Topological Proof of Bézout's Theorem

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November 28, 2025

In algebraic geometry, Bézout's theorem is a rather fundamental result:

Theorem 1 (Bézout) *In the complex projective plane*

$$\mathbb{C}P^2 = \{[X : Y : Z] \mid X, Y, Z \in \mathbb{C}, |X| + |Y| + |Z| \neq 0\}$$

for two algebraic curves $C_i\colon f_i(X, Y, Z) = 0$ *(*$i = 1, 2$*) defined by irreducible homogeneous polynomials. The number of their intersection points are either infinitely or* $\deg f_1 \cdot \deg f_2$*(counting with multiplicities).*

We additionally assume that C_1 and C_2 are non-singular, meaning they are manifolds. This is to avoid going too far into algebra: in fact, one could consider parameterizing singular algebraic curves using some complex manifold M.

In our statement, the term "multiplicity" of an intersection point is a little ambiguous, and there are several ways to define it. Here we adopt a geometric version: for an intersection point x of C_1 and C_2, if after a small perturbation of C_1 or C_2, the point x splits into m transversely intersecting points, then the multiplicity of x is m. This definition coincide with the definition of local intersection numbers in differential topology. This makes the topological tools available for our problem.

A natural approach to prove Bézout's theorem is to transform it into a problem of solving polynomial equations: we only need to count the number of triples $[X : Y : Z]$ satisfying the equations

$$\begin{cases} f_1(X, Y, Z) = 0 \\ f_2(X, Y, Z) = 0 \end{cases}$$

This idea can be generalized to arbitrary algebraically closed fields.

However, here we provide a more topological proof, with a completely different perspective: two algebraic curves intersect due to their topological constraints.

Let us first introduce some notations. For an oriented submanifold X in $\mathbb{C}P^2$, let $[X] \in H_*(\mathbb{C}P^2)$ denote its fundamental class. We denote by $c \in H^2(\mathbb{C}P^2)$ the Poincaré dual of $[\mathbb{C}P^1]$. It is the generator of the cohomology ring $H^*(\mathbb{C}P^2)$ of the projective plane. Note that c^2 is the generator of the group $H^4(\mathbb{C}P^2)$, so we have

$$\int_{[\mathbb{C}P^2]} c^2 = 1.$$

The tool for our proof is the intersection number of two submanifolds, which is an integer depending only on the homotopy classes of the submanifolds. For any two oriented submanifolds X and Y of $\mathbb{C}P^2$, we denote their intersection number by

$$I(X,Y) = \int_{[\mathbb{C}P^2]} \mathrm{P.\,D.}[X] \smile \mathrm{P.\,D.}[Y].$$

Here $\smile$ is the cup product in cohomology. Results from differential topology tell us that when $\dim X + \dim Y = \dim \mathbb{C}P^2$ and X and Y intersect transversely, $I(X,Y)$ can be expressed as a signed sum over the intersection points:

$$I(X,Y) = \sum_{p \in X \cap Y} \mathrm{sgn}(p; X, Y),$$

Here $\mathrm{sgn}(p; X, Y) \in \{1, -1\}$ is determined by the orientations of X, Y and $\mathbb{C}P^2$.

For two algebraic curves defined by irreducible polynomials, we have the following lemma:

Lemma 1 *For two algebraic curves X and Y defined by irreducible polynomials (additionally assumed to be manifolds), suppose they intersect transversely. Then the sign* $\mathrm{sgn}(p; X, Y)$ *is the same at every intersection point p.*

We postpone the proof of this lemma to the end of this article. This lemma tells us that the intersection number of two algebraic curves is exactly equal to the number of intersection points. Thus, to prove Bézout's theorem, we only need to compute the intersection number of two algebraic curves.

We compute $I(C_1, C_2)$ by its definition. We first need to compute the fundamental classes $[C_i]$ of C_i. Note that $[\mathbb{C}P^1]$ is the generator of $H_2(\mathbb{C}P^2)$, so we can write $[C_i] = k_i[\mathbb{C}P^1]$ ($k_i \in \mathbb{Z}$). Considering the intersection number of C_i and $\mathbb{C}P^1$, we have

$$\begin{aligned}
&\text{Number of intersections of } C_i \text{ and } \mathbb{C}P^1 \\
&\qquad = \pm \int_{[\mathbb{C}P^2]} \mathrm{P.\,D.}[C_i] \smile \mathrm{P.\,D.}[\mathbb{C}P^1] \\
&\qquad = \pm \int_{[\mathbb{C}P^2]} k_i \cdot \mathrm{P.\,D.}[\mathbb{C}P^1] \smile \mathrm{P.\,D.}[\mathbb{C}P^1] \\
&\qquad = \pm \int_{[\mathbb{C}P^2]} k_i \cdot c^2 = \pm k_i
\end{aligned}$$

The number of intersections of $\mathbb{C}P^1$ and C_i is easy to compute, as it amounts to counting the intersections of a line with an algebraic curve. For example, after change of coordinate, we may assume that the intersection points are of the form $[x : 1 : 0] \in \mathbb{C}P^1$. Then $x \in \mathbb{C}$ satisfies the equation

$$f_i(x, 1, 0) = 0$$

This is a polynomial equation, and the number of its solutions is $\deg f_i$. Therefore, we obtain

$$[C_i] = \pm \deg f_i \cdot [\mathbb{C}P^1], \qquad i = 1, 2.$$

Knowing $[C_i]$, we can quickly compute the intersection number of C_1 and C_2:

$$\begin{aligned}
I(C_1, C_2) &= \int_{[\mathbb{C}P^2]} \mathrm{P.\,D.}[C_1] \smile \mathrm{P.\,D.}[C_2] \\
&= \pm \int_{[\mathbb{C}P^2]} \deg f_1 \cdot \deg f_2 \cdot \mathrm{P.\,D.}[\mathbb{C}P^1] \smile \mathrm{P.\,D.}[\mathbb{C}P^1] \\
&= \pm \int_{[\mathbb{C}P^2]} \deg f_1 \cdot \deg f_2 \cdot c^2 \\
&= \pm \deg f_1 \cdot \deg f_2
\end{aligned}$$

By the previous lemma, we have thus proved Bézout's theorem.

Finally, we provide the proof of Lemma 1.

proof (Proof of Lemma 1) *Since changing the orientations of X or Y changes all the signs* $\mathrm{sgn}(p; X, Y)$ *simultaneously, we only need to find orientations of X and Y such that* $\mathrm{sgn}(p; X, Y)$ *is always* 1.

For an (almost) complex manifold M, its complex structure naturally induces an orientation: we arbitrarily choose a basis $e_1, e_2, \ldots, e_n$ for complex vector space

T_pM. Then the basis for T_pM as a real vector space,

$$e_1, i \cdot e_1, e_2, i \cdot e_2, \ldots, e_n, i \cdot e_n$$

defines an orientation of T_pM, and it can be verified that this orientation does not depend on the choice of basis. The local orientations on each T_pM then defines an orientation on M. We choose the orientations of X, Y, and $\mathbb{C}P^2$ to be those induced by their complex structures.

For $p \in X \cap Y$, the sign $\operatorname{sgn}(p; X, Y)$ is determined using the direct sum decomposition

$$T_pX \oplus T_pY = T_p\mathbb{C}P^2.$$

The sign $\operatorname{sgn}(p; X, Y)$ will be 1 if the orientation induced by direct sum is the same as the orientation of $\mathbb{C}P^1$; otherwise, it will -1.

In our case, take a basis e_X for T_pX and a basis e_Y for T_pY. Then the orientation of their direct sum is given by

$$e_X, i \cdot e_X, e_Y, i \cdot e_Y$$

which matches the orientation of $T_p\mathbb{C}P^2$. Therefore, we always have

$$\operatorname{sgn}(p; X, Y) = 1.$$