Image reconstruction by two-dimensional PCA

Abstract

Two-dimensional principal component analysis (2DPCA) is based on 2D image matrices rather than 1D image vectors. In this report, I first review the original 2DPCA algorithm which works in the row direction of the images. Then the 2DPCA technique is further extended to work in the column direction and simultaneously work in both directions of the face images. To evaluate the performance of the three types of 2DPCA, a series of experiments were performed on the Yale Face Databases. The 2DPCA working simultaneously in the row and the column directions of the face image outcompetes the other two in terms of the feature extraction and image reconstruction. In addition, the one working in the row direction is better than the column direction.

1 Introduction

Principal component analysis (PCA) is a widely used technique for feature extraction, data representation, and pattern recognition. In the classical PCA, the 2D image matrices (i.e. $m\times n$) must be initially converted to 1D vectors ($\mathbf{R}^{\mathrm{mn}}$), usually resulting in a high dimensional image vector space. Consequently, there are difficulties to evaluate the covariance matrix ($mn\times mn$) accurately when the matrix has large size and the number of the training samples is relatively small. Furthermore, computing the eigenvectors of a large size covariance matrix is time-consuming. Yang et al. (2004) developed the two-dimensional PCA (2DPCA) approach, which is based on 2D matrices rather than 1D vectors and directly computes eigenvectors of the so-called image covariance matrix ($n\times n$). Because the size of the image covariance matrix of 2DPCA is significantly smaller than that of the corresponding covariance matrix of PCA, 2DPCA improves the accuracy and efficiency when evaluating the image covariance matrix and computing the corresponding eigenvectors, respectively.

In this project, the idea of the 2DPCA method and its algorithm will be reviewed in great detail, followed by the description of the 2DPCA-based image reconstruction. Furthermore, some experiments on a set of face images taken from the Yale Face Database (http://vision.ucsd.edu/content/yale-face-database) will be performed. Finally, some conclusions will be drawn, and future work will be proposed.

2 Two-dimensional principal component analysis

2.1 Idea and algorithm

Consider image $\mathbf{A}$, an $m\times n$ random matrix. Let $\mathbf{X}\in\mathbf{R}^{\mathrm{n}\times\mathrm{d}}$ be a matrix with orthonormal columns, $n\geq d$. The projection of $\mathbf{A}$ onto $\mathbf{X}$ yields an m by d matrix $\mathbf{Y}=\mathbf{AX}$. The total scatter of the projected samples was used as a measure of the goodness of the projection vector $\mathbf{X}$. From this point of view, the following criterion is adopted:

$$\begin{aligned}\mathrm{J}(\mathbf{X}) &= \mathrm{trace}\{\mathrm{E}[(\mathbf{Y}-\mathrm{E}\mathbf{Y})(\mathbf{Y}-\mathrm{E}\mathbf{Y})^{\mathrm{T}}]\}\\ &= \mathrm{trace}\{\mathrm{E}\big[\big(\mathbf{AX}-\mathrm{E}(\mathbf{AX})\big)(\mathbf{AX}-\mathrm{E}(\mathbf{AX}))^{\mathrm{T}}\big]\}\\ &= \mathrm{trace}\{\mathbf{X}^{\mathrm{T}}\mathrm{E}[(\mathbf{A}-\mathrm{E}\mathbf{A})^{\mathrm{T}}(\mathbf{A}-\mathrm{E}\mathbf{A})]\mathbf{X}\}. \qquad (1)\end{aligned}$$

Define the matrix $\mathbf{G}=\mathrm{E}[(\mathbf{A}-\mathrm{E}\mathbf{A})^{\mathrm{T}}(\mathbf{A}-\mathrm{E}\mathbf{A})]$ as the image covariance matrix. $\mathbf{G}$ is an n by n nonnegative definite matrix. Suppose that there are M training images, denoted by m by n matrices $\mathbf{A}_{\mathrm{j}}$ (j = 1, 2, …, M), and the average image is denoted as

$$\overline{\boldsymbol{A}}=\frac{1}{M}\sum_{k=1}^{M}\mathbf{A}_k. \qquad (2)$$

Then, **G** can be evaluated by

$$\mathbf{G} = \frac{1}{\mathrm{M}}\sum_{\mathrm{k}=1}^{\mathrm{M}}(\mathbf{A}_{\mathrm{k}} - \overline{\mathbf{A}})^{\mathrm{T}}(\mathbf{A}_{\mathrm{k}} - \overline{\mathbf{A}}). \quad (3)$$

The criteria (1) can be expressed by the quadratic form as

$$\mathrm{J}(\mathbf{X}) = \mathbf{X}^{\mathrm{T}}\mathbf{G}\mathbf{X}, \quad (4)$$

and we have obtained the following problem

$$\max_{\mathrm{x}:||\mathrm{x}||=1} \mathbf{X}^{\mathrm{T}}\mathbf{G}\mathbf{X}. \quad (5)$$

By applying Rayleigh quotient theorem, the optimal direction for projecting the data is the largest eigenvector of the image covariance matrix. In practice, it is not enough to have only one optimal projection direction but a set of projection axes, $\mathbf{X}_1, \dots, \mathbf{X}_d$ which are the orthonormal eigenvectors of **G** corresponding to the first *d* largest eigenvalues. Because **G** has a small size of *n* by *n*, computing its eigenvectors is very efficient. Additionally, like in classical PCA the value of *d* can be set by the following threshold:

$$\frac{\sum_{i=1}^{d}\lambda_i}{\sum_{i=1}^{n}\lambda_i} \geq p, \quad (6)$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the *n* largest eigenvalues of **G** and p is a pre-set threshold, typically p=0.95, or 0.99.

2.2 Alternative 2DPCA

Let $\mathbf{A}_k = [(\mathbf{A}_k^{(1)})^{\mathrm{T}}(\mathbf{A}_k^{(2)})^{\mathrm{T}} \dots (\mathbf{A}_k^{(m)})^{\mathrm{T}}]$ and $\overline{\mathbf{A}} = [(\overline{\mathbf{A}}^{(1)})^{\mathrm{T}}(\overline{\mathbf{A}}^{(2)})^{\mathrm{T}}\dots(\overline{\mathrm{A}}^{(\mathrm{m})})^{\mathrm{T}}]$, where $\mathbf{A}_k^{(i)}$ and $\overline{\mathbf{A}}^{(\mathrm{i})}$ are the i^{th} row vector of $\mathbf{A}_k$ and $\overline{\mathbf{A}}$, respectively (Zhang and Zhou, 2005). Then Eq. (3) can be rewritten as

$$\mathbf{G} = \frac{1}{\mathrm{M}}\sum_{\mathrm{k}=1}^{\mathrm{M}}\sum_{\mathrm{i}=1}^{\mathrm{m}}(\mathbf{A}_{\mathrm{k}}^{(\mathrm{i})} - \overline{\mathbf{A}}^{(\mathrm{i})})^{\mathrm{T}}(\mathbf{A}_{\mathrm{k}}^{(\mathrm{i})} - \overline{\mathbf{A}}^{(\mathrm{i})}). \quad (7)$$

Eq. (7) reveals that the image covariance matrix **G** can be computed as the outer product of row vectors of the images. Therefore, the original 2DPCA can be looked as working in the row direction of the images.

Another way to construct **G** is to use the outer product between column vectors of the images. Let $\mathbf{A}_k = [(\mathbf{A}_k^{(1)})(\mathbf{A}_k^{(2)})\dots(\mathbf{A}_k^{(n)})]$ and $\overline{\boldsymbol{A}} = [(\overline{\boldsymbol{A}}^{(1)})^{T}(\overline{\boldsymbol{A}}^{(2)})^{T}\dots(\overline{\boldsymbol{A}}^{(n)})]$, where $\mathbf{A}_k^{(j)}$ and $\overline{\boldsymbol{A}}^{(j)}$ are the j^{th} column vector of $\mathbf{A}_k$ and $\overline{\mathbf{A}}$, respectively. Then an alternative definition of the image covariance matrix **G** is:

$$\mathbf{G} = \frac{1}{\mathrm{M}}\sum_{\mathrm{k}=1}^{\mathrm{M}}\sum_{\mathrm{j}=1}^{\mathrm{n}}(\mathbf{A}_{\mathrm{k}}^{(\mathrm{j})} - \overline{\mathbf{A}}^{(\mathrm{j})})(\mathbf{A}_{\mathrm{k}}^{(\mathrm{j})} - \overline{\mathbf{A}}^{(\mathrm{j})})^{\mathrm{T}}. \quad (8)$$

Similarly, the optimal projection axis can be obtained by computing the eigenvectors $\mathbf{Z}_1, \dots, \mathbf{Z}_q$ which are the orthonormal eigenvectors of Eq. (8) corresponding to the first *q* largest eigenvalues. The parameter *q* can also be set by a threshold as shown in Eq. (6). Because the eigenvectors of Eq. (8) only reflect the information between columns of the images, this alternative 2DPCA can be looked as working in the column direction of the images.

2.3 Feature extraction

The optimal projection vectors of 2DPCA working in the row direction of the images, $\mathbf{X}_1, \dots, \mathbf{X}_d$, are used for feature extraction. For a given image **A**, let

$$\mathbf{Y}_{\mathrm{k}} = \mathbf{A}\mathbf{X}_{\mathrm{k}}, \mathrm{k} = 1, 2, \dots, \mathrm{d}, \quad (9)$$

where $\mathbf{Y}_i$ is called the i^{th} principal component of the sample image **A**. The principal component vectors are used to form an *m* by *d* matrix $\mathbf{B} = [\mathbf{Y}_1, \dots, \mathbf{Y}_d]$, which is called the feature matrix or feature image of the image sample **A**.

Similarly, we can obtain the feature matrix of the 2DPCA working in the column direction of the images as $[\mathbf{Z}_1^T\mathbf{A}, \ldots, \mathbf{Z}_q^T\mathbf{A}]$. Another way to construct the feature matrix is to sign each feature direction as the sum of row feature and column feature with half of the fraction each as $[1/2(\mathbf{A}\mathbf{X}_1+\mathbf{Z}_1^T\mathbf{A}), \ldots, 1/2(\mathbf{A}\mathbf{X}_p+\mathbf{Z}_p^T\mathbf{A})]$.

3 2DPCA-based image reconstruction

The principal components and eigenvectors can be combined to reconstruct the original sample image. For a given image $\mathbf{A}$, the reconstructed image is:

$$\widetilde{\mathbf{A}} = \sum_{k=1}^{d} \mathbf{A}\mathbf{X}_k\mathbf{X}_k^T \,, \qquad (10)$$

$$\widetilde{\mathbf{A}} = \sum_{k=1}^{q} (\mathbf{A}^T\mathbf{Z}_k\mathbf{Z}_k^T)^T, \qquad (11) \text{ and}$$

$$\widetilde{\mathbf{A}} = \sum_{k=1}^{p} \frac{1}{2}(\mathbf{A}\mathbf{X}_k\mathbf{X}_k^T + (\mathbf{A}^T\mathbf{Z}_k\mathbf{Z}_k^T)^T), \qquad (12)$$

for the 2DPCA which are working in the row, column, and row-column combined directions, respectively. That is, image $\mathbf{A}$ can be approximated by adding up the first d, q, and p subimages for the three directions, respectively. The corresponding reconstruction error can be computed as $\frac{||\mathbf{A}-\widetilde{\mathbf{A}}||_F}{||\mathbf{A}||_F}$.

4 Experiment and analysis

The 2DPCA method was used for face reconstruction on the Yale database, which contains 165 images of 15 subjects. And each person has 11 different images with varied facial expressions and illumination and each image has the size of *243* by *320*. In this report, the 11 different images of the subject 15 are chosen as the sample images (Figure 1) and the reconstruction work is performed on the 3rd image (expression and illumination are both normal) of subject 15 (Figure 2). The implementation of the 2DPCA algorithm and the following analysis and visualization work were all performed by the python jupyter notebook. In particular, numpy (https://numpy.org/) was used for matrix operation and matplotlib (https://matplotlib.org/) was used for visualization.

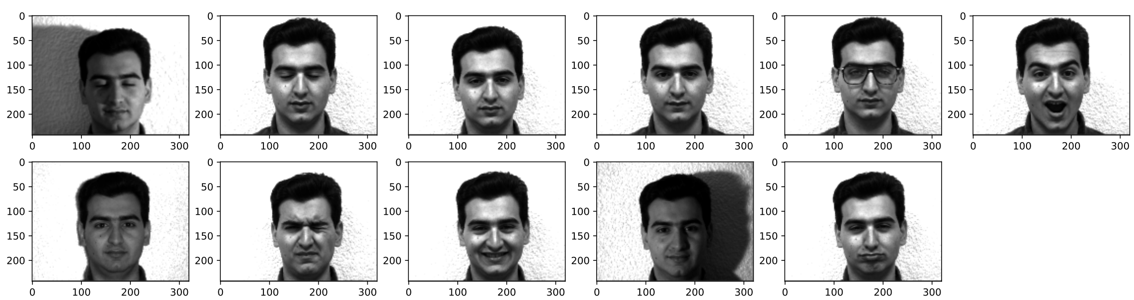

Figure 1. Sample images of subject 15 of the Yale databases.

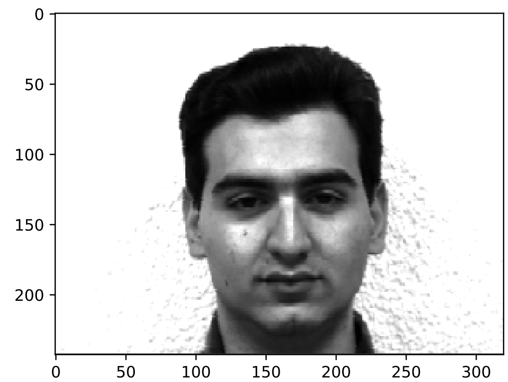

Figure 2. The image used for the reconstruction analysis.

The 2DPCA algorithm was first used for feature extraction in row and column directions. The image covariance matrices have the size of *320* by *320*, and *243* by 2*43*, respectively. It was easy to compute the eigenvectors. Figure 3 shows that the magnitude of the eigenvalues quickly converges to zero. In order to preserve 95% of the total variance, we just need the first 16 and 15 largest eigenvalues of the row and column directions, respectively (Figure 4).

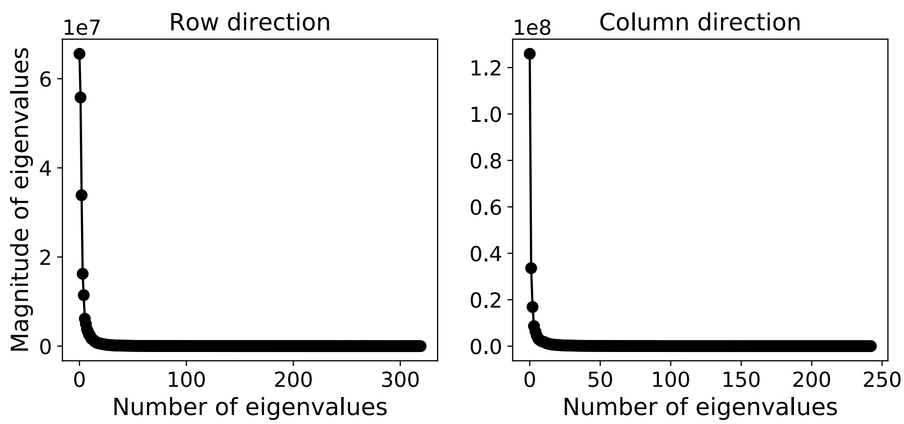

Figure 3. The plot of the magnitude of the eigenvalues in decreasing order of the 2DPCA working in the row (left) and column (right) directions.

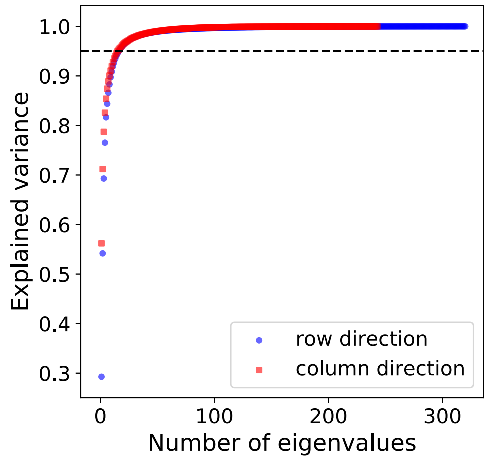

Figure 4. The explained variance by the number of eigenvalues of the 2DPCA working in the row (blue circle) and column (red square) directions.

The eigenvectors corresponding to 20 largest eigenvalues, $\mathbf{X}_1, \ldots, \mathbf{X}_{20}$, as the projection axes for the 2DPCA working in the row direction and $\mathbf{Z}_1, \ldots, \mathbf{Z}_{20}$, as the projection axes for the 2DPCA working in the column direction. I then projected the image sample onto these axes, and obtained twenty principal components for each method, $\mathbf{AX}_1, \ldots, \mathbf{AX}_{20}$, and $\mathbf{Z}_1^{\mathrm{T}}\mathbf{A}, \ldots, \mathbf{Z}_{20}^{\mathrm{T}}\mathbf{A}$. Taking the image in Figure 2 as an example, I reconstructed its 60 subimages as $\widetilde{\mathbf{A}_{\mathrm{k}}} = \mathbf{AX}_{\mathrm{k}}\mathbf{X}_{\mathrm{k}}^{\mathrm{T}}$, $\widetilde{\mathbf{A}_{\mathrm{k}}} = \mathbf{A}^{\mathrm{T}}\mathbf{Z}_{\mathrm{k}}\mathbf{Z}_{\mathrm{k}}^{\mathrm{T}}$, and $\widetilde{\mathbf{A}_{\mathrm{k}}} = \frac{1}{2}(\mathbf{AX}_{\mathrm{k}}\mathbf{X}_{\mathrm{k}}^{\mathrm{T}} + (\mathbf{A}^{\mathrm{T}}\mathbf{Z}_{\mathrm{k}}\mathbf{Z}_{\mathrm{k}}^{\mathrm{T}})^{\mathrm{T}})$, $\mathrm{k} = 1, 2, \ldots, 20$, and 20 for each. Some of these reconstructed subimages are shown in grayscale in Figure 5. The first subimage contains most of the information of the original image while the other ones show some detailed local information with different levels. Comparing the 2DPCA in row and column directions, the first subimage of column direction seems contain more energy than that of the row direction, which is exactly consistent with the results of Figure 4. In the figure, more than 50% of total variance can be explained by the first eigenvalue in column direction 2DPCA while only 30% of the total variance explained by the first eigenvalue in row direction 2DPCA. On the other hand, the row-column combination seems work the best in terms of displaying the global and local feature of the original image.

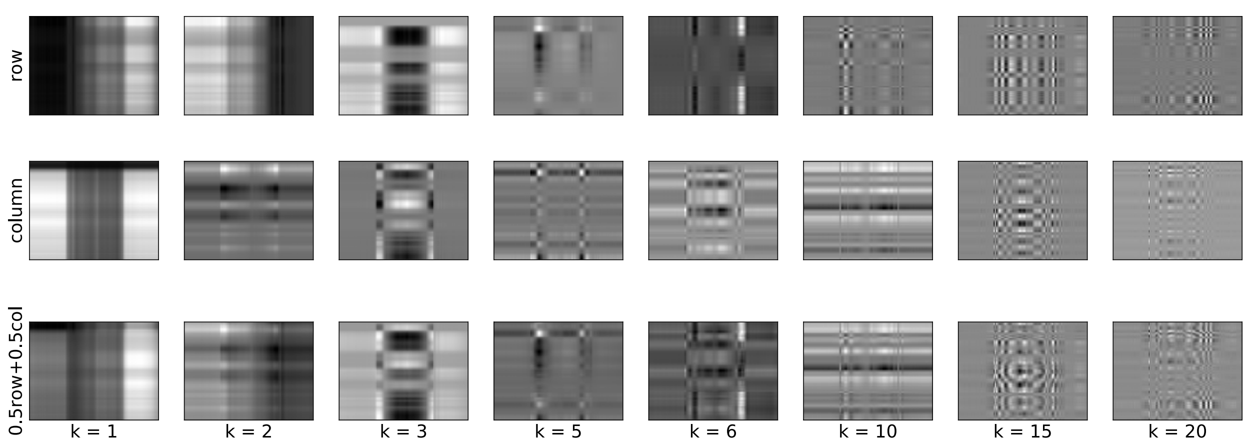

Figure 5. Some reconstructed subimages by row, column, and row-column directions in the top, middle, and bottom panels, respectively.

Further, an approximate reconstruction of the original image can be obtained by adding up the first d (d = 1, 2, 3, 5, 6, 10, 15, 20) subimages together. Figure 6 shows 24 reconstructed images of the face image in Figure 2 for row, column, and row-column directions and 8 for each direction. The reconstructed images become clearer as the number of subimages increases for all three methods. By adding up the first 10 subimages together, the face image has already been reasonably good, especially in the row direction of the 2DPCA. With adding up the first 20 subimages, the images are reconstructed with high resolution, showing that some detail face features are captured.

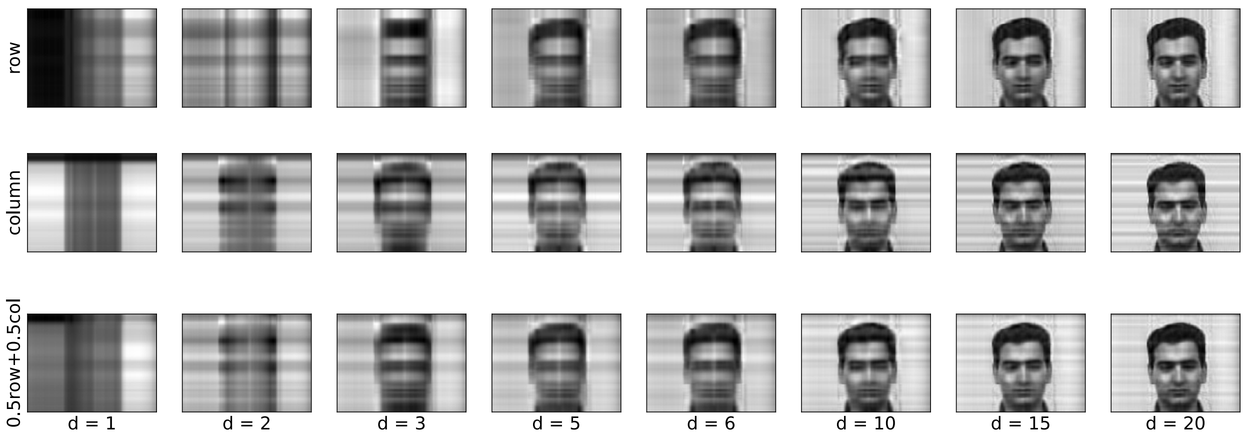

Figure 6. Some reconstructed images by row, column, and row-column directions in the top, middle, and bottom panels, respectively.

In order to evaluate the performance of 2DPCA, the reconstruction error was investigated, and the results are shown in Figure 7. The reconstruction error of the three 2DPCA methods all decreases as the number of principal components or the number of subimages cumulated increases. The error of the 2DPCA decreases much faster at the beginning of the experiment when working in the row direction in contrast with the column direction even though its error is greater when $d \leq 2$. At the end of the experiment (d = 20), the error of the row and column directions is around 10% and 18%, respectively. The error of the row-column combine direction 2DPCA shows similar trend with that of the row direction 2DPCA. And the overall error of the combine method is generally lowest among the three during the course of the experiment. At d = 20, the error of combine method converges to the error of row direction method.

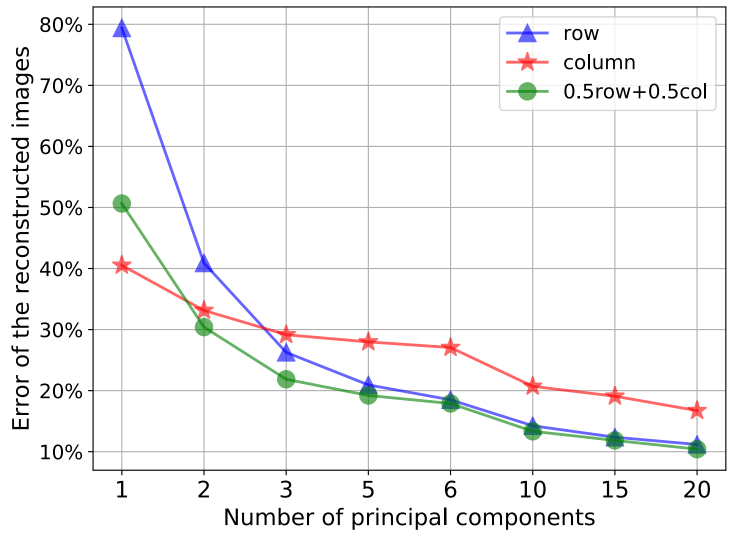

Figure 7. Reconstruction error as percentages of the 2DPCA working in the row (blue triangle), column (red star), and row-column (green circle) directions.

5 Conclusion and future work

In this report, the idea and algorithm of the 2DPCA has been reviewed and its relation to the row direction of face image has been discussed. Two alternative 2DPCA was investigated and one works in the column direction of the face image and the other works in the row-column combine direction. Both the feature extraction and image reconstruction by 2DPCA in column direction outperform 2DPCA in row direction when only considering the first one or two principal components. As we select more principal components for the face image reconstruction with acceptable resolution, the 2DPCA in row direction outcompete the one working in the column direction. Nonetheless, 2DPCA working simultaneously in the row and the column directions of the face images is the best with respect to the row and column only.

This report only investigated the feature extraction and image reconstruction of one image of subject 5 which was annotated as “normal” facial expression with right illumination. Ten other face images of subject 5 with different facial expressions and different level of the illumination can be used to further study the performance of the three types of 2DPCA.

References

Yang, J., Zhang, D., Frangi, A. F., and Yang, J., Two-dimensional PCA: a new approach to appearance-based face representation and recognition, IEEE Trans. Pattern Ana. March. Intell. 26 (1), 2004, 131-137.

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