

数学物理方法课程作业

Homework of Mathematical Physics Methods

丁毅

中国科学院大学，北京 100049

Yi Ding

University of Chinese Academy of Sciences, Beijing 100049, China

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序言

本文为笔者本科时的“数学物理方法”课程作业 (Homework of Mathematical Physics Methods, 2024.9-2025.1)。由于个人学识浅陋, 认识有限, 文中难免有不妥甚至错误之处, 望读者不吝指正, 在此感谢。

我的邮箱是 dingyi233@mails.ucas.ac.cn。

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Homework 1: 2024.8.26 - 2024.9.1

1.1 计算

(1) $\left(\frac{1+i}{2-i}\right)^2$

$$\left(\frac{1+i}{2-i}\right)^2 = \left(\frac{(1+i)(2+i)}{5}\right)^2 = \left(\frac{1+3i}{5}\right)^2 = \frac{-8+6i}{25}$$

(2) $(1+i)^n + (1-i)^n$

首先得到:

$$1+i = \sqrt{2}e^{i\frac{\pi}{4}}, \quad 1-i = \sqrt{2}e^{i(-\frac{\pi}{4})}$$

$$\Rightarrow I = 2^{\frac{n}{2}} (e^{i\frac{n\pi}{4}} + e^{-i\frac{n\pi}{4}})$$

于是有:

$$I = \begin{cases} 2^{\frac{n}{2}+1}, & n = 0 + 4k \\ 2^{\frac{n+1}{2}}, & n = 1 + 4k \\ 0, & n = 2 + 4k \\ -2^{\frac{n}{2}+1}, & n = 3 + 4k \end{cases}, \quad k \in \mathbb{N}$$

习题课补:

$$I = 2^{\frac{n}{2}} (e^{i\frac{n\pi}{4}} + e^{-i\frac{n\pi}{4}})$$

$$= 2^{\frac{n}{2}} \left(\cos\left(\frac{n\pi}{4}\right) + i \sin\left(\frac{n\pi}{4}\right) + \cos\left(-\frac{n\pi}{4}\right) + i \sin\left(-\frac{n\pi}{4}\right) \right)$$

$$= 2^{\frac{n}{2}+1} \cos\left(\frac{n\pi}{4}\right)$$

(3) $\sqrt[4]{1+i}$

$$\sqrt[4]{1+i} = \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{\frac{1}{4}} = 2^{\frac{1}{8}}e^{i\frac{\pi}{16}}$$

习题课补: 在复数域中, 开根号是多值函数, 这里四次根在复数域中应有四个复根, 设 $x = \sqrt[4]{1+i}$, 则原式等价于方程:

$$x^4 = 1+i = \sqrt{2}e^{i\frac{\pi}{4}} \Rightarrow |x| = 2^{\frac{1}{8}}, \quad \arg x = \frac{\pi}{16} + k\frac{\pi}{2}, k = 0, 1, 2, 3$$

1.2 将复数化为三角或指数形式

(1) $\frac{5}{-3+i}$

$$\frac{5}{-3+i} = \frac{5e^{i0}}{\sqrt{10}e^{i(\arctan(-\frac{1}{3})+\pi)}} = \sqrt{\frac{5}{2}} \cdot e^{-i(\arctan(-\frac{1}{3})+\pi)}$$

(2) $\left(\frac{2+i}{3-2i}\right)^2$

$$\left(\frac{2+i}{3-2i}\right)^2 = \left(\frac{\sqrt{5}e^{i\arctan(\frac{1}{2})}}{\sqrt{13}e^{i\arctan(-\frac{2}{3})}}\right)^2 = \frac{5}{13}e^{2i(\arctan(\frac{1}{2})-\arctan(-\frac{2}{3}))}$$

1.3 求极限 $\lim_{z \rightarrow i} \frac{1+z^6}{1+z^{10}}$

作不完全因式分解:

$$\begin{aligned}
1 + z^6 &= z^6 - i^6 = (z^3 - i^3)(z^3 + i^3) \\
&= (z - i)(z^2 + iz + i^2)(z^3 + i^3) \\
1 + z^{10} &= z^{10} - i^{10} = (z^5 - i^5)(z^5 + i^5) \\
&= (z - i)(z^4 + iz^3 + i^2z^2 + i^3z + i^4)(z^5 + i^5) \\
\Rightarrow L &= \lim_{z \rightarrow i} \frac{1 + z^6}{1 + z^{10}} \\
&= \lim_{z \rightarrow i} \frac{(z - i)(z^2 + iz + i^2)(z^3 + i^3)}{(z - i)(z^4 + iz^3 + i^2z^2 + i^3z + i^4)(z^5 + i^5)} \\
&= \lim_{z \rightarrow i} \frac{(z^2 + iz + i^2)(z^3 + i^3)}{(z^4 + iz^3 + i^2z^2 + i^3z + i^4)(z^5 + i^5)} \\
&= \frac{(-3) \times (-2i)}{5i} = \frac{3}{5}
\end{aligned}$$

事实上, 实数域上的洛必达法则 (L'Hospital) 可以推广到复数域的解析函数, 下面给出 $\frac{0}{0}$ 型的证明。设复变函数 $f(z), g(z)$ 在 $z = z_0$ 解析, 且 $f(z_0) = g(z_0) = 0$, 则有:

$$\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} = \lim_{z \rightarrow z_0} \frac{\frac{f(z) - f(z_0)}{z - z_0}}{\frac{g(z) - g(z_0)}{z - z_0}} = \lim_{z \rightarrow z_0} \frac{f'(z)}{g'(z)}$$

特别地, 若 $f'(z_0)$ 与 $g'(z_0)$ 存在且不为零, 就有 $\lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$

1.4 讨论函数在原点的连续性

$$(1) f(z) = \begin{cases} \frac{1}{2i}(\frac{z}{z^*} - \frac{z^*}{z}), & z \neq 0 \\ 0, & z = 0 \end{cases}$$

令 $z = x + iy, x, y \in \mathbb{R}$, 则 $\forall (x, y) \neq (0, 0)$:

$$f(x, y) = \frac{1}{2i} \left(\frac{x + iy}{x - iy} - \frac{x - iy}{x + iy} \right) = \frac{1}{2i} \cdot \frac{4ixy}{x^2 + y^2} = \frac{2xy}{x^2 + y^2}$$

令 $k = \frac{y}{x}$, 则:

$$L = \lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \lim_{(x, y) \rightarrow (0, 0)} \frac{2k}{1 + k^2}$$

显然, L 随着 k 的变化而变化, 因此极限不存在, $f(z)$ 在 0 处不连续。

$$(2) f(z) = \begin{cases} \frac{\operatorname{Im} z}{1 + |z|}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

令 $z = x + iy$ 和 $k = \frac{y}{x}$, 则 $\forall (x, y) \neq (0, 0)$:

$$f(x, y) = \frac{y}{1 + \sqrt{x^2 + y^2}} \Rightarrow \lim_{(x, y) \rightarrow (0, 0)} f(x, y) = \frac{0}{1 + 0} = 0 = f(0, 0)$$

因此 $f(z)$ 在 0 处连续。

$$(3) f(z) = \begin{cases} \frac{\operatorname{Re} z^2}{|z|^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

同理令 $z = x + iy$ 和 $k = \frac{y}{x}$, 则 $\forall (x, y) \neq (0, 0)$:

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} = \frac{1 - k^2}{1 + k^2}$$

因此 $f(z)$ 在 0 处不连续。

1.5 恒等式证明 (附加题)

$$\left| \sum_{i=1}^n a_i b_i \right|^2 = \sum_{i=1}^n |a_i|^2 \cdot \sum_{i=1}^n |b_i|^2 - \sum_{1 \leq i < j \leq n} |a_i b_j^* - a_j b_i^*|^2$$

Homework 2: 2024.9.2 - 2024.9.8

2.1 下列函数在何处可导, 何处解析

(1) $f(z) = z \cdot \operatorname{Re} z$

设 $z = x + iy$, 则 $f(z) = u(x, y) + iv(x, y) = x^2 + ixy$. $\forall z \in \mathbb{C}$, $u(x, y) = x^2$ 和 $v(x, y) = xy$ 在 \mathbb{C} 上有连续一阶偏导, 因此 f 在 \mathbb{C} 上解析 (解析必可导)。

后补:

(2) $f(x, y) = (x - y)^2 + 2i(x + y)$

$\forall z \in \mathbb{C}$, $u(x, y) = (x - y)^2$ 和 $v(x, y) = 2(x + y)$ 在 \mathbb{C} 上有连续一阶偏导, 因此 f 在 \mathbb{C} 上解析 (解析必可导)。

后补:

2.2 求下列函数的解析区域

(1) $f(z) = xy + iy$

$$\frac{\partial u}{\partial x} = y, \quad \frac{\partial u}{\partial y} = x, \quad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 1$$

欲满足 C-R 条件, 则:

$$y = 1, x = 0 \implies f \text{ 在全平面不解析}$$

不在点 $(0, 1)$ 上解析是因为在某点解析是指在此点的有心领域上解析, 显然这里不满足。

(2) $f(z) = \begin{cases} |z| \cdot z, & |z| < 1 \\ z^2, & |z| \geq 1 \end{cases}$

设 $z = x + iy$, 则:

$$f(z) = u(x, y) + iv(x, y) = \begin{cases} (x\sqrt{x^2+y^2}) + i(y\sqrt{x^2+y^2}), & \sqrt{x^2+y^2} < 1 \\ (x^2 - y^2) + i(2xy), & \sqrt{x^2+y^2} \geq 1 \end{cases}$$

$$\iff u(x, y) = \begin{cases} x\sqrt{x^2+y^2}, & \sqrt{x^2+y^2} < 1 \\ x^2 - y^2, & \sqrt{x^2+y^2} \geq 1 \end{cases}, \quad v(x, y) = \begin{cases} y\sqrt{x^2+y^2}, & \sqrt{x^2+y^2} < 1 \\ 2xy, & \sqrt{x^2+y^2} \geq 1 \end{cases}$$

分别求偏导得到:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{2x^2+y^2}{\sqrt{x^2+y^2}}, & \frac{\partial u}{\partial y} = \frac{xy}{\sqrt{x^2+y^2}} \\ \frac{\partial v}{\partial x} = \frac{xy}{\sqrt{x^2+y^2}}, & \frac{\partial v}{\partial y} = \frac{x^2+2y^2}{\sqrt{x^2+y^2}} \end{cases}, \quad \sqrt{x^2+y^2} < 1$$

$$\begin{cases} \frac{\partial u}{\partial x} = 2x, & \frac{\partial u}{\partial y} = -2y \\ \frac{\partial v}{\partial x} = 2y, & \frac{\partial v}{\partial y} = 2x \end{cases}, \quad \sqrt{x^2+y^2} \geq 1$$

偏导要满足 C-R 条件, 代入得到:

$$\begin{aligned} x^2 &= y^2, \quad 2xy = 0, \quad \forall \sqrt{x^2+y^2} < 1 \\ 2x &= 2x, \quad -2y = -2y, \quad \forall \sqrt{x^2+y^2} \geq 1 \\ \implies f(z) &\text{ 在 } \{z \in \mathbb{C} \mid |z| \geq 1\} \text{ 上解析} \end{aligned}$$

不在点 $(0, 0)$ 上解析是因为在某点解析是指在此点的有心领域上解析, 显然这里不满足。

后补:

2.3 已知解析函数 $f(z)$ 的实部如下, 求 $f(z)$

(1) $u(x, y) = x^2 - y^2 + x$

$$\begin{aligned} v'_x = -u'_y &= 2y, \quad v'_y = u'_x = 2x + 1 \\ \Rightarrow v(x, y) &= \int 2y \, dx + \int dy = 2xy + y + C \\ \Rightarrow f(x, y) &= (x^2 + y^2 + x) + i(2xy + y) + C \end{aligned}$$

(2) $u(x, y) = e^y \cos x$

$$\begin{aligned} v'_x = -u'_y &= -e^y \cos x, \quad v'_y = u'_x = -e^y \sin x \\ \Rightarrow v(x, y) &= \int -e^y \cos x \, dx + \int 0 \, dy = -e^y \sin x + C \\ \Rightarrow f(x, y) &= (e^y \cos x) + i(-e^y \sin x) + C \end{aligned}$$

2.4 f 解析, 且 $u - v = (x - y)(x^2 + 4xy + y^2)$, 求 $f(z)$

两边分别对 x, y 求导, 得到:

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = 3x^2 + 6xy - 3y^2, \quad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = 3x^2 - 6xy - 3y^2$$

联立 C-R 条件, 可以解出:

$$\begin{aligned} v'_x &= -3x^2 + 3y^2, \quad v'_y = 6xy \\ u'_x &= 6xy, \quad u'_y = 3x^2 - 3y^2 \\ \Rightarrow v(x, y) &= -x^3 + 3xy^2 + C_1, \quad u(x, y) = 3x^2y - y^3 + C_2 \\ \Rightarrow f(x, y) &= (3x^2y - y^3) + i(-x^3 + 3xy^2) + C \end{aligned}$$

2.5 极坐标 C-R 条件

证明极坐标下的 C-R 条件为:

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

极坐标变换:

$$\begin{aligned} x &= x(r, \theta) = r \cos \theta, \quad y = y(r, \theta) = r \sin \theta \\ \Rightarrow \frac{\partial x}{\partial r} &= \cos \theta, \quad \frac{\partial x}{\partial \theta} = -r \sin \theta, \quad \frac{\partial y}{\partial r} = \sin \theta, \quad \frac{\partial y}{\partial \theta} = r \cos \theta \end{aligned}$$

由复合函数的求导法则:

$$\begin{aligned} \frac{\partial}{\partial r} u(x(r, \theta), y(r, \theta)) &= \frac{\partial u(x, y)}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u(x, y)}{\partial y} \cdot \frac{\partial y}{\partial r} = \frac{\partial u(x, y)}{\partial x} \cdot \cos \theta + \frac{\partial u(x, y)}{\partial y} \cdot \sin \theta \\ \frac{\partial}{\partial \theta} u(x(r, \theta), y(r, \theta)) &= \frac{\partial u(x, y)}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u(x, y)}{\partial y} \cdot \frac{\partial y}{\partial \theta} = -\frac{\partial u(x, y)}{\partial x} \cdot r \sin \theta + \frac{\partial u(x, y)}{\partial y} \cdot r \cos \theta \end{aligned}$$

联立 C-R 条件, 化简得到:

$$v'_r = -u'_y \cos \theta + u'_x \sin \theta = \frac{1}{r} u'_\theta$$

同理, 由偏导关系:

$$\begin{aligned} \frac{\partial}{\partial \theta} u(x(r, \theta), y(r, \theta)) &= \frac{\partial u(x, y)}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u(x, y)}{\partial y} \cdot \frac{\partial y}{\partial \theta} = -\frac{\partial u(x, y)}{\partial x} \cdot r \sin \theta + \frac{\partial u(x, y)}{\partial y} \cdot r \cos \theta \\ \frac{\partial}{\partial r} v(x(r, \theta), y(r, \theta)) &= \frac{\partial v(x, y)}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial v(x, y)}{\partial y} \cdot \frac{\partial y}{\partial r} = \frac{\partial v(x, y)}{\partial x} \cdot \cos \theta + \frac{\partial v(x, y)}{\partial y} \cdot \sin \theta \end{aligned}$$

联立 C-R 条件, 化简得到:

$$u'_r = u'_x \cos \theta + u'_y \sin \theta = \frac{1}{r} v'_\theta$$

反之也可以化为原 C-R 条件, 因此 C-R 条件在极坐标下的形式为:

$$\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \cdot \frac{\partial u}{\partial \theta} \quad \square$$

2.6 证明 $f(z)$ 和 $\overline{f(\bar{z})}$ 同解析或同不解析

(1) $f(z)$ 解析 $\implies \overline{f(\bar{z})}$ 解析

假设 $f(z)$ 在点 $z = z_0$ 解析, 即 $f(z) = u(x, y) + iv(x, y)$ 在有心邻域 $U_\delta(z_0)$ 上解析, 这等价于 $f(z)$ 有一阶导, 且在邻域内满足 C-R 条件. 设 $g(z) = \overline{f(\bar{z})} = u(x, -y) - iv(x, -y)$, 也即:

$$g(z) = u_g(x, y) + iv_g(x, y), \quad u_g(x, y) = u(x, -y), \quad v_g(x, y) = -v(x, -y)$$

容易验证 $g(z)$ 有一阶偏导, 下面验证 C-R 条件:

$$\begin{aligned} \frac{\partial u_g}{\partial x} &= \frac{\partial u}{\partial x}(x, -y), & \frac{\partial u_g}{\partial y} &= \frac{\partial u(x, -y)}{\partial(-y)} \cdot \frac{\partial(-y)}{\partial y} = -\frac{\partial u}{\partial y}(x, -y) \\ \frac{\partial v_g}{\partial x} &= -\frac{\partial v}{\partial x}(x, -y), & \frac{\partial v_g}{\partial y} &= -\frac{\partial v(x, -y)}{\partial(-y)} \cdot \frac{\partial(-y)}{\partial y} = \frac{\partial v}{\partial y}(x, -y) \end{aligned}$$

联立 u 和 v 的 C-R 条件, 得到:

$$\begin{aligned} \frac{\partial u_g}{\partial x} - \frac{\partial v_g}{\partial y} &= \frac{\partial u}{\partial x}(x, -y) - \frac{\partial v}{\partial y}(x, -y) = 0 \implies \frac{\partial u_g}{\partial x} = \frac{\partial v_g}{\partial y} \\ \frac{\partial u_g}{\partial y} + \frac{\partial v_g}{\partial x} &= -\left[\frac{\partial u}{\partial y}(x, -y) + \frac{\partial v}{\partial x}(x, -y) \right] = 0 \implies \frac{\partial u_g}{\partial y} = -\frac{\partial v_g}{\partial x} \end{aligned}$$

因此 $g(z) = \overline{f(\bar{z})}$ 也解析。

(2) $f(z)$ 解析 $\longleftarrow \overline{f(\bar{z})}$ 解析

假设 $\overline{f(\bar{z})}$ 解析, 令 $g(z) = \overline{f(\bar{z})}$, 则 $f(z) = \overline{g(\bar{z})}$, 由 (1) 的结论, $g(z)$ 解析 $\implies f(z) = \overline{g(\bar{z})}$ 也解析。证毕。□

Homework 3: 2024.9.9 - 2024.9.15

3.1 若 $f(z)$ 解析, $\arg f(z)$ 是否为调和函数?

3.2 从已知的实虚部求出解析函数 $f(z)$

(1) $u(x, y) = e^x(x \cos y - y \sin y) + 2 \sin x \cdot \sin hy + x^3 - 3xy^2 + y$

(2) $v = \ln(x^2 + y^2) + x - 2y$

3.3 求下列函数的值

(1) $\cos(2 + i)$

(2) $\operatorname{Ln}(w - 3i)$

(3) $\operatorname{Arccos}\left(\frac{3+i}{4}\right)$

(4) $\operatorname{Arctan}(1 + 2i)$

3.4 判断下列函数是单值还是多值函数

(1) $\sin \sqrt{z}$

(2) $\frac{\sin \sqrt{z}}{\sqrt{z}}$

(3) $\frac{\cos \sqrt{z}}{\sqrt{z}}$

3.5 解方程: $2 \cos^2 hz - 3 \cos hz + 1 = 0$

3.6 求下列多值函数的交点

Homework 4: 2024.9.16 - 2024.9.22

Homework 5: 2024.9.23 - 2024.9.29