Quantitative Method in Finance Final Project – Euro Option Pricing Methods

Professor: Daniel Totouom

Baiji (Packy) Ye
Charlie (Changlin) Yi
Ruibin Jacky Chen

Abstract:

Our paper provides an analysis of three key option pricing models: Black-Scholes-Merton (BSM), Heston, and Merton Jump Diffusion. We apply Python code to compare their predictions for European call and put option prices under a particular parameter according to random assumption. The analysis reveals that the Heston Model, with its stochastic volatility, and the Merton Model, incorporating jump risks, predict higher option values near the strike price compared to BSM. These results underscore the significance of volatility dynamics and sudden price jumps in accurate option pricing. Our paper concludes with a comparative analysis, highlighting the models' effectiveness in reflecting real-world market behaviors.

Keywords:

Option Pricing, Black-Scholes-Merton Model, Heston Model, Merton Jump Diffusion Model;

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1. Introduction

Option Pricing is one of the main focuses of the current financial industry, and the concept of continuous-time modeling occupies significant importance within Option Pricing. The purpose of this project report is to discuss three of the famous option pricing models – Black Scholes Model, which serves as an important foundation as the other two models. Heston Model, and Merton's Jump-Diffusion Model – in multiple facades. The project will create a mathematical intuition and potential pricing scheme (using python) for each of the three models. In this particular section, the necessary background knowledge of the three methods, both conceptually and mathematically, are discussed.

1.1 Introduction of the Black Scholes Merton Method (BSM)

The Black Scholes Merton (BSM) model represents a cornerstone in modern financial theory, particularly in the realm of option pricing. The Black-Scholes Formula-The Black-Scholes formula calculates the price of a European call option, which is an option to buy an asset at a specified strike price at a specified future date. The formula is given by

$$C(S,t) = S_t N(d_1) - Ke^{-r(T-t)} N(d_2)$$

Where C(S,t) is the price of the call option, S_t is the current stock price,K is the strike price of the option, r is the risk-free interest rate, t is the time to expiration,T is the time to maturity of the option , N is the cumulative distribution function of the standard normal distribution, d_1 and d_2 are calculated as follows with σ is the volatility of the stock's returns.

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

The Black-Scholes formula for a put option can be derived similarly.

$$P(S,t) = Ke^{-r(T-t)} N(-d_2) - S_t N(-d_1)$$

The model has been extended to include dividend-paying stocks, where the stock price is adjusted for dividends.In essence, the BSM model simplifies the complex nature of option pricing into a manageable formula, making it feasible for traders and investors to estimate the fair value of options with relative ease.

1.2 Introduction of the Heston Method

The Heston Model is one of the complex financial models used for option pricing, Developed and named after Steven Heston, The Heston Model is particularly advantageous for capturing the volatility smile of a particular option and it accurately describes the progression of volatility. It takes into account how volatility is changing given the market conditions, which an easier model such as Black Scholes Model cannot achieve. The underlying assumption of Heston Method assumes that the volatility is not constant and volatility itself follows a stochastic process. Heston Method also assumes that the stock price S_t follows a Geometric Brownian Motion such that:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^S$$

Where μ is the drift rate of the asset, S_t is the asset price at time t, V_t is the instantaneous variance, and dWt(S)is the brownian motion. Brownian motion has the property of having an initial value of 0, and that every increments of brownian motion from t to t+1 is independent, and Var(B(t)-B(s)) = t-s.

Then the other underlying assumption of Heston Method is how the variance will change with respect to dt and dW, named as Variance Dynamic, which assumes the change of Variance dVt equals:

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^v$$

In this particular equations, K is the mean reversion rate, theta is the mean of long term variance, sigma is the rate at which the volatility is changing, or can be also referred to as the changing rate of the volatility, dWt(V) just assumes another Brownian motion processes, but a process that has special correlation with the previous dWt(S), and the correlations between the two processes will be denoted as p.

1.3 Introduction of Jump Merton's Diffusion Method

The Jump Merton's Diffusion Method, introduced by Robert C. Merton in 1976.This method extends the classic Black-Scholes model by incorporating random jumps in asset prices, thus providing a comprehensive framework for option pricing and risk assessment.The method is grounded in the theory of stochastic calculus, particularly the concepts of Brownian motion and Poisson processes. The asset price S(t) at time t is modeled as follows:

$$S(t) = S(0) \exp\left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W(t) + \sum_{i=1}^{N(t)} (Y_i - 1) \right\}$$

μ is the drift rate of the asset, σ is the volatility, W(t) represents a standard Brownian motion, N(t) is a Poisson process with intensity λ, Y_i random variables representing the jump sizes.

The Jump Merton model is a jump-diffusion model where the logarithm of the asset price follows a diffusion process with jumps. The dynamics of the asset price under this model are given by the stochastic differential equation (SDE):

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) + S(t-) \sum_{i=1}^{N(dt)} (Y_i - 1)$$

This model addresses the limitations of the Black-Scholes model, which assumes a continuous and smooth evolution of asset prices. By incorporating jumps, the Jump Merton model can more accurately capture the sudden and significant changes in market prices, often observed in real-world financial markets.

2. Reference of Previous Research

The landscape of option pricing models has been extensively explored in academic literature, with significant contributions made towards understanding and developing the Black Scholes model, Heston model, and Jump Merton's Diffusion model.

The seminal work by Black and Scholes (1973) laid the foundation for modern financial derivatives pricing with their development of a theoretical valuation formula for options, highlighting the impossibility of sure profits from portfolios of long and short positions in options and their underlying stocks [1]. This model was further expanded upon by Hull and White (1987), who addressed the pricing of options on assets with stochastic volatilities, a critical extension of the Black Scholes model [2].

Heston (1993) made a pivotal contribution by presenting a closed-form solution for options with stochastic volatility, which included applications to bond and currency options [3]. This model allowed for arbitrary correlation between volatility and spot-asset returns, addressing a key limitation in earlier models.

The complexity of financial markets, particularly the occurrence of sudden jumps in asset prices, led to the development of Jump Diffusion models. The work by Merton on jump diffusion processes provided a framework for incorporating these jumps into option pricing models. This approach was further explored through various studies, including those by Broadie and Kaya (2004), who derived Monte Carlo simulation estimators for option price derivatives under stochastic volatility and jump diffusion models [4], and by Sepp (2003), who developed a methodology for pricing European-style options under jump diffusion processes with stochastic volatility [5].

The non-normality of asset returns and its implications for option pricing have been a topic of significant interest. Mozumder, Sorwar, and Dowd (2013) provided a comparative analysis of option pricing under non-normality [6]. Additionally, the incorporation of transaction costs into option pricing models has been explored, with Averbuj (2013) extending the Black-Scholes model to include these costs and proposing a unique convex solution to the corresponding evolution Dirichlet problem [7].

Recent advancements have focused on enhancing the computational efficiency and accuracy of these models. Sun (2015) examined the performance of jump-risk mitigation under a double Heston stochastic volatility jump-diffusion model [8], while Liu, Zhou, Wu, and Ge (2019) extended the Heston model to a hybrid option pricing model driven by multiscale stochastic volatility and jump diffusion process [9]. The use of quadrature methods for solving complex option problems was presented by Su, Chen, and Newton (2017), demonstrating the application from plain-vanilla Black–Scholes to Bermudan and American options under a stochastic volatility jump-diffusion returns process [10].

The calibration of nonlinear feedback option pricing models, as discussed by Sanfelici (2007), further highlights the evolving nature of these models in capturing market dynamics [11]. Boyarchenko and Levendorskii (2002) contributed to the understanding of barrier options and touch-and-out options under regular Lévy processes of exponential type, offering explicit formulas for these options [12].

These studies collectively underscore the continuous evolution and refinement of option pricing models, reflecting the dynamic nature of financial markets and the ongoing quest for more accurate and robust pricing methodologies.The paper is structured in the following way. Section 3 deals with the Model Assumption and Analysis. Section 4 considers Code Analysis. Our empirical find- ings are discussed in Section 5. Section 6 concludes.

3. Model Assumption and Analysis

This section of the paper is dedicated to present the mathematics that are relevant for each of the models and the detailed underlying assumptions of the models. Since BSM is the fundamentals of the other two models, the mathematics of BSM will be presented first:

3.1 Black Scholes Merton Method

Under the Black Scholes Model, we have the following five main assumptions:

- During the life of the option, the underlying stock pays constant dividends. Equivalently, we may assume that the dividend rate remains unchanged.
- The risk-free rate remains constant during the entire time intervals.
- The Stock prices follow a geometric Brownian motion, as previously described in the introduction sections.
- Assumes no transaction costs for option.
- There are absolutely no arbitrage opportunities for the duration of the options.

These five assumptions are essential for the Black Scholes Model to hold true, it won’t take long to realize that these assumptions are very unrealistic. (In reality, the rate will change, the stock price might not necessarily follow GBM, and there will be all kinds of additional fees and transaction costs)

Assuming that these assumption holds, then the following mathematical formula will hold true under the Black-Scholes World, noticed that for simplicity of derivation, we assume the dividend rate to be 0, but in fact, Black Scholes also hold under constant dividend rate, in which the rate will be (r-q) instead of r, where q is the dividend rate.

$$C(t) = S \cdot N(d1) - Ke^{-rT}N(d2)$$
$$P(t) = Ke^{-rT}N(-d2) - S \cdot N(-d1)$$

Where d1 and d2 are defined as follow:

$$d1 = \frac{\log(S/K) + (T-t)(r + \sigma^2/2)}{\sigma\sqrt{T-t}}$$

$$d2 = \frac{\log(S/K) + (T-t)(r - \sigma^2/2)}{\sigma\sqrt{T-t}}$$

which can also be expressed as:

$$d2 = d1 - \sigma\sqrt{T-t}$$

Consider the underlying process to be log(S(T)) (Intuitions derived from GBM), and consider Ito's formula:

$$df = \left[\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right] dt + \left[\sigma S \frac{\partial f}{\partial S}\right] dW$$

plugging in f = log(S)
Note that df/dt = 0, df/dS = 1/S and $d^2S/dS^2 = -1/S^2$
Then:

$$d(\log(S(T))) = \left[rS\left(\frac{1}{S}\right) + \frac{1}{2}\sigma^2 S^2 \left(-\frac{1}{S^2}\right)\right] dt + \sigma S\left(\frac{1}{S}\right) dW$$

$$d(\log(S(T))) = (r - \frac{1}{2}\sigma^2)dt + \sigma dW$$

which yields a traditional Geometric Brownian Motion expressions, from some further mathematical deduction we can obtained that:

$$S(T) = S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}z}$$

where Z is the standard normal distribution $Z \sim N(0, 1)$ and the simple application of Ito's Lemma can obtain the stock price expression for the Geometric Brownian Motion. Notice that for the sake of European Call option assumptions, S(T) > K, then solve for Z from equations and the relationship S(T) > K above:

$$S(t)e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma\sqrt{T-t}z} > K$$

$$Z \geq \left[\log\left(\frac{K}{S(t)}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right]\frac{1}{\sigma\sqrt{T-t}}$$

and notice that $Z > \quad - N(d2)$ after manipulations.

For simplicity and demonstrational purposes, only the mathematical deductions for European Call options are presented. Consider that the payoff for a particular European call: $max(E(S(T) - K, 0)$, convert that into the formula obtained:

$$\int_{-d2}^{\infty}(S(T) - K)q(Z)\,dZ$$

The intuition of the formula stems from:

$$E(x) = \int xq(x)\,dx$$

q(Z) is the density function of standard normal distribution, S(T) is the expression above, and K is the strike price, and since K is a regular constant. The formula can be slightly simplified to:

$$\int_{-d2}^{\infty} S(T)q(Z)\,dZ - KN(d2)$$

where K*N(d2) is the second half of the Black Scholes Merton Equation for pricing European put and call. Then expanding S(T) and changing variables, we can eventually obtained:

$$e^{-r(T-t)}\left[S(t)e^{r(T-t)}N(d1) - K * N(d2)\right]$$
$$S(t)N(d1) - Ke^{-r(T-t)}N(d2)$$

which yields the Black Scholes pricing for European call options, put options derivation can be performed in a very similar way.

3.2 Heston Model

The heston method adds another layer of complexity compared to the BSM method, it assumes that both the stock price and the volatility are under variations and follow a certain process. While BSM assumes only for delta hedging, Heston Model accounts for both delta and sigma hedging. The mathematical assumptions of Heston Model, also previously defined in the introduction section, are the three following equations:

$$
\begin{aligned}
dS_t &= \mu S_t \, dt + \sqrt{v_t} S_t \, dW_1 \\
dv_t &= \kappa(\theta - v_t) \, dt + \sigma \sqrt{v_t} \, dW_2 \\
dW_1 dW_2 &= \rho \, dt
\end{aligned}
$$

For the first equation, it describes that the stock price in the Heston Model completely follows Geometric Brownian Motions, where μ = average rate of return, S_t is the stock price, $\sqrt{v_t}$ = square root of the variance, which is the standard deviation, and dW1 follows a Standard Brownian Motion.

The second equation describes a particular method that the change of variance follows, known as the Cox-Ingersoll-Ross Process, a process specifically used to describe the evolution of market risks. In the equation, κ refers as the rate of mean inversion, which is the rate of how variance can return to its long term average, θ is the long term average that the mean will invert to. dW2 follows another Standard Brownian Motion.

The final equation illustrates that the two brownian motion that are included in the two previous equations have a correlation between them, ρ represents the correlation coefficient between the two brownian motions.

Below demonstrates a slight mathematical manipulation of Heston PDE using
Then, for any V= V(t, S, V) with a slight modification of Ito's formula:

$$dV = \left(\frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2} \sigma^2 v \frac{\partial^2 V}{\partial v^2} + \rho \sigma v S \frac{\partial^2 V}{\partial v \partial S} \right) dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial v} dv$$

Which, when slightly rewrite this equation, we can obtain:

$$dV = (LV)(t, s, v) \, dt + \frac{\partial V}{\partial S} \, dS + \frac{\partial V}{\partial v} \, dv$$

Now given the portfolio assumption under Heston Model that:

$dV = \Delta dS + \Sigma dU + \alpha dB$ and $dB = rBdt$

And given that we could also write another expression of dU with exactly the same setup as dV:

$$dU = (LU)(t,s,v)\,dt + \frac{\partial U}{\partial S}\,dS + \frac{\partial U}{\partial v}\,dv$$

Then we can rewrite dV as the following expression, notice that we are substituting the equation using the two expressions below.

$$\Sigma(LU)(t,s,v)\,dt + \left(\Delta + \Sigma\frac{\partial U}{\partial S}\right)dS + \Sigma\frac{\partial U}{\partial v}\,dv + \alpha r B\,dt$$

$$\frac{\partial V}{\partial S} = \Delta + \Sigma\frac{\partial U}{\partial S}\ ,\ \frac{\partial V}{\partial v} = \Sigma\frac{\partial U}{\partial v}$$

Now we can finally plug everything into the original equations. First using the portfolio assumption equations from above we can get:

$$\alpha B = V - \Delta S - \Sigma U$$

Then expand everything and perform a shift in parameter, we get:

$$(LV)(t,s,v)\,dt = \Sigma(LU)(t,s,v)\,dt + r\left(V - \frac{\partial V}{\partial S}S + \Sigma\frac{\partial U}{\partial S}S - \Sigma U\right)dt$$

Which, by further manipulations, we can obtain the important milestones for obtaining the Heston's PDE:

$$(LV)(t,s,v) - rV = -rS\frac{\partial V}{\partial S} - f(t,s,v)\frac{\partial V}{\partial v}$$

In this particular expression, LV are as defined as before, f(t,s,v) are the drift of the options that can be any arbitrary functions containing variables t, s, and v. Then the Heston's PDE are demonstrated by applying complex Ito's and manipulating different variables.

3.3 Merton's Jump Diffusion Method

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

This is the geometric brownian for the stock price follow a diffusion process.But in the most cases, when one plots the return distribution it usually has fatter tails and taller compared to the normal distribution also the actual price show jump. So we get

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + \text{Jump}$$

This formula adds a "jump" term to the original geometric Brownian motion equation. Such models are often used to describe sudden and large price changes caused by specific events (such as economic news releases, major company announcements, etc.). This model is called a jump diffusion model.
Jump is Related to arrival(possion), Size(Log-Normal)

In the poisson process for the independent increment

$$P[N(t+\Delta t) - N(t) = 1] = \lambda \Delta t + o(\Delta t)$$
$$P[N(t+\Delta t) - N(t) > 1] = o(\Delta t)$$

In the poisson process

dN_t = 1 for lambda dt
dN_t = 0 for 1- lambda dt

This formula is a geometric Brownian motion model with jumps, which combines conventional Brownian motion and Poisson jump processes. In this model, Y_t usually represents the amplitude of the jump, while dN_t is a Poisson jump process, representing a small time interval dt

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + (Y_t - 1) dN_t$$

Find the Expected value of jump part

$$E[(Y_t - 1)dN_t] = E[Y_t - 1]E[dN_t] = \lambda k dt$$

The pure random jumps formulation with multi jump be:

$$dS = (\mu - \lambda\kappa)Sdt + \sigma S dW + S\left(\prod_{j=1}^{dN} Y_j - 1\right)$$

The differential of dS is

$$df = \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS^2 + f\left(S\left[\prod_{j=1}^{dN} Y_j\right] - f(S)\right)$$

The derivation of log of S is

$$d\ln S = -\frac{dS}{S} - \frac{1}{2}\frac{dS^2}{S^2} + \sum_{j=1}^{dN} \ln Y_j$$

Now we focus on the continuous part: $dS = (\mu - \kappa)Sdt + \sigma S dW$

First derivation : $\frac{dS}{S} = (\mu - \kappa)dt + \sigma dW$

Second derivation: $\frac{dS^2}{S^2} = \sigma^2 dt$ minus both side we get $-\frac{1}{2}\frac{dS^2}{S^2} = -\frac{1}{2}\sigma^2 dt$

Now we can make the replacement:

$$d\ln S = \left(\mu - \lambda\kappa - \frac{1}{2}\sigma^2\right) dt + \sigma dW + \sum_{j=1}^{dN} \ln Y_j$$

Then we integrate from zero to t:

$$\ln S_t - \ln S_0 = \left(\mu - \lambda\kappa - \frac{1}{2}\sigma^2\right) t + \sigma W_t + \sum_{j=1}^{N_j} \ln Y_j = \ln S_t - \ln S_0 =$$
$$\left(-\lambda\kappa + n\mu + \frac{n\sigma^2}{2}\right) + \left(\mu - \frac{1}{2}\sigma_n^2\right) t + \sigma_n W_t$$

Let exponentiate the two side we get:
Stock Price under Jump-Diffusion Model:
$S_t = S_0^{(n)} e^{\left(r - \frac{1}{2}\sigma_n^2\right)t + \sigma_n W_t}$
The formulation is like Black Scholes model except the definition of initial stock price and the volatility are different,this is because we embed the jump into the process.
Using the iterated expectation we get, the number of jump can be 0 to a large number
European Call Option Pricing under Jump-Diffusion Model:

$$C\left(S_0^{(n)}, T\right) = \sum_{n=0}^{\infty} C\left(S_0^{(n)}, T | N_T = n\right) \frac{(\lambda T)^n e^{-\lambda T}}{n!}$$

Adjusted Volatility *σ_n*

$$\sigma_n = \sqrt{\sigma^2 + \frac{n\sigma_y^2}{T}}$$

Expected Percentage Change in Price K

$$k = e^{\mu + \frac{1}{2}\sigma_y^2} - 1$$

Adjusted Initial Stock Price $S_0^{(n)}$

$$S_0^{(n)} = S_0 e^{\left(-k + n\mu + \frac{n\sigma_y^2}{2}\right)}$$

4. Code and Result Analytics

4.1 Black-Scholes Model

In this section of the research paper, we delve into the practical application of the Black-Scholes model for option pricing through a Python code implementation. This model, a seminal work in financial mathematics, is used to estimate the prices of European call options and put options. The idea of writing our code and the results of it are as follows.

First, we defined the parameters essential to the Black-Scholes formula: the maturity time of the option (T), the option's strike price (K), the risk-free interest rate (r), the dividend yield (q), and the volatility (sigma) of the underlying asset. These parameters are critical in determining the theoretical value of the options.

Next, we wanted to compare the situation when the model has different asset prices. We established a range of underlying asset prices (S). This array of prices is crucial for analyzing how the option value fluctuates with varying underlying asset prices. By considering a spectrum of prices, the model provides a comprehensive view of the option's valuation under different market conditions. The initialized parameters are shown in **Chart1**.

Description	**Symbol**	**Value**
Maturity	T	1
Strike	K	100
Interest Rate	r	0.05
Dividend Rate	q	0.03
Volatility	sigma	0.1
Underlying Asset Price Range	S	50-150

Chart1.Black-Scholes model initialized parameters

Following this, the calculation of d1 and d2, integral components of the Black-Scholes formula, is performed. These variables are a function of the initial parameters and are pivotal in the model, encapsulating the dynamics of the option's price relative to its underlying asset. They serve as standardized variables in the log-normal framework on which the Black-Scholes model is built.

Finally, we used the derived d1 and d2 to calculate the values for both call and put options. The call option value is obtained by evaluating the present value of the expected benefits from holding the call option. In contrast, the put option value is calculated as the present value of the expected payoffs from owning the put option. These calculations embody the core principle of the Black-Scholes model - providing a theoretical estimate of option prices in a frictionless market. And we visualized the result in **Fig1**.

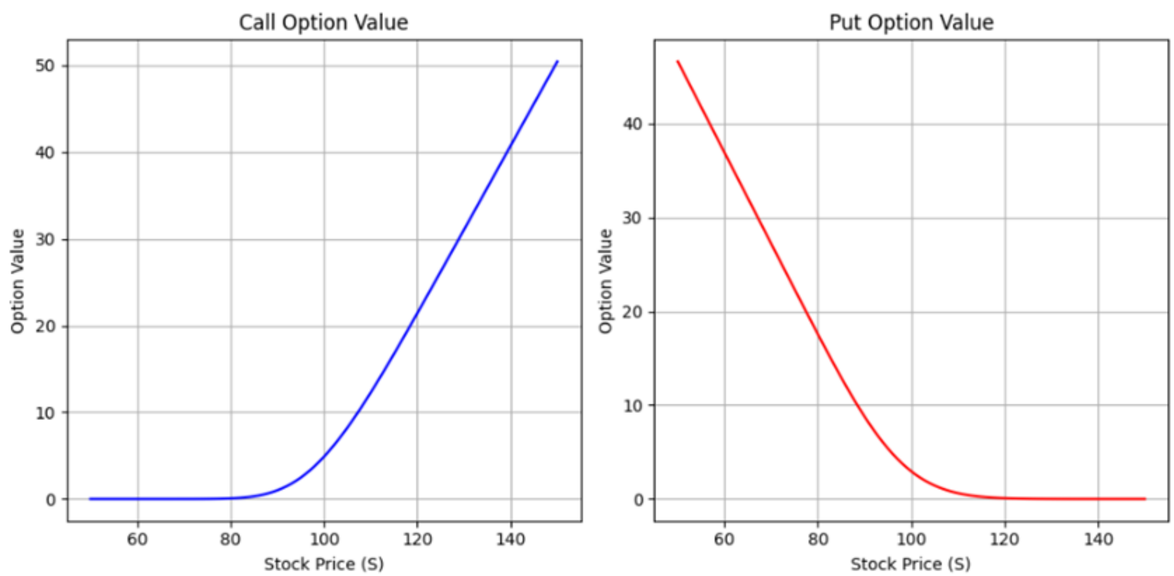


Fig1. Black-Scholes model Call-put option values

The left graph illustrates the value of a call option. As the stock price increases, the call option's value also rises, a relationship that is consistent with financial theory. This increase is non-linear, reflecting the option's intrinsic value and the time value diminishing as the option moves further in-the-money. When the stock price is below the strike price, the value of the call option primarily consists of time value, since the intrinsic value is zero. The curvature of the plot is a consequence of the option's "delta", which itself varies with the stock price. As the stock price significantly exceeds the strike price, the option's delta approaches 1, and the option value starts to increase nearly dollar-for-dollar with the stock price, which is indicative of a deep-in-the-money call option.

The right graph delineates the value of a put option. Contrary to the call option, the put option's value decreases as the stock price increases, which is aligned with the protective nature of put options against declines in stock price. The value is highest when the stock price is well below the strike price, signifying that the put option is deep in-the-money. The slope of the curve is steepest when the stock price is near the strike price, where the sensitivity of the option's price to changes in the stock price (the "gamma") is highest. As the stock price moves further above the strike price, the option becomes out-of-the-money, and its value asymptotically approaches zero since the likelihood of profit at expiration decreases.

The Black-Scholes model assumes a log-normal distribution of stock prices, constant volatility, and continuous compounding. These assumptions are embedded in the smoothness and shape of the plotted curves. However, it is important to note that actual market conditions may deviate from these assumptions, introducing model risk. The model does not account for jumps in stock prices or changes in volatility, which can lead to mispricing of options in practice.

In summary, the Black-Scholes model offers a foundational framework for option pricing, translating complex financial concepts into an accessible graphical form. The curves depicted are emblematic of the logical expectations set forth by option pricing theory, and they serve as a baseline for further empirical and theoretical exploration in the field of financial derivatives.

4.2 Heston Model

In this portion of the paper, we examine an implementation of the Heston model for option pricing, a significant advancement from the Black-Scholes model. The Heston model accounts for stochastic volatility, acknowledging that volatility is not constant but varies over time, which is a more realistic approach compared to the constant volatility assumption in the Black-Scholes model.

We begin by defining the parameters necessary for the Heston model. These include not only the standard parameters like the strike price (K), risk-free rate (r), and dividend yield (q) but also specific Heston parameters such as the initial variance (v0), the long-term variance (theta), the mean-reversion rate of the variance (kappa), the vol of vol (sigma), and the correlation between the asset price and its variance (rho). The parameter lambda (lmbda) is set to zero, indicating no jump component in this model version. The initialized parameters are as follows.

Following parameter initialization, we calculates call option prices using the `Heston_call_price` function. This function iterates over a range of underlying asset prices (S0) and calculates the option price for each. It does this by computing two probabilities (p1 and p2) using the `p_Heston` function, which are integral parts of the Heston model's option pricing formula. The initialized parameters are shown in **Chart2**.

Description	Symbol	Values

Maturity	T	1
Strike	K	100
Underlying Asset Price Range	S	50-150
Interest Rate	r	0.05
Dividend Rate	q	0.03
Init Volatility	v0	0.04
Correlation	rho	-0.7
Reversion Rate	kappa	2
Long-run variance	theta	0.04
Volatility	sigma	0.3
Jump Intensity	lmbda	0.1

Chart2. Heston model Initialized parameters

The `p_Heston` function calculates these probabilities by performing numerical integration over the Heston model's characteristic function. The characteristic function, defined in `f_Heston`, is complex and incorporates all the dynamics of the Heston model, including the stochastic nature of the volatility.

Furthermore, we employed the put-call parity to calculate the put option prices. The put-call parity is a fundamental relationship in options pricing that connects the prices of puts and calls. And we visualized the result in **Fig2.**.

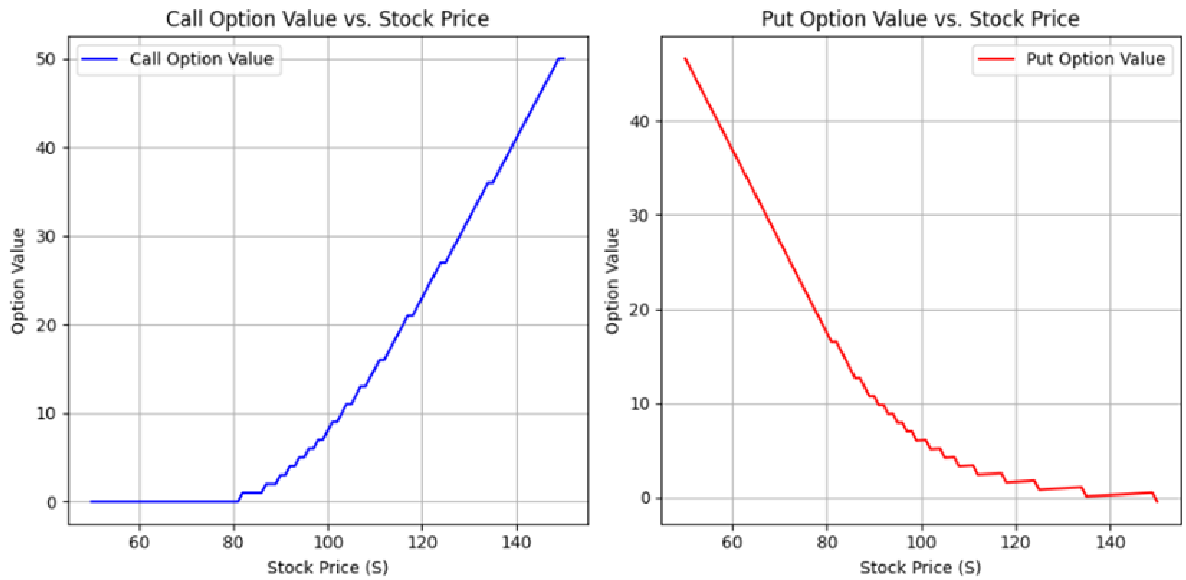


Fig2. Heston model Call-put option values

The left graph portrays the valuation curve of a call option as the stock price ascends. A prominent characteristic of the Heston model is the more pronounced curvature in the option value as the stock price approaches the strike price from below. This heightened curvature reflects the market's anticipation of volatility swings, which can significantly affect the option's delta and gamma (the first and second derivatives of the option value with respect to the stock price). As the underlying asset's price exceeds the strike price, the option transitions into the money, and the value increases steeply, an indication that the model is capturing the amplified probability of the option finishing in the money due to volatility fluctuations.

The right graph delineates the put option value diminishing as the stock price inflates. Notably, the value does not descend smoothly but in a stepped fashion, which may imply discrete recalibrations of the market's volatility outlook as new information is assimilated. This behavior is characteristic of real-market conditions where volatility surfaces are not flat and shift in response to market dynamics. When the stock price is well below the strike price, the put option is deeply in the money, and its value is at its zenith. As the stock price increases, the model reflects the diminishing likelihood of the option expiring in the money, which is why the value tapers off.

The Heston model's ability to simulate stochastic volatility is particularly important in environments where volatility is not stable and can lead to significant price changes. The 'smile' effect, often observed in implied volatilities from market prices of options, can be more accurately captured with the Heston model compared to the Black-Scholes model, which assumes constant volatility. The discrete steps in the put option graph suggest a latticed estimation of volatility states or could be an artifact of the numerical method employed to approximate the Heston model's equations, which often involve complex numerical techniques such as finite difference methods or Monte Carlo simulations. These steps differ from the smooth curves predicted by the Black-Scholes model, underscoring the Heston model's nuanced approach to incorporating market realities such as volatility clustering and mean reversion.

In conclusion, the plots reveal the nuanced behavior of option prices in the presence of stochastic volatility, as captured by the Heston model. This model provides a more comprehensive framework for option valuation by considering the dynamic nature of market volatility, which is a significant factor in the pricing of financial derivatives and risk management.

4.3 Merton Jump Diffusion Model

This segment of the paper focuses on the Merton jump-diffusion model, an extension of the Black-Scholes model that incorporates jumps in the price of the underlying asset. The Merton model is particularly useful in capturing the abrupt price movements often observed in financial markets, which are not adequately addressed by the Black-Scholes model's assumption of continuous asset price changes. The initialized parameters are in chart3.

Description	**Symbol**	**Value**
Time to Maturity (years)	T	1
Strike Price	K	100
Risk-Free Interest Rate	r	0.05
Dividend Yield	q	0.03
Volatility	sigma	0.1
Average Jump Size	mu_j	-0.1
Jump Volatility	sigma_j	0.2

Jump Intensity	lmbda	0.5
Underlying Asset Price Range	S	50-150

Chart3. Merton jump-diffusion model Initialized parameters

Initially, we defined the `bs_price` function to calculate the Black-Scholes price for a call or put option. It follows the standard Black-Scholes formula, taking into account the spot price of the asset (S), the strike price (K), time to maturity (T), risk-free rate (r), dividend yield (q), and volatility of the asset (sigma). The function differentiates between call and put options based on the boolean parameter `call`.

Next, we defined the `merton_jump` function to extend the Black-Scholes model to include a jump component, as per the Merton jump-diffusion model. In addition to the standard parameters, it incorporates the parameters for the jump part: the average jump size or jump mean (`mu_j`), the standard deviation of the jump size (`sigma_j`), and the jump intensity (`lmbda`). The function iteratively calculates the option price by summing the contributions of all possible jump counts (up to `max_iter`), with each term weighted by the Poisson probability of that number of jumps occurring.

In this model, the risk-neutral drift rate (`r_k`) and the effective volatility (`sigma_k`) are adjusted for each jump count, reflecting the additional risk and uncertainty introduced by the jumps. The Black-Scholes price for each jump scenario is then computed using the `bs_price` function.

This implementation showcases the Merton jump diffusion model's capability to incorporate sudden, discontinuous changes in the underlying asset price, offering a more comprehensive and realistic approach to option pricing. This is especially relevant in markets characterized by large, infrequent price jumps, providing a more accurate valuation tool compared to models assuming continuous price paths.

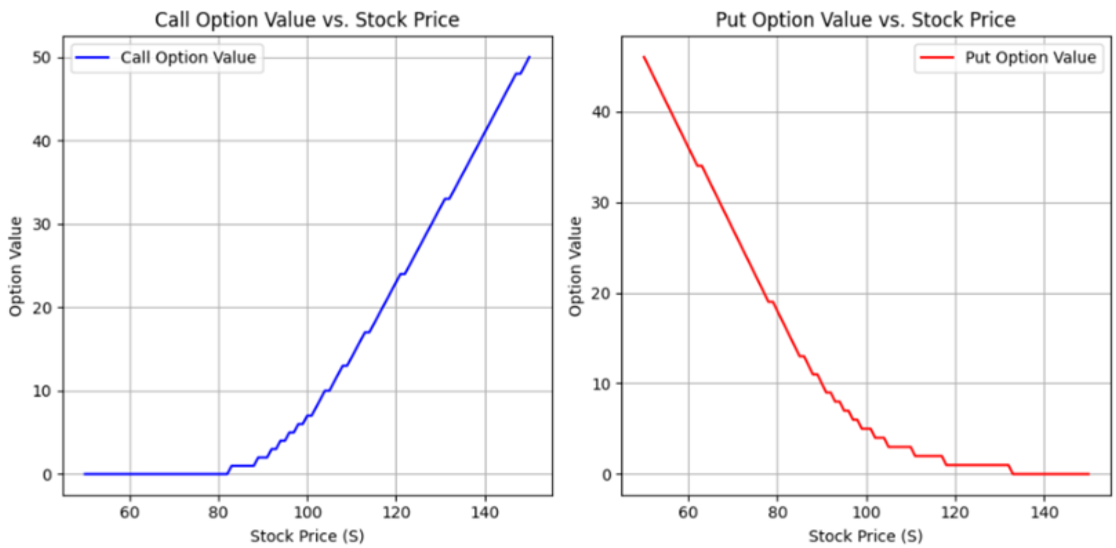


Fig3. Merton jump-diffusion model Call-put option values

The left-hand graph displays the call option value as a function of the underlying stock price. The salient feature in the call option graph is the presence of step-like discontinuities. These steps illustrate the model's jump component, which accounts for the possibility that the stock price can experience large jumps, thus altering the option value significantly. In the absence of jumps, the model would revert to the classic Black-Scholes prediction of a smooth curve. The increasing option value in response to the rising stock price reflects the inherent leverage provided by a call option: as the stock price ascends above the strike price, the value of the call option increases, reflecting the greater probability of a profitable exercise at maturity.

The right graph illustrates the put option value, which inversely correlates with the stock price. The model anticipates higher values for the put option when the stock price is low, representing the option's intrinsic value when it is "in the money." The steps in the graph are indicative of the model's accommodation for sudden decreases in stock price, which can significantly enhance the put option's value. As the stock price climbs, the value of the put option declines, owing to the reduced probability of the option ending "in the money."

The discontinuities in the valuation curves are characteristic of the Merton model's jump diffusion process and are not present in the Black-Scholes or Heston model graphs. These jumps reflect the model's integration of a Poisson process to account for the randomness of jumps in the stock price, with the jump intensity parameter (lambda) influencing the frequency of these events.

The Merton model's inclusion of jump risk is particularly pertinent for assets that are prone to sharp price movements, which could be precipitated by economic announcements, earnings reports, or other market-moving events. By incorporating both the continuous stochastic process of Black-Scholes and discontinuous jumps, the Merton model provides a more comprehensive tool for option valuation.

In conclusion, the Merton jump diffusion model's ability to incorporate sudden jumps offers a more nuanced perspective on option pricing, which can be critical for traders and risk managers who need to hedge against or speculate on significant market moves. The plots underscore the model's potential to capture complex market behaviors, rendering it an invaluable addition to the repertoire of financial modeling techniques.

5. Comparative Analysis

5.1 Call option

From the call option **Fig4** below, we observed all three models generally predict higher call option values as the underlying asset price increases, which aligns with financial theory since the intrinsic value of a call option increases with the underlying asset's price. However, we noted that both the Heston and Merton Jump Diffusion models predict higher option values near the strike price compared to the Black-Scholes model. This observation merits a detailed exploration of the underlying causes.

The Black-Scholes model provides a foundational approach for option pricing, assuming a log-normal distribution of future stock prices and constant volatility. Near the strike price, the Black-Scholes model's valuation of the call option is typically lower because it does not account for the dynamic nature of real-world volatility or the occurrence of significant jumps in stock prices. While the Heston model extends the Black-Scholes framework by incorporating stochastic volatility. This means that the model allows volatility to vary over time, capturing the market's changing uncertainty regarding the underlying asset's future price. As the underlying asset's price approaches the strike price, the Heston model predicts higher option values than the Black-Scholes model, primarily because it accounts for the possibility of increased volatility. This higher volatility can lead to a greater probability of the option finishing in the money, thereby increasing its value. Also the Merton Jump Diffusion model introduces jumps into the stock price process, reflecting the empirical observation that stock prices can exhibit significant, sudden moves, which are not captured by the standard Brownian motion in the Black-Scholes model. As a result, near the strike price, where the sensitivity of the option's value to the stock price (delta) is high, the inclusion of jumps means there's a higher probability of the stock price leaping to levels where the option has substantial intrinsic value. This leads to higher predicted call option values compared to the Black-Scholes model, which does not accommodate such jumps.

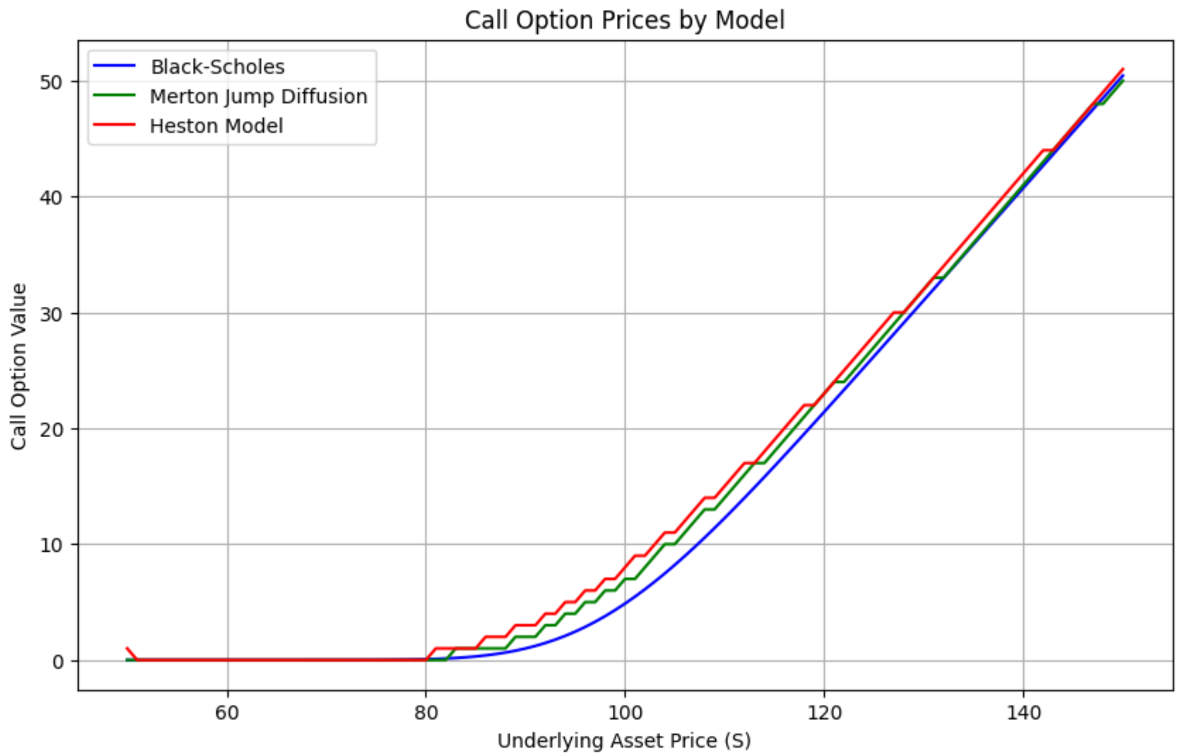


Fig4. Call option Price

The divergence between the models around the strike price is particularly telling. The Black-Scholes model's lower values can be attributed to its lack of a mechanism to account for sudden large moves or changing volatility expectations. In contrast, the Heston and Merton models, by accounting for these realistic characteristics of stock price behavior, predict higher option values. The real-world pricing of options often shows a volatility smile or skew, which these two models are better equipped to capture, particularly for options that are near or at the money.

5.2 Put option

From the put option **Fig5** below, we note that both the Heston and Merton models forecast higher put option values around the strike price compared to the Black-Scholes model. This divergence prompts an examination of the reasons behind the differences.

The Heston model, known for allowing stochastic volatility, predicts higher put option values near the strike price. This is because the model accounts for potential increases in volatility, which can raise the probability of the option ending "in the money." Higher volatility leads to greater uncertainty regarding the stock price at maturity, and since put options are a form of insurance against stock price declines, their value is increased when higher volatility is expected. Similar to the Heston model, the Merton Jump Diffusion model also predicts higher values for put options near the strike price, but for a different reason. The Merton model introduces jumps into the price process, acknowledging that stock prices can suddenly drop significantly. Such price jumps can dramatically increase the value of a put option, especially when the stock price is at or just above the strike price, as the option can quickly move from being "out of the money" to "deeply in the money."

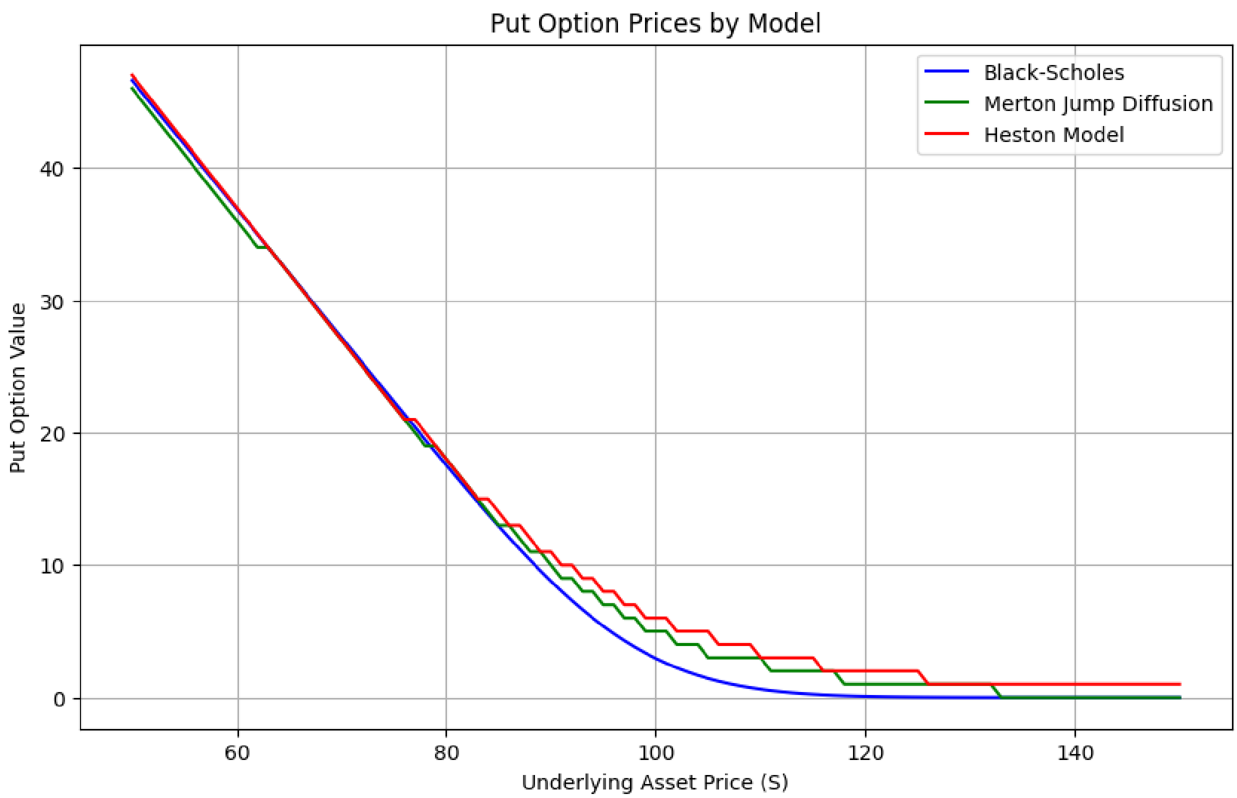


Fig5. Put option Price

The Black-Scholes model's lower valuation of put options near the strike price suggests a limitation in accounting for real market behavior where volatility is not constant and price jumps can occur. Both the Heston and Merton models address these limitations by incorporating additional market features, such as stochastic volatility and jump risks, which are particularly relevant for put options that serve as protection against declines in stock prices.

6. Conclusions and Limitations

The Black-Scholes-Merton, Heston Model, and Merton's Jump Diffusion Method are important models for option pricing. The BSM Model is the most basic model, which is based on the assumption that stock prices follow Geometric Brownian Motion and volatility is constant.

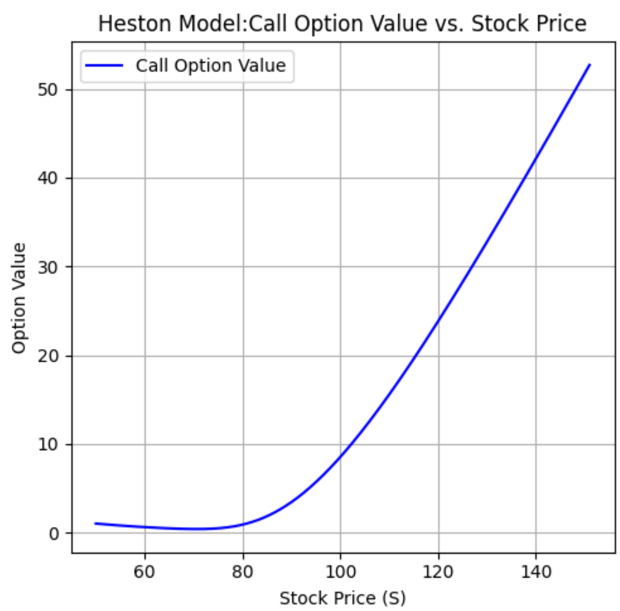
The Heston Model is optimized in the BSM model which allows volatility to change over time. In the Heston model, asset prices and volatility are stochastic processes. Such improvements make the model closer to the actual market, because volatility is not constant in the actual market. Merton's Jump Diffusion Model is another improvement of the BSM model which adds a "jump" to the original BSM to allow sudden and large changes in asset prices, which simulates emergencies in the market. In the MJD model, asset price changes are composed of continuous Brownian motion and randomly occurring jumps.

Then, From the result of the code Fig4 and Fig5, it is evident that the option pricing is ranked decreasingly from Heston, Merton's Jump-Diffusion to Black Scholes. The primary reason that accounts for that is Heston takes into account the additional increases in volatility, due to its ability to deal with the volatility smile. Then, Merton's Jump diffusion method is higher than BSM due to its attributes of "jump", which can significantly boost the pricing of a particular option. Now, the results are not so apparent when taken into account of the differences between Heston and Merton's Jump diffusion process, but the fact that Heston has two processes undergoing the model has made Heston more susceptible to price change and could potentially lead to further increase of options pricing.

7. Feedback

This section is dedicated to modify changes that Professor provided in class, there are two feedbacks that applies to our report, listed as follows:

- Convergence of Heston Model
- One additional Heston underlying assumption (the famous Heston's PDE)



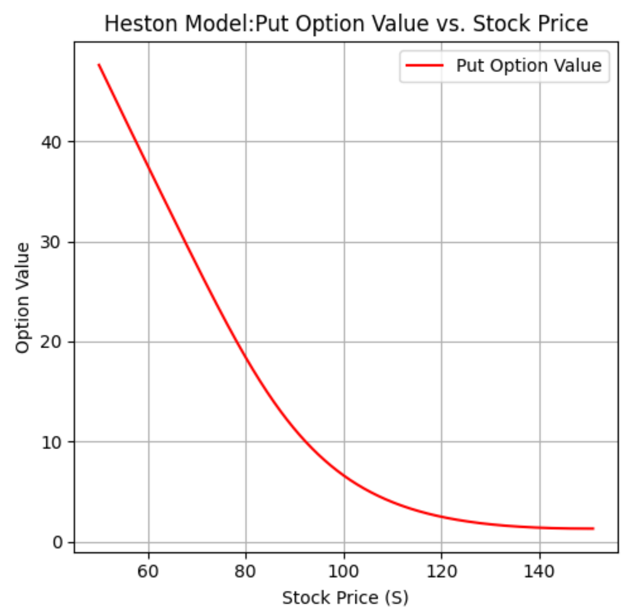


Fig5. Heston model Call-put option values (Convergence)

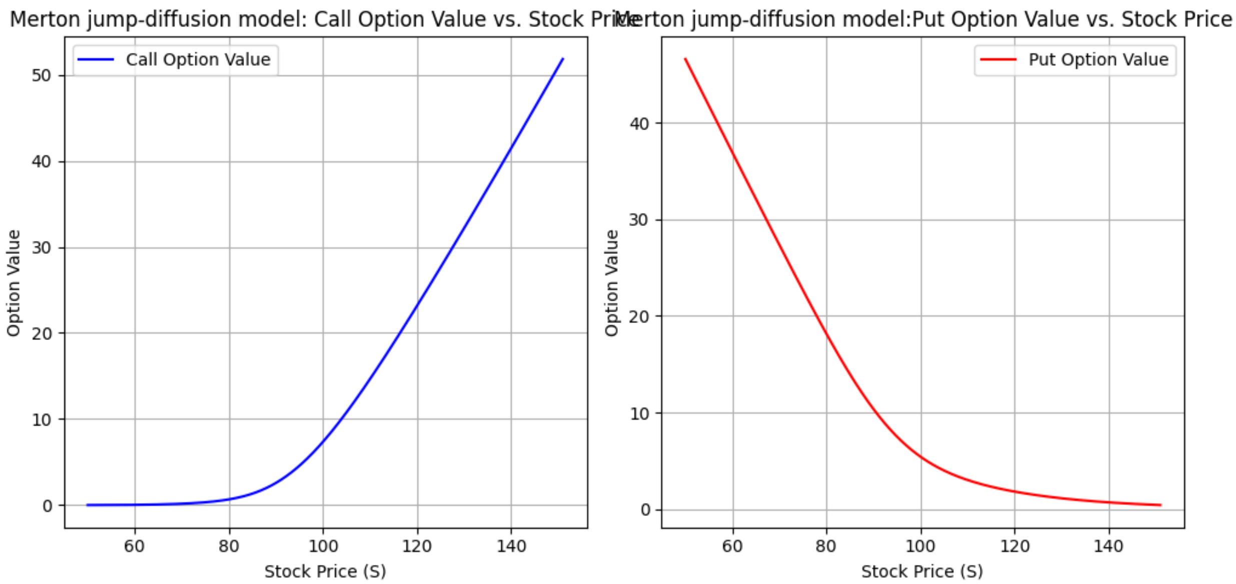


Fig6. Merton jump-diffusion model Call-put option values **(Convergence)**

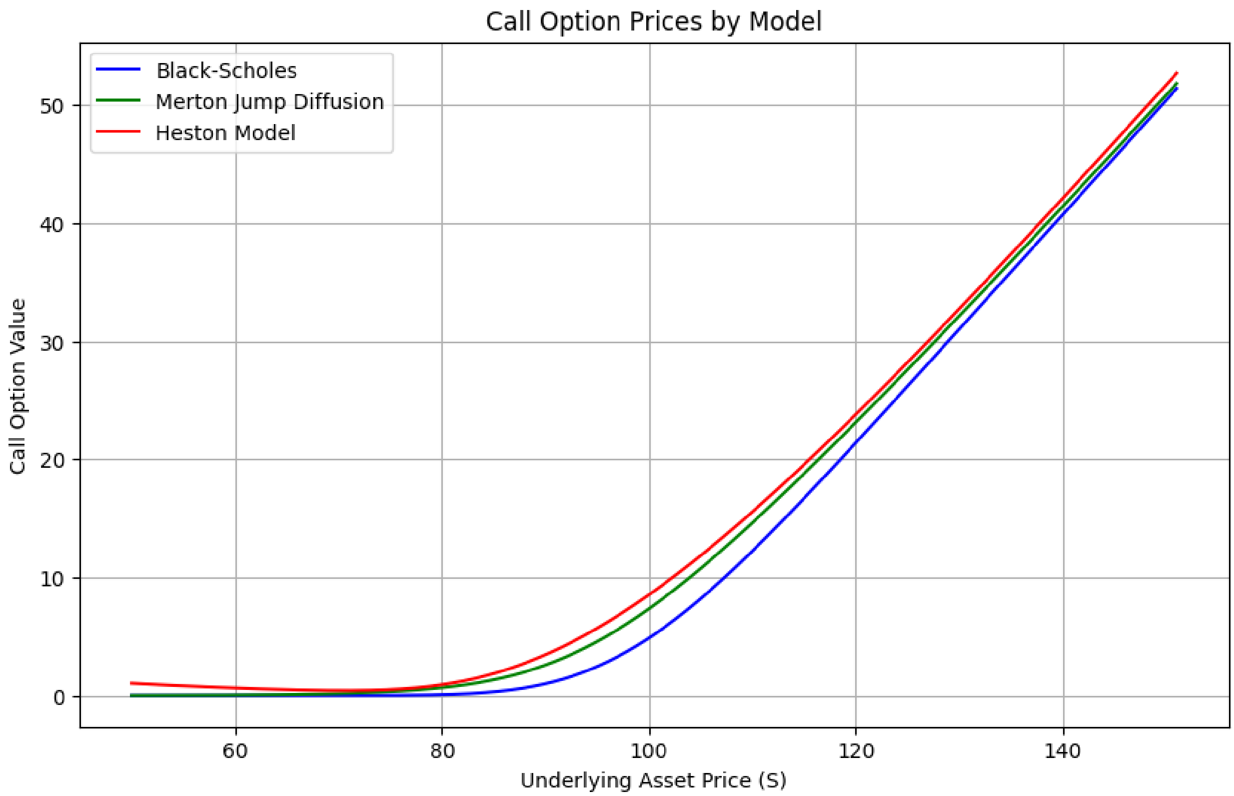


Fig7. Call option Price (Convergence)

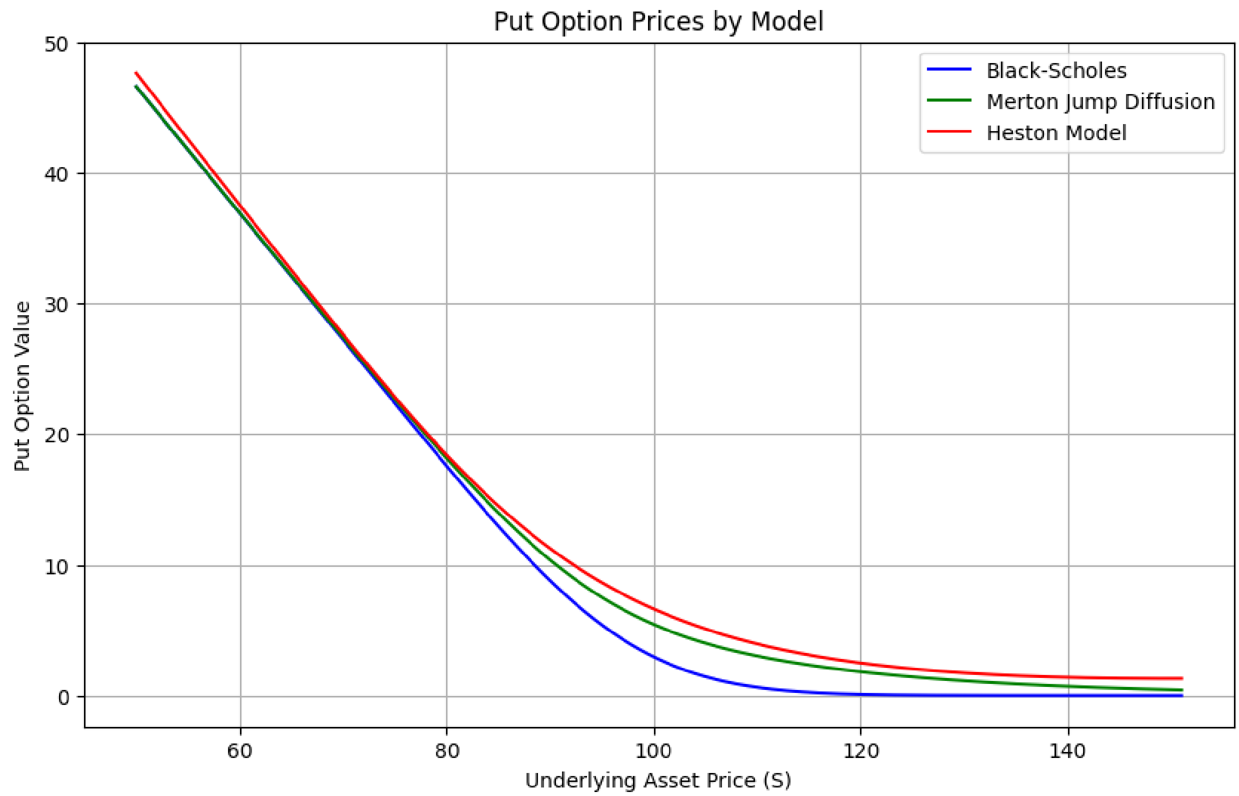


Fig8. Put option Price (Convergence)

Model/step	Convergence
Black Scholes Model	100 steps
Heston Model	200 steps
Merton jump-diffusion model	500 steps

In addition to the feedback that Professor has provided, we have implemented the convergence steps for all three of our models, and the results of steps are shown as above.

We did a convergence test on both 3 models. We tested the convergence test for 100, 200, 500 and 1000 respectively.For the Black scholes model test, the model completed convergence at 100 steps. For the Heston model test, the model completed convergence at 200 steps. For the merton jump-diffusion model test, the model completed convergence at 500 steps.

The additional assumption that Professor quoted missing is the famous PDE function that takes into account the mean reverting characteristics of the Heston Model, the actual formula are as follows:

$$\frac{1}{2}\nu S^2\frac{\partial^2 U}{\partial S^2}+\rho\nu S\frac{\partial^2 U}{\partial S\partial v}+\frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2}+rS\frac{\partial U}{\partial S}+\left[\kappa(\theta-v(t))-\lambda(S,v,t)\right]\frac{\partial U}{\partial v}-$$
$$rU+\frac{\partial U}{\partial t}=0.$$

This particular equation is partially shown in the mathematical derivation that we've shown at the top, and the component that is of utmost importance is the k(theta - Vt), which takes into account the Heston model mean-reverting characteristics. We've also implemented the change in our presentation slides.

8. Citation

[1] Black, F., & Scholes, M. S. (1973). The Pricing of Options and Corporate Liabilities. Journal of Political Economy, 81(3), 637-654.

[2] Hull, J., & White, A. G. (1987). The Pricing of Options on Assets with Stochastic Volatilities. The Journal of Finance, 42(2), 281-300.

[3] Heston, S. L. (1993). A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. Review of Financial Studies, 6(2), 327-343.

[4] Broadie, M., & Kaya, Ö. (2004). Exact Simulation of Option Greeks under Stochastic Volatility and Jump Diffusion Models. Winter Simulation Conference.

[5] Sepp, A. (2003). Pricing European-Style Options under Jump Diffusion Processes with Stochastic Volatility: Applications of Fourier Transform. SSRN Electronic Journal.

[6] Mozumder, S., Sorwar, G., & Dowd, K. (2013). Option pricing under non-normality: a comparative analysis. Journal Reference Not Available.

[7] Averbuj, C. G. (2013). Nonlinear Integral-Differential evolution equation arising in option pricing when including transaction costs: A viscosity solution approach. Journal Reference Not Available.

[8] Sun, Y. (2015). Efficient pricing and hedging under the double Heston stochastic volatility jump-diffusion model. International Journal of Computer Mathematics, 92(12), 2546-2561.

[9] Liu, S., Zhou, Y., Wu, Y., & Ge, X. (2019). Option Pricing under the Jump Diffusion and Multifactor Stochastic Processes. Mathematical Problems in Engineering, 2019.

[10] Su, H., Chen, D.-R., & Newton, D. (2017). Option Pricing via QUAD: From Black–Scholes–Merton to Heston with Jumps. The Journal of Derivatives, 24(3), 9-27.

[11] Sanfelici, S. (2007). Calibration of a nonlinear feedback option pricing model. Quantitative Finance, 7(5), 537-548.

[12] Boyarchenko, S., & Levendorskii, S. (2002). Barrier options and touch-and-out options under regular Lévy processes of exponential type. Annals of Applied Probability, 12(4), 1261-1298.

9. Appendix

9.1 BS Model

```
##BS model
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm

# Parameters
T = 1        # Maturity
K = 100       # Strike
r = 0.05      # Interest Rate
q = 0.03      # Dividend Rate
sigma = 0.1    # Volatility
S = np.arange(50, 151, 1)  # Stock Price Range

# Calculates d1 and d2
d1 = (np.log(S/K) + (r - q + 0.5 * sigma**2) * T) / (sigma * np.sqrt(T))
d2 = (np.log(S/K) + (r - q - 0.5 * sigma**2) * T) / (sigma * np.sqrt(T))

# Calculates call and put
```

```
Vc_BS = S * np.exp(-q * T) * norm.cdf(d1) - K * np.exp(-r * T) *
norm.cdf(d2)
Vp_BS = K * np.exp(-r * T) * norm.cdf(-d2) - S * np.exp(-q * T) *
norm.cdf(-d1)

# Plot
plt.figure(figsize=(10, 5))

#Call Option
plt.subplot(1, 2, 1)
plt.plot(S, Vc_BS, label='Call Value', color='blue')
plt.xlabel('Stock Price (S)')
plt.ylabel('Option Value')
plt.title('BS model:Call Option Value vs. Stock Price')
plt.grid(True)

#Put Option
plt.subplot(1, 2, 2)
plt.plot(S, Vp_BS, label='Put Value', color='red')
plt.xlabel('Stock Price (S)')
plt.ylabel('Option Value')
plt.title('BS model:Put Option Value vs. Stock Price')
plt.grid(True)

plt.tight_layout()
plt.show()
```

9.2 Heston Model

```
###Heston Model
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import quad

# Parameters
T = 1
K = 100
S = np.arange(50, 151, 1)
```

```
r = 0.05
q = 0.03
v0 = 0.04
rho = -0.7
kappa = 2
theta = 0.04
sigma = 0.3
lmbda = 0

# Arrays to store option values
Vc_Heston = np.zeros_like(S)
Vp_Heston = np.zeros_like(S)

# Calculating option prices
for i, S0 in enumerate(S):
    call_price = 0
    for j in [1, 2]:
        integral = 0
        for phi in np.linspace(0.001, 100, 500):  # Integration over phi
            if j == 1:
                u = 0.5
                b = kappa + lmbda - rho * sigma
            else:
                u = -0.5
                b = kappa + lmbda

            #calculate heston model

            d = np.sqrt((rho * sigma * phi * 1j - b)**2 - sigma**2 * (2 *
u * phi * 1j - phi**2))
            g = (b - rho * sigma * phi * 1j + d) / (b - rho * sigma * phi
* 1j - d)
            C = (r - q) * phi * 1j * T + (kappa * theta / sigma**2) * ((b
- rho * sigma * phi * 1j + d) * T - 2 * np.log((1 - g * np.exp(d * T))/(1
- g)))
            D = (b - rho * sigma * phi * 1j + d) / sigma**2 * ((1 -
np.exp(d * T)) / (1 - g * np.exp(d * T)))
            f = np.exp(C + D * v0 + 1j * phi * np.log(S0))
```

```
            integrand = np.real(np.exp(-1j * phi * np.log(K)) * f / (1j *
phi))
            integral += integrand * (100 / 500)  # Approximate the
integral

        #Calculate probability of Heston Model
        pro_Heston = 0.5 + (1 / np.pi) * integral
        if j == 1:
            call_price += S0 * np.exp(-q * T) * pro_Heston
        else:
            call_price -= K * np.exp(-r * T) * pro_Heston

    Vc_Heston[i] = call_price
    Vp_Heston[i] = call_price + K * np.exp(-r * T) - S0 * np.exp(-q * T)

# Displaying results
for i, S0 in enumerate(S):
    print(f'S0 = {S0}, Call price: {round(Vc_Heston[i], 5)}, Put price:
{round(Vp_Heston[i], 5)}')

# Plot
plt.figure(figsize=(10, 5))

# Call option
plt.subplot(1, 2, 1)
plt.plot(S, Vc_Heston, 'b-', label='Call Option Value')
plt.xlabel('Stock Price (S)')
plt.ylabel('Option Value')
plt.title('Heston Model:Call Option Value vs. Stock Price')
plt.grid(True)
plt.legend()

# Put option
plt.subplot(1, 2, 2)
plt.plot(S, Vp_Heston, 'r-', label='Put Option Value')
plt.xlabel('Stock Price (S)')
plt.ylabel('Option Value')
plt.title('Heston Model:Put Option Value vs. Stock Price')
plt.grid(True)
plt.legend()
```

```
plt.tight_layout()
plt.show()
```

9.3 Merton jump-diffusion model

```
# Merton jump-diffusion model
## parameters
T = 1
K = 100
r = 0.05
q = 0.03
S = np.arange(50, 151, 1)
sigma = 0.1

# Model parameters for the jump part
mu_j = -0.1
sigma_j = 0.2
lmbda = 0.5

# Call and Put values for each S
Vc_Merton = np.zeros_like(S)
Vp_Merton = np.zeros_like(S)

max_iter = 1000
stop_cond = 1e-16

for i, S_val in enumerate(S):
    Vc_temp = 0
    Vp_temp = 0
    for k in range(max_iter):
        r_k = r - lmbda * (np.exp(mu_j + 0.5 * sigma_j**2) - 1) + (k *
(mu_j + 0.5 * sigma_j**2)) / T
        sigma_k = np.sqrt(sigma**2 + (k * sigma_j**2) / T)

        # Black-Scholes calculations for call and put
```

```
        d1 = (np.log(S_val/K) + (r_k - q + sigma_k**2/2)*T) /
(sigma_k*np.sqrt(T))
        d2 = d1 - sigma_k * np.sqrt(T)
        bs_call = S_val * np.exp(-q*T) * norm.cdf(d1) - K *
np.exp(-r_k*T)* norm.cdf(d2)
        bs_put = K * np.exp(-r_k*T) * norm.cdf(-d2) - S_val * np.exp(-q*T)
* norm.cdf(-d1)

        sum_k_call = (np.exp(-(np.exp(mu_j + 0.5 * sigma_j**2)) * lmbda *
T) * ((np.exp(mu_j + 0.5 * sigma_j**2)) * lmbda * T)**k /
np.math.factorial(k)) * bs_call
        sum_k_put = (np.exp(-(np.exp(mu_j + 0.5 * sigma_j**2)) * lmbda *
T) * ((np.exp(mu_j + 0.5 * sigma_j**2)) * lmbda * T)**k /
np.math.factorial(k)) * bs_put

        Vc_temp += sum_k_call
        Vp_temp += sum_k_put

        if sum_k_call < stop_cond and sum_k_put < stop_cond:
            break

    Vc_Merton[i] = Vc_temp
    Vp_Merton[i] = Vp_temp

# Plot
plt.figure(figsize=(10, 5))

# Call option
plt.subplot(1, 2, 1)
plt.plot(S, Vc_Merton, 'b-', label='Call Option Value')
plt.xlabel('Stock Price (S)')
plt.ylabel('Option Value')
plt.title('Merton jump-diffusion model: Call Option Value vs. Stock
Price')
plt.grid(True)
plt.legend()

# Put option
plt.subplot(1, 2, 2)
plt.plot(S, Vp_Merton, 'r-', label='Put Option Value')
```

```
plt.xlabel('Stock Price (S)')
plt.ylabel('Option Value')
plt.title('Merton jump-diffusion model:Put Option Value vs. Stock Price')
plt.grid(True)
plt.legend()

plt.tight_layout()
plt.show()
```

4. Plot the result together to compare.

```
plt.figure(figsize=(10, 6))
plt.plot(S, Vc_BS, label='Black-Scholes', color='blue')
plt.plot(S, Vc_Merton, label='Merton Jump Diffusion', color='green')
plt.plot(S, Vc_Heston, label='Heston Model', color='red')
plt.title('Call Option Prices by Model')
plt.xlabel('Underlying Asset Price (S)')
plt.ylabel('Call Option Value')
plt.legend()
plt.grid(True)
plt.show()
```

```
plt.figure(figsize=(10, 6))
plt.plot(S, Vp_BS, label='Black-Scholes', color='blue')
plt.plot(S, Vp_Merton, label='Merton Jump Diffusion', color='green')
plt.plot(S, Vp_Heston, label='Heston Model', color='red')
plt.title('Put Option Prices by Model')
plt.xlabel('Underlying Asset Price (S)')
plt.ylabel('Put Option Value')
plt.legend()
plt.grid(True)
plt.show()
```