Mathematical Models in Finance

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MATHEMATICAL MODELS IN FINANCE

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Summary

Over the last 25 years the financial markets have gone through an enormous development. The introduction of financial derivatives such as options and futures on underlyings (stock, bond, currencies) has led to a new quality of the securitization of financial risks. The basic idea of a financial derivative is to buy insurance for risky assets on the market, i.e., to find participants in the market who are willing to share the risks and the profits of future developments in the market which are subject to uncertainty.

The pricing of these financial instruments is based on an advanced mathematical theory, called Itô stochastic calculus. The basic model for an uncertain price is described by Brownian motion and related differential equations. The pricing of a European call option by Black, Scholes and Merton in 1973 (the Nobel Prize winning Black-Scholes formula) was a breakthrough in the understanding and valuing of financial derivatives. Their approach has become the firm basis for modern financial mathematics which uses advanced tools such as martingale theory and stochastic control to find adequate solutions to the pricing of a world-wide enormously increasing number of derivatives.

1. A Tutorial on Mathematical Finance without Formulae

Financial mathematics has become one of the most recent success stories of mathematics and probability theory in particular. In contrast to many mathematical achievements which are known to specialists only, the Black-Scholes option pricing formula has gained popularity not only among practitioners in finance, but its fame has also attracted the attention of economists, physicists, econometricians, statisticians, etc. Hundreds of popular articles in economics, physics, mathematics journals, and far beyond, have been written about this topic. BBC made a documentary about the Nobel Prize winning formula. University professors explain the formula to high school students in order to convince them that mathematics is a topic worth studying.

Mathematical models have been used in economics for a long time. By now, operations research, econometrics, and time series analysis constitute major parts of the curricula of business schools and economics departments. However, this article will not focus on these more classical topics but mainly on the approach which was started by Fisher Black (1938-1995), Myron Scholes, and Robert Merton in 1973. What was so entirely new that Scholes and Merton were awarded the Nobel Prize for economics in 1997?

In 1973 the Chicago Board of Trade (CBOT) started trading so-called *options*, *futures* and other *financial derivatives*. For example, a *European call option* is a ticket (a contract) which entitles its purchaser to buy one share of a *risky asset* (such as the stock of Microsoft) at a fixed price (*strike price*) at a known date in the future (*time of expiration* or *maturity*). It is natural to ask: how much would the purchaser of the call option be willing to pay? Clearly, there are various problems. The purchaser gains a positive amount of money only if the price of Microsoft at maturity is indeed above the strike price. Then he can buy the share at the lower strike price and sell it at the higher market price to somebody else. However, since the price of Microsoft is subject to uncertainty, one does not know in advance whether this price will exceed the strike price in the future. There is a chance that the share price at maturity will reach a level below the strike price. Then it would be cheaper to buy a share of Microsoft on the stock market. However, an option (in contrast to a future) is not a contract that obliges its purchaser to buy the share. His gain at maturity would be zero in this case.

Black, Scholes and Merton approached the problem of pricing an option in a physicist's way. They started by assuming a reasonable model for the price of a risky asset. The search for such a model has a long history. Empirical (statistical, econometric) research has shown that changes of prices in the future are hardly predictable by mathematical models. In the economics literature this fact runs under the name of "random walk hypothesis". A random walk is defined at discrete equidistant instants of time. In finance, however, one is mainly interested in modeling prices at every instant of time. We call this a continuous time model. *Brownian motion* is a natural analogue of a random walk in continuous time. It is a physical model for the movement of a small particle suspended in a liquid and has been studied in the physics literature since the beginning of the 20th century. One of the famous contributors to this theory was Albert Einstein. Before Einstein, a young French PhD student, called Louis Bachelier, proposed in his 1900 thesis Brownian motion as a model for speculative prices. One of the imperfections of this model is that Brownian motion can assume negative values, and this might have been one of the reasons that his model had been forgotten for a long time. Only in the 1960s the economist Samuelson (Nobel Prize for economics in 1970) propagated the exponential of Brownian motion (so-called *geometric Brownian motion*) for modeling prices which are subject to uncertainty to his students at M.I.T.

In the work of Black, Scholes, and Merton, geometric Brownian motion is the basic mathematical model for price movements. Moreover, they realized that Brownian motion is closely related to a deep mathematical theory, *stochastic* or *Itô calculus*, named after the Japanese mathematician Kiyosi Itô who developed this theory in the 1940s. Classical calculus is about differentiation and integration of "smooth" functions. In contrast to the latter, paths of Brownian motion are extremely irregular (non-differentiable) functions and therefore classical calculus is not a suitable tool. Although Itô calculus had been known and used by certain physicists, engineers, and other applied scientists for some time, it did not become very popular outside some groups of specialists. By now, everybody working in (theoretical or practical) finance knows about the basic rules of Itô calculus.

The main contribution of the fathers of the option pricing formula, however, was a totally new idea on the economics side. They argued that the seller of a European call option (usually a big financial institution) would not wait passively until time of maturity. On the contrary, if he/she was a rational person he/she would invest a certain amount of money in the same stock (e.g. Microsoft) and in a *riskless asset* (e.g. a savings account with fixed interest rate) according to a dynamic trading strategy such that the value of the portfolio at maturity would be exactly the value of the option at maturity: either zero if the share price was below the strike price or, otherwise, the positive difference between the share and the strike prices. A trading strategy which *replicates* the value of the option at maturity is called a *hedge*. The existence of such a hedge is a justification for the price of an option, but it is important in itself for financial practice. The amount of money which the seller of the option had to invest for his/her hedge would be a fair price for the option. Moreover, Black, Scholes, and Merton argued that, if the option was sold at a price other than the Black-Scholes price, a rational person could use this fact to make unlimited profits without accompanying risk (so-called *arbitrage*).

The Black-Scholes price, expressed in the famous Black-Scholes formula, because of its convincing rationale, became a major success and was well accepted by practitioners in the financial markets. The *New York Times* (15 October, 1997) wrote: "Soon traders were valuing options in the floor of the exchange, punching half a dozen numbers into electronic calculators hard-wired with the formula. … Mr. Black and Mr. Scholes became highly regarded at the exchange that when they visited, traders would give them a standing ovation."

Despite various imperfections of the underlying mathematical model, the Black-Scholes approach was a starting point for pricing other kinds of financial derivatives. For example, in practice European calls are less frequently traded than options which can be exercised at anytime before or at time of maturity (American-type options). Moreover, an option, future, etc., does not necessarily have to be linked to a share price, but to a composite stock index such as the Dow Jones, Nikkei, Standard & Poors 500, DAX, etc., or to bond prices, foreign exchange rates, or any other underlying which is due to uncertainty. The basic aim of a financial derivative is *securitization of risks*; the Black-Scholes approach allows the seller and the purchaser of a properly priced derivative to hedge against future risks due to uncertainty of price movements.

The large variety of financial products which has been created by financial institutions became a challenge to mathematics and in particular to the specialists of Itô calculus. Since the end of the 1970s, they have pushed forward the development of financial mathematics by exploiting the most advanced tools, in particular functional analysis, martingale theory, stochastic control, partial differential equations. By now, financial mathematics is a well established theory with a great future which is taught at mathematics and economics departments all over the world.

Clearly, the Black-Scholes world is an idealization of the real financial world. For example, the mathematical assumption of geometric Brownian motion as a model for a risky price is known to be in contradiction with real-life price data. Once in a while the stock market is shaken by shocks (due to political events, recessions, bursts of economic activity, etc.) resulting in unexpected price jumps. Such a behavior cannot happen in the Black-Scholes world. Over the last 20 years several events showed the limitations of the model. In October 1987 (Black Monday) a major crash affected the New York Stock Exchange causing financial losses of several billion U.S. dollars. Although it did not have a major impact on the market, the crash of Barings Bank made the world aware of the fact that financial derivatives can be very dangerous when handled by careless management. Quite recently, in October 1998, the turmoil around Long Term Capital Management (LTCM), a hedge fund worth hundreds of billions U.S. Dollars, with both Scholes and Merton as founding members, gave the public more reasons to have less confidence in a (seemingly perfect) mathematical formula. The events around LTCM caused, within a week's time, a 13.7% loss of the U.S. Dollar against the Japanese Yen. *Newsweek* (19 October, 1998) asked: "The buck is bruised. So is another big hedge fund. What's going on?" By now, the derivatives business has exceeded an annual volume of $15 trillion. It is one of the financial fundamentals on which modern society is based.

Mathematical formulae can help to make rational decisions in finance. However, the above examples show quite clearly that too much trust in formulae, paired with wrong decisions of management, can lead to fatal consequences not only for one particular company, but for whole national economics. In view of the enormous amounts of money involved in derivatives it was realized quite early in the 1990s that the financial industry had to facilitate the measurement of risk. In 1992 the so-called Basel Committee of the Bank for International Settlements (representing 27 European members plus the U.S., Canada, Japan, Australia, and South Africa) presented proposals to estimate market risk and to define the resulting capital requirements to be implemented in the banking sector. The European Union (EEC 93/6) approved a directive, effective January 1996, that mandates banks and investment firms to set capital aside to cover market risks. In the U.S., the Securities and Exchange Commission fulfills a similar regulatory function. Measuring and estimating financial risks in its various forms has become another challenge to mathematics, in particular statistics. Based on probabilistic models, various statistical methods have been developed to quantify financial risks. Among them, the Value at Risk (VaR) has become most popular. Companies such as *RiskMetrics* have specialized in advising the financial industry how to measure and estimate risks, and government regulators control financial institutions as to whether they satisfy certain risk standards.

An exciting new development is the birth of bank-assurance: we witness a convergence of financial and actuarial thinking. One relevant buzz-word is ART: Alternative Risk Transfer. Examples are Catastrophe Bonds; the coupon payment (and possibly the principal re-payment) is contingent on the (non-)occurrence of a catastrophic event. Think for the latter of an earthquake or hurricane, say. Other examples are energy and weather derivatives.

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Biographical Sketches

Paul Embrechts is Professor of Mathematics at the ETHZ (Swiss Federal Institute of Technology, Zurich) specialising in actuarial mathematics and mathematical finance. Previous academic positions include the Universities of Leuven, Limburg and London (Imperial College). Dr. Embrechts has held visiting appointments at the University of Strasbourg, ESSEC Paris, the Scuola Normale in Pisa and the London School of Economics (Centennial Professor of Finance). He is an Elected Fellow of the Institute of Mathematical Statistics, Honorary Fellow of the Institute of Actuaries, Corresponding Member of the Italian Institute of Actuaries, Editor of the ASTIN Bulletin, on the Advisory Board of Finance and Stochastics and Associate Editor of numerous scientific journals. He is a member of the Board of the Swiss Association of Actuaries and belongs to various national and international research and academic advisory committees. His areas of specialization include insurance risk theory, integrated risk management, the interplay between insurance and finance, and the modeling of rare events. Together with

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He has organized numerous international scientific events for the Dutch Research Center EURANDOM (Netherlands), the Danish Research Network MaPhySto (Denmark), the Bernoulli Society and the Scandinavian Actuarial Associations.