Moishezon space and Moizhezon morphism Summer 2025

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1 Overview

The aim of this note is to give a brief introduction to Moishezon vairety and Moishezon morphism. The major references of this note are [Kol22], [Fuj83], and [Uen75].

Why we study the Moishezon variety/morphism? First, Moishezon variety has more functorial behavior (compared with projective variety), as we will see in Section 2. Secondly, from almost any projective variety we can construct some Moishezon variety via bimeromorphic modification so that Moishezon varieties are versatile in birational geometry. Thirdly, by Artin's fundamental theorem, the category of Moishezon varieties appears naturally in the moduli theory. And finally, let us cite a sentence by Shigefumi Mori: Projective variety like the classical paintings, after modern paining are introduced we can enjoy the classical paintings more. To me non-projective complete variety (maybe he means Moishezon variety) seems like the modern paintings.

And this series of notes is organized as follows:

Lec 1. Basic knowledge about Moishezon varieties and Moishezon morphism,

Lec 2. Fiberwise bimeromorphic problems.

Lec 3. General type locus, Moishezon locus, and projective locus.

Lec 4. Projectivity critera and behavior of projective locus.

Lec 5. Rational curves on Moishezon varieties.

Contents

2 Moishezon varieties

Definition 2.1 (Meromorphic S-map)**.** Let X, Y be a reduced complex space. We call the S-map a meromorphic S-map (not necessary morphism) if

$$\begin{tikzcd}
& \Gamma \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X \arrow[rr, dashed, "\alpha"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& S &
\end{tikzcd}$$

with $\Gamma \subset X \times_S Y$, and $p : \Gamma \to X$ is a proper bimeromosphic morphism. Moreover, if q is also a proper bimeromorphic morphism, then we all α proper bimeromorphic S-map.

Remark 2.2 (Comparison between meromorphic map and S-meromorphic map)**.** Recall that by definition, we know that

$$X \times_S Y \hookrightarrow X \times Y$$

be an inclusion. Therefore, it's easy to see by definition that

$$S\text{-meromorphic map} \implies \text{meromorphic map.}$$

Conversely, the graph of meromorphic map $\Gamma \subset X \times Y$ needs not to contain in $X \times_S Y$, so that meromorphic map needs not to be S-meromorphic map.

Remark 2.3 (Comaprison between S-meromorphic map with fiberwise meromorphic map)**.** Note that S-bimeromorphic map does not need to be fiberwise bimeromorphic map. Since the restriction of the bimeromorphic map on the subvariety (the fibers) need not to be bimeromorphic. We will see an example in Note 2.

Definition 2.4 (Moishezon variety, first definition)**.** A proper, irreducible, reduced analytic space X is Moishezon if it is bimeromorphic to a projective variety X^{p}.

Definition 2.5 (Moishezon variety, second definition)**.** A proper, irreducible, reduced analytic space X is Moishezon if

$$a(X) := \operatorname{tr}\deg_{\mathbb{C}} M(X) = \dim(X)$$

that is, it has $\dim X$ number of algebraic dependent meromorphic function.

Definition 2.6 (Moishezon variety, third definition)**.** A proper irreducible, reduced analytic space X is Moishezon if and only if it carries a big rank 1 reflexive sheaf $\mathscr{F}$. Here the big rank 1 reflexive sheaf means that the induced Kodaira map $g : X \dashrightarrow \mathbb{P}(H^0(X, \mathscr{F}))$ is bimeromorphic onto it's image

.

Proposition 2.7. Three different definitions for Moishezon varieties are equivalent.

Proof. See e.g. [Uen75]. □

The first important property for Moishezon variety is that it locally looks like quasi-projective scheme up to étale cover.

Proposition 2.8 ([Kol22, Proposition 8.2])**.** Let X be a Moishezon variety. For every $x \in X$ there is a pointed quasi-projective scheme (x', X') and an etale morphism $(x', X') \to (x, X)$.

Proof. It's quit hard, for the sake of time we omit it here. For curious reader, please read [Art70]. □

Lemma 2.9 (Existence of Galois closure)**.** Let $\pi : X' \to X$ be a finite covering between normal analytic varieties. Then there exists a finite Galois covering $\varphi : X'' \to X$ from a normal analytic variety X'' which factors through π which is universal in the following sense:

$$\begin{tikzcd} & X' \arrow[dr, "\pi"] & \\ X'' \arrow[ur] \arrow[rr, "\varphi"'] & & X \end{tikzcd}$$

For any finite Galois covering $\psi : Y \to X$ from a normal analytic variety which factors through π, there exists uniquely a Galois covering $Y \to X''$ over X'.

Proposition 2.10 ([Kol22, Proposition 8.3])**.** Let X be a Mosiehzon variety. If X is normal then there is a proper variety Y and a finite group G acting on Y such that $X \cong Y/G$. (Note that usually Y can not be chosen projective.)

Proof. First, by Proposition 2.10, there exists some etale cover of X (indeed, since the etale morphism is finite, we can find an open cover of X be the etale morphism). Since X is proper, we can find some finite cover of it. Now by the previous lemma we can take the Galois closure of the finite étale cover $X_i \to X$. We then apply the universal property of the Galois closure, thus it is possible to patch the Galois closures $\{X_i \to X\}$ together, and therefore get a finite covering of X, $Y \to X$ and thus $X \simeq Y/G$.

□

Artin [Art70] proved the following theorem, which shows the importance of the category of Moishezon varieties in the moduli theory.

Proposition 2.11 ([Art70, Theorem 7.3])**.** There is a natural functor

$$\text{an} : (\text{algebraic space of finite type over } \mathbb{C}) \to (\text{complex spaces})$$

extending the functor an on the category (schemes of finite type $/\mathbb{C}$). This functor induces an equivalence of categories

$$(\text{complex algebraic schemes of finite type}/\mathbb{C}) \to (\text{Moishezon spaces})$$

In other words, every Moishezon spaces is in an unique way an algebraic space.

Proposition 2.12 ([Nam02])**.** Let X be a Moishezon variety with 1-rational singularities. If X is Kähler, then X is projective.

Before proving the theorem, let us first state two results that will be used in the proof.

Lemma 2.13. Let X be a compact Moishezon variety with 1-rational singularities, that is, X is normal and has a resolution $\pi : Y \to X$ such that $R^1\pi_*\mathcal{O}_Y = 0$. Then an analytic homology class $b \in A_2(X, \mathbb{Q})$ is zero if it is numerically equivalent to 0 . In particular,

$$A_2(X, \mathbb{Q}) = N_1(X)_{\mathbb{Q}}$$

Lemma 2.14 (Nakai-Moishezon criterion for $\mathbb{Q}$-line bundles over Kähler Moishezon variety)**.** Let X be a Kähler Moishezon variety with a Kähler form ω. Assume that an element $L \in \mathrm{Pic}(X)_{\mathbb{Q}}$ satisfies the equality for any curve $C \subset X$:

$$(C.L) = \int_C \omega.$$

Then L is ample.

Proof of the Proposition 2.12. Since the numerical equivalence and the homological equivalence coincide for (analytic) 1-cycle by Lemma 2.13, we have a natural map

$$\alpha : N^1(X)_{\mathbb{Q}} \to (A_2(X, \mathbb{Q}))^*, \quad d \mapsto (- \cdot d),$$

and α is an isomorphism.

On the other hand, we have have the morphism

$$\alpha : N^1(X)_{\mathbb{Q}} \to (A_2(X, \mathbb{Q}))^*$$

and the theorem shows this is actually an isomorphism.

Note that $\omega \in H^2(X, \mathbb{R})$ Kähler form as an element of $(A_2(X, \mathbb{R}))^*$. By simply define

$$\alpha_\omega : A_2(X, \mathbb{R}) \to \mathbb{R}, \quad C \mapsto \omega \cdot C = \int_C \omega$$

Since $\alpha_{\mathbb{R}}$ is surjective, there is an element $d \in N^1(X)_{\mathbb{R}}$ such that

$$(C \cdot d) = \int_C \omega,$$

for every curve $C \subset X$.

Apporximate $d \in N^1(X)_{\mathbb{R}}$ by $\mathbb{Q}$-coefficient Approximate $d \in N^1(X)_{\mathbb{R}}$ by a convergent sequence $\{d_m\}$ of rational elements $d_m \in N^1(X)_{\mathbb{Q}}$.

Let us fix the basis $b_1, \ldots, b_l$ of the vector space $N^1(X)_{\mathbb{Q}}$. Each b_i is represented by an element $B_i \in \mathrm{Pic}(X)$. Here

$$\mathrm{Pic}(X)_{\mathbb{Q}} \to N^1(X)_{\mathbb{Q}}, \quad B_i \mapsto b_i,$$

Now d (respectively d_m) is represented by an element in $\mathrm{Pic}(X)_{\mathbb{R}}$. Respectively $\mathrm{Pic}(X)_{\mathbb{Q}}$,

$$D := \Sigma x_i B_i,$$

(respectively $D_m := \Sigma x_i^{(m)} B_i$) such that $\lim x_i^{(m)} = x_i$. Put $E_m := D_m - D$. Then there are d closed $(1,1)$-forms α_m corresponding to E_m such that $\{\alpha_m\}$ uniformly converge to 0 .

If m is chosen sufficiently large, then $\omega_m := \omega + \alpha_m$ is a Kähler form. Since

$$(C.D_m) = \int_C \omega_m,$$

for every curve $C \subset X$. We see that D_m is ample by Lemma 2.14 (Note that we have D_m being a $\mathbb{Q}$-divisor, so that it's possible to apply the Nakai-Moishezon criterion). In particular, X is projective. □

Remark 2.15. There exists some Kähler Moishezon variety with bad singularity that is not projective. (As we will see in the last section).

Proposition 2.16 ([Kol22, Proposition 8])**.**
(1) Let X be a Moishezon variety, if $Z \to X$ be finite then Z is Moishezon.

(2) Let X be a Moishezon variety, and $f : X \to Y$ be a surjective morphism of complex varieties. Then Y is also Moishezon.

(3) Subvarieties of a Moishezon variety are Moishezon varieties.

This is sometimes called functorial property of Moishezon variety, note that (2) needs not always be true for projective variety.

Proof of (1). By definition

$$\mathrm{trdeg}_{\mathbb{C}} K(X) = \dim X,$$

and if Z is finite map then

$$K(X) \hookrightarrow K(Z),$$

is a finite field extension and therefore by additive property for a tower of field extension, we have

$$\mathrm{trdeg}_{\mathbb{C}}(K(Z)) = \mathrm{trdeg}_{\mathbb{C}}(K(X)) + \mathrm{trdeg}_{K(X)} K(Z) = \mathrm{trdeg}_{\mathbb{C}}(K(X)).$$

□

Proof of (2). It will be generalized in to the relative version, see 3.15. □

Proof of (3). Consider the following pull back diagram.

$$\begin{tikzcd} Z^p = f^{-1}(Z) \arrow[r] \arrow[d, "f_Z"'] & X^p \arrow[d, "f"] \\ Z \arrow[r, hook] & X \end{tikzcd}$$

Clearly Z^p is projective (as subvariety of X^p), and f_Z is surjective. And therefore apply (2), we know that Z is again Moishezon. $\square$

The following proposition shows that Moishezon manifold admits strong Hodge decomposition.

Proposition 2.17. If X is a Moishezon manifold, then the Hodge decomposition holds indeed it admits strong Hodge decomposition.

Before proving the theorem, let us first define what is strong Hodge decomposition. We say that a compact manifold admits a strong Hodge decomposition if the natural maps

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^{p,q}(X,\mathbb{C}),\ [\alpha^{p,q}]_{\mathrm{BC}} \mapsto [\alpha^{p,q}]_{\bar\partial} \quad \bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^k(X,\mathbb{C}), \quad \sum[\alpha^{p,q}] \mapsto \sum \alpha^{p,q},$$

are isomorphisms.

Remark 2.18. As a direct consequence, we see that Moishezon manifold admits Du Bois property, that is

$$H^i(X,\mathbb{C}) \to H^i(X,\mathcal{O}_X),$$

is surjective for all $i \geq 0$. (which will be used in the third note).

Proof. The idea of the proof comes from [Dem97, Proposition (12.3)]. We first take the projective modification

$$\mu : \widetilde{X} \to X,$$

such that X' is a projective manifold. And therefore X' admits a strong Hodge decomposition. On the other hand

We first observe that $\mu_\star\mu^\star\beta = \beta$ for every smooth form β on Y.

In fact, this property is equivalent to the equality

$$\int_Y (\mu_\star\mu^\star\beta)\wedge\alpha = \int_X \mu^\star(\beta\wedge\alpha) = \int_Y \beta\wedge\alpha.$$

for every smooth form α on Y, and this equality is clear because μ is a biholomorphism outside sets of Lebesgue measure 0.

Consequently, the induced cohomology morphism $\mu_\star$ is surjective and $\mu^\star$ is injective (but these maps need not be isomorphisms).

$$\begin{array}{cccc}
H^{p,q}_{\mathrm{BC}}(\widetilde{X},\mathbb{C}) \longrightarrow & H^{p,q}(\widetilde{X},\mathbb{C}), & \bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(\widetilde{X},\mathbb{C}) \longrightarrow & H^k(\widetilde{X},\mathbb{C}) \\
\mu_\star\big\downarrow\big\uparrow\mu^\star & \mu_\star\big\downarrow\big\uparrow\mu^\star & \mu_\star\big\downarrow\big\uparrow\mu^\star & \mu_\star\big\downarrow\big\uparrow\mu^\star \\
H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow & H^{p,q}(X,\mathbb{C}), & \bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow & H^k(X,\mathbb{C})
\end{array}$$

Now, we have commutative diagrams with either upward or downward vertical arrows. Hence the surjectivity or injectivity of the top horizontal arrows implies that of the bottom horizontal arrows.

$\square$

We next introduce Campana's Moishezon criterion.

Proposition 2.19. Let X be a compact complex variety in the Fujiki class $\mathscr{C}$. Then X is Moishezon if and only if X is algebraically connected.

Proof. □

As a sequence consequence

Corollary 2.20. A compact Kahler manifold is projective iff it's algebraically connected.

Proposition 2.21. Let $f : X \to B$ be a fibration over an algebraically connected variety (e.g. a projective curve). Assume that X is in the Fujiki class $\mathscr{C}$ and the general fiber of f is algebraically connected, then X is Moishezon if and only if f has a multi-section.

Proof. □

3 Moishezon morphisms

Let us first recall the definition of projective morphism.

Definition 3.1 (Projective morphism, first definition)**.** Let $X \to S$ be a proper morphism between complex spaces. f is projective if there exist an f-ample line bundle $\mathcal{L}$ and a positive integer $m \in \mathbb{Z}_{>0}$ such that there exist a closed S-immersion $X \hookrightarrow \mathbb{P}_S(f_*\mathcal{L}^{\otimes m})$, in short we have the following diagram:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \mathbb{P}_S(f_*\mathcal{L}^{\otimes m}) \arrow[dl] \\ & S & \end{tikzcd}$$

Definition 3.2 (Projective morphism, second definition)**.** Let $X \to S$ be a proper morphism between complex spaces. f is projective if X can be embedded in $\mathbb{P}^N \times S$ for some N, with the following the diagram commute.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \mathbb{P}^N \times S \arrow[dl] \\ & S & \end{tikzcd}$$

Definition 3.3 (Locally projective morphism)**.** Let $f : X \to S$ be a proper morphism of complex spaces. We call f locally projective if for every relatively compact open subset Q of S the restriction $f_Q : X_Q \to Q$ is a projective morphism.

Remark 3.4. Easy to see the second definition will immediate implies the first definition. Converse direction also holds when the base is Stein.

Proof. Let $\mathcal{E} = f_*\mathcal{L}^{\otimes m}$, $f : X \to S$ and $g : Y = \mathbb{P}_S(f_*\mathcal{L}^{\otimes m}) \to S$. Let A be an g-ample line bundle. And, therefore by Serre vanishing theorem over some Stein compact subset $W \subset S$, for some sufficient large $n \gg 0$, we have

$$g^* g_*(\mathcal{E} \otimes A^{\otimes n}) \to \mathcal{E} \otimes A^{\otimes n},$$

is global generated. Since the base S is Stein, by Cartan A theorem, $g_*(\mathcal{E} \otimes A^{\otimes n})$ is global generated. And therefore so it's the pull back $g^* g_*(\mathcal{E} \otimes A^{\otimes n})$. Since the surjective map sends global generated coherent sheaf to global generated coherent sheaf. This means that $\mathcal{E} \otimes A^{\otimes n}$ is global generated.

By coherence of $\mathcal{E} \otimes A^{\otimes n}$, the cohomology group $V = H^0(Y, \mathcal{E} \otimes A^{\otimes n})$ is finite dimensional. And there is a surjection

$$V \otimes \mathcal{O}_Y \to \mathcal{E} \otimes A^{\otimes n}.$$

And therefore it will induce an embedding

$$X \hookrightarrow \mathbb{P}_B(\mathcal{E}) = \mathbb{P}_B\left(\mathcal{E} \otimes A^{\otimes m}\right) \hookrightarrow \mathbb{P}(V) \times B,$$

after shrink the base $B \subset S$. □

Remark 3.5. When the total space has only finite number of irreducible components then locally projective morphism is bimeromorphic to a projective morphism. (see [Fuj83, Lemma 1.3.1]).

In what follows, we may assume that the base S is reduced. However, in general, we do not require the total space X to be reduced or not.

Definition 3.6 (Moishezon morphism, 1st definition)**.** A proper morphism of analytic spaces $g : X \to S$ is Moishezon if $g : X \to S$ is bimeromorphic to a projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to S$.

That is, there is a closed subspace $Y \subset X \times_S X^{\mathrm{p}}$ such that the coordinate projections $Y \to X$ and $Y \to X^{\mathrm{p}}$ are bimeromorphic.

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed] \arrow[dr] & & X^p \arrow[dl] \\ & S & \end{tikzcd}$$

Definition 3.7 (Moishezon morphism, 2nd definition)**.** A proper morphism of analytic spaces $g : X \to S$ is Moishezon if There is a projective morphism of algebraic varieties $G : \mathbb{X} \to \mathbb{S}$ and a meromorphic $\phi_S : S \dashrightarrow \mathbb{S}$ such that X is bimeromorphic to $\mathbb{X} \times_{\mathbb{S}} S$, the fiber product of rational maps is defined where the maps are defined, so on a dense open set.

Remark 3.8. Let us say few words about the fiber product for a rational map $\phi_S : S \dashrightarrow \mathbb{S}$, the fiber product is defined on the place that ϕ_S is holomorphic map.

Definition 3.9 (Moishezon morphism, 3rd definition)**.** A proper morphism of analytic spaces $g : X \to S$ is Moishezon if there is a rank 1, reflexive sheaf L on X such that the natural map $X \dashrightarrow \mathrm{Proj}_S(g_* L)$ is bimeromorphic onto the closure of its image.

Proposition 3.10. Three definitions of Moishezon morphism are equivalent.

Proof. Definition 3.7 equivalent to the Definition 3.6 is clear (using Proposition 3.17). Conversely, if there exists a projective family $X^p \to S$ that bimeromorphic to given $f : X \to S$, then by generic flatness we know $g^p : X^p \to S$ is flat over S^o for some Zariski open subset $S^o \subset S$, and therefore using the definition of projective family, there exist a morphism

$$S^o \to \mathrm{Hilb}(\mathbb{P}^N)$$

such that the projective family is the pull back

$$\begin{tikzcd} X^p \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ S \arrow[r, dashed] & \mathrm{Hilb}(\mathbb{P}^N) \end{tikzcd}$$

We now show that the first definition and third definition are equivalent. From third definition to first definition is clear since $\mathrm{Proj}_S(f_*L)$ is projective over S. Conversely, if $f : X \to S$ is bimeromorphic to a projective morphism $X^p \to S$. Then since we assume X is normal, therefore the meromorphic map $X \dashrightarrow X^p$ is morphism outside codimension 2 subset. And the pull back $(\phi^o)^*\mathcal{O}_X(1)$ is a big line bundle defined on a big open subset, which extend uniquely to a big rank 1 reflexive sheaf. $\square$

Remark 3.11. The termiology in different paper are different, we can summarize it as below.

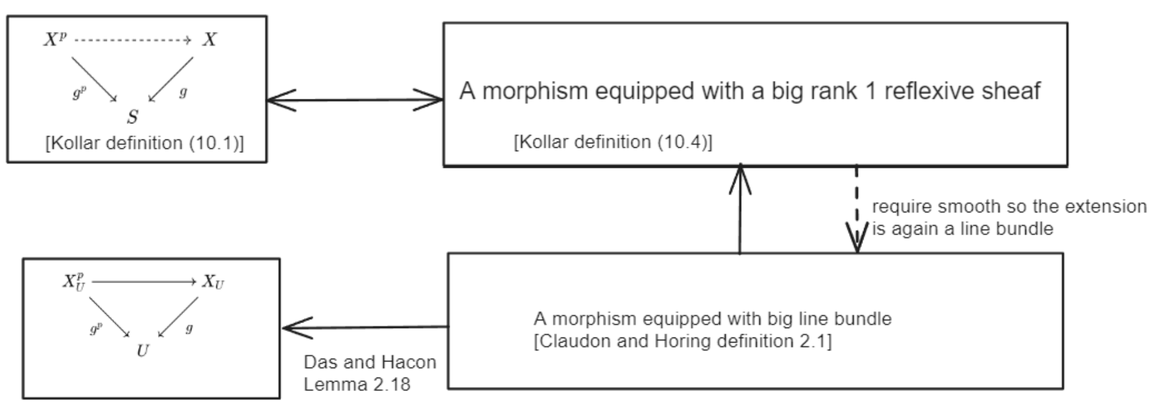

Figure 1: Definitions in different papers

Moishezon morphism satisfies the following Chow type lemma (which can be viewed as the determinisctic property of Moishezon morphism). Before proving the result, let us first recall the relative Iitaka fibration theorem (which appears implicitly in some proof in [**Naka04**] but not explicitly stated in the book).

Theorem 3.12 (Relative Iitaka fibration)**.**

Theorem 3.13 ([DH20, Lemma 2.18])**.** Let $f : X \to S$ be a proper surjective morphism of analytic varieties, and let L be a f-big line bundle on X (more slightly weaker, f is a Moishezon morphism) and D a $\mathbb{Q}$-divisor.

Then over any relatively compact open subset $V \subset S$, there exists a proper bimeromorphic morphism $\alpha : W \to f^{-1}V$ from a smooth analytic variety W such that $\beta = f|_{f^{-1}V} \circ \alpha : W \to V$ is a projective morphism and $\left(W, \alpha_*^{-1}\left(D|_{f^{-1}V}\right) + \mathrm{Ex}(\alpha)\right)$ is a log smooth pair.

First, let us compare the theorem above with the Definition 3.6, in the Definition, we only assume the existence of some bimeromorphic S-map, the Chow lemma allow us to choose some bimeromorphic morphism (which actually can be choosen to be a projective morphism).

Proof. Let $\phi : X \to Y$ be the relative Iitaka fibration of L over S and $g : Y \to S$ the induced projective morphism. Since L is f-big, $\phi : X \to Y$ is bimeromorphic. Let $p : \Gamma \to X$ and $q : \Gamma \to Y$ be the resolution of indeterminacy of ϕ so that p is proper.

$$\begin{tikzcd} & \Gamma \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\phi"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

Now fix a relatively compact open subset $V \subset S$. Choose another relatively compact open set $U \subset S$ containing V such that $\bar{V} \subset U$. Note that U is σ-compact, since it is relatively compact. Since f and g are both proper morphisms, it follows that $X_U := f^{-1}U$ and $Y_U := g^{-1}U$ are both σ-compact. Let $\Gamma_U := q^{-1}\left(g^{-1}U\right) = p^{-1}\left(f^{-1}U\right)$. Then from the commutative diagram above it follows that $q|_{\Gamma_U} : \Gamma_U \to g^{-1}U$ is a proper morphism. In particular, Γ_U is σ-compact. Note that $q|_{\Gamma_U}$ is bimeromorphic. Therefore there is a projective bimeromorphic morphism $h : Z \to \Gamma_U$ from an analytic variety Z such that $q|_{\Gamma_U} \circ h : Z \to Y_U$ is a projective bimeromorphic morphism. Since g is projective, so is $Z \to U$.

Now we replace U by our previously fixed open set V. Then $Z_V := (g \circ\, q \circ h)^{-1}V$ is a relatively compact open subset of Z. Let $r : W \to Z_V$ be the log resolution of $\left(Z_V, (p \circ h)_*^{-1}\left(D|_{f^{-1}V}\right)\right)$. Let $\alpha := p|_{\Gamma_V} \circ h\Big|_{h^{-1}\Gamma_V} \circ r$ and $\beta := g|_{g^{-1}V} \circ q\Big|_{\Gamma_V} \circ h\Big|_{h^{-1}\Gamma_V} \circ r$, where $\Gamma_V := p^{-1}\left(f^{-1}V\right) = q^{-1}\left(g^{-1}V\right)$. Note that β is a projective morphism, since it is a componsition of projective morphisms over relatively compact bases.

Then $\alpha : W \to f^{-1}V$ is a proper bimeromorpic morphism and $\beta : W \to V$ is a projective morphism such that $\beta = f|_{f^{-1}V} \circ \alpha$ and $\left(W, \alpha_*^{-1}\left(D|_{f^{-1}V}\right) + \mathrm{Ex}(\alpha)\right)$ is a log smooth pair. $\square$

Proposition 3.14 ([Fuj83, Proposition 1.5.(4)])**.** Suppose that there exist a locally projective morphism $g : Y \to S$ and a generically finite meromorphic S-map $h : X \dashrightarrow Y$. Then f is Moishezon.

$$\begin{tikzcd} X \arrow[rr, dashed, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

Proof. First since being Moishezon is stable under bimeromorphic change, without lose of generality we can assume that h is a Moishezon. And since Moishezon morphism and locally projective morphism are proper. So that h is proper. Apply the Stein factorization theorem, such that h_2 is projective and h_1 is proper. And thus the composition $g \circ h_2$ is locally projective. And thus by definition $X \to S$ is a Moishezon morphism.

$$\begin{tikzcd} & X^* \arrow[dr, "h_2"] & \\ X \arrow[ur, "h_1"] \arrow[rr, "h"] \arrow[dr] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

□

Proposition 3.15 ([Fuj83, Proposition 1.7])**.** Let $f : X \to S$ be a Moishezon morphism, and $g : Y \to S$ a proper morphism, of reduced complex spaces. Suppose that there is a generically surjective meromorphic S-map $h : X \dashrightarrow Y$. Then g also is Moishezon.

Proof. This Proposition can be viewed as a generalization of the Proposition 2.16. The proof is a bit involving, and we omit it here. □

Proposition 3.16 ([Fuj83, Proposition 1.5])**.**
(1) The morphism $f : X \to S$ is Moishezon if and only if for each irreducible component X_i of X the restriction $f = f|_{X_i} : X_i \to S$ is Moishezon.

(2) Let $f : X \to S$ be a Moishezon morphism. Then: For every reduced analytic subspace $X' \subseteq X$ the induced morphism $f' = f|_{X'} : X' \to S$ is Moishezon.

Proof. For (1), let's take the normalization

$$\nu : X^\nu \to X,$$

recall that for a reduced complex space with finite many irreducible component, the normalization is a bimeromorphic map. So that $f : X \to S$ is Moishezon iff the restriction on each component X_i are Moishezon.

For (2), by the Chow lemma (Theorem 3.13), we can find some locally projective morphism such that X^* is smooth and h is a bimeromorphic S-morphism.

$$\begin{tikzcd} X^* \arrow[rr, "h"] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

We then take the inverse image of the analytic subspace X' denote it $Z = h^{-1}(X')$. (we can assume the inverse image has reduced structure). Since the restriction of the projective morphism on $g|_Z : Z \to S$ is still locally projective. And by construction, clearly the morphism $Z \to X'$ is surjective. And therefore, by Proposition 3.15, we know that $X' \to S$ is a Moishezon morphism. $\square$

Restriction on the image side will also preserve the Moishezon condition.

Proposition 3.17 (A morphism is Moishezon iff it's Moisheozn onto its image)**.** Let $f : X \to S$ be a proper morphism between analytic spaces, let $f' : X \to f(X) = Y \subset S$ be the restriction, then f is Moishezon (resp. projective) iff f' is Moishezon (resp. projective).

Proof. It's enough to prove the case for projective morphism case (and Moishezon morphism case follows easily).

To see this, $\square$

Proposition 3.18. When the base is Moishezon then the total space is Moishezon iff the morphism is Moishezon.

Proof. We first prove that morphism between Moishezon space is a Moishezon morpihsm. Let us define the graph embedding to be

$$\iota : X \to X \times S, \quad x \mapsto (x, f(x)),$$

since X is Moishezon it's bimeromorphic to a projective variety, as the diagram below shows

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & X \times S \arrow[r, dashed] \arrow[d, "\pi"] & X^p \times S \arrow[dl, "\pi^p"] \\ & S & \end{tikzcd}$$

Clearly, π^p is a projective morphism. And consequently π is a Moishezon morphism. And finally by Proposition 3.16, the morphism $f : X \to S$ is again Moishezon.

Conversely, if the morphism is Moishezon, and S is Moishezon variety. Then there exist bimeromorphic modifications such that the following diagram commute

$$\begin{tikzcd} X^p \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[dl] \\ S^p \arrow[r] & S & \end{tikzcd}$$

Where $X' \to S$ is a projective morphism and S^p is a projective variety. Since the base change preserve the projective condition, easy to see that $X^p \to S^p$ is a projective morphism over S^p. And therefore X^p is a projective variety. By Proposition 3.15, X' is a Moishezon variety. Since $X' \to X$ is bimeromorphic, this implies that X is also projective. $\square$

Proposition 3.19 ([Kol22, Lemma 15])**.** Let $g : X \to S$ be a proper, generically finite, dominant morphism of normal, complex, analytic spaces. Then $\mathrm{Ex}(g) \to S$ is Moishezon.

Proof. We will prove the result under the additional assumption that S is Stein. By the geometric Neother normalization theorem, there exist a finite morphism

$$S \to \mathbb{C}^{\dim S}.$$

After replacing the base by $\mathbb{C}^{\dim S}$, we can assume that smooth locus of S is dense in $g(\mathrm{Ex}(g))$. Note that, by Proposition 3.14, if the restriction on $\mathbb{C}^{\dim S}$ is Moishezon, then so will the restriction on S. We will prove the result by induction on dimension.

We first define the base case $(g_0 : X_0 \to S_0) := (g : X \to S)$. Let E_0 be a g_0 exceptional divisor, with the image $Z_0 = g_0(E_0)$. We then inductivly define the morphism $g_{i+1} : X_{i+1} \to S_{i+1}$ as follows. Assume that we already construct $g_i : X_i \to S_i$, we then blow up S_i along Z_i. We then blow up S_i along Z_i and let S_{i+1} be the normalization of the blow-up $\mathrm{Bl}_{Z_i} S_i$. Since S_i is reduced, this will induce a generic finite map $X_i \dashrightarrow$. So that by the universal property of the normalization, the generic finite morphism $g_i : X_i \to S_i$ lift to a generic finite morphism $g_{i+1} : X_{i+1} \to S_{i+1}$, where X_{i+1} is the normalization of the graph of the map $X_{i+1} \dashrightarrow S_{i+1}$.

$$\begin{tikzcd}
X_{i+1} \arrow[r, "g_{i+1}"] \arrow[d] & S_{i+1} = \mathrm{Bl}_{Z_i}(S_i)^{\nu} \arrow[d] \\
\Gamma_{\phi} \arrow[ur] \arrow[d] & \mathrm{Bl}_{Z_i}(S_i) \arrow[d] \\
X_i \arrow[r, "g_i"'] \arrow[uur, dashed, "\phi"] & S_i
\end{tikzcd}$$

Let $E_{i+1} \subset X_{i+1}$ denote the bimeromorphic transform of E_i. (Note that $X_{i+1} \to X_i$ is an isomorphism over an open subset of E_i). We then compute the vanishing order $a(E_i, S_i)$ of Jacobian of g_i along E_i. We claim that

$$a\left(E_{i+1}, S_{i+1}\right) \leq a\left(E_i, S_i\right) + 1 - \operatorname{codim}\left(Z_i \subset S_i\right).$$

Thus eventually we reach the situation when $\operatorname{codim}\left(Z_i \subset S_i\right) = 1$, indeed if $\operatorname{codim}\left(Z_i \subset S_i\right) \geq 2$ then the Jacobian of g_i along E_i will eventually goes to zero. Contradiction.

Thus by comparing the dimension we know when restrict the morphism $X_i \to S_i$ to $E_i \to Z_i$ it will become a generic finite morphism. Since $S_{i+1} \to S_i$ is projective, the composition $Z_i \to Z_0$ will be a locally projective morphism.

$$\begin{tikzcd}
E_i \arrow[rr] \arrow[dr] & & Z_i \arrow[dl] \\
& Z_0 &
\end{tikzcd}$$

By Proposition 3.14, we know that $E_i \to Z_0$ is a Moishezon morphism. Since the strict transform $E_i \to E_0$ is a dominant morphism, by Proposition 3.15, we know that $E_0 \to Z_0$ is also Moishezon morphism. Finally, by Proposition 3.16 and Proposition 3.17, we know that $\mathrm{Ex}(f) \to S$ is Moishezon. $\square$

Theorem 3.20 (Fibers of the Moishezon morphism are Moishezon varieties, [Kol22, Corollary 16])**.** The fibers of a proper, Moishezon morphism are Moishezon.

Proof. Let $g : X \to S$ be a proper, Moishezon morphism. It is bimeromorphic to a projective morphism $X^{\mathrm{p}} \to S$. We may assume X^{p} to be normal. Let Y be the normalization of the closure of the graph of $X \dashrightarrow X^{\mathrm{p}}$.

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed] \arrow[dr] & & X^p \arrow[dl] \\ & S & \end{tikzcd}$$

Fix now $s \in S$. Let $Z_s \subset X_s$ be an irreducible component, since given a proper dominant morphism, there exist at least one irreducible component dominant the base, there exist $W_s \subset Y_s$ an irreducible component that dominates Z_s. And by Proposition 3.15 and Proposition 3.16, it's enough to show that W_s is Moishezon. We divide the problem into two cases:

If $\pi : Y \to X^{\mathrm{p}}$ is generically an isomorphism along W_s, then W_s is bimeromorphic to an irreducible component of X_s^{p}, hence Moishezon.

Otherwise $W_s \subset \mathrm{Ex}(\pi)$. Now $\mathrm{Ex}(\pi) \to X^{\mathrm{p}}$ is Moishezon by Proposition 3.19. And by induction on dimension, since $\dim \mathrm{Ex}(\pi) < \dim X = \dim Y$, the fiber W_s is Moishezon. □

Proposition 3.21 ([Kol22, Example 13])**.** Let Z be a normal, projective variety with discrete automorphism group. Let $g : X \to S$ be a fiber bundle with fiber Z over a connected base S. Then g is Moishezon $\Leftrightarrow$ g is projective $\Leftrightarrow$ the monodromy is finite.

Remark 3.22. The monodromy here is different from the cohomological monodromy. Here the monodromy is refered as the fiber bundle monodromy

$$\rho : \pi_1(S) \to G,$$

where $G \subset \mathrm{Aut}(Z)$ is the structure group of the fiber (e.g. when the fiber bundle is principal G-bundle, then the structure group is simply the group G, for example if you consider the Hodge bundle, then the structure group is $GL(H^i)$ and thus recover back to the standard cohomological monodromy). Finite monodromy condition means that $\mathrm{im}(\rho) \subset G$ is a finite subgroup.

Before proving the theorem, let us state a lemma from fiber bundle theory, that is useful in what follows.

Lemma 3.23. Let $f : X \to S$ be a fiber bundle with trivial monodromy group, then the fiber bundle is actually a trivial fiber bundle.

The proof of the theorem is provided by Professor Kollár, thanks him for telling me this beautiful proof.

Proof of the theorem. Only needs to show that (1) implies (3) and (3) implies (2). For (3) implies (2), we try to take the etale base change so that the fiber bundle becomes trivial bundle. We can do as follows, Consider $\rho(\pi_1(S)) \subset \mathrm{Aut}(Z)$ is finite, let $S' \to S$ be the corresponding finite (unbranched) cover that kills the monodromy. Indeed since we have the

$$\rho : \pi_1(S) \to G$$

then the kernel of $\ker(\rho)$ is a subgroup of $\pi_1(S)$ is finite index, therefore by the Galois correspondence for covering, there exist finite etale cover of the base

$$\tilde{S} \to S,$$

such that monodromy of the fiber bundle under the base change becomes trivial (by the previous lemma). Since after the base change

$$Z \times \tilde{S} \to \tilde{S},$$

clearly the morphism is projective and admits an relative ample line bundle L (since Z is projective). And therefore if we define

$$L' = \bigotimes_{g \in G} g^* L,$$

since it's monodromy invariant. The ample line bundle will descend to the original family $g : X \to S$ and thus g is a projective morphism.

For (1) implies (3). Since $g : X \to S$ is Moishezon, by Definition 3.9, there exists a g-big (rank 1 reflexive sheaf) H on X (since it's fiber bundle so the restriction on Z is again big and denote it also as H). Given an ample line bundle L on the fiber Z, we consider the monodromy action on L, which pulls back the ample line bundle to another ample line bundle $L_\gamma = \rho(\gamma)^* L$.

Note that under the monodromy action, the intersection

$$d := H \cdot (L_\gamma)^{n-1},$$

remain the same for all γ.

We then consider the linear functional

$$\ell : N^1(Z)_{\mathbb{R}} \to \mathbb{R}, \quad M \mapsto M^{n-1} \cdot H,$$

if we restrict the linear functional on the ample cone $\mathrm{Amp}(Z)$, then

$$S_d = \{M \in N^1(Z)_{\mathbb{R}} \mid M^{n-1} \cdot H = d\},$$

is a bounded slice. To see this, by the Khovanskii-Teissier inequality we have

$$\left(H \cdot M^{n-1}\right)^n \geq (H^n)\,(M^n)^{n-1},$$

thus we get

$$\mathrm{vol}(M) \leq \left(\frac{d^n}{\mathrm{vol}(H)}\right)^{\frac{1}{n-1}},$$

so that it's bounded. Thus it contains only finite many lattice point of $\mathrm{NS}(Z)$. In particular

$$\#\{M \in \mathrm{NS}(Z) \cap \mathrm{Amp}(Z) \mid \ell(M) = d\} < \infty.$$

In particular the ample line bundle on the monodromy orbit is finite

$$\{L_\gamma \mid \gamma \in \pi_1(S)\} \subset \{M \in \mathrm{NS}(Z) \cap \mathrm{Amp}(Z) : \ell(M) = d\}.$$

This will force the monodromy to be finite, indeed we count how may automorphism can send L to a fixed L_γ? If for each L_γ this is finite, then the automorphism is actually a finite group. Consider the stablizer

$$\mathrm{Stab}(L) = \mathrm{Aut}(Z, L) = \{\phi \in \mathrm{Aut}(Z) \mid \phi^* L = L\}.$$

The idea is to show that $\mathrm{Stab}(L)$ lies in the PGL_{N+1}, as discrete subgroup sit inside the projective space it must be finite.

Since L is ample line bundle, for some mL, there exists an embedding induced by the very ample line bundle mL such that

$$X \hookrightarrow \mathbb{P}^N,$$

where $N + 1 = \dim H^0(Z, mL)$, denote $V = H^0(Z, mL)$. For some projective space. It will then induce a homomorphism

$$\alpha : \mathrm{Aut}(Z, L) \to GL(V), \quad \sigma \mapsto \sigma^*$$

We claim it's an injection and it descends to an injection

$$\bar{\alpha} : \mathrm{Aut}(Z, L) \to PGL(V), \quad \sigma \mapsto [\sigma^*].$$

Indeed, since $X \hookrightarrow \mathbb{P}^N$ is embedding, so that for any two points $z_1 \neq z_2 \in Z$, there exist a setion $s \in V$ seperate them. If by contradiction $\sigma^* = id$ but $\sigma \neq id$, then there exist some $z \in Z$ such that $z \neq \sigma(z)$. So that there exist a section $s(z) \neq s(\sigma(z)) = (\sigma^*(s))(z)$, a contradiction.

Since $\mathrm{Aut}(Z)$ is discrete, thus stablizer must be finite. Thus by orbit stablizer theorem the monodromy

$$|\Gamma| = |\Gamma \cdot L||\mathrm{Stab}(L)| < +\infty.$$

□

4 Examples

In this section, we will present varies examples related to the Moishezon space and Moishezon morphism.

4.1 The Hironaka's example

Hironaka discovered a bunch of complete non-projective 3-fold which is called the Hironaka's varieties. Note that based on the construction of Hironaka, we can from almost all the projective varieties construct some Moishezon varieties, that is why we said at the beginning that Moishezon varieties are versatile in birational geometry. (However this is not true in dimension 2, since all the smooth Moishezon surface are actually projective, see e.g. [GPR94]). The major reference of this part of note is the paper by Ulrich Thiel (see `https://ulthiel.com/math/wp-content/uploads/other/hironakas_example.pdf`).

Given a smooth projective theefold, which contains two rational curves transversely intersection at two points. Assume two rational curves are C and D that intersect at point P, Q.

We then take two steps blow up

$$X_1 = \mathrm{Bl}_{(D\backslash P)'}\left(\mathrm{Bl}_{C\backslash P}(X\backslash P)\right) \xrightarrow{\pi_2} \mathrm{Bl}_{C\backslash P}(X\backslash P) \xrightarrow{\pi_1} X\backslash P$$
$$X_2 = \mathrm{Bl}_{(C\backslash Q)'}\left(\mathrm{Bl}_{D\backslash Q}(X\backslash Q)\right) \xrightarrow{\sigma_2} \mathrm{Bl}_{D\backslash Q}(X\backslash Q) \xrightarrow{\sigma_1} X\backslash Q,$$

Note that if we define $U = X - \{P, Q\}$, then $\pi^{-1}(U) \cong \sigma^{-1}(U)$. In particular, we can glue X_1 and X_2 along $\pi^{-1}(U)$ and $\sigma^{-1}(U)$. In the picture below, we glue the red exceptional surface on the right hand side with the black exceptional surface on the left hand side (denote it S_1) and blue exceptional surface on the left hand side with the black exceptional surface on the right hand side (denote it S_2). (see the pictrue 2). By gluing lemma, there exists a morphism $f : H \to X$ and the restriction of the morphism on S_1, S_2 as $f_1 = f|_{S_1} : S_1 \to C$ and $f_2 = f|_{S_2} : S_2 \to C$.

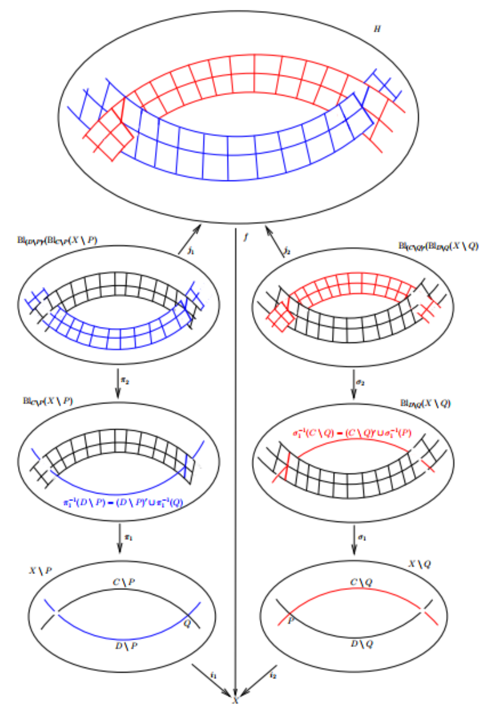

Figure 2: Construction of the Hironaka's variety

We claim that Hironaka's variety is non-projective.

The idea is to find some divisor on the surface $S = S_1 \cup S_2$ on the Hironaka's variety H.

The idea to prove the non-projective is to find some curve on the surface $S = S_1 \cup S_2$. Which has positive degree but add up to 0.

The key observation is that $f^{-1}(P)$ (resp. $f^{-1}(Q)$) decompose into two split projective line L_Q and L'_Q in S_1 (resp. L_P and L'_P in S_2). (see the precise statement below).

Choose two points $A \in C - \{P, Q\}$ and $B \in D - \{P, Q\}$. Since all the point on a rational curve are linear equivalent, therefore

$$\begin{aligned} A \sim_C Q &\Longrightarrow f_1^{-1}(A) \sim_{S_1} f_1^{-1}(Q) = L_Q + L'_Q \\ B \sim_D P &\Longrightarrow f_2^{-1}(B) \sim_{S_2} f_2^{-1}(P) = L_P + L'_P \end{aligned}$$

and Push forward of cycle, we get equivalence on S.

$$\begin{aligned} I &: f^{-1}(A) \sim_S f^{-1}(Q) = L_Q + L'_Q \\ II &: f^{-1}(B) \sim_S f^{-1}(P) = L_P + L'_P \end{aligned}$$

On the other hand we also that B, Q lies in the same rational curve, so that

$$III : B \sim_D Q \Rightarrow f_2^{-1}(B) \sim_{S_2} f_2^{-1}(Q) \Rightarrow f^{-1}(B) \sim_S L'_Q$$

and combined then together, we get

$$\begin{aligned} f^{-1}(A) + f^{-1}(B) \sim_S f^{-1}(A) + f^{-1}(B) &\Rightarrow L_Q + L'_Q + L_P + L'_P \sim_S L'_Q + L'_P \\ &\Rightarrow L_Q + L_P \sim_S 0 \end{aligned}$$

If there exist some ample divisor on A, then both $L_Q \cdot A > 0$ and $L_P \cdot A > 0$ contradict the linearly trivial relation above. Therefore the only possible case is Hironaka's variety is non-projective.

4.2 Locally Moishezon morphism which is not globally Moishezon

There exist some rational K3 surfaces with

4.3 Singular Kähler Moishezon variety needs not to be projective

This happens even for surface case.

4.4 Flip rational curves on quntic threefold produce Moishezon variety

4.5 Fibwesie projective morphism needs not to be projective morphism

Let $S_0 := (g = 0) \subset \mathbb{P}^3_{\mathrm{x}}$ and $S_1 := (f = 0) \subset \mathbb{P}^3_{\mathrm{x}}$ be surfaces of the same degree. Assume that S_0 has only ordinary nodes, S_1 is smooth $\operatorname{Pic}(S_1)$ is generated by the restriction of $\mathcal{O}_{\mathbb{P}^3}(1)$ and S_1 does not contain any of the singular points of S_0. Fix $m \geq 2$ and consider

$$X_m := (g - t^m f = 0) \subset \mathbb{P}^1_{\mathbf{x}} \times \mathbb{A}^1_t.$$

The singularities are locally analytically of the form $xy + z^2 - t^m = 0$. Thus X_m is locally analytically factorial if m is odd. If m is even then X_m is factorial since the general fiber has Picard number 1, but it is not locally analytically factorial; blowing up $\left(x = z - t^{m/2} = 0\right)$ gives a small resolution. Thus we get that (4.1) X_m is bimeromorphic to a proper, smooth family of projective surfaces iff m is even, but (4.2) X_m is not bimeromorphic to a smooth, projective family of surfaces.

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