Nonexistence of Tensor Products for Hilbert Spaces

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November 8, 2025

Contents

1 Introduction

In linear algebra, we can define the tensor product $V \otimes_{alg} W$ of two vector spaces V, W, identified by a universal property:

Theorem 1.1 *For two vector spaces V, W (possibly infinite-dimensional), there exists a vector space $V \otimes_{alg} W$ and a bilinear map $q \colon V \times W \to V \otimes_{alg} W$ satisfying the following universal property: For any bilinear map $b \colon V \times W \to H$, there exists a unique linear map $\tilde{b} \colon V \otimes_{alg} W \to H$ such that the following diagram commutes:*

$$\begin{tikzcd}
V \times W \arrow[r, "b"] \arrow[d, "q"'] & H \\
V \otimes_{alg} W \arrow[ur, dashed, "\tilde{b}"'] &
\end{tikzcd}$$

Furthermore, the universal property implies that the vector space $V \otimes_{alg} W$ is unique up to linear isomorphism.

Note that the definition of the space $V \otimes_{alg} W$ is independent of the topology of V, W, and only depends on their algebraic properties; hence we call it the algebraic tensor product of V and W.

For two infinite-dimensional Hilbert spaces, we would like to describe the tensor product of two Hilbert spaces in a similar way. It is very natural to require all maps in the above diagram to be bounded, yielding the following definition:

Definition 1.1 ("Naive" Universal Property) *For two Hilbert spaces V, W, a Hilbert space X is called a tensor product of them if there exists a bounded bilinear map $q \colon V \times W \to X$ satisfying the following universal property: For any bounded bilinear map $b \colon V \times W \to H$, there exists a unique bounded linear map $\tilde{b} \colon X \to H$ such that the following diagram commutes:*

$$\begin{tikzcd}
V \times W \arrow[r, "b"] \arrow[d, "q"'] & H \\
X \arrow[ur, dashed, "\tilde{b}"'] &
\end{tikzcd}$$

However, this fails to define a tensor product. We will prove that such a tensor product cannot exist, i.e., the following theorem:

Theorem 1.2 *For any two infinite-dimensional Hilbert spaces V, W, there does not exist a Hilbert space X and a bounded bilinear map $q\colon V \times W \to X$ satisfying the Definition 1.1.*

Here we cite Garrett's note[1] to illustrate the significance of this theorem:

The non-existence of tensor products of infinite-dimensional Hilbert spaces is important in practice, not only as a cautionary tale about naive category theory, insofar as it leads to Grothendiecks idea of nuclear spaces. A main feature of nuclear spaces is that they do admit tensor products. The original explicit example of this was Schwartz Kernel Theorem, although earlier discussions of extending differential operators to subspaces of L^2 can be recast in such terms using Sobolev spaces.

2 Preparation: Trace Class Operators

In this section, we realize the algebraic tensor product as operators on Hilbert space and provide the definition and basic properties of trace class operators for readers unfamiliar with them. Proofs for some properties (not all of them) can be found in the appendix.

For a Hilbert space V, we can naturally embed

$$V \otimes_{alg} V^* = \{\sum_{i=1}^{n} v_i \otimes v_i^* \mid v_i \in V, v_i^* \in V^*(i = 1, 2, \dots, n), n \in \mathbb{N}\}$$

into the set $\mathcal{L}(V)$ of bounded linear operators on V:

$$\begin{aligned} \sum_{i=1}^{n} v_i \otimes v_i^*\colon V &\longrightarrow V \\ u &\longmapsto \sum_{i=1}^{n} v_i \cdot v_i^*(u) \end{aligned}$$

[1]Garrett also gives a proof of this theorem in the same note, but there is a flaw: he additionally assumes that the map $q\colon V \times W \to X$ preserves orthogonality, i.e., if $v_1 \perp v_2$ or $w_1 \perp w_2$, then $q(v_1, w_1) \perp q(v_2, w_2)$. However, the note still provide some good insight in the theorem.

Thus, we always treat $V \otimes_{alg} V^*$ as operators on V. In fact, $V \otimes_{alg} V^*$ is exactly the set of all finite-rank operators on V, i.e.,

$$V \otimes_{alg} V^* = \{T \in \mathcal{L}(V) \mid \dim \operatorname{im} T < +\infty\}.$$

Informally, we can think of finite-rank operators as finite-sized matrices.

For a finite-rank operator T, we define its norm to be:

$$\|T\|_1 = \inf\{\sum_{i=1}^{n} \|v_i\| \|v_i^*\| \mid T = \sum_{i=1}^{n} v_i \otimes v_i^*\}$$

It is not difficult to verify that this does be a norm on $V \otimes_{alg} V^*$[2], and it is not less than the operator norm $\|\cdot\|_{op}$. We denote the completion of $V \otimes_{alg} V^*$ under this norm by $\mathcal{L}^1(V)$, which is a subspace of $\mathcal{L}(V)$. We call the operators in $\mathcal{L}^1(V)$ trace class operators.

From the definition of trace class operators, it is not difficult to verify that for any bounded linear operator A and a trace class operator T, AT is still trace class, and $\|AT\|_1 \leqslant \|A\|_{op} \|T\|_1$ (similarly for TA). Thus $\mathcal{L}^1(V)$ is actually an ideal of $\mathcal{L}(V)$, and we therefore also call $\mathcal{L}^1(V)$ the trace ideal.

The reason why we call operators in $\mathcal{L}^1(V)$ trace class is that we can actually define their trace. Let $B \subseteq V$ be any complete orthonormal basis of V. For any finite-rank operator $T \in V \otimes_{alg} V^*$, we define

$$\operatorname{tr} T = \sum_{e \in B} \langle e, Te \rangle .$$

One can show that the linear functional $\operatorname{tr}\colon V \otimes_{alg} V^* \to \mathbb{C}$ is bounded and independent of the choice of basis B, so it uniquely extends to a linear functional on $\mathcal{L}^1(V)$. One can further prove that the equality $\operatorname{tr} T = \sum_{e \in B} \langle e, Te \rangle$ holds for any trace class operator T, hence the linear functional tr is the trace for infinite-dimensional case.

Let $\mathcal{K}(V)$ denote the set of all compact operators on V, which is a Banach space under the operator norm. We will later use the following properties of the trace ideal:

[2] We will later prove that this is the trace norm of T as a trace class operator.

Theorem 2.1 (Dual Spaces) *For any Hilbert space V, the bilinear map*

$$(A, T) \longmapsto \operatorname{tr}(AT), \qquad \forall T \in \mathcal{L}^1(V), A \in \mathcal{K}(V) \text{ or } \mathcal{L}(V)$$

yields isometric isomorphisms between Banach spaces:

$$(\mathcal{K}(V))^* \cong \mathcal{L}^1(V), \qquad \big(\mathcal{L}^1(V)\big)^* \cong \mathcal{L}(V).$$

In particular, when V is of infinite dimension, $(\mathcal{K}(V))^{**} \ncong \mathcal{K}(V)$, i.e., $\mathcal{K}(V)$ is not reflexive.

Theorem 2.2 (Universal Property) *The bounded bilinear map $V \times V^* \to \mathcal{L}^1(V)$ satisfies the following universal property: For any Banach space H and bounded bilinear map $b\colon V \times V^* \to H$, there exists a unique bounded linear map $\hat{b}\colon \mathcal{L}^1(V) \to H$ such that the following diagram commutes:*

$$\begin{tikzcd} V \times V^* \arrow[r, "b"] \arrow[d, "\iota"'] & H \\ \mathcal{L}^1(V) \arrow[ur, dashed, "\hat{b}"'] & \end{tikzcd}$$

where ι is defined by $\iota(v, v^*) = v \otimes v^*$.

We will also use a corollary of this:

Corollary 2.1 *For bounded bilinear maps $b_i\colon V \times V^* \to X_i, (i = 1, 2)$ and a bounded linear map $f\colon X_1 \to X_2$, we have:*

$$\begin{tikzcd} V \times V^* \arrow[r, "b_2"] \arrow[d, "b_1"'] & X_2 \\ X_1 \arrow[ur, "f"'] & \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \mathcal{L}^1(V) \arrow[r, "\hat{b}_2"] \arrow[d, "\hat{b}_1"'] & X_2 \\ X_1 \arrow[ur, "f"'] & \end{tikzcd}$$

3 Proof of the Main Theorem

We prove the main theorem by contradiction. For two infinite-dimensional Hilbert spaces V, W, assume that there exists a Hilbert space X and a bounded bilinear map $q\colon V \times W \to X$ satisfying Definition 1.1.

Take a countable set of orthonormal vectors $\{e_i\}_{i\in\mathbb{N}}$ in V and consider the Hilbert subspace $H = \overline{\operatorname{span}\{e_i \mid i \in \mathbb{N}\}}$ spanned by them. After taking a countable set of orthonormal vectors $\{f_j\}_{j\in\mathbb{N}}$ in W, we can realize H^* as a subspace $\overline{\operatorname{span}\{f_j \mid j \in \mathbb{N}\}}$ of W. We will show that the tensor product of H, H^* exists.

Let P_H, P_{H^*} denote the orthogonal projections onto H, H^* respectively. Take the Hilbert space $Y \subset X$ to be the closure of the following space:

$$\operatorname{span}\{q\left(P_H v, P_{H^*} w\right) \mid v \in V, w \in W\}$$

and consider the bounded bilinear map $q(P_H \cdot, P_{H^*} \cdot)\colon H \times H^* \to Y$ (still denoted by q). For any bounded bilinear map $b\colon H \times H^* \to Z$, using the universal property of X we obtain the following commutative diagram:

$$\begin{tikzcd}
H \times H^* \arrow[r, hook] \arrow[d] \arrow[rr, bend left, "\mathrm{id}"] & V \times W \arrow[r, "P_H \times P_{H^*}"] \arrow[d, "q"] & H \times H^* \arrow[d, "b"] \\
Y \arrow[r, hook, "i_Y"] & X \arrow[r, dashed, "B"'] & Z
\end{tikzcd}$$

This shows that the map $\tilde{b} = B \circ i_Y$ satisfies the commutative diagram:

$$\begin{tikzcd}
H \times H^* \arrow[r, "b"] \arrow[d, "q"] & Z \\
Y \arrow[ur, "\tilde{b}"'] &
\end{tikzcd}$$

For two maps $\tilde{b}, \tilde{b}'\colon Y \to Z$ that make the above diagram commute, they take the same value at $q(h, h^*) \in Y$, hence they are identical on the dense subspace $\operatorname{span}\{q\left(P_H v, P_{H^*} w\right) \mid v \in V, w \in W\}$ of Y, so necessarily $\tilde{b} = \tilde{b}'$. Thus, the map $\tilde{b}$ making the above diagram commute exists and is unique, which shows that Y is indeed the tensor product of H, H^*.

Next, we compare Y with $\mathcal{L}^1(H)$. According to the universal properties of $\mathcal{L}^1(H)$

and Y, for any Hilbert space Z, we have one-to-one correspondences:

$$\begin{aligned}
\mathcal{L}(Y,Z) &\xrightarrow{1:1} \{\text{bounded bilinear map } H\times H^* \to Z\}\\
\tilde{\eta} &\longmapsto \eta = \tilde{\eta}\circ q;\\
\mathcal{L}\left(\mathcal{L}^1(H),Z\right) &\xrightarrow{1:1} \{\text{bounded bilinear map } H\times H^* \to Z\}\\
\hat{\eta} &\longmapsto \eta = \hat{\eta}\circ \iota.
\end{aligned}$$

Here $\mathcal{L}(X_1,X_2)$ is the set of all bounded linear maps from the normed space X_1 to X_2. So we can define a one-to-one correspondence from $\mathcal{L}(Y,Z)$ to $\mathcal{L}\left(\mathcal{L}^1(H),Z\right)$. For $\tilde{\eta}\in\mathcal{L}(Y,Z)$, its corresponding $\hat{\eta}\in\mathcal{L}\left(\mathcal{L}^1(H),Z\right)$ is identified by the following commutative diagram:

$$\begin{tikzcd}
H\times H^* \arrow[rr, bend left, "\hat{\eta}\circ\iota=\tilde{\eta}\circ q"] \arrow[dr, "\iota"] \arrow[ddr, "q"'] & & \mathbb{C} \\
& \mathcal{L}^1(H) \arrow[ur, "\hat{\eta}"] & \\
& Y \arrow[uur, "\tilde{\eta}"'] &
\end{tikzcd}$$

Now, using the corollary 2.1 in the previous section, we obtain the commutative diagram:

$$\begin{tikzcd}
\mathcal{L}^1(H) \arrow[r, "\hat{\eta}"] \arrow[d, "\hat{q}"'] & Z \\
Y \arrow[ur, "\tilde{\eta}"'] &
\end{tikzcd}$$

That is, $\hat{\eta}=\tilde{\eta}\circ\hat{q}$, where $\hat{q}\colon \mathcal{L}^1(H)\to Y$ is the bounded linear map induced by $q\colon H\times H^*\to Y$, and it is independent of $\hat{\eta},\tilde{\eta}$.

In particular, take the space Z to be $\mathbb{C}$ and we obtain a bounded linear bijection:

$$\begin{aligned}
Y^* &\longrightarrow \left(\mathcal{L}^1(H)\right)^* \cong \mathcal{L}(H)\\
\tilde{\eta} &\longmapsto \tilde{\eta}\circ\hat{q}.
\end{aligned}$$

Thus $\mathcal{L}(H)$ is homeomorphic to some Hilbert space Y^* due to the open mapping theorem. In particular it must be reflexive. Then $\mathcal{K}(H)$, as its closed subspace, would also be reflexive, but we have pointed out that this is impossible, leading to a contradiction. So far we complete the proof of Theorem 1.2.

4 Supplementary Topics

4.1 Tensor Product of Hilbert Spaces

We have to apologize to the reader: the title of this note is somewhat exaggerated, because actually we can define the tensor product of two Hilbert spaces V, W in the following way:

Step 1: Take the algebraic tensor product $V \otimes_{alg} W$ of V, W, on which we can assign a unique inner product satisfying[3]

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \langle v_1, v_2 \rangle \langle w_1, w_2 \rangle, \qquad \forall v_1, v_2 \in V, w_1, w_2 \in W.$$

Step 2: Define $V \otimes W$ as the completion of $V \otimes_{alg} W$ with respect to the above inner product.

Specifically, elements of $V \otimes W$ can be represented by square-summable series in $V \otimes_{alg} W$:

$$V \otimes W = \{\sum_{n=1}^{+\infty} v_i \otimes w_i \mid \sum_{n=1}^{\infty} \|v_i\|^2 \|w_i\|^2 < +\infty\}$$

Remark 4.1 *Roughly speaking, when defining the trace ideal $\mathcal{L}^1(V)$, we used the L^1 norm for completion, while when defining the tensor product we used the L^2 norm, which are not equivalent when V is of infinite dimension. This partly explains why we cannot use the universal property 1.1 to define the tensor product: the universal property requires the tensor product to take the L^1 norm, which cannot be realized by an inner product.*

The tensor product defined in this way satisfies a universal property different from that of $\mathcal{L}^1(V)$, which requires more concepts to state:

Lemma 4.1 *The bilinear map $V \times W \to V \otimes W$ is weakly Hilbert-Schmidt, and for any weakly Hilbert-Schmidt bilinear map $b\colon V \times W \to H$, there exists a unique bounded linear map such that the following diagram commutes:*

[3] We need to prove that this inner product is well-defined and positive definite, but we omit this proof.

$$\begin{tikzcd}
V \times W \arrow[r, "b"] \arrow[d] & H \\
V \otimes W \arrow[ur, dashed] &
\end{tikzcd}$$

Here, weakly Hilbert-Schmidt means that there exists a constant $C > 0$ such that for any complete orthonormal bases $B_v \subseteq V, B_w \subseteq W$ and any $h \in H$, we have:

$$\sum_{e \in B_v, f \in B_w} |\langle b(e, f), h \rangle|^2 \leqslant C \|h\|^2$$

We will not prove this lemma.

4.2 Trace Norm

For the trace ideal $\mathcal{L}^1(V)$, some textbooks define its norm using the trace, and the norm defined in this way is called the trace norm. We will give the definition of the trace norm and prove that for finite-rank operators, it is the same as the norm $\|\cdot\|_1$.

For an operator T on a Hilbert space, we can define its polar decomposition:

$$T = U|T|$$

where $|T| = \sqrt{T^*T}$, and U is a partial isometry (which is an isometry from $\overline{\operatorname{im}|T|}$ to $\overline{\operatorname{im} T}$). Then we can define the trace norm of T as:

$$\operatorname{tr}|T| := \sum_{e \in B} \langle e, Te \rangle$$

Here B is any complete orthonormal basis. And we can define the trace ideal in another way:

$$\mathcal{L}^1(V) := \{T \in \mathcal{L}(V) \mid \operatorname{tr}|T| < +\infty\}$$

If we know that finite-rank operators are dense in the new $\mathcal{L}^1(V)$, then together with the following lemma we know that the trace ideals defined by the two ways are the same. But we will not prove the density of finite-rank operators.

Lemma 4.2 *For a finite-rank operator T, we have the equation:*

$$\operatorname{tr}|T| = \inf\{\sum_{i=1}^n \|v_i\| \|v_i^*\| \mid T = \sum_{i=1}^n v_i \otimes v_i^*\}$$

Proof *First, for any representation of T as a sum of rank-one operators:*

$$T(\cdot) = \sum_{i=1}^{n} v_i \otimes v_i^* = \sum_{i=1}^{n} \langle u_i, \cdot \rangle\, v_i, \qquad (v_i^* = \langle u_i, \cdot \rangle\,, \forall 1 \leqslant i \leqslant n).$$

Let the polar decomposition of T be $T = U\,|T|$, then

$$|T| = U^*T = \sum_{i=1}^{n} \langle u_i, \cdot \rangle\, U^* v_i$$

Let $B \subseteq V$ be a complete orthonormal basis of V, then

$$\begin{aligned}\operatorname{tr}|T| &= \sum_{e\in B}\sum_{i=1}^{n} \langle u_i, e\rangle \langle e, U^* v_i\rangle = \sum_{i=1}^{n}\sum_{e\in B} \langle u_i, e\rangle \langle e, U^* v_i\rangle = \sum_{i=1}^{n} \langle u_i, U^* v_i\rangle \\ &\leqslant \sum_{i=1}^{n} \|U^*\|_{op} \|u_i\|\,\|v_i\| = \sum_{i=1}^{n} \|v_i\|\,\|v_i^*\|\end{aligned}$$

Take the infimum and we have:

$$\operatorname{tr}|T| \leqslant \inf\{\sum_{i=1}^{n} \|v_i\|\,\|v_i^*\| \mid T = \sum_{i=1}^{n} v_i \otimes v_i^*\}$$

Next we prove the inverse inequality. Take an orthogonal decomposition $V = V_1 \oplus V_2$ such that V_1 is a finite-dimensional invariant subspace of T and $T|_{V_2} \equiv 0$. Then it suffices to prove the inverse inequality for the operator $T|_{V_1}$ on the finite-dimensional space. In this case, T can be identified with an $n \times n$ complex matrix. We take the singular value decomposition of T[4]:

$$T = \sum_{i=1}^{n} s_i \langle u_i, \cdot \rangle\, v_i,$$

where $s_i \in \mathbb{R}$ are the singular values of T, $\{u_i\}, \{v_j\}$ are two orthonormal bases of V_1, and we have:

$$|T| = \sum_{i=1}^{n} s_i \langle v_i, \cdot \rangle\, v_i.$$

Using the orthonormal basis $\{v_i\}$ to compute the trace, we get:

$$\operatorname{tr}|T| = \sum_{i=1}^{n} s_i = \sum_{i=1}^{n} s_i \|u_i\|\,\|v_i\| \geqslant \inf\{\sum_{i=1}^{n} \|v_i\|\,\|v_i^*\| \mid T = \sum_{i=1}^{n} v_i \otimes v_i^*\}$$

Combining the two inequalities, we obtain the proof of the lemma.

[4]The above equality is equivalent to the usual singular value decomposition: we can first take the singular value decomposition $V \operatorname{diag}(s_1, \ldots, s_N)U$ of the matrix of T in some orthonormal basis $\{e_i\}$, then set $v_i = Ve_i, u_i = U^*e_i$.

Remark 4.2 *Using the continuity of the trace operator* tr *and the spectral decomposition of compact self-adjoint operators, we can slightly modify the proof to show that the equation in the lemma holds for any trace class operators T (in this case, requiring the infinite sum $\sum_{i=1}^{+\infty} v_i \otimes v_i^*$ to converge to T in the trace norm).*

A Properties of Trace Class Operators

We provide proofs for some properties in Section 2.

Proposition A.1 *Let $T \in V \otimes_{alg} V^*$ be a finite-rank operator, then*

(1) *If $T(\cdot) = \sum_{i=1}^{n} \langle u_i, \cdot \rangle v_i$, where $u_i, v_i \in V$, then*

$$\operatorname{tr} T = \sum_{i=1}^{n} \langle u_i, v_i \rangle$$

In particular, tr *is independent of the choice of basis.*

(2) *We have the inequality*

$$|\operatorname{tr} T| \leqslant \|T\|_1 .$$

So $\operatorname{tr}\colon V \otimes_{alg} V^* \to \mathbb{C}$ is a bounded linear operator.

Proof *For* (1)*, take a complete orthonormal basis B of V, then we can compute:*

$$\sum_{e \in B} \langle e, Te \rangle = \sum_{e \in B} \sum_{i=1}^{n} \langle u_i, e \rangle \langle e, v_i \rangle = \sum_{i=1}^{n} \sum_{e \in B} \langle u_i, e \rangle \langle e, v_i \rangle = \sum_{i=1}^{n} \langle u_i, v_i \rangle ,$$

where the last equality uses Parseval's identity.

For (2)*, for any representation of T as $T(\cdot) = \sum_{i=1}^{n} \langle u_i, \cdot \rangle v_i$, using the result from* (1)*, we have:*

$$|\operatorname{tr} T| = \left| \sum_{i=1}^{n} \langle u_i, v_i \rangle \right| \leqslant \sum_{i=1}^{n} |\langle u_i, v_1 \rangle| \leqslant \sum_{i=1}^{n} \|u_i\| \|v_i\|$$

Taking the infimum over all possible representations of T yields $|\operatorname{tr} T| \leqslant \|T\|_1$.

Proposition A.2 *For any trace class operator $T \in \mathcal{L}^1(V)$ and any complete orthonormal basis B, we have:*

$$\operatorname{tr} T = \sum_{e \in B} \langle e, Te \rangle$$

Proof *For any finite subset N of B, let V_N be the subspace spanned by N, and let P_N be the orthogonal projection onto V_N. Take a sequence of finite-rank operators T_m ($m \in \mathbb{N}$) converging to T with respect to the norm $\|\cdot\|_1$. For any finite subset N of B and any $m \in \mathbb{N}$, set:*

$$t_{m,N} = \sum_{e \in N} \langle e, T_m e \rangle = \operatorname{tr}(T_m P_N).$$

By the definition of the functional tr*, we have:*

$$\lim_m \lim_N t_{m,N} = \lim_m \operatorname{tr} T_m = \operatorname{tr} T.$$

And due to the continuity of tr*, we have:*

$$\begin{aligned}\lim_N \lim_m t_{m,N} &= \lim_N \lim_m \operatorname{tr}(T_m P_N) = \lim_N \operatorname{tr}(T P_N) \\ &= \lim_N \sum_{e \in N} \langle e, Te \rangle = \sum_{e \in B} \langle e, Te \rangle .\end{aligned}$$

So we only need to show that we can switch the order of limits. It suffices to show that $t_{m,N}$ converges uniformly to $\operatorname{tr}(TP_N)$ as m tends to infinity.

Indeed, we have:

$$\begin{aligned}|t_{m,N} - \operatorname{tr}(TP_N)| &= \left|\operatorname{tr}\big((T - T_m)P_N\big)\right| \\ &\leqslant \|(T - T_m)P_N\|_1 \\ &\leqslant \|P_N\|_{op} \|T - T_m\|_1 = \|T - T_m\|_1 .\end{aligned}$$

This is sufficient to show that $t_{m,N}$ converges uniformly to $\operatorname{tr}(TP_N)$.

The following parts are proofs of Theorem 2.2 and Corollary 2.1.

Proof (Proof of Theorem 2.2) *The uniqueness of the bounded linear map $\hat{b}$ is easy. If bounded linear maps $\hat{b}, \hat{b}'$ both make the diagram commute, then their restrictions to $V \otimes_{alg} V^*$ satisfy the commutative diagram:*

$$\begin{tikzcd} V \times V^* \arrow[r, "b"] \arrow[d] & H \\ V \otimes_{alg} V^* \arrow[ur, "B"'] & \end{tikzcd} \qquad B = \hat{b} \text{ or } \hat{b}'$$

By the universality of $V \otimes_{alg} V^*$, we know that $\hat{b}$ and $\hat{b}'$ are identical on $V \otimes_{alg} V^*$, and since $V \otimes_{alg} V^*$ is dense in $\mathcal{L}^1(V)$, we know that $\hat{b} = \hat{b}'$.

Now consider the existence of $\hat{b}$. Since $V \otimes_{alg} V^*$ is dense in $\mathcal{L}^1(V)$, we only need to define $\hat{b}$ on $V \otimes_{alg} V^*$ and verify that it is bounded and makes the diagram commute. In fact, from the universal property of the algebraic tensor product, we easily obtain the map $\hat{b}$ that makes the diagram commute:

$$\begin{aligned} \hat{b} : V \otimes_{alg} V^* &\longrightarrow H \\ \sum_{i=1}^{n} v_i \otimes v_i^* &\longmapsto \sum_{i=1}^{n} b(v_1, v_i^*) \end{aligned}$$

And for each finite-rank operator $T \in V \otimes_{alg} V^*$, for any $\varepsilon > 0$, we can assume $T = \sum_{i=1}^{n} v_i \otimes v_i^*$ such that:

$$\sum_{i=1}^{n} \|v_i\| \, \|v_i^*\| \leqslant \|T\|_1 + \varepsilon$$

Then we have:

$$\left\|\hat{b}(T)\right\| = \left\| \sum_{i=1}^{n} b(v_i, v_i^*) \right\| \leqslant \sum_{i=1}^{n} \|b(v_i, v_i^*)\| \leqslant C \sum_{i=1}^{n} \|v_i\| \, \|v_i^*\| \leqslant C \, \|T\|_1 + C\varepsilon.$$

Here $C = \sup_{(v,v^*) \in V \times V^*} \frac{\|b(v,v^*)\|}{\|v\|\|v^*\|}$ is a finite constant depending only on b. Noting that ε is arbitrary, we obtain:

$$\left\|\hat{b}(T)\right\| \leqslant C \, \|T\|_1 , \qquad \forall \, T \in V \otimes_{alg} V^*.$$

That is, $\hat{b}$ is bounded. This completes the proof.

Proof (Proof of Corollary 2.1) *We assume the left diagram commutes and prove that the right one commutes, i.e., $\hat{b}_2 = f \circ \hat{b}_1$. Due to the universality of $\mathcal{L}^1(V)$, we only need to prove the following commutative diagram:*

$$\begin{tikzcd} V \times V^* \arrow[r, "b_2"] \arrow[d, "\iota"'] & X_2 \\ \mathcal{L}^1(V) \arrow[ur, "f \circ \hat{b}_1"'] & \end{tikzcd}$$

That is, $b_2 = f \circ \hat{b}_1 \circ \iota$. And from the commutative diagram:

$$
\begin{tikzcd}
V \times V^* \arrow[dr, "\iota"] \arrow[drr, bend left, "b_2"] \arrow[ddr, bend right, "b_1"'] & & \\
& \mathcal{L}^1(V) \arrow[d, "\hat{b}_1"] & X_2 \\
& X_1 \arrow[ur, "f"'] &
\end{tikzcd}
$$

it is easy to see that $f \circ \hat{b}_1 \circ \iota = f \circ b_1 = b_2$. *This completes the proof.*