Find the eigenvalues and the corresponding eigenvectors of this matrix

$$\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

Solution:

From the definition of the eigenvector v corresponding to the eigenvalue λ, we have

$$Av = \lambda v$$
$$Av - \lambda v = (A - \lambda I)v = 0$$
$$(A - \lambda I) = \begin{bmatrix} 1-\lambda & 3 \\ 3 & 9-\lambda \end{bmatrix}$$

The equation has a nonzero solution if and only if

$$\det(A - \lambda I) = 0$$
$$\det(A - \lambda I) = (1-\lambda)(9-\lambda) - 9 = \lambda^2 - 10\lambda = \lambda(\lambda - 10) = 0$$
$$\lambda_1 = 0 \ ; \ \ \lambda_2 = 10$$

1. $\lambda_1 = 0$

$$(A - \lambda I) = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

Perform Gaussian Elimination

$$\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$v = \begin{bmatrix} -3v_2 \\ v_2 \end{bmatrix}$$

2. $\lambda_2 = 10$

$$(A - \lambda I) = \begin{bmatrix} -9 & 3 \\ 3 & -1 \end{bmatrix}$$

Perform Gaussian Elimination

$$\begin{bmatrix} 1 & -1/3 \\ 3 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$v = \begin{bmatrix} 1/3v_2 \\ v_2 \end{bmatrix}$$