Projectivity Criterira and it's deformation behavior Summer 2025

Note 4 — 2025-07-09 (draft version)

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1 Overview

The aim of this note is to give some projectivity critera for Moishezon/Kähler morphism and study the deformation bahavior of projectivity. The ultimate goal is to finish the proof of [Kol22, Theorem 2].

Theorem 1.1 ([Kol23, Theorem 2])**.** Let $g : X \to S$ be a proper morphism of complex analytic spaces and $S^* \subset S$ a dense, Zariski open subset such that g is flat over S^*. Assume that

(1) X_0 is projective for some $0 \in S$,
(2) the fibers X_s have rational singularities for $s \in S^*$, and
(3) g is bimeromorphic to a projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to S$.

Then there is a Zariski open neighborhood $0 \in U \subset S$ and a locally closed, Zariski stratification $U \cap S^* = \cup_i S_i$ such that each $g|_{X_i} : X_i := g^{-1}(S_i) \to S_i$ is projective.

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2 Several projectivity critera

In this section, we try to summarize the projectivity critera related to Moishezon varieties (we will use the Moishezon variety or algebraic space interchangeably).

2.1 Kodaira's projectivity criterion

Proposition 2.1. Let X be a compact Kähler variety with rational singularities such that $H^2(X, \mathcal{O}_X) = 0$, then X is projective.

Proof. Take the resolution $\nu : X' \to X$, where X' is Kähler manifold. Since X has rational singularity, $R^i\nu_*\mathcal{O}_{X'} = 0$ with $i > 0$. Thus, by the Leray spectral sequence arguement, $H^2(X, \mathcal{O}_X) = H^2(X', \mathcal{O}_{X'}) = 0$ and therefore by Kodaira's projectivity criterion for smooth manifold, X' is projective. And therefore X is a Kähler Moishezon variety with rational singularity. By the result we proved in the 1st time, X is a projective variety. □

2.2 Nakai-Moishezon ampleness critera

Proposition 2.2 ([Kol90, Theorem 3.11])**.** Let X be a proper algebraic space and let H be a line bundle on X. Then H is ample on X if an only if for every irreducible closed subspace $Z \subset X$, the intersection product $H^{\dim(Z)} \cdot Z$ is positive.

2.3 Seshadri criterion of projectivity, line bundle version

Seshadri constant was first introduced by Demailly in the early 90s, when he tried to prove Fujita's conjecture.

Conjecture 2.3. Let X be a smooth projective variety of dimension n, with L being ample. Then

(a) $K_X + (n+1)L$ is global generated,

(b) $K_X + (n+2)L$ is very ample.

Demailly define the Seshadri constant as

Definition 2.4. Given a proper analytic space X and a line bundle L, the Seshari constant is defined to be

$$\epsilon(L, x) := \inf_{C \ni x} \frac{L \cdot C}{\operatorname{mult}_x C}.$$

Then he showed the following result holds true.

Theorem 2.5.
(a) If $\epsilon(L, x) > \frac{n}{n+1}$ then $K_X + (n+1)L$ is global generated,

(b) If $\epsilon(L, x) > \frac{2n}{n+2}$ then $K_X + (n+2)L$ is very ample.

For the reader want to know more about this, please read the [Dem82].

Proposition 2.6. Let X be a proper algebraic space, and D a divisor on X (the same also true for $\mathbb{Q}$, $\mathbb{R}$ dvisor). Then D is ample if and only if there exists a positive number $\varepsilon > 0$ such that

$$\frac{(D \cdot C)}{\text{mult}_x C} \geq \varepsilon$$

for every point $x \in X$ and every irreducible curve $C \subseteq X$ passing through x.

2.4 Seshadri criterion of projectivity, cohomology class version

For cohomology class, the Seshadri criterion is also true.

Lemma 2.7. Let X be a normal compact complex space with rational singularities of dimension n in Fujiki class $\mathcal{C}$. Then the canonical map

$$\Phi : N^1(X) \to N_1(X)^\vee, \quad [\omega] \mapsto \lambda_\omega$$

is an isomorphism. Here we define

$$\lambda_\omega : N_1(X) \to \mathbb{R}, \quad [T] \mapsto T(\omega)$$

Remark 2.8. Here

$$N^1(X) := H^{1,1}_{\text{BC}}(X),$$

and $N_1(X)$ to be the vector space of real closed currents of bidimension $(1,1)$ modulo the following equivalence relation: $T_1 \equiv T_2$ if and only if

$$T_1(\eta) = T_2(\eta),$$

for all real closed $(1,1)$-forms η with local potentials. When X is assumed to be Moishezon.

Proposition 2.9. Let X be a proper algebraic space over $\mathbb{C}$ with 1-rational singularities (see Definition 11). Then X is projective iff there is a cohomology class $\Theta \in H^2(X(\mathbb{C}), \mathbb{Q})$ and an $\epsilon > 0$ such that

$$\Theta \cap [C] \geq \epsilon \cdot \text{mult}_p C$$

for every integral curve $C \subset X$ and every $p \in C$.

Proof. Note that the cup product induce a $\mathbb{Q}$-bilinear form

$$(-) \cap (-) : H^2(X, \mathbb{Q}) \times H_2(X, \mathbb{Q}) \to \mathbb{Q},$$

which will induce a $\mathbb{Q}$-linear functional on $H_2(X, \mathbb{Q})$. If $C \mapsto [C]$ gives an injection $N_1(X, \mathbb{Q}) \hookrightarrow H_2(X(\mathbb{C}), \mathbb{Q})$, then we can view $C \mapsto \Theta \cap [C]$ as a linear map

$$\Theta \cap N_1(X, \mathbb{Q}) \to \mathbb{Q}.$$

By the previous lemma, line bundles span the dual space of $N_1(X, \mathbb{Q})$, so there is a line bundle L on X and an $m > 0$ such that $\deg(L|_C) = m \cdot \Theta \cap [C]$ for every integral curve $C \subset X$. Thus

$$\deg(L|_C) = m \cdot \Theta \cap [C] \geq m\epsilon \cdot \text{mult}_p C,$$

for every integral curve $C \subset X$ and every $p \in C$. Then L is ample by the line bundle version Seshadri criterion, so X is projective.

It remains to show that $C \mapsto [C]$ gives an injection $N_1(X, \mathbb{Q}) \hookrightarrow H_2(X(\mathbb{C}), \mathbb{Q})$ if X has 1-rational singularities. (see Note-1). □

2.5 Klieman's ampleness criterion for Moishezon spaces

Proposition 2.10 ([VP21])**.** Suppose that Y is a Moishezon space with $\mathbb{Q}$-factorial, log terminal singularities and that L is a Cartier divisor on Y. Then L is ample if and only if L has positive degree on every irreducible curve on Y and L induces a strictly positive function on $\overline{\mathrm{NE}}(Y)$.

Remark 2.11. It remains open if the result is still true without the $\mathbb{Q}$-factorial KLT assumption.

Proof. We will prove it in Note-5. □

3 Approximate the Chow-Barlet cycle space by countable many morphisms

In this Section, we will study the main technical tools: Chow-Barlet cycle space, which allows to approximate the Chow-Barlet cycle space using countable many projective morphism. The tool in this section is crucial for the proof of Theorem 6.1.

Definition 3.1 (Chow functor with m-marked points, [Kol96, Definition I.3.20])**.** Let X be a scheme over S (algebraic space/complex analytic space over S). Let

$$\mathrm{Chow}_m(X/S)(Z) = \left\{ \begin{array}{c} \text{Well defined families of nonnegative,} \\ \text{proper, algebraic cycles } \mathcal{C} \text{ of } X \times_S Z/Z, \\ s_1, \ldots, s_m : Z \to X, s_i(z) \in \mathcal{C}_z \text{ for all } z \in Z \end{array} \right\}.$$

We call the data in the bracket the Chow data with marked points. We say C is a pointed curve if it is 1-cycle that has one marked point.

Lemma 3.2 (representative of the Chow functor with marked points)**.** Let $X \to S$ be a proper morphism between complex analytic spaces. The relative Chow functor with m-marked points is representable by a complex analytic space denoted it $\mathrm{Chow}_m(X/S)$.

Proof. We claim that Chow functor with marked points is actually represented by a closed subspace of the original Chow-Barlet cycle space. Let

$$\mathcal{U} \to \mathrm{Chow}(X/S),$$

be the universal family of the Barlet-Chow cycle space (with $\mathcal{U} \subset X \times_S \mathrm{Chow}(X/S)$ as closed complex subspace). We then define the m-fold fiber product to be $X^{(m)} = \underbrace{X \times_S X \times_S ... \times_S X}_{m\text{-times}}$.

We then define $P = \mathrm{Chow}(X/S) \times_S X^{(m)}$, the incident complex subspace to be

$$I = \{(s, x_1, ..., x_m) \in P \mid x_i \in \mathcal{U}_s, \text{ for all } i\}.$$

We first claim that $I \subset P$ is a closed complex subspace. Indeed, we define the natural projective

$$p_i : P \to \mathrm{Chow}(X/S) \times_S X, \quad (c, x_1, ..., x_m) \mapsto (c, x_i),$$

and easy to check that the incidence variety can be represented as

$$I = \bigcap_{i=1}^{m} p_i^{-1}(\mathcal{U}),$$

since $\mathcal{U}$ is closed complex subspace, and therefore as a finite intersection I is a closed complex subspace in P.

We then show that I is the representative of the Chow functor with marked points that is

$$\mathrm{Hom}_S(T, I) \simeq \mathrm{Chow}_m(X/S)(T).$$

To see this, we first show that given a S-morphism $T \to I/S$ it will induce a Chow data with marked points over S. Indeed, since $I \subset \mathrm{Chow}(X/S) \times_S X^{(m)}$, so that the first projection

$$\pi_1 : T \to I \to \mathrm{Chow}(X/S),$$

will induce a family over T via pull back. And the second projection

$$\sigma_i = \pi_{2,i} : T \to I \xrightarrow{q_i} X,$$

will defines the section we want. Conversely, given the Chow data $(\mathcal{Z}, \sigma_1, ..., \sigma_m)$ with marked point, it will induce a morphism. To see this, by the representative of the standard Chow functor, we know that there exists a morphism $\phi_{\mathcal{Z}} : T \to \mathrm{Chow}(X/S)$ such that $\mathcal{Z}$ is the pull back family. And the section $\sigma_i : T \to X^{(m)}$ defines the second factor. And easy to check the induced morphism maps into I,

$$\phi_{\mathcal{Z}} \times \sigma_i : T \to I.$$

And therefore, we checked that I actually represent the Chow functor with marked points. □

The following upper semi-continuity result is needed in the proof.

Lemma 3.3 (upper semi-continuity of the multiplicities, [BM19, Proposition 4.3.10])**.** Let $(X_s)_{s\in S}$ be an analytic family of n-cycles of a complex space M. Then the function

$$S \times M \longrightarrow \mathbb{N}, (s, z) \mapsto \mathrm{mult}_z\,(X_s)$$

is upper semicontinuous in the Zariski topology of $S \times M$.

Proof. The proof of the lemma is a big complicated and we omit it here. □

Remark 3.4. In particular, let $f : X \to S$ be a proper flat morphism of relative dimension 1, assume that there is a holomorphic section $\sigma : S \to X$. Then the multiplicity

$$\mathrm{mult} : S \to \mathbb{Z}, \quad s \mapsto \mathrm{mult}_{\sigma(s)} X_s$$

is Zariski upper-semicontinuous.

Proof. Since the fibers $\{X_s\}$ clearly forms an anlytic family of cycles in X. Since the section map $\sigma : S \to X$ is holomorphic, thus

$$S \to S \times X \to \mathbb{N}, \quad s \mapsto (s, \sigma(s)) \mapsto \mathrm{mult}_{\sigma(s)} X_s,$$

as composition of continuous function and upper semi-continuous function is upper semi-continuous. □

Theorem 3.5 (Approximation Chow-Barlet cycle space using countable many projective morphisms)**.** Let $g : X \to S$ be a proper morphism of complex analytic spaces that is bimeromorphic to a projective morphism. Fix $m \in \mathbb{N}$. Then there are countably many diagrams of complex analytic spaces over S,

$$\begin{tikzcd} C_i \arrow[r, hook] \arrow[d, shift right, "w_i"'] & W_i \times_S X \\ W_i \arrow[u, shift right, "\sigma_i"'] \end{tikzcd}$$

indexed by $i \in I$, such that

(1) the $w_i : C_i \to W_i$ are proper, of pure relative dimension 1 and flat over a dense, Zariski open subset $W_i^\circ \subset W_i$,

(2) the fiber of w_i over any $p \in W_i^\circ$ has multiplicity m at $\sigma_i(p)$,

(3) the W_i are irreducible, the structure maps $\pi_i : W_i \to S$ are projective, and

(4) the fibers over all the W_i° give all irreducible curves that have multiplicity m at the marked point.

Proof. By assumption, there is a bimeromorphic morphism $r : Y \to X$ such that Y is projective over S.

$$\begin{tikzcd} Y \arrow[rr, "r"] \arrow[dr, "f"'] & & X \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

The Barlet-Chow cycle space of curves with marked points on Y/S exist and its irreducible components W_i are projective over S. And the universal family

$$\mathcal{C} \to \mathrm{Chow}^1(Y/S),$$

parameterize all pointed curves on Y in the fiber direction. Let W be any irreducible component of $\mathrm{Chow}^1(Y/S)$, We restrict the universal family on that component $\mathcal{C}^Y \to W$.

We then map back the family of curves on Y

$$\begin{tikzcd} C^Y \arrow[r, hook] \arrow[d, shift right, "w_Y"'] & W \times_S Y \\ W \arrow[u, shift right, "\sigma_Y"'] \end{tikzcd}$$

to family of curves on X.

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, shift right, "w"'] & W \times_S X \\ W \arrow[u, shift right, "\sigma"'] & \end{tikzcd}$$

Note that the family $w : C \to W$ is no longer flat, as curves can be contracted by $Y \to X$. However it's still proper flat over some dense Zariski open subset $W^o \subset W$.

By the lemma we just proved, since the family is flat over W^o, the multiplicity of a fiber C_w at the section s is an upper semi-continuous function on W°. For each $m \in \mathbb{N}$, let $W^m \subset W$ denotes the closure of the set of points $p \in W^\circ$ for which $\mathrm{mult}_{\sigma(p)} C_p = m$. Thus the restriction $W^m \to S$ is still a projective morphism.

We finally going back to the original Moishezon morphism $g : X \to S$. Let $X^\circ \subset X$ be the largest open set over which r is an isomorphism. The above procedure gives all irreducible pointed curves that have nonempty intersection with X°. Equivalently, all curves with a marked point that are not contained in $X \backslash X^\circ$.We can now use dimension induction to get countably many diagrams that give us all curves on $g : (X \backslash X^\circ) \to S$ (note that by the result we proved the first time the restriction is a proper Moishezon morphism, so that we can repeat the same arguement as above). □

4 Projectivity of very general fiber

We can now prove the following theorem, which is the key step to deduce the main theorem.

Theorem 4.1 (Projectivity on very general fiber, [Kol22, Proposition 14])**.** Let $g : X \to S$ be a proper morphism of complex analytic spaces and $S^* \subset S$ a dense, Zariski open subset such that g is flat over S^*. Assume that
(1) X_0 is projective for some $0 \in S$,
(2) the fibers X_s have rational singularities for $s \in S^*$, and
(3) g is bimeromorphic to a projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to S$.

Then there is a Euclidean open neighborhood $0 \in U \subset S$ and countably many nowhere dense, closed, analytic subsets $\{H_j \subset U : j \in J\}$, such that X_s is projective for every $s \in U \backslash \cup_j H_j$.

Proof. First choose $0 \in U \subset S$ such that X_U retracts to X_0. Since X_0 is projective, it has an ample line bundle L. Let $\Theta \in H^2(X_U, \mathbb{Q})$ be the pull-back of $c_1(L)$ to X_U. Note that Θ is a topological cohomology class that is usually not the Chern class of a holomorpic line bundle.

Using Theorem 3.5, we can find countable many diagram as in the theorem. Let $J \subset I$ be the index such that $H_i := \pi_i(W_i) \subset U$ for $i \in J$ is nowhere dense in U. And thus the rest of the $\pi_i(W_i) \subset U$ for $i \in I \setminus J$ will contains some open subset of U. Since $\pi_i(W_i)$ is proper, and thus it's an analytic subset that contains an open subset. By identity principal, it must equal to U. In particular, $0 \in \pi_i(W_i)$.

Let $s \in U \setminus \cup_{j \in J} H_j$, then by assumption, there is an $i \in I \backslash J$ and a diagram as in

$$\begin{tikzcd} C_i \arrow[r, hook] \arrow[d, shift right, "w_i"'] & W_i \times_S X \\ W_i \arrow[u, shift right, "\sigma_i"'] & \end{tikzcd}$$

such that the following conditions hold.

(a) (C_s, p_s) is one of the fibers of w_i over W_i°,

(b) $\mathrm{mult}_{\sigma(p)} C_p = m$ for all $p \in W_i^\circ$, and

(c) $\pi_i : W_i \to S$ is projective and its image contains $0, s \in S$ (say $\pi_i(0) = 0, \pi_i(w) = s$)

Note that preimage of 0 may not lies in the interior W_i^o. Since W_i are irreducible, there exist a holomorphic curve $\tau : D \to W_i$ connecting the point $0, w$ (with $\tau(0) = 0, \tau(1) = w$ and the radius of $D\ > 1$). We then pull back the family back to the disc D.

Note that

$$\mathrm{mult}_{p_t} C_t^D = \mathrm{mult}_{p_1} C_1^D = \mathrm{mult}_{p_s} C_s \text{ for all } t \in D^\circ,$$

On the other hand, by the Lemma 3.4, we have

$$\mathrm{mult}_{p_0} C_0^D \geq \mathrm{mult}_{p_t} C_t^D = \mathrm{mult}_{p_s} C_s.$$

(Here the pull back family $C^0 \to D$ is flat, since).

$$\mathrm{mult}_{p_0} C_0^D \geq \mathrm{mult}_{p_t} C_t^D = \mathrm{mult}_{p_s} C_s.$$

Here C_0^D is a 1-cycle on the projective scheme X_0, and Θ_0 is the Chern class of an ample line bundle on X_0. Thus

$$\Theta \cap \left[C_0^D\right] \geq \epsilon \cdot \mathrm{mult}_{p_0} C_0^D$$

by the easy direction of Theorem 2.9, where ϵ depends only on X_0 and Θ_0.

Since C_0^D and C_1^D lies in the same irreducible component of Chow-Barlet cycle space, they are algebraic equivalent. Thus the cup product with Θ are the same, and putting these together gives that

$$\Theta_s \cap [C_s] = \Theta \cap \left[C_1^D\right] = \Theta \cap \left[C_0^D\right] \geq \epsilon \cdot \mathrm{mult}_{p_0} C_0^D \geq \epsilon \cdot \mathrm{mult}_{p_s} C_s.$$

Thus X_s is projective by Theorem 2.9. □

5 The alternating property of projective locus

In this section, we will finish the proof of the alternating property about the projective locus. The following Thom Whitney stratification theorem is useful in our setting.

Proposition 5.1 (Thom Whitney stratification theorem, [Kol22, Lemma 15])**.** Let $f : X \to S$ be a proper morphism of complex analytic spaces. There exist finite Whitney stratifications $\mathcal{X}$ of X and $\mathcal{S} = \{S_l\}_{l \leq d}$ of S by locally closed subsets S_l of dimension l, with $d = \dim S$, such that for each connected component S (a stratum) of S_l. The following condition holds.

(a) $f^{-1}S$ is a topological fibre bundle over S, union of connected components of strata of $\mathfrak{X}$, each mapped submersively to S

(b) For all $v \in S$, there exist an open neighborhood $U(v)$ in S and a stratum preserving homeomorphism $h : f^{-1}(U) \simeq f^{-1}(v) \times U$ s.t. $f_{|U} = p_U \circ h$ where p_U is the projection on U.

In particular, there is a dense, Zariski open subset $S^\circ \subset S$ such that $g^\circ : X^\circ \to S^\circ$ is a topologically locally trivial fiber bundle. Moreover. If $S = \Delta$, if we shrink the disc then $f : X^* \to \Delta^*$ is topologically fiber bundle.

Under this assumption, we can prove the local system $R^i g_* \mathbb{Z}_X$ is constructible in the analytic Zariski topology for a proper morphism between complex analytic spaces.

Corollary 5.2 ([Kol22, Corollary 16])**.** Let $g : X \to S$ be a proper morphism of complex analytic spaces. Then the sheaves $R^i g_* \mathbb{Z}_X$ are constructible in the analytic Zariski topology.

When consider the global section of a local system, the following result is helpful.

Lemma 5.3. Let $\mathscr{L}$ be a local system on a complex manifold S, the global section

$$H^0(S, \mathscr{L}) = L^\rho := \{a \in L|\ \rho(\alpha)(a) = a, \forall \alpha \in \pi_1(S, v)\},$$

where L is the fiber of the local system on the reference point $v \in S$. And $\rho : \pi_1(S, v) \to GL(L)$ be the monodromy action. In particular if the base S is simply connected, then $H^0(S, \mathscr{L}) = L$

Proposition 5.4 ([Kol22, Proposition 17])**.** Let $g : X \to S$ be a proper morphism of normal, irreducible analytic spaces. Then there is a dense, Zariski open subset $S^\circ \subset S$ such that
(1) either X is locally projective over S°,
(2) or $\mathrm{PR}_S(X) \cap S^\circ$ is locally contained in a countable union of Zariski closed, nowhere dense subsets.

If g is bimeromorphic to projective morphism, then X is projective over S^o.

Proof. If we restrict out attention on the main strata of the Whitney stratification, the direct image $R^2 g_* \mathbb{Z}_X$ is locally constant. And further restrict on some Zariski open subset, we can also assume that $R^2 g_* \mathcal{O}_X$ is locally free. By passing to universal cover, we may also assume $R^2 g_* \mathbb{Z}_X$ is a constant sheaf.

Let Θ be a global section of $R^2 g_* \mathbb{Z}_X$. By Lemma 5.3, we know it's actually $\Theta \in H^2(X, \mathbb{Z})$).

We decompose the cohomology into two disjoint part,

$$H^2(X, \mathbb{Z}) = V_1 \sqcup V_2$$

where

$$V_1 = \{\Theta \in H^2(X, \mathbb{Z}) \mid \partial\Theta \equiv 0\},\ V_2 = \{\Theta \in H^2(X, \mathbb{Z}) \mid \partial\Theta \not\equiv 0\}.$$

We claim that if

$$PR_S(X) \subset \cup_{\Theta \in V_2} V(\Theta),$$

is not true, then the morphism is locally projective. Note that, we assume that $R^2 g_* \mathcal{O}_X$ is a vector bundle, thus the vanishing locus is a Zariski closed nowhere dense subset we denote $H_\Theta = V(\Theta)$ for $\Theta \in V_2$.

That is if there exists some $s \in PR_S(X)$, such that $s \notin \cup_{\Theta \in V_2} V(\Theta)$, then the morphism is locally projective. To see this, since $s \in PR_S(X)$, there exist some ample line bundle L_s on X_s, and thus under the exact sequence

$$\mathrm{Pic}(X_s) \to H^2(X_s, \mathbb{Z}) \xrightarrow{\partial} H^2(X_s, \mathcal{O}_{X_s})$$

then L_s maps to some zero element $\partial(\Theta_{L_s}) = 0$.

Since by assumption

$$\mathrm{res}_s : H^2(X_, \mathbb{Z}) \xrightarrow{\cong} H^2(X_s, \mathbb{Z}),$$

one can lift the class Θ_{L_s} to a class Θ. And then divide into two cases.

If $\partial\Theta$ is identically zero, it then lift to a line bundle $L \in \mathrm{Pic}(X)$, such that $L|_{X_s} = L_s$, which is ample and therefore by Grothendieck ampleness theorem. We know that the morphism is locally projective.

If $\partial\Theta$ is not identically 0, then $\partial\Theta = 0$ defines a Zariski closed, nowhere dense subset $H_\Theta \subset S$. In this case we know that

$$\Theta \in V_2$$

and by commutative diagram, we know $s \in V(\Theta)$. And thus

$$PR_S(X) \subset \cup_{\Theta \in V_2} V(\Theta)$$

□

Sometimes we need the global projective condition, which is guranteed if the Moishezon condition holds. Before proving the proposition, we need to the following statement on the local system of Neron-Sever group.

Proposition 5.5.

Proof. □

Proposition 5.6. Assume that $g : X \to S$ be a proper Moishezon morphism of normal irreducible analytic spaces. Assume that there exist a dense Zariski open subset $S^o \subset S$ such that X is locally projective over S^o then it's actually global projective.

Proof. □

6 Kollár's stratification of Moishezon morphism to projective morphisms

Now we can prove the main theorem of this note.

Theorem 6.1 (Openess of projectivity, [Kol22, Theorem 2]). Let $g : X \to S$ be a proper morphism of complex analytic spaces and $S^* \subset S$ a dense, Zariski open subset such that g is flat over S^*. Assume that

(1) X_0 is projective for some $0 \in S$,

(2) the fibers X_s have rational singularities for $s \in S^*$, and

(3) g is bimeromorphic to a projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \rightarrow S$.

Then there is a Zariski open neighborhood $0 \in U \subset S$ and a locally closed, Zariski stratification $U \cap S^* = \cup_i S_i$ such that each

$$g|_{X_i} : X_i := g^{-1}(S_i) \rightarrow S_i \text{ is projective.}$$

Proof. By Theorem 4.1, we know $\mathrm{PR}_S(X)$ contains the complement of a countable union of Zariski closed, nowhere dense subsets. Thus by the Baire category theorem, $\mathrm{PR}_S(X)$ is not contained in a countable union of closed, nowhere dense subsets. And by Theorem 5.4, Thus we are in case (1) i.e. $g : X \rightarrow S^o$ is locally projective over a dense, Zariski open subset $S^\circ \subset S$.

Since the morphism is Moishezon, thus by Theorem 5.6, the morphism $g : X \rightarrow S$ ia actually global projective over S^o. And we repeat the process on $S \setminus S^o$ gives the stratification of $g : X \rightarrow S$ into projective morphisms $g|_{X_i} : X_i = g^{-1}(S_i) \rightarrow S_i$. □

7 Bingener's stratification theorem

Bingener proved a similar statement about the distribution of projective locus.

Proposition 7.1 ([Bin83, (1.6) Theorem])**.** Let $p : X \rightarrow S$ and $q : Y \rightarrow S$ be proper holomorphic maps, $f : X \rightarrow Y$ a holomorphic S-map and let T be the set of points $s \in S$ such that $f_s : X_s \rightarrow Y_s$ is projective. Then there is a countable family $S_v, v \in \mathbb{N}$, of locally closed analytic subspaces of S with: (1) $T = \bigcup_{v \in \mathbb{N}} S_v$. (2) $f_{S_v} : X_{S_v} \rightarrow Y_{S_v}$ is locally projective relative S_v for all ν.

If p is cohomologically flat in dimension $\leqq 2$ and $\mathrm{R}^2 p_* (\mathbb{Z}_X)$ is constant, we can moreover achieve: (3) S_v is constructible in S for all v. (4) $\operatorname{codim}(S_y, S) \leqq l := \sup \left\{\dim_{\mathbb{C}} \mathrm{H}^2 (X_s, \mathcal{O}_{X_s}) : s \in S\right\}$ for all v.

8 Claudon-Höring's projectivity criterion for Kähler morphism

In this section, we try to prove the following projectivity criterion for Kähler morphism (due to Claudon-Höring).

Theorem 8.1 ([CH24, Theorem 3.1])**.** Let $f : X \rightarrow Y$ be a fibration between normal compact Kähler spaces. Assume that X has strongly $\mathbb{Q}$-factorial klt singularities. Assume one of the following:
(a.1) The normal space Y has klt singularities and the natural map

$$f^* : H^0\left(Y, \Omega_Y^{[2]}\right) \longrightarrow H^0\left(X, \Omega_X^{[2]}\right)$$

is an isomorphism.
(a.2) The morphism f is Moishezon.

Then f is a projective morphism.

Proof. If time permit, we will dicuss it in the next note. □

Final words, there are other projectivity critera developed by Kollár in [Kol90] (which we does not cover in this note).

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