MRC Fibrations Reading Notes Fall 2025

Note I.3 — 22, 09, 2025 (in progress draft)

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The aim of this note is to introduce the MRC fibrations.

Contents

1 Construction of MRC (and MRCC) Fibration

Existence of MRC Fibration for Projective manifold.

Theorem 1. Given any smooth projective variety X, there exists a rational map $\phi : X \dashrightarrow B$ which has the following properties:

(1) (Maximality) For a very general point $x \in X$, any rational curve $C \subset X$ passing through a general point x' of the fiber X_x of ϕ through x is contained in X_x,

(2) (Rational Connected of the fiber) The fibers of ϕ are rationally connected,

(3) (Almost Holomorphic of the Rational Map) The rational map ϕ is almost holomorphic, which means that it is well-defined along the general fiber.

PROOF IDEA 2. The idea is not hard, we will divide the proof into 3 steps.

Step 1. Finding a rationally chain connected surfaces passing through the very general point $x \in S$. Given a very general point $x \in X$, we first try to construct a rationally chain connected surfaces S. To do this, we will first pick a free rational curve C_x passing through x. We then repeat the process for each $x' \in C_x$, and hopefully, we will get a surface

$$S = \bigcup_{x' \in \mathbb{P}^1} C'_{x'}$$

as the picture below shows

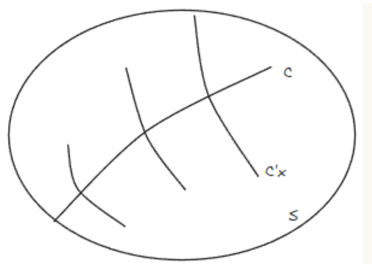

Figure 1: Rationally chain connected surfaces passing through $x \in X$.

which is rationally chain connected (and may not be rationally connected since the surface can be singular).

Step 2. Smoothing the chain of free rational curves. We then find the rational connected surfaces using a "smoothing chain of free rational curves lemma".

We then repeat this process treating S_x as C_x. And therefore find a fiber X_x, that satisfies (1) and (2).

Step 3. Prove almost holomorphic of the MRC Fibration. That is we want to prove that $\phi : X \dashrightarrow B$ generically over the base B the morphism is well defined. To prove this by construction, via sequence of blow up $\tilde{X} \to X$, we can define a morphism $\tilde{X} \to B$. Then to show the almost holomorphic of $X \dashrightarrow B$, We need to show if the exceptional divisor of $\tilde{X} \to X$ does not dominant the base then $E \to B$ factor through $\tilde{X} \to X$.

Existence of MRC Fibration for compact Kähler manifold.

Theorem 3.

When the variety is singular, it's also possible to define the MRC Fibration.

Proposition 4.

2 GHS Theorem for MRC Fibration

3 Applications of MRC Fibrations

References