Hacon-Popa-Schnell's Readings Notes Fall 2025

Note III.3 — 2025 09 12 (draft version 0)

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The aim of this note is to prove the Iitaka conjecture when the base has maximal Albanese dimension.

Theorem 0.1 ([HPS18, Theorem 1.1])**.** Let $f : X \to Y$ be an algebraic fiber space with general fiber F. Assume that Y has maximal Albanese dimension, then $\kappa(X) \geq \kappa(F) + \kappa(Y)$.

There are three essential ingredients appears in the proof.

(a) Green-Lazarsfeld-Simpson's structure theorem on cohomological support loci and generic vanishing theorem;
(b) Hacon-Popa-Schnell's construction of singular Hermitian metric on $f_*\omega_X^{\otimes n}$;
(c) Kawamata's Iitaka conjecture when the base is of general type and structure theorem for finite cover of Abelian varieties.

We will first introduce these three ingredients in more details and finally finish the proof of the Hacon-Popa-Schnell's Iitaka conjecture.

Contents

1 Green-Lazarsfeld-Simpson's Generic Vanishing theorem

1.1 Basic Properties of GV sheaves

Definition 1.1 (GV sheaves)**.**

1.2 Green-Lazarsfeld-Simpson's generic vanishing theorem

Theorem 1.2.

2 Hacon-Popa-Schnell's construction of semi-positive singular Hermitian metric on $f_*\omega_X^{\otimes n}$

3 A brief review of Kawamata's work on Iitaka conjecture when base is of general type

3.1 Several Critera for Albanese map to be fibration

Theorem 3.1 (Albanese map as fibration when Kodaira dimension 0,[Kaw81, Theorem 1])**.** Let X be a non-singular and projective algebraic variety and assume that $\kappa(X) = 0$. Then the Albanese map $\alpha : X \to A(X)$ is an algebraic fiber space.

3.2 Structure theorem for finite cover of Abelian varieties

Theorem 3.2 ([Kaw81, Theorem 13])**.**

3.3 Kawamata's proof of Iitaka conjecture when base is of general type

4 Proof of the Iitaka conjecture when the base has maximal Albanese dimension

Let us finish this note by the proof of Hacon-Popa-Schnell's result.

Theorem 4.1. Let $f : X \to Y$ be an algebraic fiber space with general fiber F. Assume that Y has maximal Albanese dimension, then $\kappa(X) \geq \kappa(F) + \kappa(Y)$.

4.1 Reduce the problem to the case when the base is Abelian Varieties and $\kappa(X) = 0$

Proposition 4.2. To prove Theorem 4.1, it's sufficient to prove the case when $\kappa(X) = 0$ and Y is an Abelian variety.

PROOF IDEA 4.3. We will divide the proof into 3 cases.
Case 1. We will prove the result in the case when $\kappa(X) = -\infty$. Green-Lazarsfeld-Simpson's generic vanishing theorem is used.
Case 2. We assume that $\kappa(Y) = 0$. Using Theorem 3.1, we can reduce the base to an Abelian variety Y. We then take a Iitaka fibration on the total space

$$h : X \to Z,$$

with the general fiber G. We then restrict the fibration $f : X \to Y$ to a new fibration

$$G \to B',$$

with $\kappa(G) = 0$ and B' is an Abelian variety and H be the general fiber. which is exactly the situation of Proposition 4.1, that is

$$\kappa(G) \geq \kappa(H) + \kappa(B'),$$

thus $\kappa(H) = 0$.

$$\begin{tikzcd}
H = F \cap G \arrow[r, hook] \arrow[d, hook] & F \arrow[r] \arrow[d, hook] & h(F) \arrow[d, hook] \\
G \arrow[r, hook] \arrow[d] & X \arrow[r, "h"] \arrow[d, "f"] & Z \\
B \arrow[r, hook] & Y &
\end{tikzcd}$$

We then apply the easy addition formula to $F \to h(F)$, so that

$$\kappa(F) \leq \kappa(H) + \dim h(F) = \dim h(F),$$

which complete the proof of this case.
Case 3. When the base Y has maximal Albanese dimension. Then by the Kawamata's structure theorem for finite cover of Abelian varieties, we can decompose the base Y into a product of Abelian variety K and a variety of general type z.

$$\begin{tikzcd}
F \arrow[r, hook] \arrow[dr, hook] & E \arrow[d, hook] \arrow[dr] & \\
& X \arrow[r] \arrow[d] \arrow[dr, "f"] & K \\
& Z & Y = Z \times K \arrow[l] \arrow[u]
\end{tikzcd}$$

The key point is to observe that the general fiber F of $f : X \to Y$ is also the general fiber of the induced $E \to K$. We then apply the Iitaka conjecture when base Z is of general type so that

$$\kappa(X) = \kappa(E) + \kappa(Z)$$

on the other hand since K is Abelian variety, so that it's possible to prove the Iitaka conjecture for $E \to K$ that is

$$\kappa(E) \geq \kappa(F).$$

This complete the proof of the 3rd case.

We can apply the result of Iitaka conjecture when the base is of general type to finish the proof.

4.2 Analytic Proof of Iitaka conjecture when base is Abelian Varieties and $\kappa(X) = 0$

We will finish the proof by showing that

Proposition 4.4. If $f : X \to A$ is an algebraic fiber space over an abelian variety, with $\kappa(X) = 0$, then we have

$$\mathscr{F}_m = f_* \omega_X^{\otimes m} \simeq \mathscr{O}_A$$

for every $m \in \mathbb{N}$ such that $H^0\left(X, \omega_X^{\otimes m}\right) \neq 0$.

PROOF IDEA 4.5. The point

References

[HPS18] Christopher Hacon, Mihnea Popa, and Christian Schnell. "Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Păun". In: *Local and global methods in algebraic geometry*. Vol. 712. Contemp. Math. Amer. Math. Soc., [Providence], RI, [2018] ©2018, pp. 143–195. ISBN: 978-1-4704-3488-5. DOI: 10.1090/conm/712/14346. URL: https://doi.org/10.1090/conm/712/14346.

[Kaw81] Yujiro Kawamata. "Characterization of abelian varieties". In: *Compositio mathematica* 43.2 (1981), pp. 253–276.