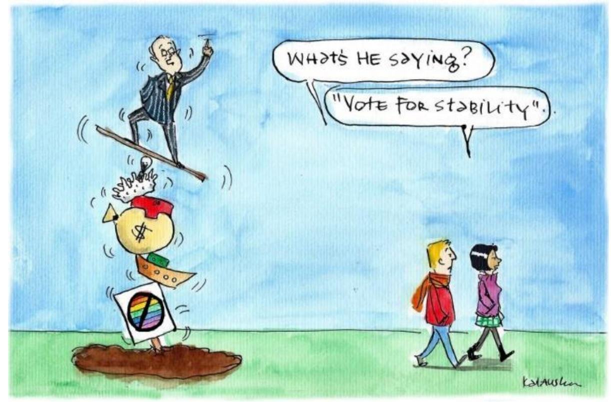
EULER METHOD. NUMERICAL STABILITY



The stability of a numerical solution by Forward Euler method depends on the equation. To help understand the stability of this method, let's examine a particular problem, i.e., a linear IVP, given by [Eq#1]

$$\frac{dy}{dt} = -ay; \quad y(0) = 1; \;\; with\; a > 0$$

As we know, the exact solution is $y^e = \exp(-at)$, which is stable and has a very smooth solution with $y^e(0) = 1$ and $y^e(\infty) = 0$. The discrete equation obtained by applying the forward Euler method to this IVP is [Eq#2]:

$$y_{i+1} = y_i - ahy_i = (1 - ah)y_i$$

Apply the recurrence formula above for the last node, $n + 1$ [Eq#3]:

$$y_{n+1} = y_n - ahy_n = (1 - ah)y_n$$

Apply the recurrence formula to y_n [Eq#4]:

$$y_n = (1 - ah)y_{n-1}$$

Apply the recurrence formula to y_{n-1} [Eq#5]:

$$y_{n-1} = (1 - ah)y_{n-2}$$

Plug Eq#4 and Eq#5 into Eq#3 to yield [Eq#6]

$$y_{n+1} = y_n - ahy_n = (1 - ah)y_n = (1 - ah)^2 y_{n-1} = \cdots = (1 - ah)^n y_1 = \cdots = (1 - ah)^{n+1} y_0$$

$$y_{n+1} = (1 - ah)^{n+1} y_0$$

which can be generalized for any index [Eq#7]:

$$y_{i+1} = (1 - ah)^{i+1} y_0$$

To prevent the amplification of the errors in the iteration process, we require that in [Eq#7] $|1 - ah| < 1$, which have two possible expressions due the absolute value term:

(a) $1 - ah < 1$, or
(b) $-(1 - ah) < 1$ for stability of the forward Euler method.

The first one (a) yields $-ah < 0$, next since $(a > 0)$ then (-$h < 0$) which is true for any positive value of h and bring **no** light to the problem (remember Maria).

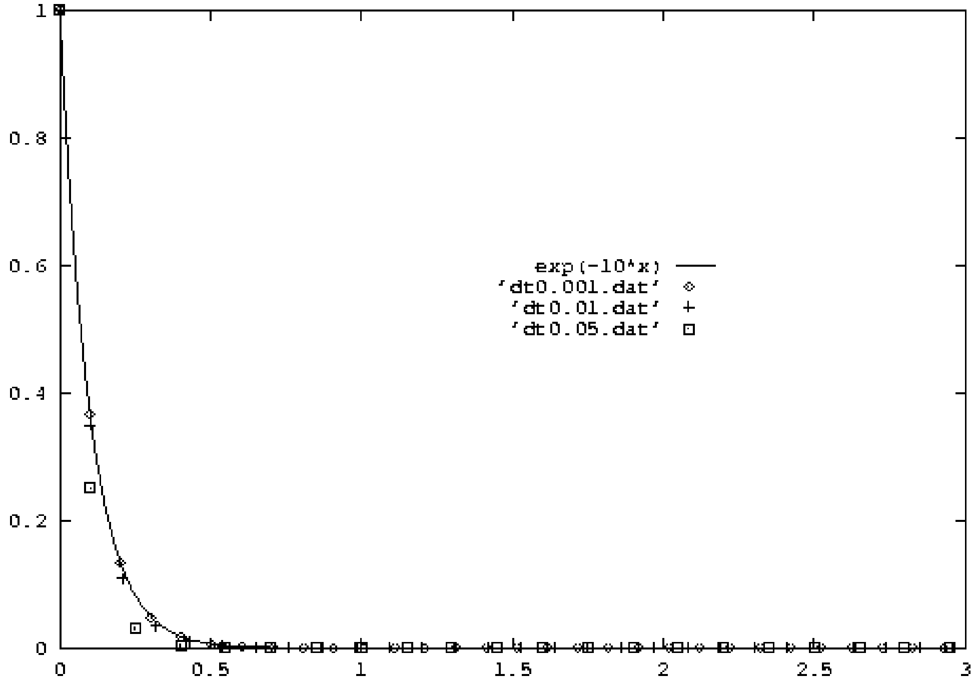
Then, the second equation (b) yields: $-(1 - ah) < 1$, then $(-1 + ah) < 1$, further massage yields $ah < 2$, therefore we should have $h < 2/a$ where $(a > 0)$ as stability criterion.

These results can be better perceived from Figures 1 and 2 below. The test problem is the IVP given by **{ dy/dt = -10y, y(0)=1}**, which has the exact solution $y^e = \exp(-10t)$. Comparing the test problem with ours previous:

$$\frac{dy}{dt} = -ay; \quad y(0) = 1; \;\; with\; a > 0$$

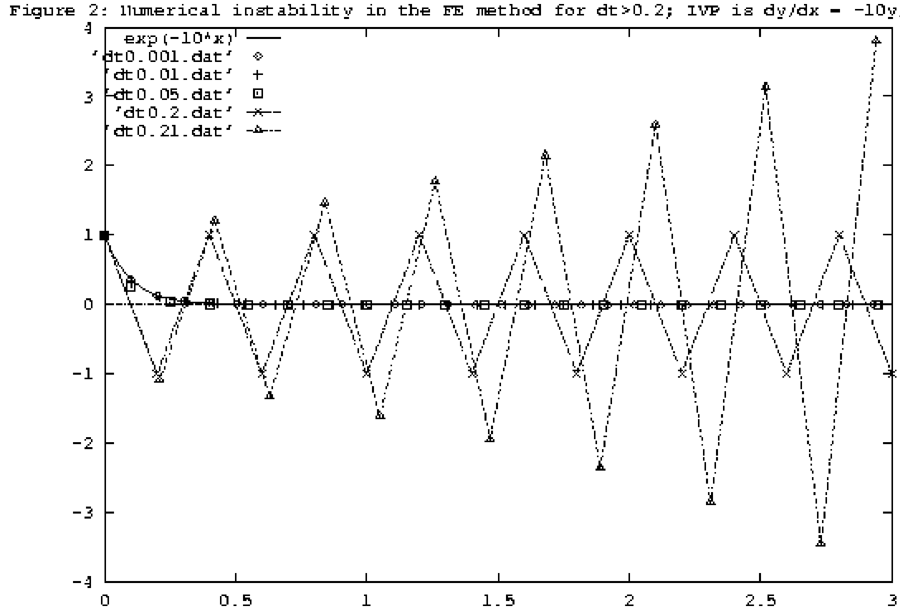
Then we have $a = 10$. The stability criterion for the forward Euler method requires the step size **h** to be less than $\left(h < \frac{2}{a}\right)$ which with a=10, yields $h < \frac{2}{10} = 0.2$. In Figure 1, we have shown the computed solution for h=0.001, 0.01 and 0.05 along with the exact solution1.

Figure 1: Convergence of the FE method: dt = 0.05, 0.01 & 0.001



As seen from there, the method is numerically stable for these values of h and becomes more accurate as h decreases. However, based on the stability analysis given above, the forward Euler method is stable only for h < 0.2 for our test problem. The numerical instability which occurs for $h \geq 0.2$ is shown in Figure 2. For h =0.2, the instability is oscillatory between $y = \pm 1$, whereas for h>0.2, the amplitude of the oscillation grows in time without bound, leading to an explosive numerical instability.

Figure 2: Numerical instability in the FE method for dt>0.2; IVP is dy/dx = -10y, y(0



The stability criterion for a different ODE must be investigated independently. Not general stability criterium can be stablished for all ODE types. This document has been modified from: http://web.mit.edu/10.001/Web/Course_Notes/Differential_Equations_Notes/node3.html