Flipping contraction in Kähler MMP reading notes Spring 2025

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Yi Li

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The aim of this note

2 Das-Hacon's approach to flipping contraction for Kähler 3-fold MMP

In this section, we will prove the following theorem.

Theorem 1 ([DH24, Theorem 6.9])**.** Let (X, B) be a strong $\mathbb{Q}$-factorial Kähler 3-fold KLT pair. With the following condition holds

1. $K_X + B$ is pseudo-effective
2. $\alpha = [K_X + B + \beta]$ is nef and big class such that β is Kähler,
3. The negative extremal ray $R = \overline{\mathrm{NA}}(X) \cap \alpha^{\perp}$ is flipping type. (that is $\dim \mathrm{Null}(\alpha) < n - 1$)

Then there exists an α-trivial flipping contraction

$$f : X \to Z,$$

such that there exist some Kähler form α_Z on Z such that $\phi^*(\alpha_Z) = \alpha$. And the flip exist.

Remark 2. Before proving the theorem, let us briefly sketch the idea of the proof. By definition, the null locus has high codiemsnion, therefore α as a big and nef class is actually modified Kähler. Using Fujita approximation type modification, we will find some

$$\nu : X' \to X,$$

such that

$$\alpha' = \nu^*(\alpha) = \omega' + G',$$

for some $G' \geq 0$ and $\mathrm{supp}(G') = \mathrm{Ex}(\nu) = \lfloor \Delta' \rfloor$. Our next goal is try to construct a α'-trivial $(K_{X'} + \Delta' + t\omega' + a\alpha')$-MMP:

$$X' \dashrightarrow X^1 \dashrightarrow \dots \dashrightarrow X^+,$$

satisfying the following properties:

(1) The MMP terminate at a place such that for all $0 < t \leq \epsilon$, there exist some $a(t) \gg 0$ with

$$K_{X^+} + \Delta^+ + t\omega^+ + a(t)\alpha^+,$$

is nef,

(2) The MMP descend to a small bimeromorphic map $\psi : X \dashrightarrow X^+$,

(3) The small bimeromorphic map $\psi : X \dashrightarrow X^m$ is isomorphism on $U = X - \mathrm{Null}(\alpha)$.

Our next goal is taking the normalization of the graph of the bimeromorphic map $\psi : X \dashrightarrow X^+$, and trying to prove

$$\mathrm{Ex}(p) = \mathrm{Ex}(q) = E,$$

and that $-E|_{\mathrm{supp}(E)}$is ample. Therefore by Fujiki blowing down, we can contract the expectional, and therefore by rigidity lemma it will induce two morphisms

$$f : X \to Z, \quad f^+ : X^+ \to Z.$$

And finally we try to show the base point freeness.

2.1 Apply the DLT modification

Differ from the divisorial contraction, we can not insert the null locus as divisor into the boundary. Instead, we resolve the modified Kähler class onto the higher model $\nu : X' \to X$, and some effective divisor G' out, such that it record all the exceptional information $\mathrm{supp}(G') = \mathrm{Ex}(\nu)$. To be more precise, we have the following proposition.

Lemma 3. Let (X, B) be an strong $\mathbb{Q}$-factorial compact normal KLT pair. satisfies condition (1)-(3) in the Theorem 1. Then there exists a modification

$$\nu : X' \to X,$$

with $\Delta' = \nu_*^{-1}(B) + \mathrm{Ex}(\nu)$ such that the following condition holds,

(1) (X', Δ') is a DLT pair, and X' is strong $\mathbb{Q}$-factorial variety,

(2) The pull back class

$$\alpha' = \nu(\alpha) = \omega' + G',$$

such that $G' \geq 0$ and $\mathrm{supp}(G') = \mathrm{Ex}(\nu) = \lfloor \Delta' \rfloor$,

(3) The class ω' is Kähler and α' is nef,

(4) There exist some $a > 0$ and $t > 0$ such that $K_{X'} + \Delta' + t\omega' + a\alpha'$ is Kähler.

(5) Under the log resolution

$$K_{X'} + \Delta' = \nu^*(K_X + B) + E,$$

with E being effective and $\mathrm{supp}(E) = \mathrm{Ex}(\nu)$.

The idea is simple, since $\dim \mathrm{Null}(\alpha) = 1$ the non-Kahler locus contains no divisor. And therefore α is actually a modified Kähler class, we can take some modification so that

$$\alpha' = \nu^*(\alpha) = \omega' + G'.$$

We will discussion more on how to find such modification in the proof below. Since ω' is Kahler, given any $a > 0$, we can always find some $t \gg 0$ such that $K_{X'} + \Delta' + t\omega' + a\alpha'$ is nef.

2.2 Find a negative extremal ray on one of the component $S' \subset \lfloor \Delta' \rfloor$

Let $S' \subset \lfloor \Delta' \rfloor$, we define the

$$\tau_{S'} = \inf\{s \geq 0 \mid K_{S'} + \Delta' + s\omega' + a\alpha' \text{ is nef for } s > 0\}.$$

Before proving the theorem, let us first briefly discuss what infimum means? By definition of infimum, the following two conditions are satisfied:

(1) If $t' < \tau_{S'}$, then for any $a' > 0$, the $K_{S'} + \Delta_{S'} + t'\omega' + a'\alpha'$ is not nef,

(2) For any $t' > \tau_{S'}$, there exist some $\tau_{S'} \leq t'' < t'$ such that $K_{S'} + \Delta_{S'} + t''\omega' + a(t')\alpha'$ is nef.

These two conditions for infimum will be important in the proposition below.

Proposition 4. Let $\tau_{S'} > 0$, then there exists a $(K_{S'} + \Delta_{S'})$-negative extremal ray R, such that

$$(K_{S'} + \Delta_{S'} + \tau_{S'}\omega_{S'}) \cdot R = \alpha_{S'} \cdot R = 0,$$

and for any $a' \gg 0$, the $K_{S'} + \Delta_{S'} + \tau_{S'}\omega_{S'} + a'\alpha_{S'}$ is nef.

PROOF IDEA 5. The idea is the apply the cone theorem on the surface S'. The geometry picture looks as follows

we first apply the cone theorem on S', so that for $\tau_{S'} > 0$, the generalized Mori cone decompose into

$$\overline{\mathrm{NA}}(S') = \overline{\mathrm{NA}}(S')_{K_{S'} + \Delta_{S'} + \frac{\tau_{S'}}{2}\omega_{S'}} + \sum_{i=1}^{r} \mathbb{R}_+[\Sigma_i]$$

then we define try to find the extremal place for the α'-supporting class say $\sigma_{S'}$.

We first define the set of negative extremal ray lies on the α'-supporting class

$$I = \{i \mid \alpha_{S'} \cdot \Sigma_i = 0\}$$

Here

$$\sigma_{S'} = \min\{s \geq \frac{\tau_{S'}}{2} \mid (K_{S'} + \Delta_{S'} + s\omega_{S'}) \cdot \Sigma_i \geq 0, \quad \Sigma_i \in I\}$$

(By finiteness in the cone theorem, the minimal is attainable). Note that since α' is nef, this in particular will imply that

$$(K_{S'} + \Delta_{S'} + \sigma_{S'}\omega_{S'} + a'\alpha') \cdot \Sigma_i \geq 0, \quad \forall\ 1 \leq i \leq r$$

Easy to see by definition that $\sigma_{S'} \leq \tau_{S'}$ (by definition of infimum $\tau_{S'}$), only need to show that $\tau_{S'} = \sigma_{S'}$. Then easy to see from the picture above, that there exist some negative extremal ray R such that

$$(K_{S'} + \Delta_{S'} + (\tau_{S'}/2) \cdot \omega_{S'}) \cdot R = 0 = (K_{S'} + \Delta_{S'} + \tau_{S'}\omega_{S'}) \cdot R = \alpha'_{S'} \cdot R$$

which is precisely what we want.

Only needs to show that $\sigma_{S'} = \tau_{S'}$. We will prove it by contradiction. If $\sigma_{S'} < \tau_{S'}$, we try to show that

(1) $K_{S'} + \Delta_{S'} + \sigma_{S'}\omega_{S'} + a'\alpha_{S'}$ is non-negative on $\sum_{i=1}^{k} \mathbb{R}_+[\Sigma_i]$ for $a' \gg 0$,

(2) $K_{S'} + \Delta_{S'} + (\tau_{S'} - \epsilon)\omega_{S'} + a'\alpha_{S'}$ is non-negative on the part $\overline{\mathrm{NA}}(S')_{K_{S'}+\Delta_{S'}+\frac{\tau_{S'}}{2}\omega_{S'}}$.

Combined these two, immediate implies the nefness of $K_{S'} + \Delta_{S'} + \tau_{S'}\omega_{S'} + a'\alpha_{S'}$ for $a' \gg 0$.

Proof. We first prove that $\sigma_{S'} \leq \tau_{S'}$. For otherwise by definition of nef threshold $\tau_{S'}$, there exist some $\tau_{S'} \leq t' < \sigma_{S'}$ such that for some a' we have

$$K_{S'} + \Delta_{S'} + t'\omega_{S'} + a'\alpha'$$

is nef. In particular, we have

$$(K_{S'} + \Delta_{S'} + t'\omega_{S'}) \cdot \Sigma_i \geq 0, \quad \forall i \in I$$

In particular, this means that

$$\sigma_{S'} \leq t' < \sigma_{S'}$$

a contradiction. Therefore $\sigma_{S'} \leq \tau_{S'}$.

We then prove $K_{S'} + \Delta_{S'} + (\tau_{S'} - \epsilon)\omega_{S'} + a'\alpha_{S'}$ is non-negative on the part $\overline{\mathrm{NA}}(S')_{K_{S'}+\Delta_{S'}+\frac{\tau_{S'}}{2}\omega_{S'}}$ using some convex combination argument. Let

$$\eta_{x,y} = K_{S'} + \Delta_{S'} + x\omega_{S'} + y\alpha_{'},$$

note that by our previous discussion, $\eta_{t,a}$ is nef.

Since we can express

$$\eta_{\tau_{S'}-\epsilon,a'} = A\eta_{s,t} + B\eta_{\frac{\tau_{S'}}{2},0} + C\alpha_{S'},$$

for some $A, B, C > 0$ when $a' \gg 0$.

This immdediatly implies that $\eta_{\tau_{S'}/2-\epsilon,a'}$ is non-negative on $\overline{\mathrm{NA}}(S')_{K_{S'}+\Delta_{S'}+\frac{\tau_{S'}}{2}\omega_{S'}}$.

On the other hand, we know that $(K_{S'} + \Delta_{S'} + \sigma_{S'}\omega_{S'} + a'\alpha_{S'}) \cdot \Sigma_i \geq 0, \quad \forall 1 \leq i \leq k$ when $a' \gg 0$.

Therefore if $\sigma_{S'} < \tau_{S'}$ we will get contradiction. The only possible case, therefore is $\tau_{S'} = \sigma_{S'}$, which clearly implies the result.

□

Let $\tau = \max_{S'} \tau_{S'}$, we claim

Proposition 6. If $\tau > 0$, then there exists some $a' \gg 0$, we have $K_{X'} + \Delta' + \tau\omega' + a'\alpha'$ is nef.

Remark 7. Why this result is interesting? Since it relates the nef threshold for S' with the nef threshold for the ambient space. We will apply some convex combination trick in the proof.

PROOF IDEA 8. We will prove it by contradiction. If for all fixed $a' \gg 0$, the $K_{X'}+\Delta'+\tau\omega'+a'\alpha'$ is not nef. Then (by characterization theorem of nefness) there exists some subvariety Z' such that the restriction $(K_{X'} + \Delta' + \tau\omega' + a'\alpha)|_Z$ is not pseff.

Now we claim that the subvariety Z' lies in some component $S' \subset \mathrm{Ex}(\nu)$. Indeed we try to show that $Z' \subset \mathrm{supp}\ G'$.

We try to show that $\tau_{S'} > 0$ for that S' contains Z'. For otherwise, this will contradict to the fact that $\tau > 0$, using the infimum of $\tau_{S'}$.

Proof. We prove it by contradiction, if for all $a' \gg 0$, that $K_{X'} + \Delta' + \tau\omega' + a'\alpha'$ is not nef. We choose a' as follows:

(1) First for those $S' \subset \mathrm{supp}(\Delta')$, with $\tau_{S'}$, there exist some $a_{S'}$ such that $a' \geq a_{S'}$ then $K_{S'} + \Delta_{S'} + \tau_{S'}\omega_{S'} +$

(2) For those $\tau_{S'} = 0$, since $\tau > 0$, there exist some $a_{S',\tau}$ such that $a' \geq a_{S',\tau}$ then $K_{S'} + \Delta_{S'} + \tau\omega$

We choose $a' \geq \max\{a_{S',\tau}, a_{S'}\}$. Then for any such a', we have $K_{X'} + \Delta' + \tau\omega' + a'\alpha'$ is not nef. Therefore, there exist some $Z' \subset X$ such that

$$(K_{X'} + \Delta' + a'\alpha' + \tau\omega')|_Z,$$

is not pseff.

Since $\alpha' = \omega' + G'$, we have

$$K_{X'} + \Delta' + a'\alpha' + \tau\omega' = (K_{X'} + \Delta' + t\omega' + (a' - t + \tau)\alpha') + (t - \tau)G',$$

note that for $a' \gg 0$, the 1st term on the RHS is nef, therefore the only possible case is $Z' \subset \mathrm{supp}(G')$. Hence, there exist some component $S' \subset \mathrm{supp}(G')$ such that $K_{S'} + \Delta_{S'} + \tau\omega_{S'} + a'\alpha_{S'}$ is not nef.

We finish the proof by contradiction:

(Case 1) $\tau_{S'} = 0$, then by definition of nef threshold, the given $\tau > 0$, there exist some $a' \geq a'' \gg 0$ such that

$$K_{S'} + \Delta_{S'} + \tau\omega_{S'} + a''\alpha_{S'}$$

is nef. Therefore gets the contradiction.

(Case 2) $\tau_{S'} > 0$, in this case, by Proposition 4, we know that

$$K_{S'} + \Delta_{S'} + \tau_{S'}\omega_{S'} + a'\alpha'$$

is nef for some sufficient larget a'. Since $\tau \geq \tau_{S'}$, this is also true for $K_{S'} + \Delta_{S'} + \tau\omega_{S'} + a'\alpha'$, and gets the contradiction.

Therefore, by Proposition 4, we have

$$K_{S'} + \Delta_{S'} + \tau_{S'}\omega_{S'} + a'\alpha_{S'}$$

is nef. Since $\tau \geq \tau_{S'}$, this will imply that $K_{S'} + \Delta_{S'} + \tau\omega_{S'} + a'\alpha_{S'}$ is nef. Contradiction again. □

2.3 Run the MMP $\phi : X' \dashrightarrow X^+$

Next, we try to extend to the contraction from $S' \to T'$ to $X' \to Z'$.

Theorem 9. There exists a sequence of MMP

$$(X', \Delta') \dashrightarrow (X^1, \Delta^1) \dashrightarrow \cdots \dashrightarrow (X^+, \Delta^+),$$

such that when terminates $\tau_{X^+} = 0$.

PROOF IDEA 10. The idea is not hard, we try to apply the PLT contraction theorem. By Proposition 4, if $\tau > 0$, then we can find some negative extremal ray on some surfaces S', we can apply usual trick to replace the surface to some new S' such that $S' \cdot R < 0$.

and the extremal contraction $S' \to T'$, then under certain condition we can extend the contraction onto X'. To achieve this, we need to check the following conditions hold:

(1) The pair $(X', (1-\epsilon)(\Delta' - S') + S' + \omega')$ is PLT pair,

(2) Kahlerness of $-(K_{X'} + (1-\epsilon)(\Delta' - S) + S')|_{S'}$ over T, since intersection $(K_{X'} + \Delta') \cdot R_i < 0$.

(3) ampleness $-S'|_{S'}$ over T'. (Since $S' \cdot R < 0$).

Proof. We prove it by induction on the steps of the MMP. Assume that we already run the MMP to i-th step

$$(X', \Delta') \dashrightarrow (X^1, \Delta^1) \dashrightarrow \cdots \dashrightarrow (X^i, \Delta^i).$$

If $\tau_{X_i} = 0$, then we are done. Otherwise, by definition of τ_{X_i}, there exist some components $S^i \subset \lfloor \Delta^i \rfloor$, such that $\tau_{S'} >$. By Proposition 4, there exists some negative extremal ray $R_i \in \overline{\mathrm{NA}}(S')$ such that

$$(K_{S'} + \Delta_{S'} + \tau_{S'}\omega_{S'}) \cdot R_i = \alpha_{S'} \cdot R_i = 0.$$

We try to replace S^i by $\bar{S}^i$ such that R_i is $K_{\bar{S}^i} + \Delta_{\bar{S}^i}$-negative and $\bar{S}^i \cdot R_i < 0$. (The similar arguement appears also in the divisorial contraction case). Since $\alpha' \cdot \Sigma_i = (\omega' + G') \cdot \Sigma_i = 0$ this means that $G' \cdot \Sigma_i < 0$ and therefore $\Sigma_i \subset \bar{S}^i \subset \mathrm{Supp}(G')$. Note that $F = \alpha_{\bar{S}'} \cap \overline{\mathrm{NA}}(\bar{S}')$ is negative extremal face. And $\Sigma \in F$. Therefore there exist a set of extremal ray of F span Σ_i. And therefore exists some extremal ray

$$\bar{\Sigma}_i \cdot \bar{S}' < 0,\ \alpha' \cdot \bar{\Sigma}_i = 0 \implies (K_{\bar{S}'} + \Delta_{\bar{S}'}) \cdot \bar{\Sigma}_i < 0$$

Therefore it follows that $-(K_{\bar{S}'} + \Delta_{\bar{S}'})$ is ample and $-\bar{S}|_{\bar{S}'}$ is ample. And we can apply the PLT contraction theorem.

□

The following properties hold for the MMP above

Proposition 11.
(1) There exist some $\epsilon > 0$, for all $0 \leq t < \epsilon$, $K_{X^+} + \Delta^+ + t\omega^+ + a(t)\alpha^+$ is nef for some $a(t)$.

(2) The MMP step is proper over $U = X - \mathrm{Null}(\alpha)$, that is in the following diagram ν_i is proper morphism over U, and the i-th step of the MMP is ν_i-vertical,

(3) The induced bimeromorphic map $\psi : X^+ \dashrightarrow X$ is small, and it's isomorphism over U.

Proof. Now we prove (3). The idea is to use the Zariski decomposition. We want to show that

$$N_\sigma(K_{X'} + \Delta' + a\alpha' + \epsilon\omega') = \mathrm{Ex}(\nu),$$

Since we have shown that the MMP will terminate at some □

2.4 Find the contraction $f : X \to Z$

In this step, we take the normalization on the graph of the map $\psi : X \dashrightarrow X^+$. Let $\eta := \alpha + \delta\,(K_X + B)$, we try to show that

$$E = p^*\eta - q^*\eta^+,$$

satisfies the condition that $-E|_E$ is ample. So that we can apply the Grauert contraction theorem, and get a contraction $g : W \to Z$.

We then try to show that this contraction will induce $f : X \to Z$ and $f^+ : X^+ \to Z$ as the diagram below shows.

$$\begin{tikzcd}
X' \arrow[r, dashed, "\phi"] \arrow[d, "\nu"'] & X^+ \arrow[ddr, "f^+"] & \\
X \arrow[ur, dashed, "\psi"] \arrow[drr, "f"'] & W \arrow[l, "p"] \arrow[u, "q"'] \arrow[dr, "g"] & \\
& & Z
\end{tikzcd}$$

We then show that $f : X \to Z$ will contract every curves in the negative extremal ray R and it's a $(K_X + B)$-negative contraction.

2.5 Prove the base point freeness

Just as what we did in the divisorial contraction case, we try to use that fact that proper bimeromorphic morphism $f : X \to Z$ between Kähler varieties with rational singularity satisfies the condition

$$\mathrm{im}(f^*) = \{\alpha \in H^{1,1}_{\mathrm{BC}}(X) \mid \alpha \cdot C = 0,\ \forall\ C \in N_1(X/Z)\}.$$

Then apply singular version Demailly-Păun Kählerness criterion to the big and nef class α_Z.

2.6 Prove the existence of flips in any dimension

We finally prove that flip exist in any dimension.

Theorem 12. Let $f : X \to Z$ be a flipping contraction, then

3 Höring-Peternell's approach for flipping contraction

References

[DH24] Omprokash Das and Christopher Hacon, *On the minimal model program for kähler 3-folds*, 2024.