BDPP reading notes Spring 2025

Lecture Supplement 3 — 17, 02, 2025 (version 0.0)

Yi Li

1 Overview

This aim of this note is to introduce the BDPP theorem for projective [BDPP13] and Kahler manifold [Ou25]. Varies applications of the BDPP theorem are shown.

2 Transcendentla cone

On a compact Kähler manifold, there may not have plently of divisors. To make sense varies positivities, it is necessary to introduce the transcendentla cones.

3 Duality between varies cones

The following theorem shows the duality between pseudo-effective cone and movable cone on the projective manifold.

Lemma 1. Let X be a projective manifold. Let $\gamma \in N_1(X)$ be a movable class. Then given any prime divisor E, there exist a representative γ_E such that γ_E intersect E properly and $\gamma \equiv \gamma_E$.

Remark 2. I am not pretty sure, if the result is also true for Kähler manifold?

Proof. □

Theorem 3. Let X be a projective manifold, then the pseudo-effective cone is dual to the cone of movable curves

$$\mathcal{E} = \overline{\mathrm{Mov}(X)}^{\vee}.$$

In other words, a divisor is pseudo-effective iff it has non-negative intersection with any movable curves.

Remark 4. David [WN19] proved ...

Proof. Let C be a movable curve, By Lemma 1, we can choose some C' such that $C \equiv C'$ and C' meets the given pseudo-effective divisor properly. Hence

$$\mathcal{E} \subset \overline{\mathrm{Mov}(X)}^{\vee}.$$

Conversely, if the inclusion is strict then there exist some

$$\xi \in \partial(\mathcal{E}(X)), \quad \xi \in \mathrm{int}\left(\overline{\mathrm{Mov}}(X)^*\right).$$

We want to deduce contradition. Since X is projective, we can find some ample divisor H such that $\xi - \epsilon H$ still in the movable cone. So that

$$\frac{(\xi \cdot C)}{(H \cdot C)} \geq \varepsilon, \quad \forall\, C \in \overline{\mathrm{Mov}(X)}.$$

On the other hand we can apply Fujita approximation to the class $\xi + tH$ for the ample H. And gets

$$\mu_t : X_t \to X$$

such that

$$\mu_t^*(\xi + tH) = A_t + E_t$$

choose $C = \mu_* A_t^{n-1}$, then apply the Asymptotic orthogonality of Fujita approximation to $\xi \cdot C$ and Teissier-Hovanskii inequality to deduce an upper bound

$$\delta_t \geq \frac{\xi \cdot C}{H \cdot C} \geq \epsilon$$

with $\delta_t \to 0$ when $t \to 0$ (here δ_t is a constant depend on the volume of A_t, since $\mathrm{vol}(\xi) = 0$ by Fujita approximation $\mathrm{vol}(A_t) \to 0$ when $t \to 0$).

□

4 Characterization of the projective uniruled manifold

The projective uniruled manifold is characterized by the pseudo-effectiveness of the canonical bundle.

Lemma 5. Given a movable curve C, there exist a covering family $\bigcup_{t \in S} C_t$ contains C, which covers a dense open subset of X. To be more precise, we can find a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\phi"] \arrow[d, "f"'] & X \\ S & \end{tikzcd}$$

with f a fibration, with fibers C_t and ϕ is dominant generic finite morphism, with $\{C_t\}_{t \in S}$ lies the same numerical class.

Proof. □

Theorem 6 ([BDPP13, Corollary 0.3])**.** Let X be a projective manifold. Then X is uniruled iff K_X is not pseudo-effectiveness.

Remark 7. One direction of the proof is easy, and can be adopted to the Kähler manifold. The converse direction (say K_X is not pseudo-effective) implies uniruled of X is non-trivial, which requires the Mori bend and break technique and the duality between pseudo-effective cone and movable cone.

Remark 8. Miyaoka and Mori [MM86] proved that a projective manifold is uniruled iff there exist an open subset over which there exist a K_X-negative curve passing through it. For more discussion about Miyaokao-Mori theorem (and varies properties of uniruled manifold) see my Note 15.

Proof. It's sufficient to prove that if K_X is not pseudo-effective, then X is uniruled. By duality of pseudo-effective cone and movable cone, we know that there exists a movable curve such that

$$K_X \cdot C < 0.$$

By Lemma 5, we can produce a covering family of K_X-negative irreducible curves using the movable curve C. □

We can generalize the BDPP theorem to the singular case.

Theorem 9. Let (X, B) be a $\mathbb{Q}$-factorial log pair. If $K_X + B$ is not pseudo-effective, then X is uniruled.

Remark 10. Rational curves on singular space is tricky. See more discussion on my notes note-9 Rational curves on Moishezon space, Kaehler varieties.

Proof. Taking the log resolution

$$f : X' \to X,$$

such that $f^*(K_X + B) = K_{X'} + B'$. Since being uniruled is birational invariant, if X is not uniruled, then so it is X'. Then by the BDPP theorem we just proved, $K_{X'}$ is pseudo-effective, thus K_X is pseudo-effective. Since B is effective, $K_X + B$ is pseudo-effective. □

We can characterize the uniruled variety using subsheaf of tangent sheaf

Theorem 11. Let X be a projective manifold, $\mathscr{F} \subset T_X$ be a coherent subsheaf such that $\det \mathscr{F}^* \subset T_X$ is not pseudo-effective, then X is uniruled.

Proof. □

5 Proof of BDPP conjecture for Kähler manifold

Recently, [Ou25] proved the BDPP conjecture for the compact Kähler manifold. In this section, we will briefly introduce the result that he proved.

5.1 Algebraic integrability criteria under Kähler setting

5.2 Pseudo-effectiveness of the adjoint class

5.3 Relative Albanese reduction

5.4 Proof of BDPP conjecture for compact Kähler manifold

6 Varies applications

6.1 Applications of duality of pseudo-effective cone and cone of movable curves

6.2 Producing rational curves using BDPP conjecture

6.3 Cone theorem using BDPP conjecture

References

[BDPP13] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Păun, and Thomas Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), no. 2, 201–248.

[MM86] Yoichi Miyaoka and Shigefumi Mori, *A numerical criterion for uniruledness*, Ann. of Math. (2) **124** (1986), no. 1, 65–69.

[Ou25] Wenhao Ou, *A characterization of uniruled compact kähler manifolds*, 2025.

[WN19] David Witt Nyström, *Duality between the pseudoeffective and the movable cone on a projective manifold*, J. Amer. Math. Soc. **32** (2019), no. 3, 675–689, With an appendix by Sébastien Boucksom.