Cone Theorem Readings Notes Fall 2025

Note 4 — 28, 12, 2025 (draft version round 1)

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The aim of this note is to give a introduction of cone theorem in Kähler mmp, the major references are [HP24], [HP16], [DHY23] and [DH24].

For the Kähler threefolds with terminal singularities, we will prove the following

Theorem 0.1 ([HP16, Theorem 6.3])**.** Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudoeffective. Then there exists a $d \in \mathbb{N}$ and a countable family $(\Gamma_i)_{i\in I}$ of curves on X such that

$$0 < -K_X \cdot \Gamma_i \leq d$$

and

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{K_X \geq 0} + \sum_{i\in I} \mathbb{R}^+ [\Gamma_i]$$

If the ray $\mathbb{R}^+ [\Gamma_i]$ is extremal in $\overline{NA}(X)$, there exists a rational curve C_i on X such that $[C_i] \in \mathbb{R}^+ [\Gamma_i]$.

For Kähler threefolds dlt log pair, we can prove

Theorem 0.2 ([DH24, Theorem 1.3])**.** Let (X, B) be a compact Kähler 3 -fold dlt pair. (Assume $K_X + B$ is pseudo-effective). Then there exists a countable collection of rational curves $\{C_i\}_{i\in I}$ such that $0 < -(K_X + B) \cdot C_i \leq 6$ for all $i \in I$ and

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{(K_X+B)\geq 0} + \sum_{i\in I} \mathbb{R} \geq 0 \cdot [C_i]$$

Moreover the following holds:
(1) For any Kähler class ω, there are only finitely many extremal rays $R_i := \mathbb{R} \geq 0 \cdot [C_i]$ such that $(K_X + B + \omega) \cdot R_i < 0$.
(2) For any $(K_X + B)$-negative extremal ray $R = \mathbb{R} \geq 0 \cdot [C_i]$, there is a nef class $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ such that $\alpha^\perp \cap \overline{\mathrm{NA}}(X) = R$ and $\alpha = K_X + B + \eta$ for some Kähler class η.

Hacon–Păun finished the proof of the cone theorem, using BDPP theorem proved by [Ou25].

Theorem 0.3 ([HP24, Theorem 0.5])**.** Assume that BDPP conjecture holds in dimension n (resp. in dimension $n - 1$). Let X be a compact $\mathbb{Q}$-factorial Kähler variety of dimension n such that $(X, B + \boldsymbol{\beta})$ is generalized klt.

[1]**WARNING:** (1) Round 1: sketch notes; (2) Round 2: more details but contains errors; (3) Round 3: correct version but not smooth to read; (4) Round 4: close to the published version.
To ensure a pleasant reading experience. Please read my notes from ROUND ≥ 4.

There are at most countably many rational curves $\{\Gamma_i\}_{i\in I}$ such that

$$\overline{\mathrm{NA}}(X)=\overline{\mathrm{NA}}(X)_{K_X+B+\boldsymbol{\beta}_X\geq 0}+\sum_{i\in I}\mathbb{R}^+[\Gamma_i]$$

where $0<-(K_X+B+\boldsymbol{\beta}_X)\cdot\Gamma_i\leq 2n$. Moreover, if $B+\boldsymbol{\beta}_X$ (or $K_X+B+\boldsymbol{\beta}_X$) is big, then I is finite.

Contents

1 A brief review of cone theorem for projective varieties

Let us first give a brief review of the standard cone theorem that we are familar with.

Theorem 1.1 ([HAC13, Theorem 6.4])**.** Let $(X.Z,B)$ be a projective KLT pair of dimension d.

(1) There are countable many rational curves $C_i\subset X$ such that

$$\overline{NE}(X/Z)=\overline{NE}(X/Z)_{K_X+B\geq 0}+\sum_i R_i$$

such that the negative extremal rays R_i are all span by curve class C_i (as the picture shows below).

(2) Each R_i generated by the class of some rational curve C_i satisfying

$$-2d\leq (K_X+B)\cdot C_i<0$$

(3) If $F \subset \overline{\mathrm{NE}(X)}$ is a $K_X + \Delta$-negative extreme face, then there exist a unique contraction morphism

$$\mathrm{cont}_F : X \to Z$$

such that a curve $C \subset X$ is contracted to a point iff $[C] \in F$.

(4) Let L be a Cartier divisor on X with $L \cdot R = 0$. For some negative extreme ray R. Then, there is a Cartier divisor L_Y on Y such that

$$L \sim (\mathrm{cont}_R)^* L_Y.$$

Sometimes, we will call (2) the Kawamata's length of extremal ray.

Note that in the theorem above we assume that B is a $\mathbb{R}$-divisor.

PROOF IDEA 1.2. We will divide the proof into several steps.

(1) Step 1. Perturb the divisor B and replace it by some $\mathbb{Q}$-divisor,

(b) Step 2. Given any relative ample $\mathbb{Q}$-divisor, we try to construct an associated extremal contraction.

(c) Step 3. We construct

$$\mathscr{C} = \overline{NE(X/S)_{K_X+B\geq 0} + \sum_{L_H \not\equiv_S 0} F_H}$$

and using some basic cone geometry to show that $\mathscr{C} = \overline{\mathrm{NE}(X/S)}$.

(d) Step 4. Prove discreteness of the extremal ray. This follows by boundedness of length of extremal rays.

Proof. □

2 The generalized Mori cone, some basic convex geometry

Before going into the Kähler setting, let us figure out what is the generalized Mori cone. In the Kähler setting, we define

Definition 2.1. We define the generalized Mori cone $\overline{\mathrm{NA}}(X) \subset N_1(X)$ to be the closed cone generated by the classes of positive closed currents.

We define the Mori cone

$$\overline{\mathrm{NE}}(X) \subset \overline{\mathrm{NA}}(X)$$

as the closure of the cone generated by the currents of integration T_C where $C \subset X$ is an irreducible curve.

Remark 2.2. In the complex analytic setting, it may happen that

$$\overline{\mathrm{NE}}(X) \subsetneq \overline{\mathrm{NA}}(X).$$

e.g. there exists K3 surface with Picard number 0, over which there is not curve C, however there are many closed positive current.

The inclusion can be strict even for projective manifold, e.g. for projective K3 surfaces, by Torelli theorem

$$0 \leq \dim \overline{\mathrm{NE}}(X) = \rho(X) \leq 20.$$

While $\dim \overline{\mathrm{NA}}(X) = h^{1,1} = 20$.

In the second part of this note, we will introduce the basic cone geometry.

3 Höring–Peternell's approach to the cone theorem for Kähler 3-folds

In this section, we will prove

Theorem 3.1 ([HP16, Theorem 6.3])**.** Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities such that K_X is pseudoeffective. Then there exists a $d \in \mathbb{N}$ and a countable family $(\Gamma_i)_{i\in I}$ of curves on X such that

$$0 < -K_X \cdot \Gamma_i \leq d$$

and

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{K_X \geq 0} + \sum_{i\in I} \mathbb{R}^+ [\Gamma_i]$$

If the ray $\mathbb{R}^+ [\Gamma_i]$ is extremal in $\overline{NA}(X)$, there exists a rational curve C_i on X such that $[C_i] \in \mathbb{R}^+ [\Gamma_i]$.

PROOF IDEA 3.2.

4 Das–Hacon's approach to cone theorem for DLT Kähler 3-folds

In this section we will prove

Theorem 4.1 ([DH24, Theorem 1.3])**.** Let (X, B) be a compact Kähler 3 -fold dlt pair. (Assume $K_X + B$ is pseudo-effective). Then there exists a countable collection of rational curves $\{C_i\}_{i\in I}$ such that $0 < -(K_X + B) \cdot C_i \leq 6$ for all $i \in I$ and

$$\overline{\mathrm{NA}}(X) = \overline{\mathrm{NA}}(X)_{(K_X+B)\geq 0} + \sum_{i\in I} \mathbb{R} \geq 0 \cdot [C_i]$$

Moreover the following holds:
(1) For any Kähler class ω, there are only finitely many extremal rays $R_i := \mathbb{R} \geq 0 \cdot [C_i]$ such that $(K_X + B + \omega) \cdot R_i < 0$.
(2) For any $(K_X + B)$-negative extremal ray $R = \mathbb{R} \geq 0 \cdot [C_i]$, there is a nef class $\alpha \in H^{1,1}_{\mathrm{BC}}(X)$ such that $\alpha^\perp \cap \overline{\mathrm{NA}}(X) = R$ and $\alpha = K_X + B + \eta$ for some Kähler class η.

5 Das–Hacon–Yáñez cone theorem for generalized Kähler threefolds

Theorem 5.1 ([DHY23])**.** Let $(X, B+\boldsymbol{\beta}_X)$ be a $\mathbb{Q}$-factorial generalized klt pair, where X is a compact Kähler 3-fold. Then there are at most countably many rational curves $\{\Gamma_i\}_{i\in I}$ such that

$$\overline{\mathrm{NA}}(X)=\overline{\mathrm{NA}}(X)_{K_X+B+\beta_X\geq 0}+\sum_{i\in I}\mathbb{R}^+[\Gamma_i]$$

and $-(K_X+B+\boldsymbol{\beta}_X)\cdot\Gamma_i\leq 6$. Moreover, if $\boldsymbol{\beta}_X$ is big, then I is finite.

6 Hacon–Păun's approach to cone theorem in any dimension

Assuming BDPP theorem [Ou25], one can prove cone theorem holds in any dimension.

Hacon–Păun finished the proof of the cone theorem, using BDPP theorem proved by [Ou25].

Theorem 6.1 ([HP24, Theorem 0.5])**.** Assume that BDPP conjecture holds in dimension n (resp. in dimension $n-1$). Let X be a compact $\mathbb{Q}$-factorial Kähler variety of dimension n such that $(X, B+\boldsymbol{\beta})$ is generalized klt.

There are at most countably many rational curves $\{\Gamma_i\}_{i\in I}$ such that

$$\overline{\mathrm{NA}}(X)=\overline{\mathrm{NA}}(X)_{K_X+B+\boldsymbol{\beta}_X\geq 0}+\sum_{i\in I}\mathbb{R}^+[\Gamma_i]$$

where $0<-(K_X+B+\boldsymbol{\beta}_X)\cdot\Gamma_i\leq 2n$. Moreover, if $B+\boldsymbol{\beta}_X$ (or $K_X+B+\boldsymbol{\beta}_X$) is big, then I is finite.

7 Applications of the cone theorem

7.1 Applications of length of extremal ray

8 Further discussions

(1) Recently [**<empty citation>**] proved the optimal bound for Kawamata's length of extremal ray.

(2) Fujino [**<empty citation>**] generalized the cone theorem to the log canonical case and semi-log canonical case.

References

[DH24] O. Das and C.D. Hacon. *On the Minimal Model Program for Kähler 3-folds*. 2024. arXiv: 2306.11708 [math.AG].

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