# Linear vs. nonlinear modeling of black hole ringdowns

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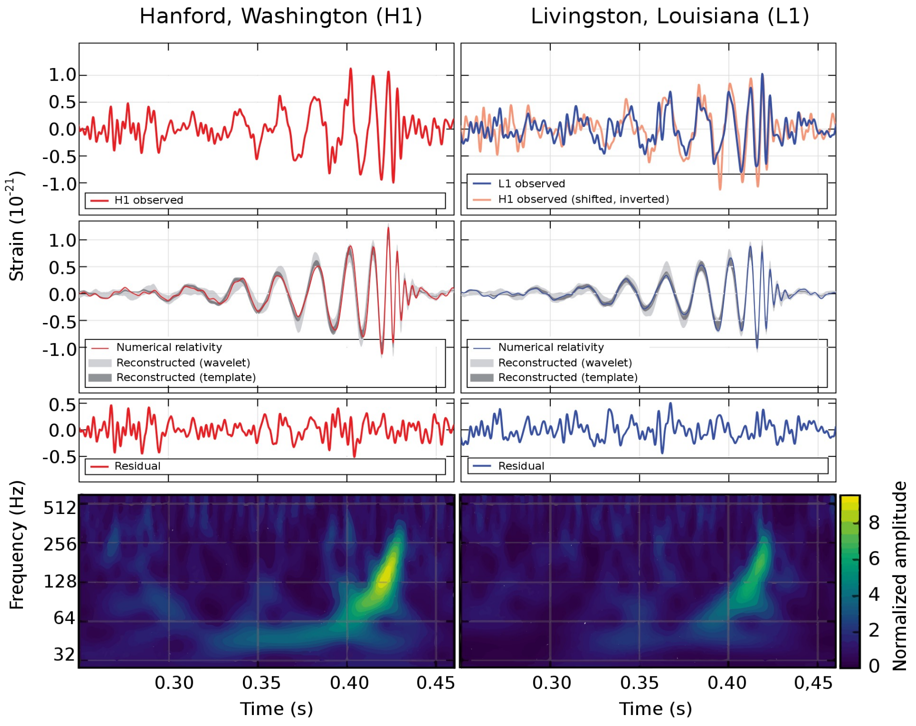
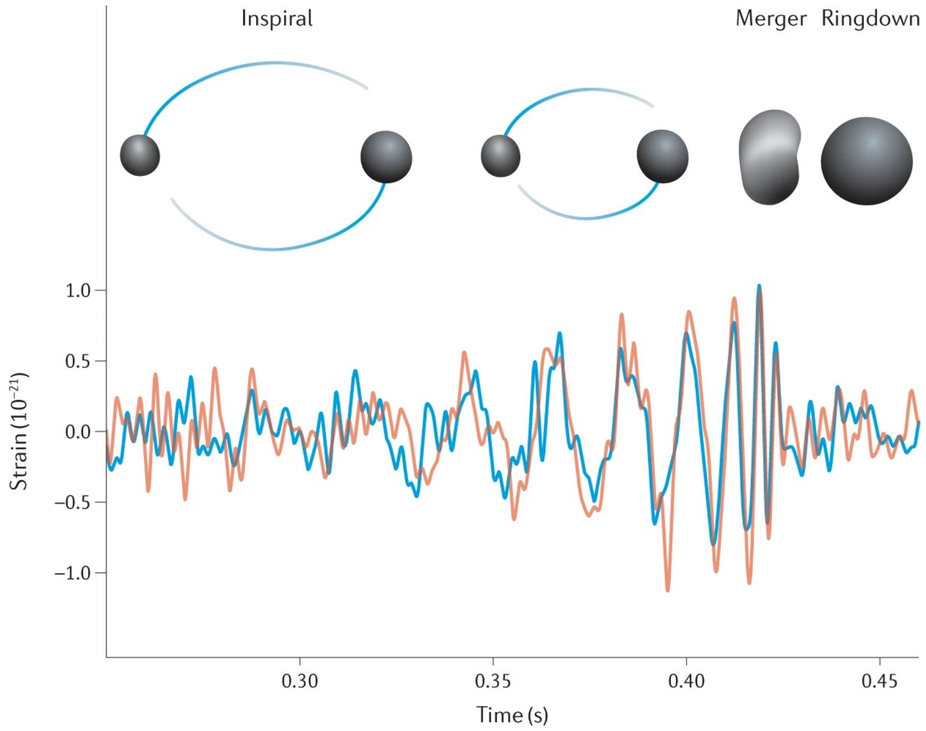
# Why ringdown test?

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- Ringdown quasi-normal-mode (QNM) frequencies uniquely determined by the final BH's mass and spin as predicted by the no-hair theorem in general relativity (GR).
- Testing no-hair theorem:
  - Blackhole spectroscopy
  - Inspiral-merger-ringdown (IMR) consistency test

# Linear vs. nonlinear

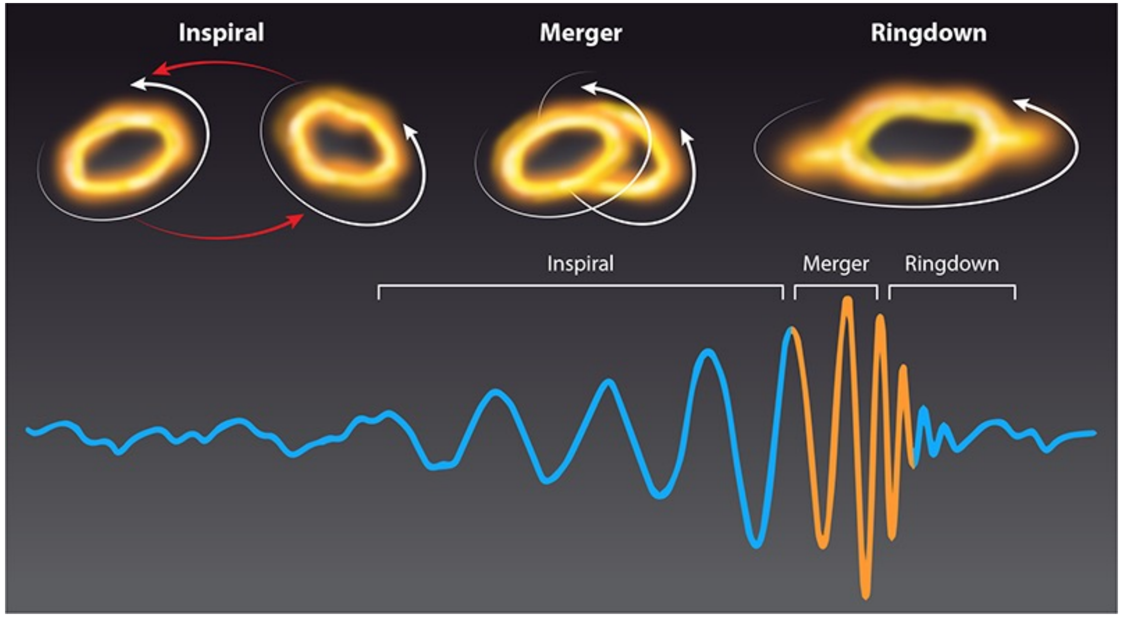



Image credit: Swetha Bhagwat, APS/articles/v16/29 (2023).

Nonlinearity in the merger phase.

- Linear
  - Linear perturbation applied from the peak strain onward.
  - Nonlinear effects observed at the merger regime become quickly irrelevant.
  - Includes a large number of tones, typically N=7 overtone model.
- Issues in "linear" models
  - Instability of the high overtone models -- overfitting.
  - Mode mixing, prompt emission, late tail effects, etc.
  - Un-modelled nonlinearity on the dominant (l, m) = (2, 2) mode.

# Linear models

- QNM overtone models
  - Teukolsky equation describes linear perturbations off a Kerr background spacetime.
  - Its solution is the countably infinite set of the complex QNMs of the final (Kerr) BH.

The complex strain at future null infinity:

$$h(t,\theta,\phi) = \sum_{l,m} h_{lm}(t)\ {}_{-2}\mathcal{Y}_{lm}(\theta,\phi;a_f),$$

${}_{-2}\mathcal{Y}_{lm}(\theta,\phi;a_f)$ are the spin-weighted spheroidal harmonics of spin weight $s=-2$. Thus, $h_{22}(t)$ for a given number $N \geq 0$ of overtones defines the linear overtone model:

$$\boxed{\mathrm{OM}_N(t) = \sum_{n=0}^{N} \mathcal{A}_n e^{-\imath(t-t_r)\omega_{22n}}.}$$

where $t_r$ is a reference time.

# Nonlinear models & non-GR model

- IMR models
  - Calibrated to NR waveforms, which depend solely on the progenitors' parameters.
  - After calibration, required fewer input parameters than the overtone models
- We use non-precessing IMRPhenomD waveform model for this work.
- 4 real parameters: $q$, $\chi_1^z$, $\chi_2^z$, and phase.

- Toy models
  - Time coordinate transformation models (TCTMs).
  - Nonlinear modification at early times.
  - Asymptotically recovering the linear model at later times.

$$\mathrm{TCTM}_N(t) \equiv \mathrm{OM}_N\left(t + Ae^{-t/\tau}\right)$$

- $A$ and $\tau$ are two parameters to be determined.
- $2N + 6$ real parameters: the final mass and spin, $A, \tau$, and $N + 1$ complex tone amplitudes.

- Non-GR models
  - Highest-tone perturbation models (HTPMs).
  - Deviation to the QNM spectrum.
  - Popular in spectroscopic studies (N=1).

$$\mathrm{HTPM}_N(t) \equiv \mathrm{OM}_{N-1}(t) + \mathcal{A}_N e^{-\iota(t-t_r)w_{22N}(1+\alpha_N)} e^{-(t-t_r)/\left(\tau_{22N}(1+\beta_N)\right)}$$

- $\alpha_N$ and $\beta_N$ are the oscillation frequency and damping time perturbation parameters
- $2N + 6$ real parameters: final mass, spin, $\alpha_N$ , $\beta_N$, and $N + 1$ complex tone amplitudes.

# Model comparison

- Mismatch (typically used in comparisons to NR data):

$$\mathcal{M} = 1 - \frac{\langle h_{\rm NR} \mid h_m \rangle}{\sqrt{\langle h_{\rm NR} \mid h_{\rm NR} \rangle \langle h_m \mid h_m \rangle}},$$

  - We compare between our model for the strain 22 mode, $h_m(t)$, and the corresponding numerical data for the strain 22 mode, $h_{\rm NR}(t)$, with

$$\langle f \mid g \rangle = \mathrm{Re} \int_{t_0}^{t_f} f^*(t) g(t) \mathrm{d}t.$$

  - $t_0$ and $t_f$ mark the fit starting and ending time, respectively. $\mathcal{M}$ varies between 0 and 1.

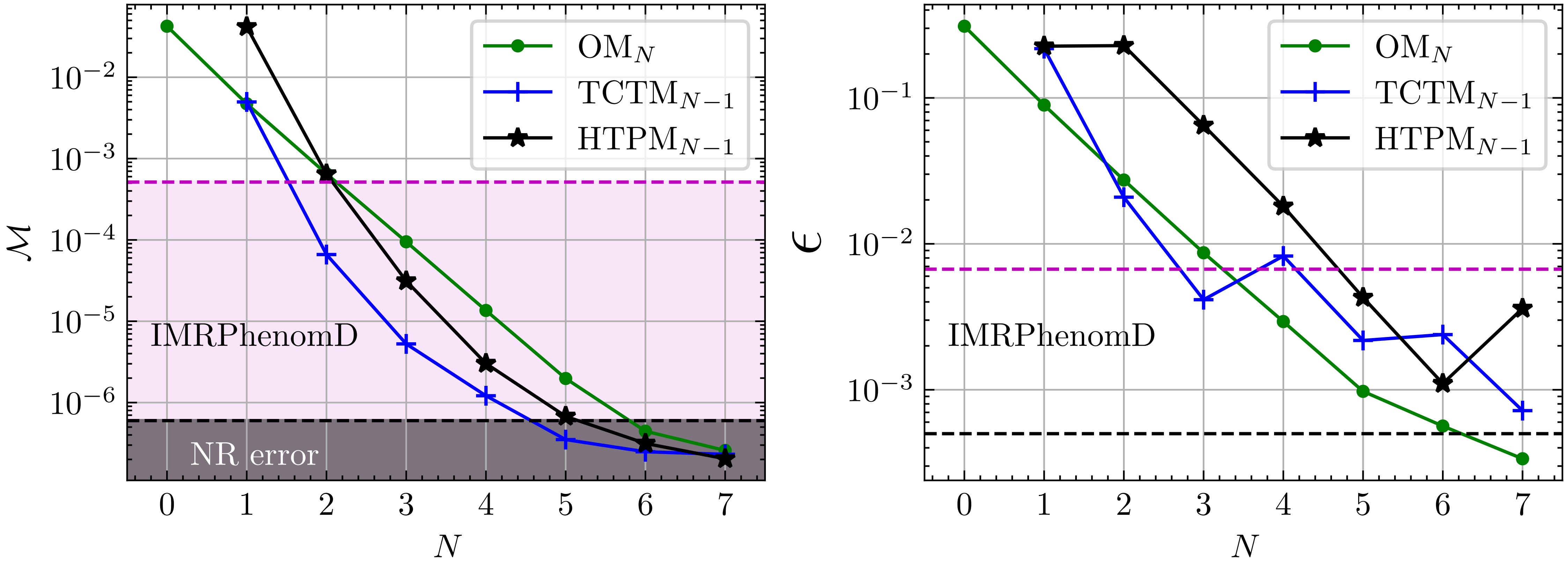
- Bias of recovery of remnant BH's mass and spin (dimensionless):

$$\epsilon = \sqrt{\left(\frac{M_f^{\rm fit} - M_f^{\rm true}}{M}\right)^2 + \left(a_f^{\rm fit} - a_f^{\rm true}\right)^2}.$$

- Posterior probablity for model $M$ with parameters $\vec{\theta}$, given the data $d$ (here $d = h_{\rm NR}$ ),

$$p(\vec{\theta} \mid d, M) = \frac{p(\vec{\theta} \mid M) p(d \mid \vec{\theta}, M)}{p(d \mid M)}.$$

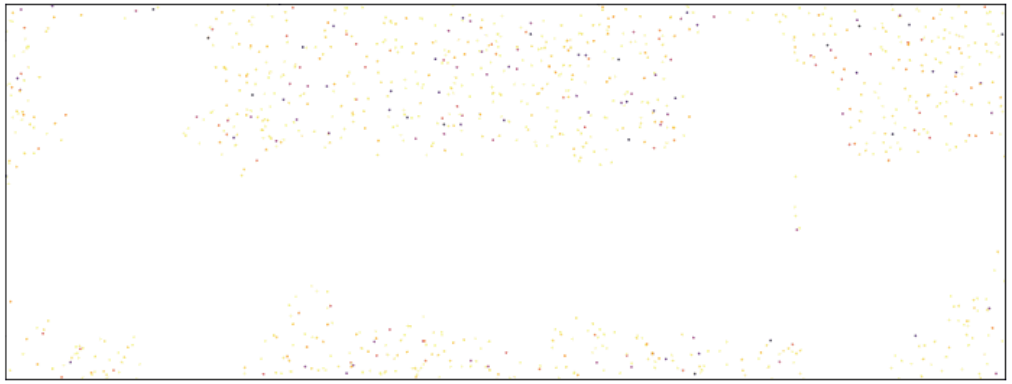
# Preliminary tests




- Grid method on mass and spin ($+A/\tau$ or $\alpha_N/\beta_N$) --- linear fitting implemented at each grid point, then find minimum mismatch using bisection.

# Parameter estimation

- Sampling using *dynesty* package.
- Likelihood function in time domain.
- Assuming constant noise spectrum in relevant frequency range. Consequently, an optimal SNR $\rho$ is set as well.



$$p(d \mid \vec{\theta}, M) = \exp\left[-\rho^2 \frac{\langle d(t) - m(\vec{\theta}, t) | d(t) - m(\vec{\theta}, t) \rangle}{2 \langle d(t) | d(t) \rangle}\right].$$

where $m(\vec{\theta}, t)$ denotes the strain 22 mode from model $M$ with particular parameters $\vec{\theta}$, evaluated at time $t$, and $d(t)$ is the time-domain NR strain (zero noise realization) for the 22 mode.

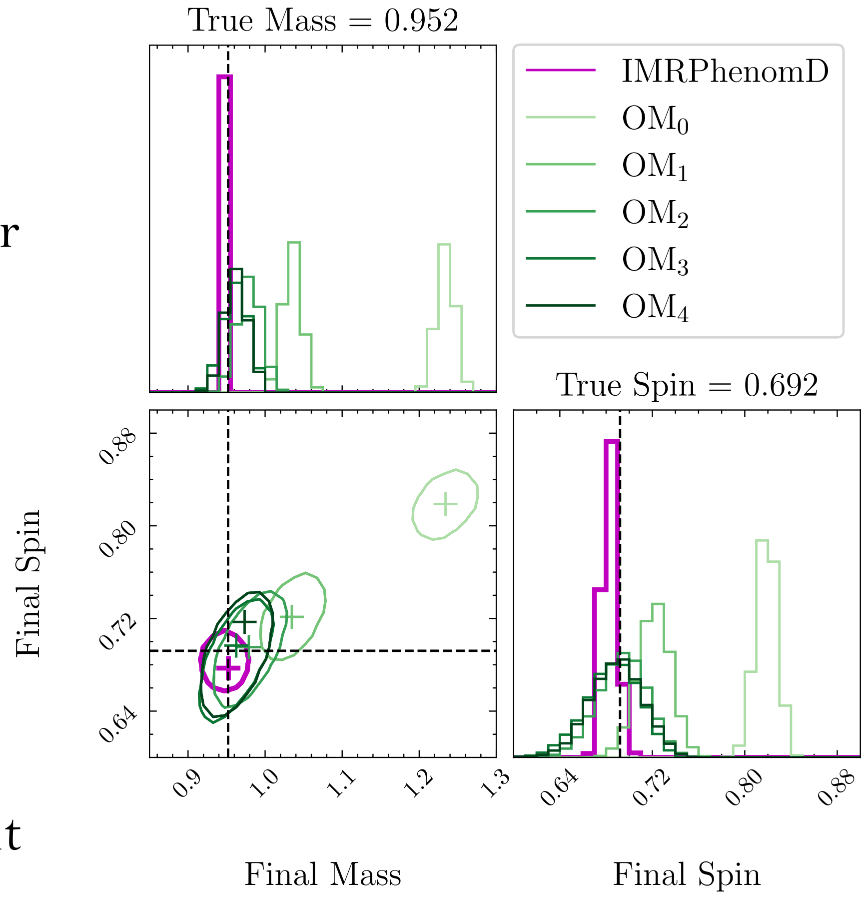
# Bayesian analysis

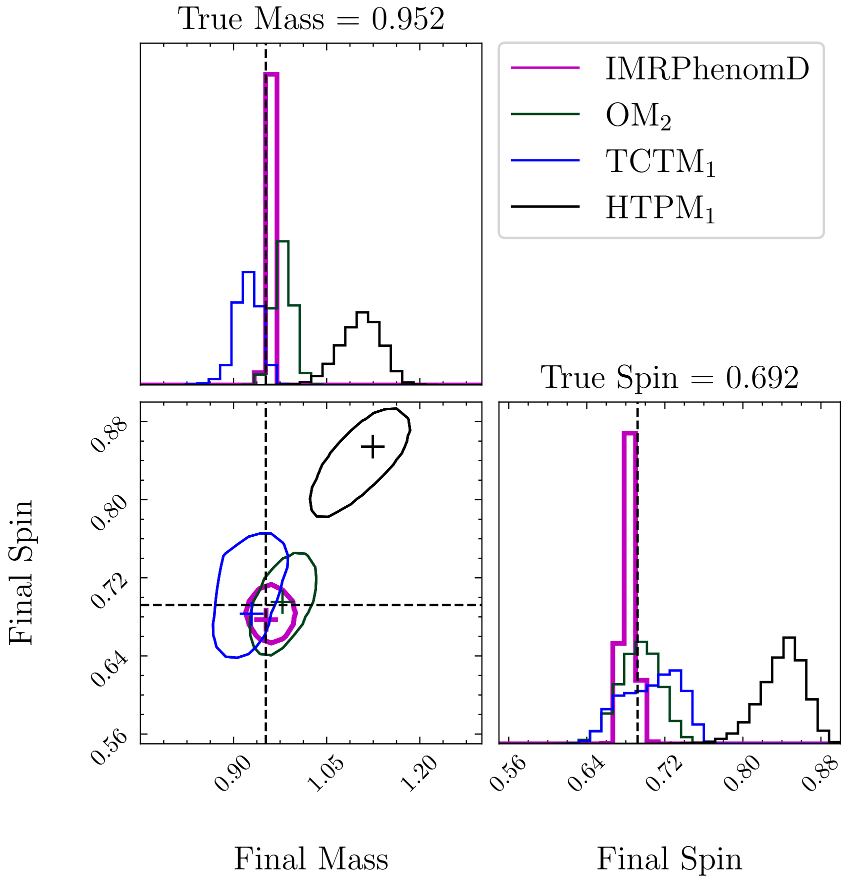
- Setup:
  - Data from SXS catalog.
  - Assuming SNR=100.
  - GW150914 like (SXS:BBH:0305).
  - Truncate the waveform at $t_0 = 0$ (strain peak).
  - Use 2000 n-live points and the stopping criterion of $\Delta(\ln \mathcal{Z}) = 0.1$ for the nested sampling.
- Results:
  - Bayes evidences favors IMRPhenomD the most.
  - Overtone model performance reach a plateau at $N = 2$.
  - Second best models are TCTMs (also nonlinear).
  - Non-GR model has similar performance as OMs.

| Model | Parameters | $\log_{10} \mathcal{Z}$ |
|---|---|---|
| $OM_0$ | 4 | -188.435 |
| $OM_1$ | 6 | -31.9132 |
| $OM_2$ | 8 | -17.0332 |
| $OM_3$ | 10 | -17.1311 |
| $OM_4$ | 12 | -17.7541 |
| $TCTM_0$ | 6 | -33.2123 |
| $TCTM_1$ | 8 | -14.4255 |
| $TCTM_2$ | 10 | -14.2678 |
| $HTPM_1$ | 8 | -17.1984 |
| $HTPM_2$ | 10 | -17.3353 |
| IMRPhenomD | 4 | -9.57672 |

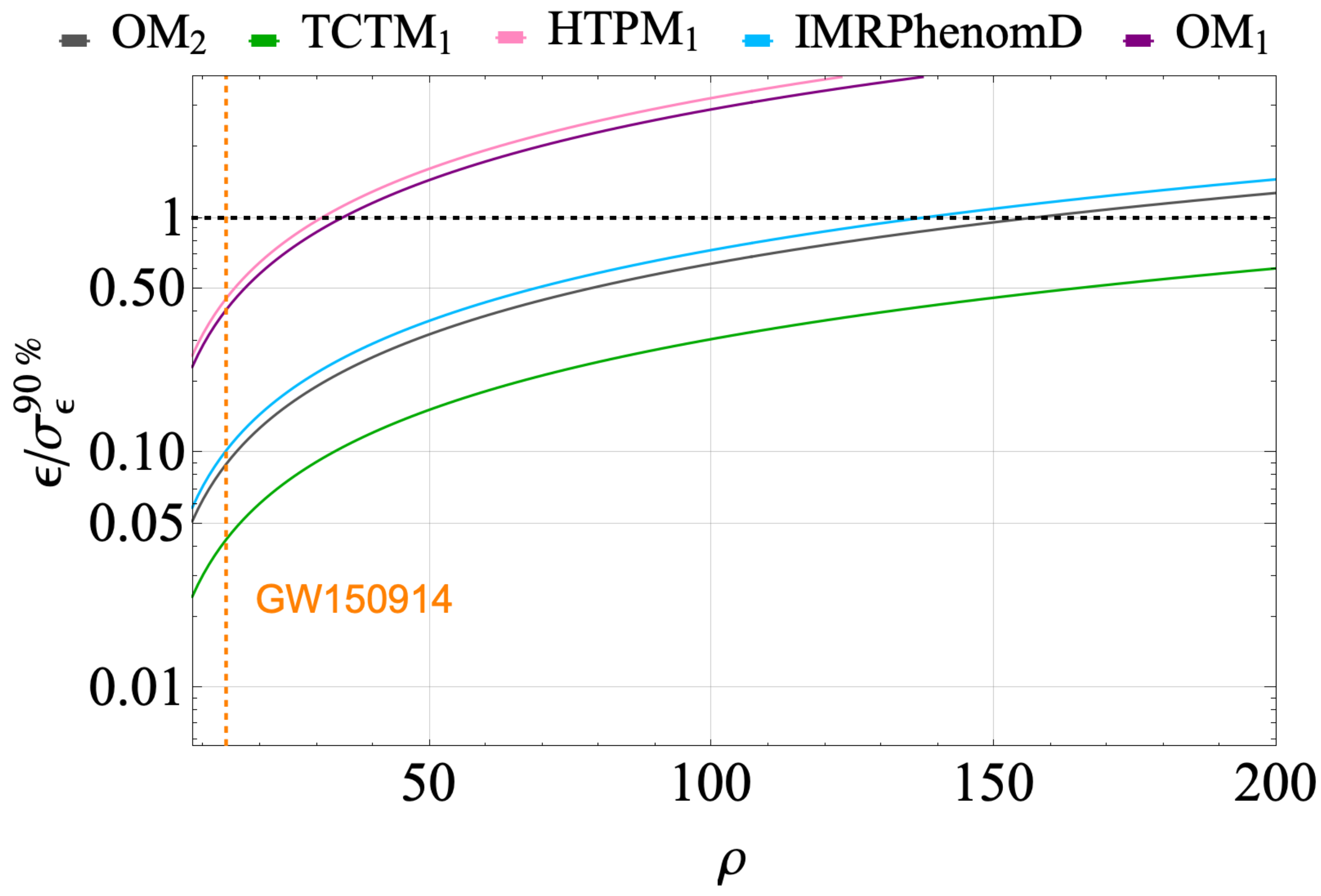
# Posterior distribution

- Posterior distributions are similar for $\mathrm{OM}_{n>=2}$.
- While it shows large offsets for the models $\mathrm{OM}_{n=0,1}$.
- Tighter mass and spin contours in IMRPhenomD than all the OMs.
- $\mathrm{TCTM}_1$ shows comparable performance as $\mathrm{OM}_2$ (i.e. having the same number of parameters).
- $\mathrm{HTPM}_1$ displaying a significant bias.



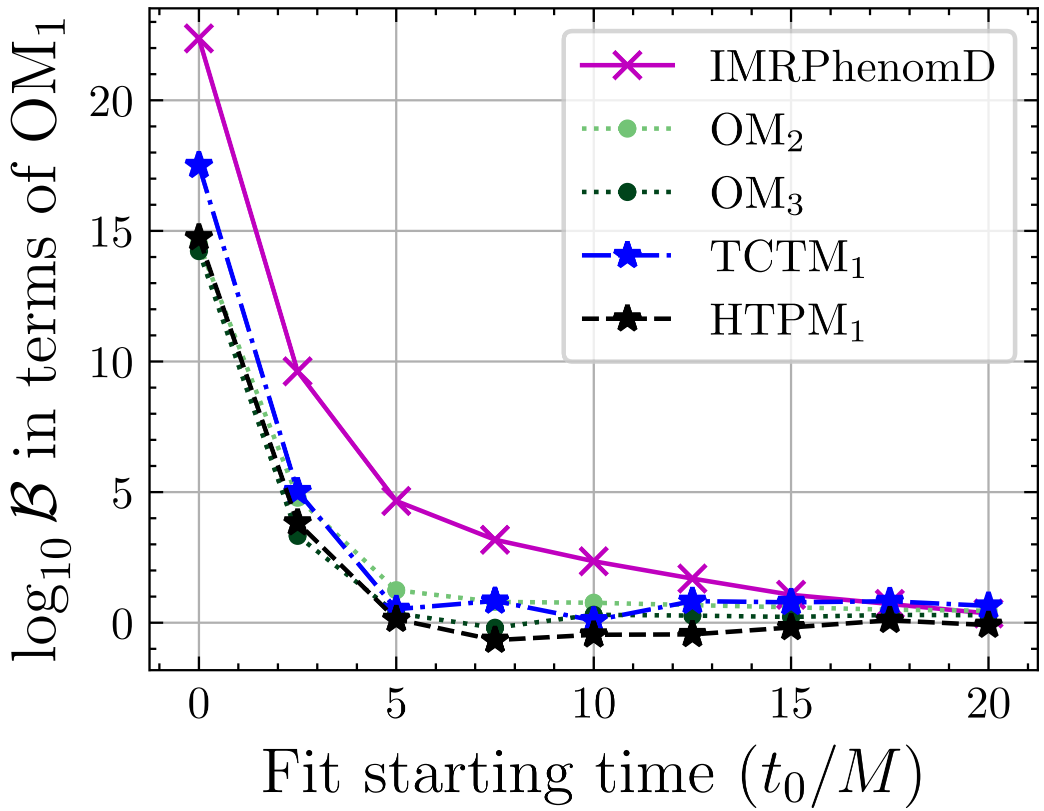
# Bias vs. SNR



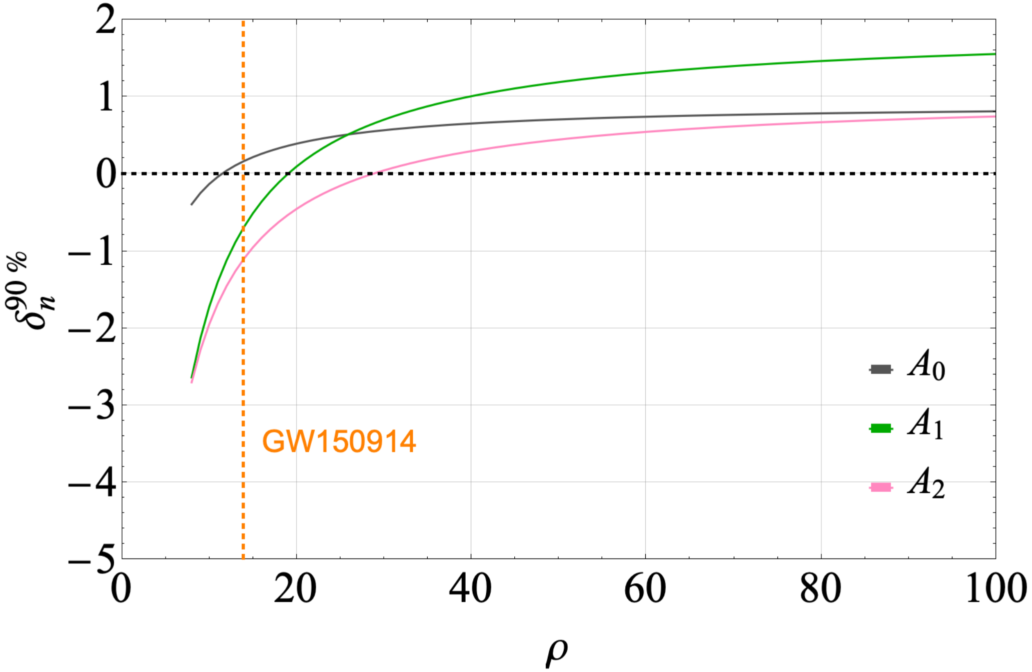

- $OM_1$ and $HTPM_1$ start to be dominated by systematic bias at SNR $\gtrsim$ 30-40.

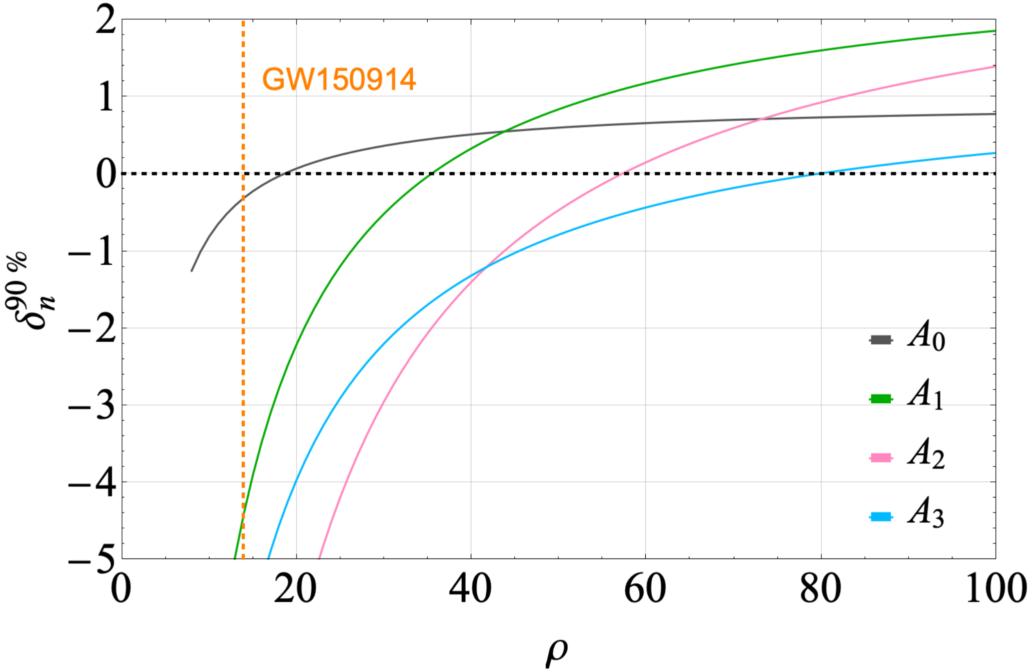
# Ringdown starting time dependence

- The two nonlinear models considered in this work, IMRPhenomD and $\mathrm{TCTM}_1$ provide larger Bayes evidences over the linear models $\mathrm{OM}_2$ and non-GR model $\mathrm{HTPM}_1$ for low $t_0$ values.
- IMRPhenomD is supported over OMs at early starting time up to $t_0/M = 15$.
- All other models become comparable from $t_0/M = 5$ onward.



# Higher tone amplitudes observability




For overtone models, confident observations of the N = 2, 3 overtones require large SNRs in the ringdown O(30-100).

# Robustness test

| Catalog | Waveform | $q$ | $\chi_{1,z}$ | $\chi_{2,z}$ | IMRPhenomD | $OM_2$ | $OM_3$ | $TCTM_1$ | $HTPM_1$ |
|---|---|---|---|---|---|---|---|---|---|
| Main | BBH:0150 | 1 | 0.2 | 0.2 | (-8.215, 0.0080) | (-19.916, 0.0283) | (-22.449, 0.0095) | (-14.771, 0.0119) | (-19.314, 0.2059) |
| | BBH:0305 | 1.221 | 0.33 | -0.44 | (-9.558, 0.0150) | (-17.032, 0.0270) | (-17.121, 0.0115) | (-14.447, 0.0246) | (-17.208, 0.2367) |
| | BBH:1221 | 3 | 0 | 0 | (-7.988, 0.0074) | (-16.652, 0.0494) | (-18.539, 0.0456) | (-15.071, 0.0503) | (-15.944, 0.2531) |
| | BBH:0300 | 8.5 | 0 | 0 | (-6.207, 0.0092) | (-17.000, 0.1209) | (-18.583, 0.0851) | (-15.456, 0.0323) | (-15.647, 0.3011) |
| Ext-CCE | BBH:0002 | 1 | 0.2 | 0.2 | (-44.57, 0.0238) | (-80.87, 0.0282) | (-81.44, 0.0238) | (-77.01, 0.0592) | (-79.67, 0.1880) |

- Testing other waveforms form SXS main catalog and ExtCCE catalog.
- Table showing $\log_{10}$ Bayes evidence and bias in (,).
- Different waveform results are consistent.
- IMRPhenomD is preferred over OMs.
- Prior tests also imply robustness of our prior choices for different models.

# Conclusions

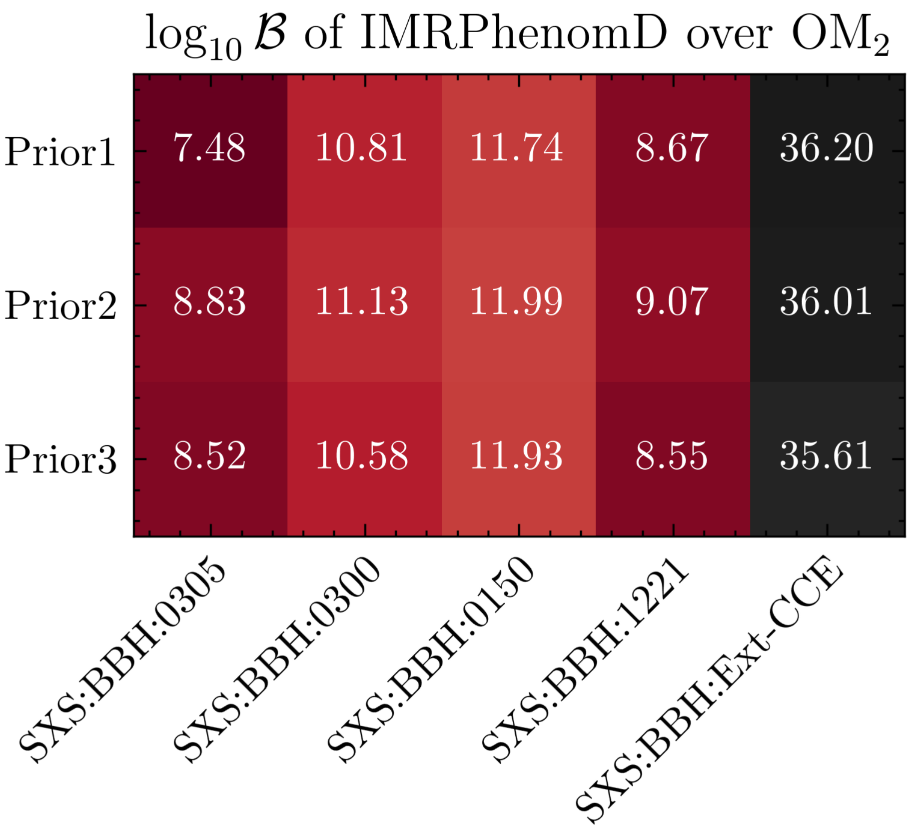
- Nonlinear models provide comparable/better performance in both Bayes analysis and bias analysis when doing parameter estimation using numerical data.
- Overtone models $\mathrm{OM}_1$ and $\mathrm{HTPM}_1$ start to be dominated by systematic errors at SNR $\gtrsim$ 30-40.
- IMR tests at the ringdown regime much less affected by systematic errors in final BH's parameter recoveries.
- Assuming overtone model, confident observation of the overtones N = 2, 3 requires large SNRs in the ringdown O(30-100).

# Thank you

# Backup slide: prior test

| Model | Parameter | Prior |
|---|---|---|
| $OM_0$ | Remnant mass | (0.5, 1.3) |
| | Remnant spin | (0, 0.99) |
| | Amplitudes | (0, 2) |
| | Phases | (0, $2\pi$) |
| $OM_{1\sim2}$ | Remnant mass | (0.5, 1.3) |
| | Remnant spin | (0, 0.99) |
| | Amplitudes | (0, 10) |
| | Phases | (0, $2\pi$) |
| $OM_{3\sim4}$ | Remnant mass | (0.5, 1.1) |
| | Remnant spin | (0, 0.99) |
| | Amplitudes | (0, 10) |
| | Phases | (0, $2\pi$) |
| $TCTM_n$ | Remnant mass | (0.5, 1.3) |
| | Remnant spin | (0, 0.99) |
| | Amplitudes | (0, 2) |
| | Phases | (0, $2\pi$) |
| | $\log_{10} A$ | (-5, 5) |
| | $\tau$ | (0, 100) |
| $HTPM_n$ | Remnant mass | (0.5, 1.3) |
| | Remnant spin | (0, 0.99) |
| | Amplitudes | (0, 2) |
| | Phases | (0, $2\pi$) |
| | $\alpha$ | (-0.5, 1) |
| | $\beta$ | (-0.5, 1) |
| IMRPhenomD | Mass ratio | (1,8) |
| | Initial spin 1 | (-0.99, 0.99) |
| | Initial spin 2 | (-0.99, 0.99) |
| | Phases | (0, $2\pi$) |

"Prior1", "Prior2", "Prior3" are used to denote different ranges for the final mass prior of, (0.5, 1.3), (0.7, 1.2) and (0.8, 1.1)




Different prior ranges tests also prove the robustness of our choices in parameter estimation.

# Backup slide: bias at max likelihood sample point

| Model | Parameters | $\log_{10} \mathcal{Z}$ | $\epsilon$ |
|---|---|---|---|
| $OM_0$ | 4 | -188.435 | 0.311261 |
| $OM_1$ | 6 | -31.9132 | 0.087670 |
| $OM_2$ | 8 | -17.0332 | 0.027019 |
| $OM_3$ | 10 | -17.1311 | 0.011475 |
| $OM_4$ | 12 | -17.7541 | 0.032821 |
| $TCTM_0$ | 6 | -33.2123 | 0.216384 |
| $TCTM_1$ | 8 | -14.4255 | 0.024559 |
| $TCTM_2$ | 10 | -14.2678 | 0.004472 |
| $HTPM_1$ | 8 | -17.1984 | 0.236650 |
| $HTPM_2$ | 10 | -17.3353 | 0.096609 |
| IMRPhenomD | 4 | -9.57672 | 0.014981 |

We append the column of bias of recovery of final BH's mass and spin for SXS:BBH:0305 test. Nonlinear models have lower epsilons than the OMs in most cases.