Divisorial contraction in Kähler MMP reading notes Spring 2025

Lecture 4 — 25, 02, 2025 (draft version)

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2 Das-Hacon's approach to divisorial contraction for Kähler 3-fold MMP

In this section, we will prove the following theorem.

Theorem 1 ([DH24, Theorem 6.9]). Let (X, B) be a strong $\mathbf{Q}$-factorial Kähler 3-fold KLT pair. With the following condition holds

1. $K_X + B$ is pseudo-effective
2. $\alpha = [K_X + B + \beta]$ is nef and big class such that β is Kähler,
3. The negative extremal ray $R = \overline{\mathrm{NA}}(X) \cap \alpha^{\perp}$ is divisorial.

Then there exists an α-trivial divisorial contraction

$$f : X \to Z,$$

such that there exist some Kähler form α_Z on Z such that $\phi^*(\alpha_Z) = \alpha$.

Before going to the proof let us briefly sketch the idea. We first try to prove that the null locus $\mathrm{Null}(\alpha)$ is the Moishezon surface whose smooth model is projective uniruled. We then take a DLT modification

$$\varphi : (X', \Delta') \to (X, \Delta)$$

of the pair $(X, \Delta = B + (1-b)S)$ (note that this pair (X, Δ) differs from the original pair (X, B) and it is not a KLT pair).

We show that the DLT modification φ preserve the geometry outside the null locus $\mathrm{Null}(\alpha)$. We then run the relative Kähler MMP for (X', Δ') over (X, Δ), which becomes the core of the proof. Since it's Kähler 3-fold MMP, the termination is known. So that it's possible to produce positivity (say $K_{X^m} + \Delta^m$ is nef over (X, Δ)) by the termination theorem.

We need to control the divisors being contracted in the MMP.

So that the induced bimeromorphic map $f : X \dashrightarrow X^m$ is a morphism, and this is the divisorial contraction we want.

In the final step, we will show that the base point freeness holds for the divisorial contraction, say α as pull back of some Kähler form α_Z down stairs.

2.1 The null locus is a Moishezon surface whose smooth model is projective uniruled

In this subsection, we will proof the following lemma.

Lemma 2. In the same setting as Theorem 1. The null locus $\mathrm{Null}(\alpha)$ is a irreducible Moishezon surface, whose smooth model is projective uniruled. Such that the curves in the negative extremal ray R covers the surface S with

$$R \cdot S < 0.$$

Remark 3. Let us breifly sketch the idea. The class $\alpha|_S, \alpha|_{S^\nu}, \alpha|_{S'}$ play important role in this lemma (for simplicity let us assume for now that S is a smooth surface). The idea is to try to use that if a smooth surface is not pseudo-effective, then it's uniruled projective surface. The non-pseudo-effectiveness comes from some intersection number analysis. To be more precise, we will use that $S = \mathrm{Null}(\alpha)$, so that volume $\mathrm{vol}(\alpha|_S) = (\alpha|_S)^2 = 0$ (by definition of null locus). In particular, the restriction $\alpha|_S$ can not be a big class. On the other hand, we can apply adjunction to

$$\alpha|_S = (K_X + B + \beta)|_S.$$

If the coefficient of S in B is 1, then everything is nice and we get

$$\alpha|_S = (K_X + B' + S + \beta)|_S = K_S + B'|_S + \beta|_S.$$

Since $B'|_S \geq 0$ and $\beta|_S$ Kahler, this will imply that K_S can not be pseudo-effective.

However, the coefficient of S in B is not 1, we need some tricky arguement. Let ϵ small enough, so that $\alpha - \epsilon\omega$ is still big. Then apply the divisorial Zariski decomposition to

$$\alpha - \epsilon\omega = \sum c_i S_i + P.$$

We restrict the class on the surface S. If $S_i \neq S$ for all i, then

$$\alpha|_S = \epsilon\omega|_S + \sum c_i(S_i \cap S) + P|_S.$$

The right hand side is big, which contradict to the fact S is null locus of α. Thus there exists some component say $S_1 = S$. We try to make the coefficient of S in α is 1. So that we take the **scaling** that

$$(1 + \frac{1-b}{c_1})\alpha|_S = (K_X + B + \beta + \frac{1-b}{s_1}(\sum s_i S_i + P))|_S.$$

Note that in this case, the coefficient of S in α is 1. So that we can apply the adjunction

$$(K_X + B' + \beta + S + \sum_{j\geq 2} s_j S_j + P)|_S = K_S + B'_S + \beta|_S + \sum_{j\geq 2} s_j S_j \cap S + P|_S,$$

which will imply that K_S is not pseudo-effective. And by the classification theorem of complex surfaces, we know that S is uniruled (and projective as S is assumed to be smooth).

What nice on the projective uniruled surface is that the (0,2)-Hodge number is 0, so that the Bott-Chern class $\alpha|_S$ can be realized as a $\mathbf{R}$-divisor (which is also a $\mathbf{R}$-curve on the surface).

Finally, we need to prove that $R \cdot S < 0$. To do this, Batyrev cone theorem for movable curve is applied. So that $\alpha|_S = C_\epsilon + \sum a_j H_j$. We try to prove that there exist a movable curve H_k in the component such that $\alpha \cdot H_k = 0$ and it generates the negative extremal ray R. So that apply it to the Zariski decomposition of $\alpha - \epsilon\omega = s_1 S + \sum_{j\geq 2} s_j S_j + P$, we get

$$s_1 H_i \cdot S = (\alpha - \epsilon\omega) \cdot H_k - (\sum s_j S_j \cdot H_k + P|_S \cdot H_k) < 0,$$

using that ω is Kähler, H_k meets S_j properly, and $P|_S$ is pseudo-effective and thus intersection with movable curve is non-negative.

Proof. □

2.2 Take DLT modification

2.3 Run the relative MMP

2.4 Control the set of divisors being contracted

2.5 Proof of the base pointness

To prove the base point freeness result, we need the following lemma, the first one says that image of pull back of Bott-Chern is those classes that curves being contracted are trivial on it (assume the singularity is nice and).

Lemma 4. Let $f : X \to Y$ be a morphism between normal compact complex spaces with rational singularity. If in addition one of the following two conditions hold,

1. f is a proper bimeromorphic morphism between Fujiki varieties,

2. f is surjective, there exist some boundary divisor B such that (X, B) is KLT. Moreover $(K_X + B)$ is f-big and nef.

Then the pull back

$$f^* : H^{1,1}_{\mathrm{BC}}(Y) \to H^{1,1}_{\mathrm{BC}}(X),$$

is injective, and the image

$$\mathrm{im}(f^*) = \{\alpha \in H^{1,1}(Y) \mid \alpha \cdot C = 0,\ \forall\ C \in N_1(X/Z)\}.$$

Thus if the contraction morphism $f : X \to Z$ is α-trivial, then there exists some $\alpha_Z \in H^{1,1}_{\mathrm{BC}}(Z)$ with $f^*\alpha_Z = \alpha$.

Remark 5. Before proving the lemma, let us compare this result with the projective contraction theorem. Recall that in the projective setting, if D is a Cartier divisor supporting some negative extremal ray R, then D comes from the pull back (i.e. $D \in \mathrm{im}(f^* : \mathrm{NS}(Y) \to \mathrm{NS}(X))$). The proof requires the base point free theorem to show that mD is base point free. Thus, mD is the pull back of Serre twisted line bundle via map associated to mD. Finally, using rigidty lemma to show that the Kodaira map coincides with the contraction $f : X \to Z$. Thus the divisor mD is also pull back via $f : X \to Z$.

On the other hand, the transcendental case is relatively easier. Since ...

Proof. □

To check α_Z is Kähler, we need the following (singular version) Demailly-Păun Kählerness criterion.

Lemma 6. Let X be a compact normal complex variety. Let $\{\alpha\} \in H^{1,1}_{\mathrm{BC}}(X)$ be a big and nef class. Then $\{\alpha\}$ is Kähler iff for any positive dimensional subvariety (or reduced analytic subset) W, the following holds true

$$\int_W (\alpha|_W)^{\dim W} > 0.$$

Proof. □

Now we can prove the base freeness for the divisorial contraction $f : X \to Z$, using Lemma 4 and Lemma 6

Proof of base point freeness. □

3 Höring-Peternell's approach for Kähler 3-fold MMP

References

[DH24] Omprokash Das and Christopher Hacon, *On the minimal model program for kähler 3-folds*, 2024.