数学物理方法课程作业

Homework of Mathematical Physics Methods

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序言

本文为笔者本科时的“数学物理方法”课程作业（Homework of Mathematical Physics Methods, 2024.9-2025.1）。由于个人学识浅陋，认识有限，文中难免有不妥甚至错误之处，望读者不吝指正，在此感谢。

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目录

Homework 1: 2024.8.26 - 2024.9.1

1.1 计算

(1) $(\frac{1+\mathrm{i}}{2-\mathrm{i}})^2$

$$\left(\frac{1+\mathrm{i}}{2-\mathrm{i}}\right)^2 = \left(\frac{(1+i)(2+i)}{5}\right)^2 = \left(\frac{1+3i}{5}\right)^2 = \frac{-8+6i}{25}$$

(2) $(1+i)^n + (1-i)^n$

首先得到：

$$1+i = \sqrt{2}e^{i\frac{\pi}{4}},\ 1-i = \sqrt{2}e^{i(-\frac{\pi}{4})}$$
$$\Longrightarrow I = 2^{\frac{n}{2}}\left(e^{i\frac{n\pi}{4}} + e^{-i\frac{n\pi}{4}}\right)$$

于是有：

$$I = \begin{cases} 2^{\frac{n}{2}+1}, & n = 0+4k \\ 2^{\frac{n+1}{2}}, & n = 1+4k \\ 0\quad, & n = 2+4k \\ -2^{\frac{n}{2}+1}, & n = 3+4k \end{cases}\quad,\ \ k \in \mathbb{N}$$

习题课补：

$$\begin{aligned} I &= 2^{\frac{n}{2}}\left(e^{i\frac{n\pi}{4}} + e^{-i\frac{n\pi}{4}}\right) \\ &= 2^{\frac{n}{2}}\left(\cos(\frac{n\pi}{4}) + i\sin\frac{n\pi}{4} + \cos(-\frac{n\pi}{4}) + i\sin(-\frac{n\pi}{4})\right) \\ &= 2^{\frac{n}{2}+1}\cos(\frac{n\pi}{4}) \end{aligned}$$

(3) $\sqrt[4]{1+i}$

$$\sqrt[4]{1+i} = \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{\frac{1}{4}} = 2^{\frac{1}{8}}e^{i\frac{\pi}{16}}$$

习题课补：在复数域中，开根号是多值函数，这里四次根在复数域中应有四个复根，设 $x = \sqrt[4]{1+i}$，则原式等价于方程：

$$x^4 = 1+i = \sqrt{2}e^{i\frac{\pi}{4}} \Longrightarrow |x| = 2^{\frac{1}{8}},\quad \arg x = \frac{\pi}{16} + k\frac{\pi}{2}, k = 0,1,2,3$$

1.2 将复数化为三角或指数形式

(1) $\frac{5}{-3+i}$

$$\frac{5}{-3+i} = \frac{5e^{i0}}{\sqrt{10}e^{i(\arctan(-\frac{1}{3})+\pi)}} = \sqrt{\frac{5}{2}}\cdot e^{-i(\arctan(-\frac{1}{3})+\pi)}$$

(2) $\left(\frac{2+i}{3-2i}\right)^2$

$$\left(\frac{2+i}{3-2i}\right)^2 = \left(\frac{\sqrt{5}e^{i\arctan(\frac{1}{2})}}{\sqrt{13}e^{i\arctan(-\frac{2}{3})}}\right)^2 = \frac{5}{13}e^{2i\left(\arctan(\frac{1}{2})-\arctan(-\frac{2}{3})\right)}$$

1.3 求极限 $\lim_{z\to i}\frac{1+z^6}{1+z^{10}}$

作不完全因式分解：

$$
\begin{aligned}
1+z^6 &= z^6 - i^6 = (z^3-i^3)(z^3+i^3) \\
&= (z-i)(z^2+iz+i^2)(z^3+i^3) \\
1+z^{10} &= z^{10} - i^{10} = (z^5-i^5)(z^5+i^5) \\
&= (z-i)(z^4+iz^3+i^2z^2+i^3z+i^4)(z^5+i^5) \\
\Longrightarrow L &= \lim_{z\to i} \frac{1+z^6}{1+z^{10}} \\
&= \lim_{z\to i} \frac{(z-i)(z^2+iz+i^2)(z^3+i^3)}{(z-i)(z^4+iz^3+i^2z^2+i^3z+i^4)(z^5+i^5)} \\
&= \lim_{z\to i} \frac{(z^2+iz+i^2)(z^3+i^3)}{(z^4+iz^3+i^2z^2+i^3z+i^4)(z^5+i^5)} \\
&= \frac{(-3)\times(-2i)}{5i} = \frac{3}{5}
\end{aligned}
$$

事实上，实数域上的洛必达法则（L'Hospital）可以推广到复数域的解析函数，下面给出 $\frac{0}{0}$ 型的证明。设复变函数 $f(z), g(z)$ 在 $z=z_0$ 解析，且 $f(z_0)=g(z_0)=0$，则有：

$$
\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{g(z)-g(z_0)} = \lim_{z\to z_0} \frac{\frac{f(z)-f(z_0)}{z-z_0}}{\frac{g(z)-g(z_0)}{z-z_0}} = \lim_{z\to z_0} \frac{f'(z)}{g'(z)}
$$

特别地，若 $f'(z_0)$ 与 $g'(z_0)$ 存在且不为零，就有 $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$

1.4 讨论函数在原点的连续性

(1) $f(z) = \begin{cases} \frac{1}{2i}\left(\frac{z}{z^*} - \frac{z^*}{z}\right), & z \neq 0 \\ 0, & z = 0 \end{cases}$

令 $z = x+iy, x, y \in \mathbb{R}$，则 $\forall\,(x,y) \neq (0,0)$:

$$
f(x,y) = \frac{1}{2i}\left(\frac{x+iy}{x-iy} - \frac{x-iy}{x+iy}\right) = \frac{1}{2i}\cdot\frac{4ixy}{x^2+y^2} = \frac{2xy}{x^2+y^2}
$$

令 $k = \frac{y}{x}$，则：

$$
L = \lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{2k}{1+k^2}
$$

显然，L 随着 k 的变化而变化，因此极限不存在，$f(z)$ 在 0 处不连续。

(2) $f(z) = \begin{cases} \frac{\operatorname{Im} z}{1+|z|}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

令 $z = x+iy$ 和 $k = \frac{y}{x}$，则 $\forall\,(x,y) \neq (0,0)$:

$$
f(x,y) = \frac{y}{1+\sqrt{x^2+y^2}} \Longrightarrow \lim_{(x,y)\to(0,0)} f(x,y) = \frac{0}{1+0} = 0 = f(0,0)
$$

因此 $f(z)$ 在 0 处连续。

(3) $f(z) = \begin{cases} \frac{\operatorname{Re} z^2}{|z^2|}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

同理令 $z = x+iy$ 和 $k = \frac{y}{x}$，则 $\forall\,(x,y) \neq (0,0)$:

$$
f(x,y) = \frac{x^2-y^2}{x^2+y^2} = \frac{1-k^2}{1+k^2}
$$

因此 $f(z)$ 在 0 处不连续。

1.5 恒等式证明（附加题）

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 = \sum_{i=1}^{n} |a_i|^2 \cdot \sum_{i=1}^{n} |b_i|^2 - \sum_{1\leqslant i<j\leqslant n} \left|a_i b_j^* - a_j b_i^*\right|^2$$

Homework 2：2024.9.2 - 2024.9.8

2.1 下列函数在何处可导，何处解析

(1) $f(z) = z \cdot \mathrm{Re}\, z$
设 $z = x + iy$，则 $f(z) = u(x,y) + iv(x,y) = x^2 + ixy$。$\forall\, z \in C$，$u(x,y) = x^2$ 和 $v(x,y) = xy$ 在 $\mathbb{C}$ 上有连续一阶偏导，因此 f 在 $\mathbb{C}$ 上解析（解析必可导）。

(2) $f(x,y) = (x-y)^2 + 2i(x+y)$
$\forall\, z \in C$，$u(x,y) = (x-y)^2$ 和 $v(x,y) = 2(x+y)$ 在 $\mathbb{C}$ 上有连续一阶偏导，因此 f 在 $\mathbb{C}$ 上解析（解析必可导）。

2.2 求下列函数的解析区域

(1) $f(z) = xy + iy$

$$\frac{\partial u}{\partial x} = y, \quad \frac{\partial u}{\partial y} = x, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 1$$

欲满足 C-R 条件，则：

$$y = 1,\ x = 0 \Longrightarrow f \text{ 在全平面不解析}$$

不在点 $(0,1)$ 上解析是因为在某点解析是指在此点的有心领域上解析，显然这里不满足。

(2) $f(z) = \begin{cases} |z| \cdot z, & |z| < 1 \\ z^2, & |z| \geqslant 1 \end{cases}$
设 $z = x + iy$，则：

$$f(z) = u(x,y) + iv(x,y) = \begin{cases} (x\sqrt{x^2+y^2}) + i(y\sqrt{x^2+y^2}), & \sqrt{x^2+y^2} < 1 \\ (x^2 - y^2) + i(2xy), & \sqrt{x^2+y^2} \geqslant 1 \end{cases}$$

$$\Longleftrightarrow u(x,y) = \begin{cases} x\sqrt{x^2+y^2}, & \sqrt{x^2+y^2} < 1 \\ x^2 - y^2, & \sqrt{x^2+y^2} \geqslant 1 \end{cases}, \quad v(x,y) = \begin{cases} y\sqrt{x^2+y^2}, & \sqrt{x^2+y^2} < 1 \\ 2xy, & \sqrt{x^2+y^2} \geqslant 1 \end{cases}$$

分别求偏导得到：

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{2x^2+y^2}{\sqrt{x^2+y^2}}, & \frac{\partial u}{\partial y} = \frac{xy}{\sqrt{x^2+y^2}} \\ \frac{\partial v}{\partial x} = \frac{xy}{\sqrt{x^2+y^2}}, & \frac{\partial v}{\partial y} = \frac{x^2+2y^2}{\sqrt{x^2+y^2}} \end{cases}, \quad \sqrt{x^2+y^2} < 1$$

$$\begin{cases} \frac{\partial u}{\partial x} = 2x, & \frac{\partial u}{\partial y} = -2y \\ \frac{\partial v}{\partial x} = 2y, & \frac{\partial v}{\partial y} = 2x \end{cases}, \quad \sqrt{x^2+y^2} \geqslant 1$$

偏导要满足 C-R 条件，代入得到：

$$x^2 = y^2,\ 2xy = 0, \quad \forall\, \sqrt{x^2+y^2} < 1$$
$$2x = 2x,\ -2y = -2y, \quad \forall\, \sqrt{x^2+y^2} \geqslant 1$$
$$\Longrightarrow f(z) \text{ 在 } \{z \in \mathbb{C} \mid |z| \geqslant 1\} \text{ 上解析}$$

不在点 $(0,0)$ 上解析是因为在某点解析是指在此点的有心领域上解析，显然这里不满足。

2.3 已知解析函数 $f(z)$ 的实部如下，求 $f(z)$

(1) $u(x,y) = x^2 - y^2 + x$

$$
\begin{gathered}
v'_x = -u'_y = 2y, \quad v'_y = u'_x = 2x+1 \\
\Longrightarrow v(x,y) = \int 2y \,\mathrm{d}x + \int \mathrm{d}y = 2xy + y + C \\
\Longrightarrow f(x,y) = (x^2 + y^2 + x) + i(2xy + y) + C
\end{gathered}
$$

(2) $u(x,y) = e^y \cos x$

$$
\begin{gathered}
v'_x = -u'_y = -e^y \cos x, \quad v'_y = u'_x = -e^y \sin x \\
\Longrightarrow v(x,y) = \int -e^y \cos x \,\mathrm{d}x + \int 0 \,\mathrm{d}y = -e^y \sin x + C \\
\Longrightarrow f(x,y) = (e^y \cos x) + i(-e^y \sin x) + C
\end{gathered}
$$

2.4 f 解析，且 $u - v = (x-y)(x^2 + 4xy + y^2)$，求 $f(z)$

两边分别对 x, y 求导，得到：

$$
\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = 3x^2 + 6xy - 3y^2, \quad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = 3x^2 - 6xy - 3y^2
$$

联立 C-R 条件，可以解出：

$$
\begin{gathered}
v'_x = -3x^2 + 3y^2, \quad v'_y = 6xy \\
u'_x = 6xy, \quad u'_y = 3x^2 - 3y^2 \\
\Longrightarrow v(x,y) = -x^3 + 3xy^2 + C_1, \quad u(x,y) = 3x^2 y - y^3 + C_2 \\
\Longrightarrow f(x,y) = (3x^2 y - y^3) + i(-x^3 + 3xy^2) + C
\end{gathered}
$$

2.5 极坐标 C-R 条件

证明极坐标下的 C-R 条件为：

$$
\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}
$$

极坐标变换：

$$
\begin{gathered}
x = x(r,\theta) = r\cos\theta, \quad y = y(r,\theta) = r\sin\theta \\
\Longrightarrow \frac{\partial x}{\partial r} = \cos\theta,\ \frac{\partial x}{\partial \theta} = -r\sin\theta,\ \frac{\partial y}{\partial r} = \sin\theta,\ \frac{\partial y}{\partial \theta} = r\cos\theta
\end{gathered}
$$

由复合函数的求导法则：

$$
\begin{gathered}
\frac{\partial}{\partial r} u\left(x(r,\theta), y(r,\theta)\right) = \frac{\partial u(x,y)}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u(x,y)}{\partial y} \cdot \frac{\partial y}{\partial r} = \frac{\partial u(x,y)}{\partial x} \cdot \cos\theta + \frac{\partial u(x,y)}{\partial y} \cdot \sin\theta \\
\frac{\partial}{\partial \theta} v\left(x(r,\theta), y(r,\theta)\right) = \frac{\partial v(x,y)}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial v(x,y)}{\partial y} \cdot \frac{\partial y}{\partial \theta} = -\frac{\partial v(x,y)}{\partial x} \cdot r\sin\theta + \frac{\partial v(x,y)}{\partial y} \cdot r\cos\theta
\end{gathered}
$$

联立 C-R 条件，化简得到：

$$v'_r = -u'_y \cos\theta + u'_x \sin\theta = \frac{1}{r}u'_\theta$$

同理，由偏导关系：

$$\frac{\partial}{\partial\theta}u\left(x(r,\theta),y(r,\theta)\right) = \frac{\partial u(x,y)}{\partial x}\cdot\frac{\partial x}{\partial\theta} + \frac{\partial u(x,y)}{\partial y}\cdot\frac{\partial y}{\partial\theta} = -\frac{\partial u(x,y)}{\partial x}\cdot r\sin\theta + \frac{\partial u(x,y)}{\partial y}\cdot r\cos\theta$$

$$\frac{\partial}{\partial r}v\left(x(r,\theta),y(r,\theta)\right) = \frac{\partial v(x,y)}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial v(x,y)}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial v(x,y)}{\partial x}\cdot\cos\theta + \frac{\partial v(x,y)}{\partial y}\cdot\sin\theta$$

联立 C-R 条件，化简得到：

$$u'_r = u'_x\cos\theta + u'_y\sin\theta = \frac{1}{r}v'_\theta$$

反之也可以化为原 C-R 条件，因此 C-R 条件在极坐标下的形式为：

$$\frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial v}{\partial\theta},\quad \frac{\partial v}{\partial r} = -\frac{1}{r}\cdot\frac{\partial u}{\partial\theta}\quad\square$$

2.6 证明 $f(z)$ 和 $\overline{f(\bar{z})}$ 同解析或同不解析

(1) $f(z)$ 解析 $\Longrightarrow \overline{f(\bar{z})}$ 解析

假设 $f(z)$ 在点 $z=z_0$ 解析，即 $f(z)=u(x,y)+iv(x,y)$ 在有心邻域 $U_\delta(z_0)$ 上解析，这等价于 $f(z)$ 有一阶导，且在邻域内满足 C-R 条件。设 $g(z)=\overline{f(\bar{z})}=u(x,-y)-iv(x,-y)$，也即：

$$g(z)=u_g(x,y)+iv_g(x,y),\quad u_g(x,y)=u(x,-y),\ v_g(x,y)=-v(x,-y)$$

容易验证 $g(z)$ 有一阶偏导，下面验证 C-R 条件：

$$\frac{\partial u_g}{\partial x} = \frac{\partial u}{\partial x}(x,-y),\quad \frac{\partial u_g}{\partial y} = \frac{\partial u(x,-y)}{\partial(-y)}\cdot\frac{\partial(-y)}{\partial y} = -\frac{\partial u}{\partial y}(x,-y)$$

$$\frac{\partial v_g}{\partial x} = -\frac{\partial v}{\partial x}(x,-y),\quad \frac{\partial v_g}{\partial y} = -\frac{\partial v(x,-y)}{\partial(-y)}\cdot\frac{\partial(-y)}{\partial y} = \frac{\partial v}{\partial y}(x,-y)$$

联立 u 和 v 的 C-R 条件，得到：

$$\frac{\partial u_g}{\partial x} - \frac{\partial v_g}{\partial y} = \frac{\partial u}{\partial x}(x,-y) - \frac{\partial v}{\partial y}(x,-y) = 0 \Longrightarrow \frac{\partial u_g}{\partial x} = \frac{\partial v_g}{\partial y}$$

$$\frac{\partial u_g}{\partial y} + \frac{\partial v_g}{\partial x} = -\left[\frac{\partial u}{\partial y}(x,-y) + \frac{\partial v}{\partial x}(x,-y)\right] = 0 \Longrightarrow \frac{\partial u_g}{\partial y} = -\frac{\partial v_g}{\partial x}$$

因此 $g(z)=\overline{f(\bar{z})}$ 也解析。

(2) $f(z)$ 解析 $\Longleftarrow \overline{f(\bar{z})}$ 解析

假设 $\overline{f(\bar{z})}$ 解析，令 $g(z)=\overline{f(\bar{z})}$，则 $f(z)=\overline{g(\bar{z})}$，由 (1) 的结论，$g(z)$ 解析 $\Longrightarrow f(z)=\overline{g(\bar{z})}$ 也解析。证毕。□

Homework 3: 2024.9.9 - 2024.9.15

Homework 4: 2024.9.16 - 2024.9.22

Homework 5: 2024.9.23 - 2024.9.29