HEFT numerator factors

In this worksheet we present the code for computing the gauge invariant form for the HEFT numerator factor.
The code implements the algorithm described in the paper
HEFT Numerators from Kinematic Algebra
by Chih-Hao Fu, Pierre Vanhove and Yihong Wang
arXiv:https://arxiv.org/abs/2501.14523

The main routines are in the file : **functionsHEFT.wl** which needs to be executed before running this worksheet

We run some example of gauge invariant numerator factors for the emission of n gluons of momenta k_i and field-strengths F_i from a massive scalar line of momentum p

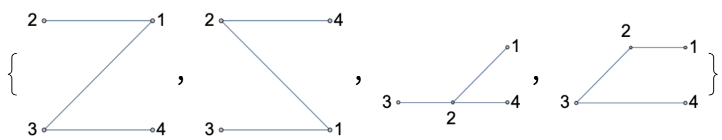
With the convention that k1+k2+...+kn=p'-p with ki^2=0, p^2=(p')^2=0 and p.(k1+...+kn)=0

We present the tree structure described in section 4.2 of the paper, and the expression for the gauge invariant numerator factors

In[ ]:=
```
Now[]
```
Out[ ]=
Fri 17 Jan 2025 14:12:07 GMT+1 [ ]

In[ ]:=
```
SetDirectory[NotebookDirectory[]]
```
Out[ ]=
/Users/vanhove/Git/Stringy-Numerator

The three gluons case

In[ ]:=
```
Ngluons = 3;
Numerator3gluonsTree = GIAOTree[Ngluons, p];
```
Displaying the graphs
```
tijplot[Ngluons, #] & /@ (#[[2]] & /@ Numerator3gluonsTree)
```
Out[ ]=


Displaying the numerator factors

In[∘]:=
```
Numerator3gluons = GINum[Ngluons, p]
```
Out[∘]=
$$-\frac{k1\odot F3\odot p\ p\odot F1\odot F2\odot p}{k1\cdot p\ k3\cdot p}-\frac{k2\odot F3\odot p\ p\odot F1\odot F2\odot p}{k1\cdot p\ k3\cdot p}-\frac{k1\odot F2\odot p\ p\odot F1\odot F3\odot p}{k1\cdot p\ k2\cdot p}-\frac{p\odot F1\odot F2\odot F3\odot p}{k1\cdot p}$$

In[∘]:=
```
listpoles3gluons = DeleteDuplicates@Denominator@(List @@ Numerator3gluons)
```
Out[∘]=
$$\{k1\cdot p\ k3\cdot p,\ k1\cdot p\ k2\cdot p,\ k1\cdot p\}$$

In[∘]:=
```
Numerator3gluons
```
Out[∘]=
$$-\frac{k1\odot F3\odot p\ p\odot F1\odot F2\odot p}{k1\cdot p\ k3\cdot p}-\frac{k2\odot F3\odot p\ p\odot F1\odot F2\odot p}{k1\cdot p\ k3\cdot p}-\frac{k1\odot F2\odot p\ p\odot F1\odot F3\odot p}{k1\cdot p\ k2\cdot p}-\frac{p\odot F1\odot F2\odot F3\odot p}{k1\cdot p}$$

In[∘]:=
```
stmp = Sum[Simplify[Coefficient[Numerator3gluons, 1 / listpoles3gluons[[itmp]]]]
    1 / listpoles3gluons[[itmp]], {itmp, 1, Length[listpoles3gluons] - 1}];
stmp + Simplify[Numerator3gluons - stmp]
```
Out[∘]=
$$-\frac{(k1\odot F3\odot p+k2\odot F3\odot p)\ p\odot F1\odot F2\odot p}{k1\cdot p\ k3\cdot p}-\frac{k1\odot F2\odot p\ p\odot F1\odot F3\odot p}{k1\cdot p\ k2\cdot p}-\frac{p\odot F1\odot F2\odot F3\odot p}{k1\cdot p}$$

In[∘]:=
```
(*  k1+k2+k3=p'-p  =Q  ---  p^2=(p')^2=M^2  → p.(k1+k2+k3)=p'.(k1+k2+k3)=0     *)
```

Comparing with the results from
A. Brandhuber, G. Chen, H. Johansson, G. Travaglini and C. Wen, Kinematic *Hopf Algebra for Bern-Carrasco-Johansson Numerators in Heavy-Mass Effective Field Theory and Yang-Mills Theory*, Phys. Rev. Lett. 128 (2022) 121601, [2111.15649].

In[∘]:=
```
QMNum[Ngluons, p]
```
Out[∘]=
$$\frac{(k1\odot F3\odot p+k2\odot F3\odot p)\ p\odot F1\odot F2\odot p}{k1\cdot p\ (k1\cdot p+k2\cdot p)}+\frac{k1\odot F2\odot p\ p\odot F1\odot F3\odot p}{k1\cdot p\ (k1\cdot p+k3\cdot p)}-\frac{p\odot F1\odot F2\odot F3\odot p}{k1\cdot p}$$

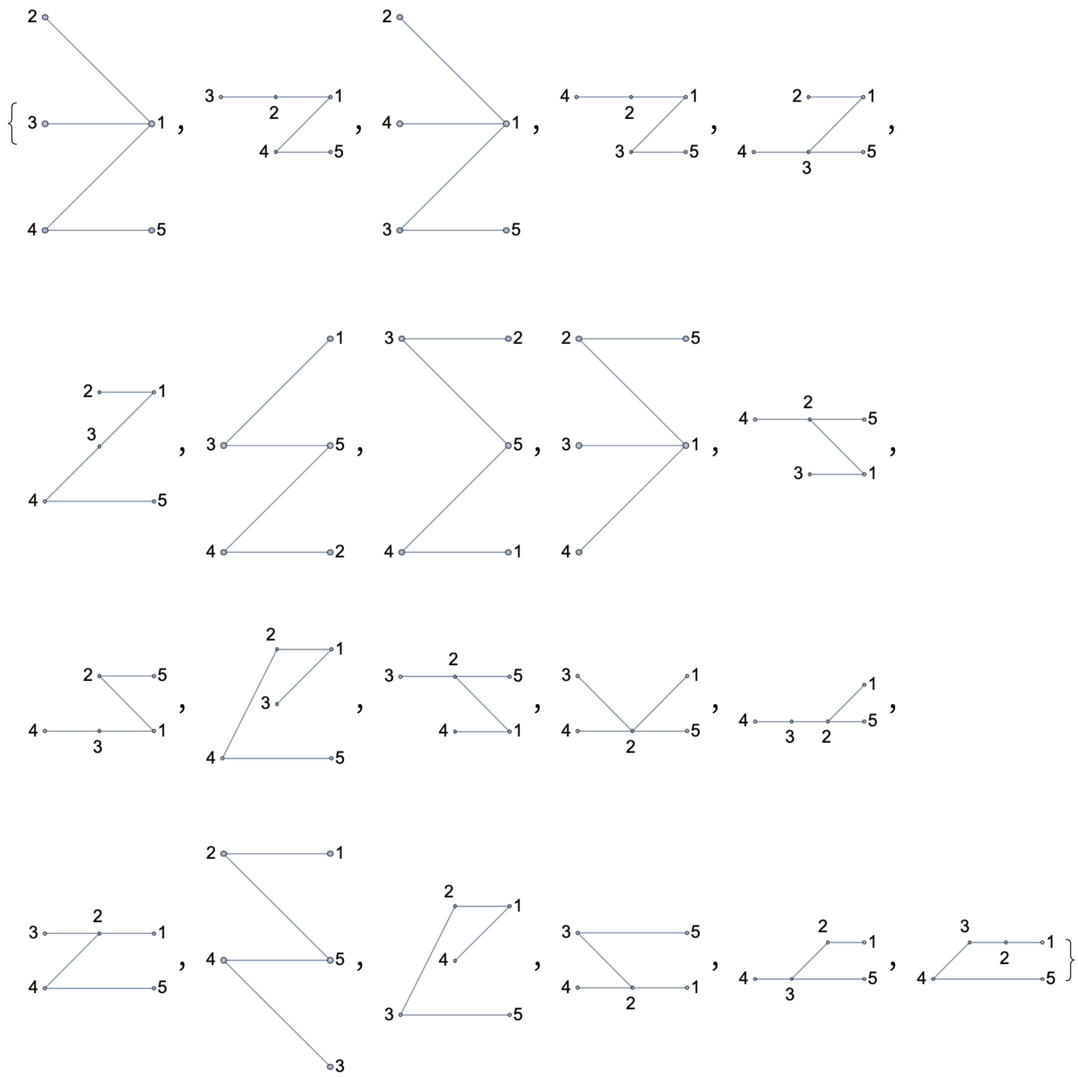
The four gluons case

In[∘]:=
```
Ngluons = 4;
Numerator4gluonsTree = GIAOTree[Ngluons, p];
```

Displaying the graphs

```
tijplot[Ngluons, #] & /@ (#[[2]] & /@ Numerator4gluonsTree)
```

Out[∘]=



Displaying the numerator factors

In[ ]:=
```
Numerator4gluons = GINum[Ngluons, p]
```

Out[ ]=

$$
\begin{aligned}
&\frac{k1\odot F3\odot p\; k1\odot F4\odot p\; p\odot F1\odot F2\odot p}{k1\cdot p\; k3\cdot p\; k4\cdot p}+\frac{k1\odot F4\odot p\; k2\odot F3\odot p\; p\odot F1\odot F2\odot p}{k1\cdot p\; k3\cdot p\; k4\cdot p}+\\
&\frac{k1\odot F3\odot p\; k2\odot F4\odot p\; p\odot F1\odot F2\odot p}{k1\cdot p\; k3\cdot p\; k4\cdot p}+\frac{k2\odot F3\odot p\; k2\odot F4\odot p\; p\odot F1\odot F2\odot p}{k1\cdot p\; k3\cdot p\; k4\cdot p}+\\
&\frac{k1\odot F3\odot p\; k3\odot F4\odot p\; p\odot F1\odot F2\odot p}{k1\cdot p\; k3\cdot p\; k4\cdot p}+\frac{k2\odot F3\odot p\; k3\odot F4\odot p\; p\odot F1\odot F2\odot p}{k1\cdot p\; k3\cdot p\; k4\cdot p}+\\
&\frac{k1\odot F2\odot p\; k1\odot F4\odot p\; p\odot F1\odot F3\odot p}{k1\cdot p\; k2\cdot p\; k4\cdot p}+\frac{k1\odot F2\odot p\; k2\odot F4\odot p\; p\odot F1\odot F3\odot p}{k1\cdot p\; k2\cdot p\; k4\cdot p}+\\
&\frac{k1\odot F2\odot p\; k3\odot F4\odot p\; p\odot F1\odot F3\odot p}{k1\cdot p\; k2\cdot p\; k4\cdot p}+\frac{k1\odot F2\odot p\; k1\odot F3\odot p\; p\odot F1\odot F4\odot p}{k1\cdot p\; k2\cdot p\; k3\cdot p}+\\
&\frac{k1\odot F2\odot p\; k2\odot F3\odot p\; p\odot F1\odot F4\odot p}{k1\cdot p\; k2\cdot p\; k3\cdot p}-\frac{k1\cdot k2\; p\odot F1\odot F4\odot p\; p\odot F2\odot F3\odot p}{k1\cdot p\; k2\cdot p\;(k1\cdot p+k4\cdot p)}-\\
&\frac{k1\cdot k2\; p\odot F1\odot F3\odot p\; p\odot F2\odot F4\odot p}{k1\cdot p\; k2\cdot p\;(k1\cdot p+k3\cdot p)}+\frac{(-(k1\cdot k3)-k2\cdot k3)\; p\odot F1\odot F2\odot p\; p\odot F3\odot F4\odot p}{k1\cdot p\;(k1\cdot p+k2\cdot p)\; k3\cdot p}+\\
&\frac{k1\odot F4\odot p\; p\odot F1\odot F2\odot F3\odot p}{k1\cdot p\; k4\cdot p}+\frac{k2\odot F4\odot p\; p\odot F1\odot F2\odot F3\odot p}{k1\cdot p\; k4\cdot p}+\\
&\frac{k3\odot F4\odot p\; p\odot F1\odot F2\odot F3\odot p}{k1\cdot p\; k4\cdot p}+\frac{k1\odot F3\odot p\; p\odot F1\odot F2\odot F4\odot p}{k1\cdot p\; k3\cdot p}+\\
&\frac{k2\odot F3\odot p\; p\odot F1\odot F2\odot F4\odot p}{k1\cdot p\; k3\cdot p}+\frac{k1\odot F2\odot p\; p\odot F1\odot F3\odot F4\odot p}{k1\cdot p\; k2\cdot p}+\frac{p\odot F1\odot F2\odot F3\odot F4\odot p}{k1\cdot p}
\end{aligned}
$$

In[ ]:=
```
DeleteDuplicates@Denominator@(List @@ Numerator4gluons)
Length[%]
```

Out[ ]=

$$
\{k1\cdot p\; k3\cdot p\; k4\cdot p,\; k1\cdot p\; k2\cdot p\; k4\cdot p,\; k1\cdot p\; k2\cdot p\; k3\cdot p,\; k1\cdot p\; k2\cdot p\;(k1\cdot p+k4\cdot p),\; k1\cdot p\; k2\cdot p\;(k1\cdot p+k3\cdot p),\; k1\cdot p\;(k1\cdot p+k2\cdot p)\; k3\cdot p,\; k1\cdot p\; k4\cdot p,\; k1\cdot p\; k3\cdot p,\; k1\cdot p\; k2\cdot p,\; k1\cdot p\}
$$

Out[ ]=

```
10
```

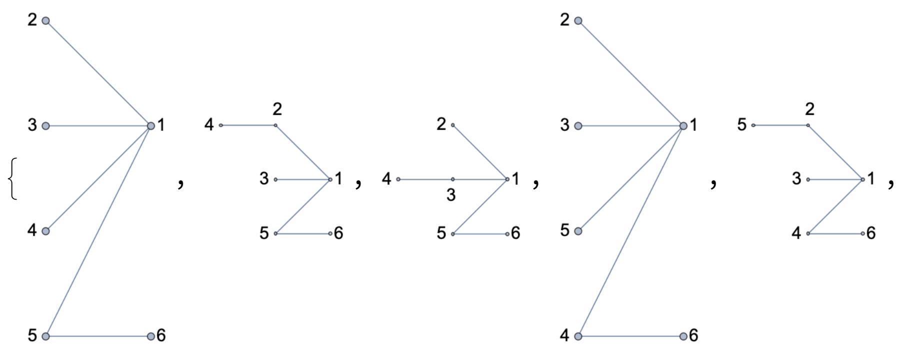
The five gluons case

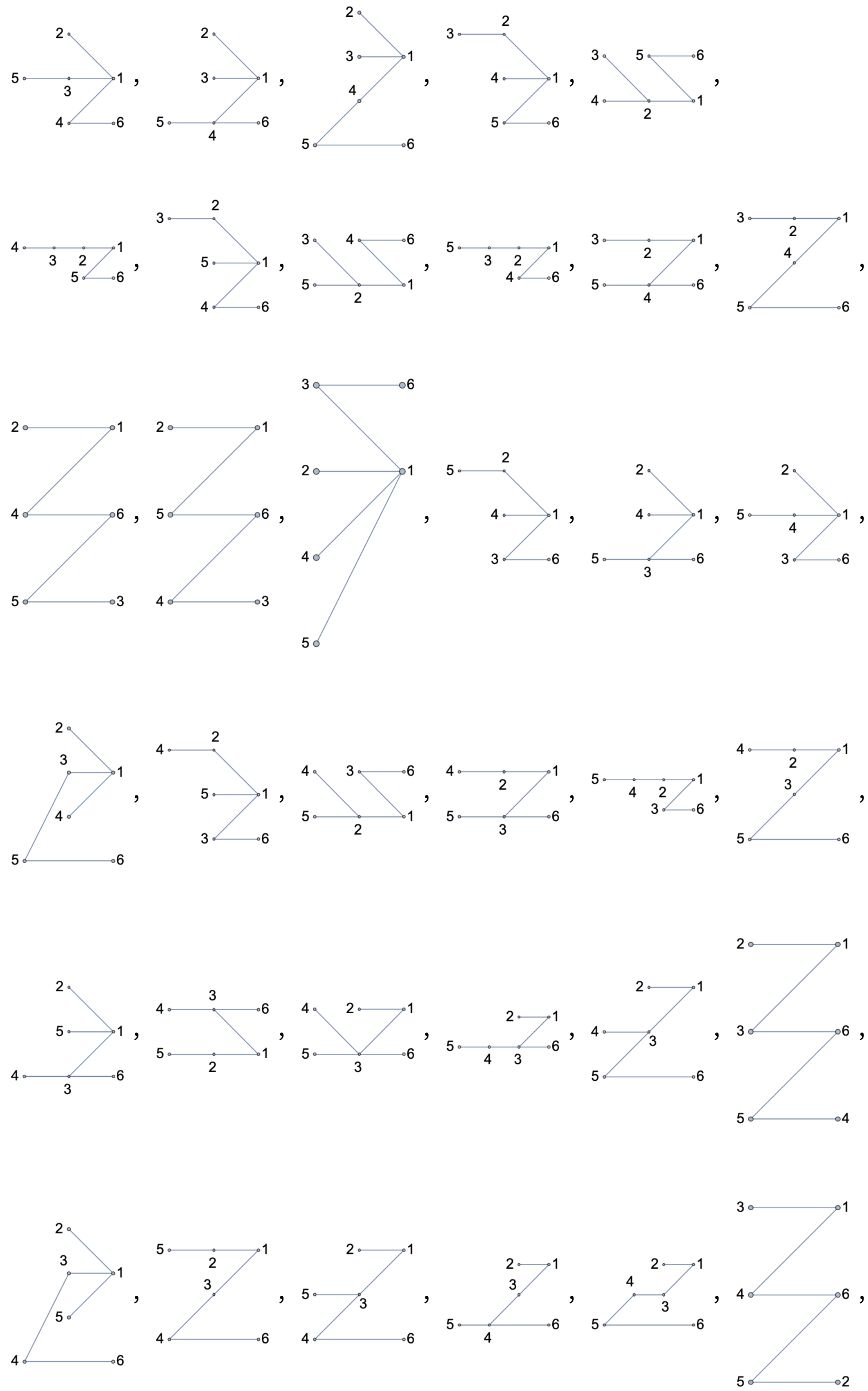
In[ ]:=
```
Ngluons = 5;
Numerator5gluonsTree = GIAOTree[Ngluons, p];
```

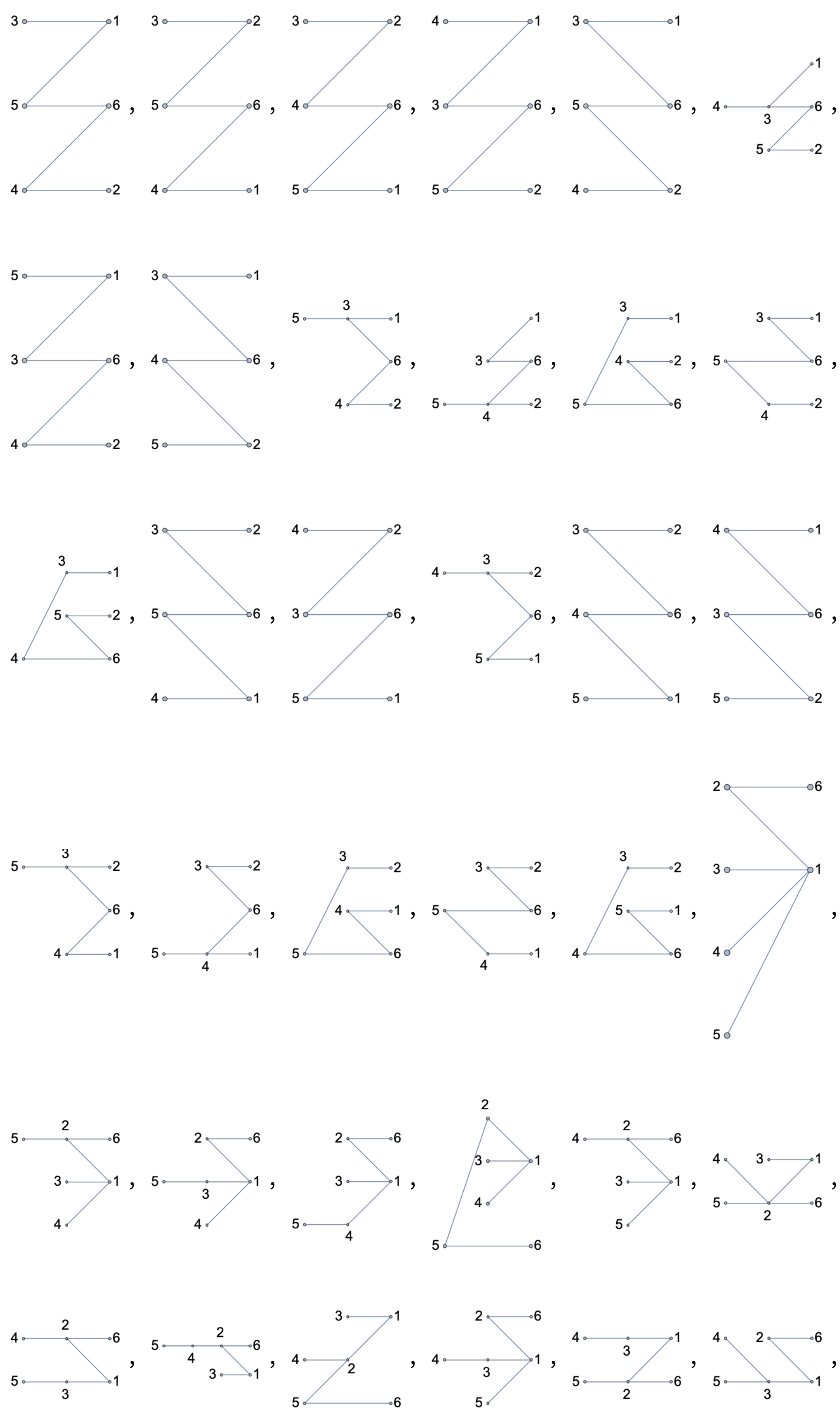
Displaying the graphs

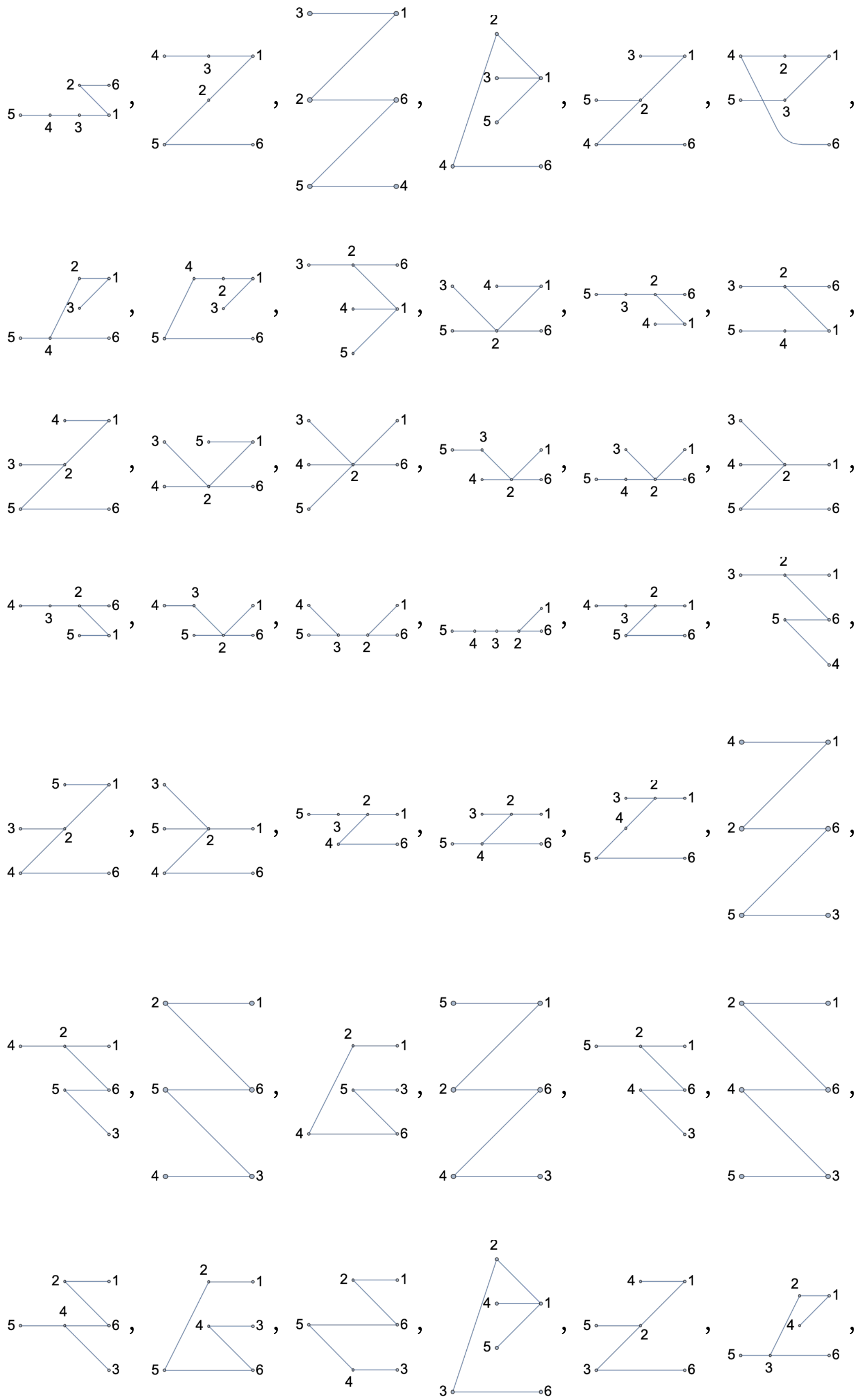
```
tijplot[Ngluons, #] & /@ (#[[2]] & /@ Numerator5gluonsTree)
```

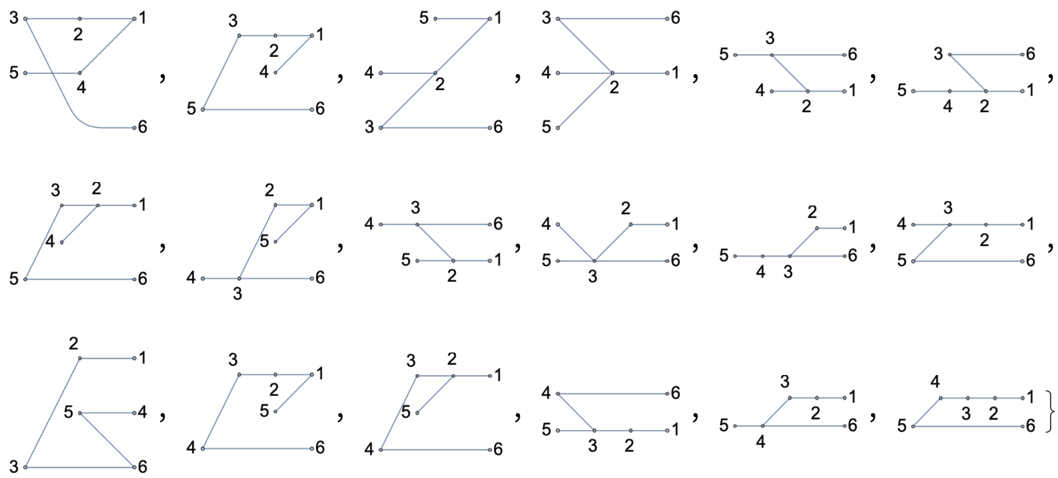
Out[ ]=











Displaying the numerator factors

```
In[ ]:= Numerator5gluons = GINum[Ngluons, p];
```

```
In[ ]:= DeleteDuplicates@Denominator@(List @@ Numerator5gluons)
        Length[%]
```

```
Out[ ]=
{k1 · p k3 · p k4 · p k5 · p, k1 · p k2 · p k4 · p k5 · p, k1 · p k2 · p k3 · p k5 · p,
 k1 · p k2 · p k3 · p k4 · p, k1 · p k2 · p k5 · p (k1 · p + k4 · p + k5 · p),
 k1 · p k2 · p (k1 · p + k4 · p) k5 · p, k1 · p k2 · p k4 · p (k1 · p + k4 · p + k5 · p),
 k1 · p k2 · p k4 · p (k1 · p + k5 · p), k1 · p k2 · p k5 · p (k1 · p + k3 · p + k5 · p),
 k1 · p k2 · p (k1 · p + k3 · p) k5 · p, k1 · p k2 · p k3 · p (k1 · p + k3 · p + k5 · p),
 k1 · p k2 · p k3 · p (k1 · p + k5 · p), k1 · p k2 · p k4 · p (k1 · p + k3 · p + k4 · p),
 k1 · p k2 · p (k1 · p + k3 · p) k4 · p, k1 · p k2 · p k3 · p (k1 · p + k3 · p + k4 · p),
 k1 · p k2 · p k3 · p (k1 · p + k4 · p), k1 · p k3 · p k5 · p (k1 · p + k2 · p + k5 · p),
 k1 · p (k1 · p + k2 · p) k3 · p k5 · p, k1 · p k2 · p k3 · p (k1 · p + k2 · p + k5 · p),
 k1 · p k3 · p k4 · p (k1 · p + k2 · p + k4 · p), k1 · p (k1 · p + k2 · p) k3 · p k4 · p,
 k1 · p k2 · p k3 · p (k1 · p + k2 · p + k4 · p), k1 · p k3 · p (k1 · p + k2 · p + k3 · p) k4 · p,
 k1 · p k2 · p (k1 · p + k2 · p + k3 · p) k4 · p, k1 · p k4 · p k5 · p,
 k1 · p (k1 · p + k2 · p + k3 · p) k4 · p, k1 · p k3 · p k5 · p,
 k1 · p k3 · p (k1 · p + k2 · p + k4 · p), k1 · p k3 · p k4 · p,
 k1 · p k3 · p (k1 · p + k2 · p + k5 · p), k1 · p k2 · p k5 · p,
 k1 · p k2 · p (k1 · p + k3 · p + k4 · p), k1 · p k2 · p k4 · p,
 k1 · p k2 · p (k1 · p + k3 · p + k5 · p), k1 · p k2 · p k3 · p,
 k1 · p k2 · p (k1 · p + k4 · p + k5 · p), k1 · p k2 · p (k1 · p + k5 · p),
 k1 · p k2 · p (k1 · p + k4 · p), k1 · p k2 · p (k1 · p + k3 · p), k1 · p (k1 · p + k2 · p) k3 · p,
 k1 · p k5 · p, k1 · p k4 · p, k1 · p k3 · p, k1 · p k2 · p, k1 · p}
```

```
Out[ ]=
45
```

The six gluons case

```
In[ ]:= Ngluons = 6;
```

In[ ]:=
```
Timing[Numerator6gluons = GINum[Ngluons, p];][[1]]
```
Out[ ]=
```
1.90963
```
In[ ]:=
```
DeleteDuplicates@Denominator@(List @@ Numerator6gluons);
Length[%]
```
Out[ ]=
```
226
```
In[ ]:=
```
Save["numerator-six-gluons.txt", Numerator6gluons]
```

The seven gluons case

In[ ]:=
```
Ngluons = 7;
```
In[ ]:=
```
Timing[Numerator7gluons = GINum[Ngluons, p];][[1]]
```
Out[ ]=
```
35.4813
```
In[ ]:=
```
DeleteDuplicates@Denominator@(List @@ Numerator7gluons);
Length[%]
```
Out[ ]=
```
1113
```
In[ ]:=
```
Save["numerator-seven-gluons.txt", Numerator7gluons]
```

The eight gluons case

In[ ]:=
```
Ngluons = 8;
```
In[ ]:=
```
Timing[Numerator8gluons = GINum[Ngluons, p];][[1]]
```
Out[ ]=
```
854.979
```
In[ ]:=
```
DeleteDuplicates@Denominator@(List @@ Numerator8gluons);
Length[%]
```
Out[ ]=
```
5230
```
In[ ]:=
```
Save["numerator-eight-gluons.txt", Numerator8gluons]
```