Assignment 1.1

Wednesday, January 31, 2024 3:33 PM

1. I. Definition of Convex: 2nd derivative should be ≥ 0

$$f(\theta) = \|y - X\theta\|_2^2 = (y - X\theta)^T (y - X\theta)$$

$$\frac{df(\theta)}{d\theta} = (y - X\theta)^T(-X) + (y - X\theta)(-X)^T$$

$$\frac{d^2 f(\theta)}{d\theta^2} = 2X^TX$$

$$\boxed{= 2\|X\|_2}$$

Since the second derivative of the function is $2\|X\|_2$, no values in the squared norm of X can be negative. Therefore, the function is Convex

II $\min_\theta \|y - X\theta\|_2^2$ st. $w^T\theta = b \Rightarrow w^T\theta - b = 0$

$$L(\theta,\beta) = \|y - X\theta\|_2^2 + \beta(w^T\theta - b)$$
$$= (y - X\theta)^2 + \beta(w^T\theta - b)$$

$$\frac{\partial L(\theta,\beta)}{\partial \theta} = -2X^T(y - X\theta) + \beta^T w = -2X^Ty + 2\|X\|_2\theta + \beta^T w$$
$$\Rightarrow \theta = \frac{2X^Ty - \beta^T w}{2\|X\|_2} \quad (1)$$

$$\frac{\partial L(\theta,\beta)}{\partial \beta} = w^T\theta - b$$
$$\overset{(1)}{=} w^T\left(\frac{2X^Ty - \beta^T w}{2\|X\|_2}\right) - b = 0$$
$$\Rightarrow 2b\|X\|_2 = 2w^TX^Ty - \beta\|w\|_2$$
$$\Rightarrow \beta = \frac{2w^TX^Ty - 2b\|X\|_2}{\|w\|_2}$$
$$= \frac{2(w^TX^Ty - b\|X\|_2)}{\|w\|_2}$$

Subbing back into (1)

$$\theta^* = \frac{2X^Ty - 2\left(\frac{w^TX^Ty - b\|X\|_2}{\|w\|_2}\right)^T w}{2\|X\|_2}$$

$$\boxed{= \frac{X^Ty - \left(\frac{w^TX^Ty - b\|X\|_2}{\|w\|_2}\right)^T w}{\|X\|_2}}$$