e}_j = \Gamma^k_{ji}\hat{e}_k, \quad i, j, k = 1, 2, \cdots, D, \tag{B19}$$

where ∇_i is the shorthand notation of $\nabla_{\hat{e}_i}$, then

$$\begin{aligned} \nabla_{\mathbf{u}}\mathbf{w} &= u^i\nabla_i(w^j\hat{e}_j) && \text{(B20)} \\ &= u^i(\nabla_i w^j)\hat{e}_j + \Gamma^k_{ji}u^i w^j\hat{e}_k && \text{(B21)} \\ &= \left(\frac{dw^k}{d\lambda} + \Gamma^k_{ji}w^j u^i\right)\hat{e}_k, && \text{(B22)} \end{aligned}$$

in which we have used $u^i\nabla_i w^k = dw^k/d\lambda$. The coefficient Γ^k_{ji} is called the **Christoffel symbol**.

A vector $\mathbf{w}$ is said to be parallel transported along the path if

$$\nabla_{\mathbf{u}}\mathbf{w} = 0. \tag{B23}$$

After parallel transport, the vector would differ by

$$\delta w^k = -\Gamma^k_{ji} w^j u^i d\lambda. \tag{B24}$$

A **geodesic** is a curve that parallel transports its own tangent vector,

$$\nabla_{\mathbf{u}}\mathbf{u} = 0. \tag{B25}$$

This leads to the **geodesic equation**,

$$\frac{du^i}{d\lambda} + \Gamma^i_{jk} u^j u^k = 0, \tag{B26}$$

$$\text{or} \quad \frac{d^2x^i}{d\lambda^2} + \Gamma^i_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} = 0. \tag{B27}$$

Its solution gives the shortest path between two points on the surface.

b. Riemann curvature

One can parallel transport a vector $\mathbf{w}$ around an infinitesimal loop, first along u^k by $d\lambda$ (step 1), then along v^l by $d\mu$ (step 2), then take two more steps to close the path. The returning vector would differ from the initial vector. We wish to find out this difference (Schutz, 1982).

First, from the definition of the covariant derivative, we have

$$\mathbf{w}(\lambda_0) = \tilde{\mathbf{w}}(\lambda_0 - d\lambda) + d\lambda\nabla_{\mathbf{u}}\mathbf{w}. \tag{B28}$$

Thus, after step 1, $\mathbf{w}(\mathbf{r}) \to \tilde{\mathbf{w}}(\mathbf{r} + d\boldsymbol{\lambda})$ with $d\lambda^i = u^i d\lambda$, we have

$$\tilde{\mathbf{w}}(\mathbf{r}+d\boldsymbol{\lambda}) \simeq \mathbf{w}(\mathbf{r}) + d\lambda\nabla_{\mathbf{u}}\mathbf{w}(\mathbf{r}) \tag{B29}$$

$$\simeq e^{d\lambda\nabla_{\mathbf{u}}}\mathbf{w}(\mathbf{r}). \tag{B30}$$

For the second step, $\tilde{\mathbf{w}}(\mathbf{r}+d\boldsymbol{\lambda}) \to \tilde{\mathbf{w}}(\mathbf{r}+d\boldsymbol{\lambda}+d\boldsymbol{\mu})$, and

$$\tilde{\mathbf{w}}(\mathbf{r}+d\boldsymbol{\lambda}+d\boldsymbol{\mu}) \simeq e^{d\mu\nabla_{\mathbf{v}}}e^{d\lambda\nabla_{\mathbf{u}}}\mathbf{w}(\mathbf{r}). \tag{B31}$$

After four steps, the returning vector differs from the initial one by $\delta\mathbf{w}$, and

$$\delta\mathbf{w} \simeq e^{d\mu\nabla_{\mathbf{v}}}e^{d\lambda\nabla_{\mathbf{u}}}\mathbf{w} - e^{d\lambda\nabla_{\mathbf{u}}}e^{d\mu\nabla_{\mathbf{v}}}\mathbf{w} \tag{B32}$$

$$\simeq -d\lambda d\mu[\nabla_{\mathbf{u}}, \nabla_{\mathbf{v}}]\mathbf{w} \tag{B33}$$

$$= -d\lambda d\mu R(\mathbf{u},\mathbf{v},\mathbf{w}), \tag{B34}$$

where (here we use coordinate bases in a manifold without torsion)

$$R(\mathbf{u},\mathbf{v},\mathbf{w}) \equiv [\nabla_{\mathbf{u}}, \nabla_{\mathbf{v}}]\mathbf{w}. \tag{B35}$$

It can be shown that,

$$R(f\mathbf{u}, g\mathbf{v}, h\mathbf{w}) = fghR(\mathbf{u},\mathbf{v},\mathbf{w}). \tag{B36}$$

Thus,

$$R(\mathbf{u},\mathbf{v},\mathbf{w}) = R(\hat{e}_k,\hat{e}_l,\hat{e}_j)u^k v^l w^j. \tag{B37}$$

The **Riemann curvature tensor** is defined as ($\hat{e}^i \cdot \hat{e}_j = \delta^i_j$),

$$R^i_{jkl} = \hat{e}^i \cdot R(\hat{e}_k,\hat{e}_l,\hat{e}_j) \tag{B38}$$

$$= \hat{e}^i \cdot [\nabla_k, \nabla_l]\hat{e}_j. \tag{B39}$$

We now have,

$$\delta w^i = -R^i_{jkl} w^j u^k v^l d\lambda d\mu. \tag{B40}$$

It is left as an exercise to show that,

$$R^i_{jkl} = \partial_k\Gamma^i_{jl} - \partial_l\Gamma^i_{jk} + \Gamma^i_{hk}\Gamma^h_{jl} - \Gamma^i_{hl}\Gamma^h_{jk}. \tag{B41}$$

One can see Nakahara, 2003 for more information.

The Riemann tensor changes sign when one exchanges i, j, or exchanges k, l. It is not changed if one changes the indices $ijkl$ to $klij$. Also, since

$$[\nabla_i,[\nabla_j,\nabla_k]]+[\nabla_j,[\nabla_k,\nabla_i]]+[\nabla_k,[\nabla_i,\nabla_j]] = 0, \tag{B42}$$

we have the (first) **Bianchi identity**,

$$R^l_{ijk} + R^l_{kij} + R^l_{jki} = 0. \tag{B43}$$

Because of these restrictions, in $D = 2$, there is only one independent component. It is nothing but the Gaussian curvature,

$$R_{1212} = gK, \tag{B44}$$

where $R_{ijkl} = g_{ih}R^h_{jkl}$, and $g = \det(g_{ij})$, determinant of metric tensor. In general, in D dimension, there are $D^2(D^2-1)/12$ components. For example, for $D = 3, 4$, there are 6 and 20 independent components respectively.

The connection and curvature discussed here resemble closely with the connection and curvature in the theory of geometric phase. Recall that the Berry connection is defined as,

$$(\mathsf{A}_k)_{\alpha\beta} = i\langle\alpha|\partial_k|\beta\rangle. \tag{B45}$$

whereas the Christoffel symbol is defined as (see Eq. (B19)),

$$\Gamma^i_{jk} = \hat{e}^i \cdot \nabla_k\hat{e}_j, \tag{B46}$$

We can rewrite the Christoffel symbol and the Riemann tensor as matrices,

$$(\mathsf{\Gamma}_k)^i_j \equiv \Gamma^i_{jk}, \tag{B47}$$

and

$$(\mathsf{R}_{kl})^i_j \equiv R_{ijkl}. \tag{B48}$$

The matrix $\mathsf{\Gamma}_k$ plays the role of the Berry connection A_k. Eq. (B41) can be rewritten as,

$$\mathsf{R}_{kl} = \partial_k\mathsf{\Gamma}_l - \partial_l\mathsf{\Gamma}_k + [\mathsf{\Gamma}_k, \mathsf{\Gamma}_l]. \tag{B49}$$

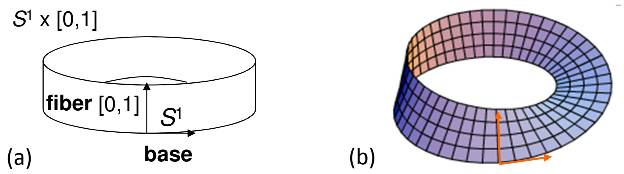

FIG. 8 (a) A product space is an example of a trivial bundle. (b) The simplest example of a non-trivial fiber bundle: locally it is a product space, but globally it is not, because of the twist.

This looks almost the same as the expression of the non-Abelian Berry curvature in Eq. (A36). Such a resemblance is first noticed by C.N. Yang when he was teaching a course on general relativity.

From the connection and the Riemann tensor, we can have the **connection 1-form** and **curvature 2-form**,

$$\Gamma \equiv \Gamma_k dx^k, \tag{B50}$$

$$\Omega \equiv \frac{1}{2}\mathsf{R}_{kl} dx^k \wedge dx^l. \tag{B51}$$

The $D \times D$ matrix Ω is antisymmetric, and each matrix element Ω^i_j is a 2-form, $\Omega^i_j = \frac{1}{2} R^i_{jkl} dx^k \wedge dx^l$. Finally, Eq. (B49) can be written as

$$\Omega = d\Gamma + \Gamma \wedge \Gamma, \tag{B52}$$

which is analogous to Eq. (A37).

APPENDIX C: Fiber bundle and characteristic class

1. Fiber bundle

Due to the resemblance between Berry curvature and Riemann curvature, the theories of holonomy angle and Berry phase can be put within the same mathematical framework: the theory of **fiber bundle**. A fiber bundle is composed of a **base space** and a **fiber space**. Locally it is product of the two spaces, but globally it may not be (see Fig. 8), due to some topological twist.

In the case of the holonomy angle, the base space is the 2D surface. The fiber is a space of tangent vectors at each point of the base space. When one parallel transports a vector around a closed loop C in the base space, the returning vector could rotate to a different direction. That is, the curvature results in a shift of coordinate along a fiber (see Fig. 9). The global topology of this fiber bundle is classified by the Euler characteristic.

In the case of the Berry phase, the base space is the parameter space (e.g., the BZ). The fiber is the U(1) phase factor (or U(2), or other non-Abelian groups) of the quantum state under consideration. When one moves a state $|\psi_{\boldsymbol{\lambda}}\rangle$ around a closed loop C in the base space with the **parallel transport condition**,

$$\langle\psi_{\boldsymbol{\lambda}}|\frac{\partial}{\partial\boldsymbol{\lambda}}|\psi_{\boldsymbol{\lambda}}\rangle = 0, \tag{C1}$$

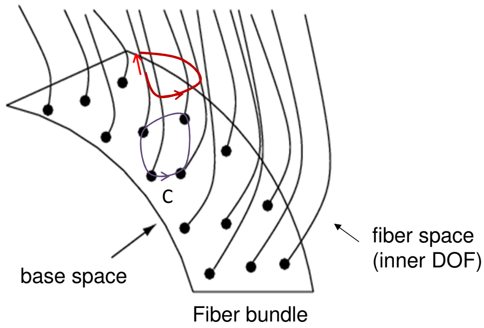

FIG. 9 Visualization of a fiber bundle. Locally it is a product of the **base space** and the **fiber space**. When one returns from a closed path C in the base space, there may be a shift in the fiber coordinate (Fig. from wiki).

the returning state acquires a Berry phase. That is, the Berry curvature results in a shift of coordinate along the U(1) fiber. In this case, the global topology of the fiber bundle is classified by the first Chern number.

Obviously, higher dimensional base space, and more complicated Lie groups can be the components of a fiber bundle. In the following, we consider some Lie groups that are of interest to physicists: U(N), O(N), and SO(N). The topology of their fiber bundles are related to the Chern class (1946), the Pontryagin class (1947), and the Euler class (Whitney, 1927), respectively. For simplicity, we do not consider base manifold with boundary.

For a nice and short account of related topics, see Chern, 1979. For more details, see Eguchi *et al.*, 1980, Nakahara, 2003.

2. Connection and curvature in fiber bundle

3. Chern class

First we show an identity: If a matrix A can be diagonalized by similarity transformation,

$$\mathsf{SAS}^{-1} = \text{diag}(x_1, x_2, \cdots, x_N), \tag{C2}$$

then since the determinant and the trace are invariant under such a transformation, we have

$$\begin{aligned} & \det(1+\mathsf{A}) \\ =\; & \det[\text{diag}(1+x_1, 1+x_2, \cdots, 1+x_N)] \\ =\; & \prod_{j=1}^{N}(1+x_j) \\ =\; & 1 + (x_1 + \cdots + x_N) + (x_1x_2 + \cdots + x_{N-1}x_N) \\ +\; & \cdots + (x_1x_2\cdots x_N) \\ =\; & 1 + \text{tr } \mathsf{A} + \frac{1}{2}\left[(\text{tr } \mathsf{A})^2 - \text{tr } \mathsf{A}^2\right] + \cdots + \det \mathsf{A}. \end{aligned} \tag{C3}$$

In ??, we have already defined the curvature 2-form F. They correspond to U(1) (Eq. (A29)) and U(N)

(Eq. (A35)) gauge groups. The **Chern classes** $c_\ell(F)$ are generated from expanding the determinant,

$$\det\left(1+\frac{i}{2\pi}F\right) = 1 + c_1(F) + c_2(F) + \cdots, \tag{C4}$$

Using the identity above, we have

$$\begin{aligned}
c_0 &= 1, && \text{(C5)}\\
c_1 &= \frac{i}{2\pi}\mathrm{tr}F, && \text{(C6)}\\
c_2 &= \left(\frac{i}{2\pi}\right)^2 \frac{1}{2!}\left[(\mathrm{tr}F)^2 - \mathrm{tr}(F^2)\right], && \text{(C7)}\\
c_3 &= \left(\frac{i}{2\pi}\right)^3 \frac{1}{3!}\left[(\mathrm{tr}F)^3 - 3(\mathrm{tr}F)^2\mathrm{tr}F + 2\mathrm{tr}(F^3)\right], && \text{(C8)}\\
&\vdots\\
c_N &= \left(\frac{i}{2\pi}\right)^N \det F, && \text{(C9)}
\end{aligned}$$

where $F^2 = F\wedge F, (\mathrm{tr}\ F)^2 = \mathrm{tr}\ F\wedge \mathrm{tr}\ F\cdots$.

In general,

$$c_\ell = \frac{1}{\ell!}\left(\frac{i}{2\pi}\right)^\ell \epsilon^{i_1 i_2 \cdot i_\ell}_{j_1 j_2\cdots j_\ell} F_{i_1 j_1}\wedge\cdots F_{i_\ell j_\ell}, \tag{C10}$$

where each subscript runs from 1 to N, and $\epsilon^{i_1 i_2 \cdot i_\ell}_{j_1 j_2\cdots j_\ell} = +1$ if $(i_1, i_2, \cdots, i_\ell)$ differ from $(j_1, j_2, \cdots, j_\ell)$ by an even permutation, -1 if they differ by an odd permutation.

Higher Chern classes $c_\ell = 0$ if $2\ell > D$, where D is the dimension of the base manifold M. Therefore, the expansion could terminate early. For example, if $D = 2$, then only c_1 is non-zero. If $D = 4$, then only c_1 and c_2 can be non-zero.

The integral of the Chern class gives the **Chern number**,

$$\int_M c_\ell = C_\ell, \tag{C11}$$

which is an integer. For example, In $D = 2$, we have the first Chern number,

$$C_1 = \frac{i}{2\pi}\int_M \mathrm{tr}\ F. \tag{C12}$$

In $D = 4$, we have two independent Chern numbers,

$$\begin{aligned}
C_2 &= \int_M c_2 = \frac{1}{8\pi^2}\int_M \left[\mathrm{tr}(F^2) - (\mathrm{tr}\ F)^2\right], && \text{(C13)}\\
C_1^2 &= \int_M c_1\wedge c_1 = -\frac{1}{4\pi^2}\int_M (\mathrm{tr}\ F)^2. && \text{(C14)}
\end{aligned}$$

There is no Chern number if the dimension of the base manifold is odd.

If the gauge group is SU(N), then $c_1 = 0$. As a result, $C_1^2 = 0$, and

$$C_2 = \frac{1}{8\pi^2}\int_M \mathrm{tr}(F^2). \tag{C15}$$

Let's introduce two mathematical terms before moving on: We say that a differential form
A is **closed** if $dA = 0$;
A is **exact** if $A = dB$.
What is special about the Chern classes is that (see Eguchi *et al.*, 1980 for proofs),
1. They are invariant under a gauge transformation,

$$c_\ell\left(gFg^{-1}\right) = c_\ell(F). \tag{C16}$$

By the way, this implies that c_ℓ is a closed form,

$$dc_\ell = 0. \tag{C17}$$

2. One can have two sets of rules for the parallel transport, so that there are two connections A, A', and two curvatures F, F'. But the Chern classes would only differ by an exact form dG,

$$c_\ell(F') = c_\ell(F) + dG. \tag{C18}$$

Therefore, after integration, they would yield the same Chern number,

$$\int_M c_\ell(F') = \int_M c_\ell(F) + \cancelto{0}{\int_M dG}, \tag{C19}$$

in which $\int_M dG = \int_{\partial M} G = 0$ since a closed manifold M has no boundary.

That is, the Chern numbers are independent of the choice of either the *gauge*, or the *connection*. Furthermore, the Chern numbers are integers, invariant under a *continuous deformation* of the manifold. Thus, they reveal the topology of the fiber bundle.

In mathematical jargon, the set of k-forms that are closed, and differ from each other only by exact forms, form a **de Rham cohomology group** $H^k(M)$,

$$H^k(M) = \frac{\{\text{closed } k \text{ forms on } M\}}{\{\text{exact } k \text{ forms on } M\}}. \tag{C20}$$

Thus, the Chern classes are group elements of $H^{2\ell}(M)$.

4. Pontryagin class

Pontryagin class to real vector bundle is like Chern class to complex vector bundle. Since the generators of the Lie algebra of O(N) are real antisymmetric matrices, the curvature Ω of a fiber bundle with O(N) fiber is real and anti-symmetric. It cannot be diagonalized, but can be put in the block-diagonal form,

$$\mathsf{S}\Omega\mathsf{S}^{-1} = \begin{pmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & 0 & x_2 & \\ & & -x_2 & 0 & \\ & & & & \ddots \end{pmatrix}. \tag{C21}$$

If N is odd, then the last diagonal element is set to 0, and the determinant of the matrix vanishes.

If imaginary numbers are allowed, then the matrix can be diagonalized,

$$\mathsf{S}F\mathsf{S}^{-1} = \begin{pmatrix} -ix_1 & 0 & & & \\ 0 & ix_1 & & & \\ & & -ix_2 & 0 & \\ & & 0 & ix_2 & \\ & & & & \ddots \end{pmatrix}. \tag{C22}$$

We now let $N = D = 2d$, the dimension of the base manifold. The **Pontryagin classes** are generated from the expansion,

$$\det\left(1 + \frac{\Omega}{2\pi}\right) = 1 + p_1(\Omega) + p_2(\Omega) + \cdots. \tag{C23}$$

Since

$$\det(1 + \frac{\Omega}{2\pi}) = \det(1 + \frac{\Omega^T}{2\pi}) = \det(1 - \frac{\Omega}{2\pi}), \tag{C24}$$

each term in the expansion is an even function of Ω. As a result, the first non-zero class p_1 is a 4-form, p_2 is a 8-form ... etc,

$$p_1(\Omega) = -\frac{1}{8\pi^2}\mathrm{tr}(\Omega^2) = -\frac{1}{8\pi^2}\Omega_{ij}\Omega_{ji}, \tag{C25}$$

$$p_2(\Omega) = \frac{1}{128\pi^4}\left[\left(\mathrm{tr}\ \Omega^2\right)^2 - 2\mathrm{tr}(\Omega^4)\right]. \tag{C26}$$

The highest Pontryagin class is,

$$p_d(\Omega) = \det\left(\frac{\Omega}{2\pi}\right). \tag{C27}$$

The integral of the Pontryagin class gives the **Pontryagin number**,

$$\int_M p_k = P_k, \tag{C28}$$

which is an integer topological invariant. It vanishes if D is not divisible by 4.

The Pontryagin classes can be written in terms of the Chern classes. Recall that,

$$\det\left(1 + \frac{i}{2\pi}F\right) = 1 + c_1(F) + c_2(F) + \cdots, \tag{C29}$$

If we relate Ω with iF (need to *complexify* the real vector bundle) (Nakahara, 2003), then

$$p_k(\Omega) = (-1)^k c_{2k}(\Omega). \tag{C30}$$

5. Euler class

The Pontryagin classes are for the gauge group $\mathrm{O}(D)$. If the gauge group is $\mathrm{SO}(D)$ and the base manifold is *orientable*, then in addition to the Pontryagin classes, there is also the Euler class. If the base manifold is not orientable (such as the Klein bottle), then the sign of the Euler class $e(\Omega)$ is ambiguous, and $e(\Omega)$ is not a topological invariant.

The square of the **Euler class** is the highest Pontryagin class,

$$e^2(\Omega) = \det\left(\frac{\Omega}{2\pi}\right). \tag{C31}$$

Recall that for an antisymmetric matrix M, one can define its **pfaffian**, which is the square root of $\det \mathsf{M}$,

$$(\mathrm{pf}\ \mathsf{M})^2 = \det\ \mathsf{M}. \tag{C32}$$

If D, the dimension of M, is odd, then $\mathrm{pf}\ \mathsf{M} = 0$ (and $e(\Omega) = 0$) since $\det \mathsf{M}$=0.

If $D = 2d$, then the Euler class is,

$$e(\Omega) = \mathrm{pf}\left(\frac{\Omega}{2\pi}\right) \tag{C33}$$

$$= \frac{1}{2^d d!}\frac{1}{(2\pi)^d}\epsilon^{i_1 i_2 \cdots i_{2d}}\Omega_{i_1 i_2}\cdots\Omega_{i_{2d-1}i_{2d}}. \tag{C34}$$

For example,

$$D = 2: \quad e(\Omega) = \frac{1}{2\pi}\Omega_{12}, \tag{C35}$$

$$D = 4: \quad e(\Omega) = \frac{1}{32\pi^2}\epsilon^{ijkl}\Omega_{ij}\Omega_{kl}. \tag{C36}$$

The integral of the Euler class is the **Euler characteristic**,

$$\int_M e(\Omega) = \chi(M). \tag{C37}$$

This is the generalization of the 2D Gauss-Bonnet theorem to higher dimensions. It applies to a *closed and orientable* manifold M in *even* dimension, The Euler characteristic of an odd-dimensional surface is zero (see App. B.5), so it's not an interesting topological invariant therein.

Unlike $\mathrm{O}(N) \to \mathrm{SO}(N)$, there is no additional class when one restricts $\mathrm{U}(N)$ to $\mathrm{SU}(N)$. The Chern classes still fully account for the topology in the $\mathrm{SU}(N)$ case (Eguchi *et al.*, 1980). For $\mathrm{SU}(N)$, one has $c_1 = 0$ automatically, but higher Chern classes can be nonzero.

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