Nakayama's Extension Theorems Summer 2025

Note I.1 — 08, 17, 2025 (draft version)

Yi Li

The aim of this note is to prove the following two extension results appear in Nakayama's book [Nak04].

Definition 1 (Settings of Nakayama's Extension Theorem)**.** Let $\pi : V \to S$ be a projective surjective morphism from a nonsingular variety with connected fibers, let the central fiber $V_0 = \bigsqcup \Gamma_i$ be a disjoint union of non-singular prime divisors Γ_i of V.

Theorem 2 ([Nak04, Theorem VI.3.7])**.** In the setting of Nakayama (Definition 1).

(1) Let L be a π-pseudo-effective $\mathbb{Z}$-divisor of V such that $L - (K_V + V_0)$ is π-nef.

(2) Let Λ be a π-nef and π-big $\mathbb{Q}$-divisor of V such that $0 \geq \langle \Lambda \rangle$ and $k\Lambda \in \mathbb{E}_{\text{big}}$ is for some $k \in \mathbb{N}$. Then

(A) The homomorphism

$$\pi_* \mathcal{O}_V \left(lL + \lfloor \Lambda \rfloor \right) \to \pi_* \mathcal{O}_X \left(lL + \lfloor \Lambda \rfloor \right)$$

is surjective for $l \gg 0$.

(B) If $L|_X$ is $(\pi|_X)$-pseudo-effective, then the homomorphism above is surjective for any $l > 0$.

Theorem 3 ([Nak04, Theorem VI.3.16])**.** In the setting of Nakayama (Definition 1).

Let L be a π-abundant divisor of V. Suppose that

(1) $\pi \left(\Gamma_i \right)$ is a prime divisor of S for any Γ_i,

(2) $L - (K_V + V_0)$ is π-nef and π-abundant,

(3) $\kappa \left(L|_{\Gamma_i} ; \Gamma_i / \pi \left(\Gamma_i \right) \right) \geq \kappa(L; V/S)$ for any i.

Then the restriction homomorphism $\pi_* \mathcal{O}_V(lL) \to \pi_* \mathcal{O}_X(lL)$ is surjective for any $l \geq 1$.

Our presentation uses slightly different notation, and several simplifications are made. For example, we may assume the divisor X in the book to be the central fiber of the family, and several assumptions becomes vacus then. We also adopt the notations in [BCHM10] and [HM10] and also [Kaw24], for example the multiplier ideal sheaves are denoted as

Contents

1 Base ideal sheaves and Multiplier ideal sheaves

The major tools that will be used is the multiplier ideal sheaves. First recall the setting of this note.

Definition 4. Let $\pi : V \to S$ be a projective surjective morphism from a nonsingular variety with connected fibers, $X = \bigsqcup \Gamma_i$ a disjoint union of non-singular prime divisors Γ_i of V.

Remark 5. In Nakayama's book, the original setting is much more general than the one above (see [Nak04, V.2.2, Situation]). However, for simplicity and without losing the essence of the theory, we stick to the simplified setting described above. For the readers who require more general setting, please refer to the original book.

The following result will be the blueprint for the extension theorem. We then define the base ideal and multiplier ideal sheaves.

Definition 6 (Base ideal)**.**

It can be proved support of the base ideal coincide with the base locus.

Proposition 7.

Proof. □

2 Nakayama's Blueprint Theorem

Theorem 8. Let L and L' be $\mathbb{Q}$-divisors of V with $\langle L\rangle \leq \Delta$ such that (L, L') satisfies one of the three conditions

(VI-3),

(VI-4), and

(VI-5).

Suppose that there exist - a rational number $0 < \beta < 1$, positive integers m, m', and an integer b, - $\mathbb{Z}$-divisors A and D of V, and - a bimeromorphic morphism $\rho : W \to V$ from a non-singular variety satisfying the following conditions:

(1) $mL, m'L$, and bL' are $\mathbb{Z}$-divisors with $mL + A \in \mathbb{E}_V, m'L + bL' \in \mathbb{E}_V$;

(2) $m\beta \leq m' + b\beta$ and $L' - \beta L$ is π-semi-ample;

(3) $\mathcal{I}[mL] \subset \mathcal{J}[mL + A]$;

(4) D is an effective divisor containing no components of X and $(V\&X, \Delta + (1/m)D)$ is log-terminal along X;

(5) ρ satisfies the conditions **E** for $mL + A$ and **E** for $m'L + bL'$ in which the inequality

$$-E(mL + A) \leq \rho^* D - E\left(m'L + bL'\right)$$

holds. Then $\pi_* \mathcal{O}_V\left(L_L L_\perp\right) \to \pi_* \mathcal{O}_X\left(L_L L_\perp\right)$ is surjective.

3 Nakayama's Extension theorem under nef and big conditions

Theorem 9 ([Nak04, Theorem VI.3.7])**.** In the setting of Nakayama (Definition 1).

(1) Let L be a π-pseudo-effective $\mathbb{Z}$-divisor of V such that $L - (K_V + X + \Delta)$ is π-nef.

(2) Let Λ be a π-nef and π-big $\mathbb{Q}$-divisor of V such that $\Delta \geq \langle\Lambda\rangle$ and $k\Lambda \in \mathbb{E}_{\text{big}}$ is for some $k \in \mathbb{N}$. Then

(A) The homomorphism

$$\pi_* \mathcal{O}_V\left(lL + \lfloor\Lambda\rfloor\right) \to \pi_* \mathcal{O}_X\left(lL + \lfloor\Lambda\rfloor\right)$$

is surjective for $l \gg 0$.

(B) If $L|_X$ is $(\pi|_X)$-pseudo-effective, then the homomorphism above is surjective for any $l > 0$.

4 Nakayama's Extension theorem under nef and abundant conditions

In this section we will give a proof of the following result.

Theorem 10 ([Nak04, Theorem VI.3.16])**.** In the setting of Nakayama (Definition 1).

Let L be a π-abundant divisor of V. Suppose that

(1) $\pi\left(\Gamma_i\right)$ is a prime divisor of S for any Γ_i,

(2) $L - (K_V + X + \Delta)$ is π-nef and π-abundant,

(3) $\kappa\left(L|_{\Gamma_i}; \Gamma_i / \pi\left(\Gamma_i\right)\right) \geq \kappa(L; V/S)$ for any i.

Then the restriction homomorphism $\pi_* \mathcal{O}_V(lL) \to \pi_* \mathcal{O}_X(lL)$ is surjective for any $l \geq 1$.

5 Applications of Nakayama's Extension Theoem

5.1 Applications in deformation of numerical Kodaira Dimension

5.2 Applications in deformation of plurigenera

References

[BCHM10] C. Birkar, P. Cascini, C.D. Hacon, and J. McKernan, *Existence of minimal models for varieties of log general type*, Journal of the American Mathematical Society **23** (2010), no. 2, 405–468.

[HM10] C.D. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type. II*, J. Amer. Math. Soc. **23** (2010), no. 2, 469–490.

[Kaw24] Yujiro Kawamata, *Algebraic varieties: minimal models and finite generation*, Cambridge Studies in Advanced Mathematics, vol. 214, Cambridge University Press, Cambridge, 2024, Translated by Chen Jiang.

[Nak04] N. Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004.