General type Locus, Moishezon Locus and Projective Locus Summer 2025

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1 Overview

The aim of this note is to study the distribution of general type locus, Moishezon locus and projective locus (see Definition 2.1) on the irreducible base. The motivation of this topic comes from the following observation on distribution of polarized (projective) K3 surfaces in the universal family of marked complex K3 surfaces.

Let $X \to D^{20}$ be a universal family of K3 surfaces. A smooth, compact surface is Moishezon iff it is projective. The projective fibers of $X \to D^{20}$ correspond to a countable union of hypersurfaces $H_{2g} \subset D^{20}$. As we can see from this example, the projective locus (which corresponds to projective K3 surfaces) is a countable union of the hypersurface in the moduli space D^{20}.

It is natural to ask how the locus of fibers that admits certain properties is distributed on the base. This note focus on the distribution of the fibers that are of projective, general type and Moishezon on the base S. The major reference of this note are [Kol22a], [RT22] and [Kol22b].

Contents

1 Overview **1**

2 The alternating property of the very big locus, general type locus **1**

3 The alternating property of the Moishezon locus **4**

4 The alternating property of the projective locus **8**

2 The alternating property of the very big locus, general type locus

We first give the definitions for the very big locus, Moishezon locus, general type locus, and the projective locus.

Definition 2.1 (Very big locus, general type locus, Moishezon locus, [Kol22a, Definition 18])**.** Let $g : X \to S$ be a proper morphism of normal analytic spaces and L a line bundle on X. Set

1. $\mathrm{VB}_S(L) := \{s \in S : L_s \text{ is very big on } X_s\} \subset S$,

2. $\mathrm{GT}_S(X) := \{s \in S : X_s \text{ is of general type }\} \subset S$,
3. $\mathrm{MO}_S(X) := \{s \in S : X_s \text{ is Moishezon }\} \subset S$,
4. $\mathrm{PR}_S(X) := \{s \in S : X_s \text{ is projective }\} \subset S$.

Here very big means the place $s \in S$ that $X_s \dashrightarrow \mathrm{Proj}_S(g_* L_s)$ is bimeromorphic onto its closure of the image.

Definition 2.2 (Locus V that satisfies the alternating property over S)**.** Let $g : X \to S$ be a proper morphism of normal irreducible analytic spaces, we say the locus

$$V := \{s \in S \mid X_s \text{ admits property } P\},$$

satisfies *the alternating property over* S if $V \subset S$ is

(1) either nowhere dense (in the analytic Zariski topology),
(2) or it contains a dense open subset of S.

Remark 2.3. In general: (a) A subset which is not nowhere dense does not need to contain an open subset of S. e.g. $\mathbb{Q} \subset \mathbb{R}$ is not nowhere dense but it clearly contains no dense open subset of $\mathbb{R}$. (b) A subset that is not nowhere dense does not need to contain a dense subset of S as well, e.g. the disc $\mathbb{D} = \{|z| < 1\} \subset \mathbb{C}$ is not nowhere dense, but it is also not dense in $\mathbb{C}$.

In the analytic Zariski topology. If S is irreducible, and $V \subset S$ is not nowhere dense, then V is dense in S. To see this, by definition, $\bar{V}$ contains a non-empty Zariski open subset of S. Since S is irreducible, all the non-empty Zariski open subset is dense and therefore $\bar{V} = S$.

Note that the property that $V \subset S$ satisfies the alternating property over S does not care about the information on the special fibers. In other words, we have the following lemma.

Lemma 2.4. If V satisfies the alternating property on some non-empty Zariski open subset $S' \subset S$, then V also satisfies the alternating property on S.

Proof. We first prove that the alternating property does not depend on the special fibers. For if there exists some non-empty Zariski open subset $S' \subset S$, such that V satisfies the alternating property on S'. Note that S is irreducible, the non-empty Zariski open subset $S' \subset S$ is Zariski dense in S. Then we have two cases:

Case 1. If V is nowhere dense in S', then V is also nowhere dense in S. By contradiction, if there exists some non-empty Zariski open subset W (of S) contained in V. Since S is irreducible, the intersection $W \cap S'$ is a non-empty Zariski open subset of S'. And therefore it contradicts to the nonwhere dense of V in S'.

Case 2. If V is dense in S' and we know that $S' \subset S$ is Zariski dense, then V is also dense in S.

$\square$

We first show that the very big locus satisfies the alternating property.

Theorem 2.5 (Alternating property of very big locus, [Kol22a, Lemma 19])**.**
Let $g : X \to S$ be a proper morphism of normal irreducible analytic spaces and L a line bundle on X. Then $\mathrm{VB}_S(L) \subset S$ is

(1) either nowhere dense (in the analytic Zariski topology),
(2) or it contains a dense open subset of S, and $g : X \to S$ is Moishezon.

Proof. We may assume that $g : X \to S$ is surjective (otherwise by properness of g, it will immediately in (1)). By Lemma 2.4, it is possible to pass to a non-empty Zariski open subset of S. Thus, we may assume that g is flat, g_*L is locally free and commutes with restriction to the fibers. We get a meromorphic map $\phi : X \dashrightarrow \mathbb{P}_S(g_*L)$. There is thus a smooth, bimeromorphic model $\pi : X' \to X$ such that $\phi \circ \pi : X' \to \mathbb{P}_S(g_*L)$ is a morphism. After replacing X by X' and again passing to an open subset of S, we may assume that g is flat, g_*L is locally free, commutes with restriction to fibers, and $\phi : X \to \mathbb{P}_S(g_*L)$ is a morphism.

Let $Y \subset \mathbb{P}_S(g_*L)$ denote its image and $W \subset X$ the Zariski closed set of points where $\pi : X \to Y$ is not smooth. Set $Y^\circ := Y \backslash \phi(W)$ and $X^\circ := X \backslash \phi^{-1}(\phi(W))$. The restriction $\phi^\circ : X^\circ \to Y^\circ$ is then smooth and proper. We divide the discussion into two cases:

Case 1. If we assume that the set of points

$$E = \{y \in Y \mid \phi^{-1}(y) \text{ is single points}\} \subset Y,$$

is not dense in Y. We claim in this case the $\mathrm{VB}_S(L)$ is nowhere dense in S. For otherwise, it will imply that $\mathrm{VS}_S(L)$ is dense in S. And, so that for dense set of fibers $\{X_s\}_{s \in \mathrm{VB}_S(L)} \subset X$, the restriction of the relative Kodaira map are bimeromorphic onto its image. And therefore this will imply relative Kodaira map is bimeromorphic. And E should be dense in Y a contradiction.

Case 2. If we assume that $\phi^{-1}(y)$ is a single point for a dense set in Y, thus it's also for a dense set in Y°. Since ϕ^o is proper and smooth, ϕ^o is a finite étale morphism of degree 1, thus it is an isomorphism.

Thus, ϕ is bimeromorphic on every irreducible fiber that has a non-empty intersection with X°. That is, if we denote $D := \{s \in S \mid X_s \cap X^o \neq \emptyset\} \bigcap \{s \in S \mid X_s \text{ is irreducible}\}$ with $g(X^o) = \{s \in S \mid X_s \cap X^o \neq \emptyset\}$, then

$$D \subset \mathrm{VB}_S(L),$$

(1) Note that irreducible of the fiber X_s is needed, if $X_s \cap X^o \neq \emptyset$ and X_s is irreducible, then $X^o \cap X_s \subset X_s$ is an non-empty Zariski open subset of X_s, which is dense on the fiber X_s. Note again since both X and S are irreducible, the generic fibers of g are irreducible, see [GW20, Exercise 6.15]. Thus adding this constraint will not change the result),
(2) Note that the very big locus is not directly defined by the restriction of $X \to \mathbb{P}_S(g_*L)$ on the fibers. Instead, it's defined by the Kodaira map $X_s \to \mathbb{P}(H^0(X_s, L_s)$. Since we assume that g_*L commutes with restriction on the fiber, these two Kodaira maps coincide.

Recall that a morphism between analytic varieties will send a dense subset to a dense subset in its image. And g is flat (by assumption at the beginning), so that g is open. Thus g will send a Zariki dense open subset to a Zariski dense open subset. Thus D is a non-empty dense Zariski open subset contained in the $\mathrm{VB}_S(L)$.

Finally, we need to show that in this case $g : X \to S$ is a Moishezon morphism, i.e. the relative Kodaira map over S induced by L is bimeromorphic onto its image. Since $\phi^o : X^o \to Y^o$ is an isomorphism for $X^o \subset X$ an non-empty dense open subset, the result follows. $\square$

As a direct consequence (combined with birational boundedness result of Hacon-Mckernan [HM06]) we see the general type locus also admits the alternating property.

Theorem 2.6 (The alternating property of general type locus, [Kol22a, Corollary 20])**.**
Let $g : X \to S$ be a proper morphism of normal, irreducible analytic spaces. Then the general type locus

$$\mathrm{GT}_S(X) = \{s \in S \mid X_s \text{ is of general type}\},$$

(1) either nowhere dense (in the analytic Zariski topology),
(2) or it contains a dense open subset of S, and $g : X \to S$ is Moishezon

Proof. Using resolution of singularities, we may assume that X is smooth. By passing to an open subset of S, we may also assume that S and g are smooth. By [HM06] there is an m (depending only on $\dim X_s$) such that $|mK_{X_s}|$ is very big whenever X_s is of general type. Thus, Theorem 2.5 applies to $L = mK_X$. $\square$

3 The alternating property of the Moishezon locus

In this section, we will prove that the Moishezon locus also admits certain alternating property. Before proving Theorem 3.3. Let us first introduce the following result, by [RT22].

Theorem 3.1 (Moishezon morphism criterion, [RT22, Proposition 3.15])**.** Let $\pi : \mathcal{X} \to \Delta$ be a one-parameter degeneration.
(1) Assume that there exists an uncountable subset B of Δ such that for each $t \in B$, the fiber X_t admits a line bundle L_t with the property that $c_1(L_t)$ comes from the restriction to X_t of some cohomology class in $H^2(\mathcal{X}, \mathbb{Z})$.
(2) Assume further that the Hodge number $h^{0,2}(X_t) := h^2(X_t, \mathcal{O}_{X_t})$ is independent of $t \in \Delta$ (the original theorem requires only Hodge (0,1) deformation invariance).

Then there exists a global big line bundle L over $\mathcal{X}$ such that $c_1(L|_{X_s}) = c_1(L_s)$ for any s in some uncoutable subset of B.

Proof. Apply the sheaf exponential exact sequence so that

$$0 \to \mathbb{Z} \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}^*_{\mathcal{X}} \to 0.$$

We claim that

$$H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \cong R^2\pi_*\mathcal{O}_X(\Delta),\ H^2(X_s, \mathcal{O}_{X_s}) \cong R^2\pi_*\mathcal{O}_{\mathcal{X}}(s).$$

Indeed:

(1) By Cartan B. we have

$$H^p(S, R^q\pi_*\mathcal{O}_X) = 0,\ p > 0, q \geq 0,$$

and the Leray spectral sequence argument implies the first one,
(2) Since we assume the cohomological dimension $h^{0,2}$ is constant, by Grauert base change theorem, the second one follows.

Thus we have the following commutative diagram.

$$\begin{tikzcd}
 & & & H^0(\mathcal{X}, R^2\pi_*\mathcal{O}_{\mathcal{X}}) \arrow[d, "\cong"] & \\
{} \arrow[r] & H^1(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}}) \arrow[r] \arrow[d] & H^2(\mathcal{X}, \mathbb{Z}) \arrow[r, "e_2"] \arrow[d] & H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H^1(X_s, \mathcal{O}^*_{X_s}) \arrow[r] & H^2(X_s, \mathbb{Z}) \arrow[r, "e_2"] & H^2(X_s, \mathcal{O}_{X_s}) \arrow[r] \arrow[d, "\cong"] & {} \\
 & & & R^2\pi_*\mathcal{O}_{\mathcal{X}}(s) &
\end{tikzcd}$$

Where we have the evaluation $\mathrm{ev}_s : H^0(\mathcal{X}, R^2\pi_*\mathcal{O}_{\mathcal{X}}) \to R^2\pi_*\mathcal{O}_{\mathcal{X}}(s)$ in the diagram above.

Let $L_s \in \mathrm{Pic}(X_s)$ such that $c_1(L_s) \in H^2(X_s, \mathbb{Z})$. By simply connectedness of Δ, $c_1(L_s) \in H^2(X_s, \mathbb{Z})$ will lift to $c' \in H^2(\mathcal{X}, \mathbb{Z})$. If we can prove the vanishing of $e_2(c) \in H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ then by the exactness of the sequence we can find some global line bundle $L \in \mathrm{Pic}(\mathcal{X})$.

Observe that the cohomology group $H^2(X_s, \mathbb{Z}) \cong H^2(\mathcal{X}, \mathbb{Z})$ is $\mathbb{Z}$ coefficient, so that it has only countable many elements. Given uncountable many L_t, it must have some $c \in H^2(\mathcal{X}, \mathbb{Z})$ such that uncountable subset of t satisfies $c_1(L_t) = c$.

Since this $c \in H^2(X_s, \mathbb{Z})$ comes from $L_s \in \mathrm{Pic}(X_s)$, we have $e_2(c) = 0 \in R^2\pi_*\mathcal{O}_{\mathcal{X}}(s)$ and thus if we lift it to $c \in H^2(\mathcal{X}, \mathbb{Z})$ the global section $e_2(c) \in H^0(\mathcal{X}, R^2\pi_*\mathcal{O}_{\mathcal{X}})$ will vanish on uncountable many points. Thus by the identity principle (since $R^2\pi_*\mathcal{O}_{\mathcal{X}}$ is locally free the vanishing locus of $e_2(c)$ is a subvariety), we have $e_2(c) = 0 \in H^2(\mathcal{X}, R^2\pi_*\mathcal{O}_{\mathcal{X}})$. Therefore, there exists a global line bundle $L \in \mathrm{Pic}(\mathcal{X})$ with the restriction $c_1(L|_{X_s}) = c_1(L_s)$. Finally, by the Proposition 3.2 deformation density of Iitaka-Kodaira dimension, we conclude that L is indeed a global big line bundle. $\square$

The following proposition is used in the proof of Theorem 3.1.

Proposition 3.2 (Deformation density of Iitaka-Kodaira dimension, see [**LiebermanSernesi**; RT22])**.** Let $\pi : \mathcal{X} \to Y$ be a flat family from a complex manifold over a one-dimensional connected complex manifold Y with possibly reducible fibers. If there exists a holomorphic line bundle L on $\mathcal{X}$ such that the Kodaira-Iitaka dimension $\kappa(L_t) = \kappa$ for each t in an uncountable set B of Y, then any fiber X_t in π has at least one irreducible component C_t with $\kappa\left(L|_{C_t}\right) \geq \kappa$.

In particular, if any fiber X_t for $t \in Y$ is irreducible, then $\kappa(L_t) \geq \kappa$.

We now turn to the proof of the alternating property of the Moishezon locus. As noted by Professor Kollár in his paper, the proof of the following theorem is inspired by the argument in the Theorem 3.1.

Theorem 3.3 ([Kol22a, Theorem 21])**.** Let $g : X \to S$ be a smooth, proper morphism of normal, irreducible analytic spaces. Then $\mathrm{MO}_S(X) \subset S$ is

(1) either contained in a countable union $\cup_i Z_i$, where $Z_i \subsetneq S$ are Zariski closed,
(2) or $\mathrm{MO}_S(X)$ contains a dense, open subset of S.

Furthermore, if $R^2 g_* \mathcal{O}_X$ is torsion free then (2) can be replaced by
(3) $\mathrm{MO}_S(X) = S$ and g is locally Moishezon.

Remark 3.4. The condition (1) is slightly differ from the nowhere dense condition compared with Lemma 2.5 and Theorem 2.6. Indeed the countable union of nowhere dense subset needs not to be nowhere dense (e.g. $\mathbb{Q}$ as countable union of nowhere dense subset is no longer nowhere dense). As we will see in the proof, this replacement is necessary. Another difference compared with Lemma 2.5 and Proposition 2.6 is here we assume the morphism is smooth.

Remark 3.5. Compared with proof of [RT22], Kollár's proof does not require the base to be the unit disc Δ. Consequently the direct image $R^2 g_* \mathcal{O}_X$ is only torsion free, which needs not to be locally free.

Proof. Assume first that $R^2 g_* \mathcal{O}_X$ is torsion free. The sheaf exponential sequence

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 1.$$

gives

$$R^1 g_* \mathcal{O}_X^* \to R^2 g_* \mathbb{Z}_X \xrightarrow{e_2} R^2 g_* \mathcal{O}_X.$$

We may pass to the universal cover of S. Note that the local system on the simply connected space is constant, thus $R^2 g_* \mathbb{Z}_X \otimes \mathcal{O}_S$ is a trivial bundle .

Let $\{\ell_i\}$ be those global sections of $R^2 g_* \mathbb{Z}_X$ such that $e_2(\ell_i) \in H^0(S,\ R^2 g_* \mathcal{O}_X)$ is identically 0, and $\left\{\ell'_j\right\}$ the other global sections (those $\{\ell_i, \ell'_i\}$ are countable since we consider the $\mathbb{Z}$-coefficient cohomology). The ℓ_i then lift back to the global sections of $R^1 g_* \mathcal{O}_X^*$. Hence to line bundles L_i on X. We then divide the problem into two cases:

Case 1. If there is an L_i such that $\mathrm{VB}_S(L_i)$ contains a dense open subset of S, then $X \to S$ is Moishezon (by Proposition 2.5) and we are done.

Case 2. If any such line bundle L_i has nowhere dense very big locus $\mathrm{VB}_S(L_i)$. We claim

$$\mathrm{MO}_S(X) \subset \cup_i \mathrm{VB}_S(L_i) \bigcup \cup_j \left(e_2\left(\ell'_j\right) = 0\right).$$

If $s \in \mathrm{MO}_S(X)$, and $s \notin \cup_j \left(e_2\left(\ell'_j\right) = 0\right)$. We claim in this case every line bundle on X_s is numerically equivalent to some $L_i|_{X_s}$. For otherwise, there exist a line bundle L_s on X_s, with $c_1(L_s)$ lift to some ℓ'_j. Since the diagram below commute, which means that $\mathrm{ev}_s(e_2(\ell'_j)) = e_2(c_1(L_s)) = 0$ must vanish, contradict to the $s \notin \left(e_2\left(\ell'_j\right) = 0\right)$.

$$\begin{tikzcd}
H^0(S,R^2g_*\mathbb{Z}) = H^2(S,\mathbb{Z}) \arrow[r, "e_2"] \arrow[d, "\simeq"'] & H^0(X,R^2g_*\mathcal{O}_X) \arrow[d, "\mathrm{ev}_s"] \\
H^2(X_s,\mathbb{Z}) \arrow[r, "e_2"'] & R^2g_*\mathcal{O}_X(s) \simeq H^0(X_s,\mathcal{O}_{X_s})
\end{tikzcd}$$

(Note that the isomorphism $H^2(X_s, \mathcal{O}_s) \simeq R^2g_*\mathcal{O}_X(s)$ at the point $s \in \mathrm{MO}_S(X)$ since locally free of $R^2g_*\mathcal{O}_X$ in neighborhood of $s \in S$ using Proposition 3.6 and the Hodge decomposition we proved in the first time).

Thus X_s has a big line bundle (as $s \in \mathrm{MO}_S(X)$) $\Leftrightarrow L_i|_{X_s}$ is big for some $i \Leftrightarrow L_i|_{X_s}$ is very big for some i (and therefore $s \in \cup_i \mathrm{VB}_S(L_i)$). This completes the case when $R^2g_*\mathcal{O}_X$ is torsion free. $\square$

We next show that fiberwise Moishezon morphism is locally Moishezon if the morphism is smooth. Before proving the result, let us give an locally free criterion of direct image when the fibers satisfy the Du Bois property.

Theorem 3.6 (Locally freeness criterion for $R^if_*\mathcal{O}_X$, [Kol22a, Theorem 24])**.** Let $f : X \to S$ be a smooth, proper morphism of analytic spaces. Assume that $H^i(X_s, \mathbb{C}) \to H^i(X_s, \mathcal{O}_{X_s})$ is surjective for every i for some $s \in S$. Then $R^ig_*\mathcal{O}_X$ is locally free in a neighborhood of s for every i.

Proof. We begin our proof by noticing by the direct image theorem it's enough to show the surjectivity of the base change morphism

$$\phi_s^i : R^if_*\mathcal{O}_X(s) \to H^i(X_s, \mathcal{O}_{X_s}),$$

for every i. Indeed the base change theorem shows that the surjectivity of the base change morphisms ϕ_s^i and ϕ_s^{i-1} implies the locally freeness of the direct image $R^if_*(\mathcal{O}_X)$ (see Hartshorne Corollary 12.9).

Next by the Theorem on Formal Functions, it is enough to prove this when S is replaced by any Artinian local scheme S_n, whose closed point is s.

By Cartan B easy to see the vanishing of $H^p(S_n, R^if_*\mathcal{O}_X) = 0,\ \forall q, \forall i > 0$ then by the Leray spectral sequence arguement we get

$$H^0\left(S_n, R^if_*\mathcal{O}_X\right) = H^i\left(X_n, \mathcal{O}_{X_n}\right), \quad \text{for } i \geq 0.$$

On the local Artinian base with the closed point s, we have the following equality

$$R^if_*\mathcal{O}_X(s) = H^0(S_n, R^if_*\mathcal{O}_X) = H^i(X_n, \mathcal{O}_{X_n}).$$

The base change morphism thus becomes

$$\psi^i : H^i\left(X_n, \mathcal{O}_{X_n}\right) \to H^i\left(X_s, \mathcal{O}_{X_s}\right).$$

Let $\mathbb{C}_{X_n}$ (resp. $\mathbb{C}_{X_s}$) denote the sheaf of locally constant functions on X_n (resp. X_s) and $j_n : \mathbb{C}_{X_n} \to \mathcal{O}_{X_n}$ (resp. $j_s : \mathbb{C}_{X_s} \to \mathcal{O}_{X_s}$) the natural inclusions. We have a commutative diagram

$$\begin{array}{ccc} H^i(X_n, \mathbb{C}_{X_n}) & \xrightarrow{\alpha^j} & H^i(X_s, \mathbb{C}_{X_s}) \\ {\scriptstyle j'_n}\downarrow & & \downarrow{\scriptstyle j'_s} \\ H^i(X_n, \mathcal{O}_{X_n}) & \xrightarrow[\psi^j]{} & H^i(X_s, \mathcal{O}_{X_s}) \end{array}$$

Note that α^i is an isomorphism since the inclusion $X_s \hookrightarrow X_n$ is a homeomorphism, and j_s^i is surjective by assumption. Thus ψ^i is also surjective. □

Using this we can prove the smooth fiberwise Moishezon morphism is locally Moishezon morphism.

Theorem 3.7 (Fiberwise Moishezon smooth morphism is locally Moishezon , [Kol22a, Corollary 22])**.** Let $g : X \to S$ be a smooth, proper morphism of normal, irreducible analytic spaces whose fibers are Moishezon. Then g is locally Moishezon.

Proof. Since we proved (in the first time) the Moishezon manifolds admit strong Hodge decomposition, thus

$$H^i(X_s, \mathbb{C}) \to H^i(X_s, \mathcal{O}_{X_s}),$$

is surjective for every $i \geq 0$. The result then follows directly by Theorem 3.6. □

4 The alternating property of the projective locus

In the last section, we will finish the proof of the alternating property about the projective locus. The following Thom Whitney stratification theorem is useful in the proof.

Proposition 4.1 (Thom Whitney stratification theorem, [Kol22b, Lemma 15])**.** Let $f : X \to S$ be a proper morphism of complex analytic spaces. There exist finite Whitney stratifications $\mathcal{X}$ of X and $\mathcal{S} = \{S_l\}_{l \leq d}$ of S by locally closed subsets S_l of dimension l, with $d = \dim S$, such that for each connected component S (a stratum) of S_l. The following condition holds.

(a) $f^{-1}S$ is a topological fibre bundle over S, union of connected components of strata of $\mathcal{X}$, each mapped submersively to S,

(b) For all $v \in S$, there exist an open neighborhood $U(v)$ in S and a stratum preserving homeomorphism $h : f^{-1}(U) \simeq f^{-1}(v) \times U$ s.t. $f_{|U} = p_U \circ h$ where p_U is the projection on U.

In particular, there is a dense, Zariski open subset $S^\circ \subset S$ such that $g^\circ : X^\circ \to S^\circ$ is a topologically locally trivial fiber bundle. Moreover, If $S = \Delta$, if we shrink the disc then $f : X^* \to \Delta^*$ is topologically fiber bundle.

Under this assumption, we can prove the local system $R^i g_* \mathbb{Z}_X$ is constructible in the analytic Zariski topology for a proper morphism between complex analytic spaces.

Corollary 4.2 ([Kol22b, Corollary 16])**.** Let $g : X \to S$ be a proper morphism of complex analytic spaces. Then the sheaves $R^i g_* \mathbb{Z}_X$ are constructible in the analytic Zariski topology.

When consider the global section of a local system, the following result is helpful.

Lemma 4.3. Let $\mathscr{L}$ be a local system on a complex manifold S, the global section

$$H^0(S,\mathscr{L}) = L^\rho := \{a \in L|\ \rho(\alpha)(a) = a, \forall \alpha \in \pi_1(S,v)\},$$

where L is the fiber of the local system on the reference point $v \in S$. And $\rho : \pi_1(S,v) \to GL(L)$ be the monodromy action. In particular if the base S is simply connected, then $H^0(S,\mathscr{L}) = L$.

Proposition 4.4 (The alternating property of projective locus, [Kol22b, Proposition 17])**.** Let $g : X \to S$ be a proper morphism of normal, irreducible analytic spaces. Then there is a dense, Zariski open subset $S^\circ \subset S$ such that
(1) either X is locally projective over S°,
(2) or $\mathrm{PR}_S(X) \cap S^\circ$ is locally contained in a countable union of Zariski closed, nowhere dense subsets.

If g is bimeromorphic to projective morphism, then X is projective over S^o.

Proof. If we restrict our attention on the main strata S^o of the Whitney stratification, the direct image $R^2g_*\mathbb{Z}_X$ is locally constant. And further restrict on some Zariski open subset, we can also assume that $R^2g_*\mathcal{O}_X$ is locally free. By passing to the universal cover, we may also assume that $R^2g_*\mathbb{Z}_X$ is a constant sheaf. Now consider the sheaf exponential sequence

$$R^1g_*\mathcal{O}_X^* \to R^2g_*\mathbb{Z}_X \xrightarrow{\partial} R^2g_*\mathcal{O}_X.$$

Let Θ be a global section of $R^2g_*\mathbb{Z}_X$. By Lemma 4.3, we know that $\Theta \in H^2(X,\mathbb{Z}) = H^0(S, R^2g_*\mathbb{Z}_X)$. We decompose the cohomology into two disjoint parts,

$$H^2(X,\mathbb{Z}) = V_1 \sqcup V_2,$$

where

$$V_1 = \{\Theta \in H^2(X,\mathbb{Z}) \mid \partial\Theta \equiv 0\},\ V_2 = \{\Theta \in H^2(X,\mathbb{Z}) \mid \partial\Theta \not\equiv 0\}.$$

We claim that if

$$PR_S(X) \subset \cup_{\Theta \in V_2} V(\Theta),$$

is not true, then the morphism is locally projective. Note that, we assume that $R^2g_*\mathcal{O}_X$ is a vector bundle, thus the vanishing locus is a Zariski closed nowhere dense subset we denote $H_\Theta = V(\Theta)$ for $\Theta \in V_2$.

That is if there exists some $s \in PR_S(X)$, such that $s \notin \cup_{\Theta \in V_2} V(\Theta)$, then the morphism is locally projective. To see this, since $s \in PR_S(X)$, there exists some ample line bundle L_s on X_s, and thus under the exact sequence

$$\mathrm{Pic}(X_s) \to H^2(X_s,\mathbb{Z}) \xrightarrow{\partial} H^2(X_s,\mathcal{O}_{X_s}),$$

then L_s maps to some zero element $\partial(\Theta_{L_s}) = 0$.

Since by assumption

$$\mathrm{res}_s : H^2(X,\mathbb{Z}) \xrightarrow{\simeq} H^2(X_s,\mathbb{Z}),$$

one can lift the class Θ_{L_s} to a class Θ. We then divide the problem into two cases.

If $\partial\Theta$ is identically zero, then it lifts to a line bundle $L \in \mathrm{Pic}(X)$, such that $L|_{X_s} = L_s$, which is ample and therefore by Grothendieck ampleness theorem. We know that the morphism is locally projective.

If $\partial\Theta$ is not identically zero, then $\partial\Theta = 0$ defines a Zariski closed, nowhere dense subset $H_\Theta \subset S$. In this case we know that

$$\Theta \in V_2,$$

and by commutative diagram, we know $s \in V(\Theta)$. And thus

$$PR_S(X) \subset \cup_{\Theta \in V_2} V(\Theta).$$

$\square$

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