Moishezon space and Moizhezon morphism Summer 2025

Note 5 — 2025-07-10 (draft version)

Yi Li

The aim of this not is to give an introduction to the rational curves on Moishezon spaces. The major references are [Kol22], [VP21], [McK17].

Contents

1 BDPP theorem on Moishezon spaces 1

2 Minimal model theory over Moishezon base 1

3 Projectivity criterion for Moishezon space using rational curves 2

4 Kleiman's projectivity criterion 5

5 Mckernan's Mori bend and break theorem on Moishezon spaces 5

1 BDPP theorem on Moishezon spaces

The first topic that we will discussed is the BDPP theorem for Moishezon space.

Theorem 1.1. Let X be a compact Moishezon manifold, then the canonical class K_X is pseudo-effective if and only if X is not uniruled.

Proof. □

It's worth mentioning that Ou recently proved the BDPP conjecture for compact Kähler manifold.

Theorem 1.2 ([Ou25])**.** Let X be a compact Kähler manifold. Then the canonical class K_X is pseudoeffective if and only if X is not uniruled (i.e. not covered by rational curves).

His proof uses some algebrac criterion for foliation, which is another story that we will not follow here.

2 Minimal model theory over Moishezon base

In this section, we introduce the minimal model theory over Moishezon base, developed by Paz [VP21]. (It's worth mentioning that Fujino's [**Fujino**] proved much stronger result where the base is not necessary a Moishezon variety). From the proof, we can see that one of the reason to introduce the Moishezon varieties and Moishezon spaces are this category allow us to do some cut and paste operations like the topology.

Proposition 2.1 ([VP21])**.** Suppose $g : (X, \Theta) \to (Y, \Delta)$ is a projective morphism of algebraic spaces of finite type over a field of characteristic 0 , where (X, Θ) is $\mathbb{Q}$-factorial and dlt.

(2)Suppose that g has exceptional divisor $E = E_1 + \cdots + E_n$ and that $K_X + \Theta \sim_{g,\mathbb{R}} E_\Theta = \Sigma e_j E_j$ for some numbers $e_j \geq 0$.

(3) Finally, suppose that H is a divisor on X, such that $K_X + \Theta + cH$ is g-ample for some number c.

Then we may run the g-relative $(K_X + \Theta)$-MMP with scaling of H.

Remark 2.2 (The r-th output of MMP)**.** We need to clarify the following termilogy that is useful in the proof of the main theorem.

The r-th output of MMP in what follows is different from the r-th step of MMP in usual sense. Here the r^{th} output of the g-relative D-MMP with scaling of H, denoted by $f^r : X \dashrightarrow X^r$, will mean the composite $X \dashrightarrow X^{r_1} \dashrightarrow \cdots \dashrightarrow X^{r_n}$ for numbers $r_1 > \cdots > r_n \geq r$, where each r_i is such that
(1) $D^{r_{i-1}} + r_i H^{r_{i-1}}$ is nef, but not ample, over Y, (as usual MMP with scaling)
(2) $D^{r_{i-1}} + (r_i + \epsilon) H^{r_{i-1}}$ is ample over Y, and
(3) $D^{r_n} + (r - \epsilon) H^{r_n}$ is ample over Y, where $\epsilon > 0$ is sufficiently small.

Proof. □

3 Projectivity criterion for Moishezon space using rational curves

Proposition 3.1. Let $f : Y \to X$ be a small bimeromorphic projective morphism of analytic varieties such that X is $\mathbb{Q}$-factorial. Then f is an isomorphism.

Remark 3.2. Note that the projective assumption is ncessary here.

Proposition 3.3 ([VP21])**.** Suppose that $\psi : (Y, \Delta) \to U$ is a proper morphism of normal algebraic spaces of finite type over a field k of characteristic 0 and that (Y, Δ) has KLT singularities. If ψ is non-projective, then
(1) either Y contains a rational curve C such that $\psi(C)$ is a point and $-[C] \in \overline{\mathrm{NE}}(Y/U)$,
(2) or Y has a small, $\mathbb{Q}$-factorial modification Y^{qf} that is projective over U (more precisely, the composite morphism $Y^{\mathrm{qf}} \to Y \to U$ is projective).

Remark 3.4. By Nakai-Moishezon ampleness criterion, the Mori cone is a pointed closed convex cone. Thus the condition $-[C] \in \overline{\mathrm{NE}}(Y/U)$ means that the rational curve is numerical trivial.

Proof. We first find a log resolution $g : X \to (Y, \Delta)$, such that X is projective over U. Write $E = E_1 + \cdots + E_n$ for the exceptional divisor of g. We can write $K_X + F_1 = g^\star (K_Y + \Delta) + F_2$, where F_1, F_2 are effective and F_2 is g-exceptional. Since (Y, Δ) is klt, then the coefficients of F_1 are all less than 1 . For $0 < \eta \ll 1$, the coefficients of $F_1 + \eta E$ are still less than 1 . Choose such a value of η, and let $\Theta = F_1 + \eta E$. Then we have $K_X + \Theta \sim_{g,\mathbb{R}} E_\Theta$, where $E_\Theta = F_2 + \eta E$ is effective and $\operatorname{Supp}(E_\Theta) = \operatorname{Ex}(g)$.

With this choice of Θ, the pair (X, Θ) is klt. Now choose a divisor A, sufficiently ample over U, such that $\operatorname{Supp}(A) \cup \operatorname{Supp}(\Theta)$ is an snc divisor, and such that $K_X + \Theta + cA$ is ample over Y, where $0 < c < 1$ is sufficiently general. We may pick such a divisor by Bertini's Theorem. Note that this choice of A implies that the pair $(X, \Theta + cA)$ is klt. Next, we perturb cA as follows: for every divisor class in some basis for $\mathrm{NS}(X)$, we pick a divisor representing that class and we add a sufficiently small, sufficiently general multiple of it to cA. Call the resulting divisor H. With these choices, we can arrange that $K_X + \Theta + H$ is ample over $Y, (X, \Theta + H)$ is klt, and the coefficients of H are linearly independent over $\mathbb{Q}(e_1, \ldots, e_n)$, where $e_1, \ldots, e_n$ are the coefficients of E_Θ.

We now run the relative $(K_X + \Theta)$-MMP with scaling of H over Y, whose steps exist and terminate by Theorem 2.6. Since (X, Θ) is klt, then this MMP terminates in a klt pair $(X^{\min}, \Theta^{\min})$ projective over Y. Since Y is klt, then by Theorem 3.52 in [32], $X^{\min}$ is a small modification of Y. Note that our choice of X was smooth, and hence $\mathbb{Q}$-factorial. Since each step of this MMP comes from the contraction of an extremal ray, then $X^{\min}$ is still $\mathbb{Q}$-factorial. If $X^{\min}$ is projective over U, then $X^{\min}$ is the claimed small, $\mathbb{Q}$-factorial modification of Y that is projective over U.

Otherwise, along the course of the MMP described above, there is a first step $X^r \dashrightarrow X^{r'}$, such that X^r is projective over U but $X^{r'}$ is not. We will focus on this MMP step; this is not necessarily the first step where the relative MMPs over Y and

over U deviate from each other. We emphasise that this is where our approach differs from that of [47] and [6].

Note first that since $K_X + \Theta + H$ is ample over U, then in particular it is relatively ample, and therefore the index r_1 corresponding to the first step in this MMP satisfies $r_1 < 1$. Since X is smooth (and hence $\mathbb{Q}$-factorial), then $(X, \Theta + r_1 H)$ is klt. Additionally, after we do the first MMP step, the output $(X^{r_1}, \Theta^{r_1} + r_1 H^{r_1})$ remains klt. This step arises from the contraction of an extremal ray in the cone of curves, so X^{r_1} is still $\mathbb{Q}$-factorial. Since the next index, say r_2, is smaller than r_1 and X^{r_1} is $\mathbb{Q}$-factorial, then in fact $(X^{r_1}, \Theta^{r_1} + r_2 H^{r_1})$ is klt.Inductively, we see that right before we do the MMP step $X^r \to X^{r'}$ that loses projectivity over U, we have a klt pair $(X^r, \Theta^r + r' H^r)$ projective over U. Since X^r is $\mathbb{Q}$-factorial, we can actually conclude that $(X^r, \Theta^r + (r' - \epsilon) H^r)$ is klt for $0 < \epsilon \ll 1$. By Lemma 3.3.1, this MMP step arises from the contraction of some extremal ray R in $\overline{\mathrm{NE}}(X^r/Y)$. Let F be the minimal extremal face of the larger cone of curves $\overline{\mathrm{NE}}(X^r/U)$ that contains R. We have two possibilities to consider:

First, suppose that F is itself a ray. If F contains only curves whose images in Y are points, then our step of the relative MMP over Y is actually a step of the relative MMP over U. However, the steps of the MMP over U preserve projectivity over U. This then gives us a contradiction, because we assumed that $X^{r'}$ is not projective U. It may also happen that F contains some curves whose images in Y are curves; in other words, our step of the MMP over Y doesn't contract every curve in F. Then there are curves C, C', such that $[C'] = \lambda[C]$ in $\overline{\mathrm{NE}}(X^r/U)$ for some $\lambda > 0$, and such that $g^r(C)$ is a point and $g^r(C')$ is a curve in Y. In fact, by Corollary 1.4 in [14] applied to the contraction morphism $\operatorname{cont}_F : X^r \to Z$ over U, we can take the curves C, C' to be rational. This implies that $[g^r(C')] = 0$ in $\overline{\mathrm{NE}}(Y/U)$, so certainly $-[g^r(C')] \in \overline{\mathrm{NE}}(Y/U)$.

The second possibility is that F has dimension greater than 1 . Since R is anextremal ray contracted by a step of some MMP, then it is spanned by the class of some rational curve C, and $[C] \in F$.

Then we can write $[C] = \sum \lambda_j v_j$, where each $v_j \in R_j$ is some vector contained in an extremal ray R_j of F, and $\lambda_j \neq 0$. Here we allow the extremal rays R_j to have non-negative intersection with the log canonical divisor, so at this point we do not know that any v_j is the class of some curve in X^r.

We have that $K_{X^r} + \Theta^r + (r' + \epsilon) H^r$ is relatively ample, and $(K_{X^r} + \Theta^r + r'H^r) \cdot C = 0$. This means that if we decrease r', we get a negative intersection product: $(K_{X^r} + \Theta^r + \ (r' - \epsilon) H^r) \cdot C < 0$, so there must exist some j_0, such that $(K_{X^r} + \Theta^r + (r' - \epsilon) H^r) \cdot v_{j_0} < 0$ for all sufficiently small $\epsilon > 0$. Since we know that $(X, \Theta^r + (r' - \epsilon) H^r)$ is klt and R_{j_0} is a negative extremal ray for this pair, then by [20] this means that $R_{j_0} = \mathbb{R}_{\geq 0} [C_{j_0}]$ for some rational curve C_{j_0} in X^r. Replacing λ_{j_0} if necessary, we can assume that $v_{j_0} = [C_{j_0}]$. Letting $\epsilon \to 0$, we see that actually $(K_{X^r} + \Theta^r + r'H^r) \cdot C_{j_0} \leq 0$. Suppose for a contradiction that $g^r(C_{j_0})$ is a point. Then $(K_{X^r} + \Theta^r + r'H^r) \cdot C_{j_0} \geq 0$ because the divisor is relatively nef. Combining our two inequalities, we deduce that $(K_{X^r} + \Theta^r + r'H^r) \cdot C_{j_0} = 0$, so that the ray $R_{j_0} \cap \overline{\mathrm{NE}}(X^r/Y)$ gets contracted by this MMP step. This is impossible because this step contracts only the ray R. Therefore, $g^r(C_{j_0})$ is a rational curve in Y, and $-[g^r(C_{j_0})] = \lambda_{j_0}^{-1} \sum_{j \neq j_0} \lambda_j [g^r(v_j)] \in \overline{\mathrm{NE}}(Y/U)$. □

We can prove the following corollary

Proposition 3.5 ([VP21]). Suppose that $\psi : Y \to U$ a proper morphism between Moishezon varieties, and that Δ is some divisor on Y, such that the pair (Y, Δ) has KLT singularities.

Assume additionally that Y is $\mathbb{Q}$-factorial. Then ψ is non-projective if and only if Y contains a rational curve C such that $\psi(C)$ is a point and $-[C] \in \overline{\mathrm{NE}}(Y/U)$.

Proof. If $\psi : Y \to U$ is projective, then Y carries some ψ-ample divisor A. Then A is positive on every curve on Y that is contracted by ψ, so Y cannot contains a rational curve C such that $\psi(C)$ is a point and $-[C] \in \overline{\mathrm{NE}}(Y/U)$.

Conversely, if ψ is non-projective, then we can run the argument in the proof of previous theorem, we can extract all the exceptional divisor of $X \to Y$ and

$$\begin{tikzcd}
X \arrow[rr, dashed] \arrow[dr] \arrow[ddr] & & X^{\min} \arrow[dl] \arrow[ddl] \\
& Y \arrow[d, dashed, "\psi"] & \\
& U &
\end{tikzcd}$$

However, we are now assuming that Y is $\mathbb{Q}$-factorial, so Y admits no non-trivial small modifications that are projective over it. This implies that $X^{\min} \cong Y$, so in particular $X^{\min}$ is not projective over U.

Then, there must have some step of MMP over Y that lose the projectivity, and the rational curve comes from that step.

□

4 Kleiman's projectivity criterion

Proposition 4.1 ([VP21])**.** Suppose that $\psi : (Y, \Delta) \to U$ is a proper morphism of normal algebraic spaces of finite type over a field k of characteristic 0 and that (Y, Δ) has $\mathbb{Q}$-factorial klt singularities.

Then ψ is projective if and only if there is a Cartier divisor L on Y such that $L \cdot C > 0$ for every irreducible curve C in Y such that $\psi(C)$ is a point. (L need not be ample over U).

Proof. If ψ is projective, then Y contains a Cartier divisor L ample over U, so we are done.

Conversely, suppose that there is a Cartier divisor L on Y that is positive on every irreducible curve contracted by ψ. Assume for a contradiction that ψ is not projective. By Corollary 3.5, Y contains a rational curve C such that $-[C] \in \overline{\mathrm{NE}}(Y)$. Since the Mori cone is dual to the nef cone for Moishezon variety, for this curve we have $L \cdot (-C) \geq 0$, which contradicts the assumption. □

5 Mckernan's Mori bend and break theorem on Moishezon spaces

This section, we will introduce the new proof of Mori bend and break on the Moishezon variety by Mckernan.

Theorem 5.1 ([McK17, Theorem 1.1])**.** Let S be a proper algebraic space and let (S, Θ) be a kawamata log terminal pair. Suppose that Θ is big.

For every point c of the stable base locus of $K_S + \Theta$ there is a rational curve $c \in M \subset S$.

Proof. Let $\pi : X \longrightarrow S$ be a log resolution of (S, Θ), so that X is a smooth projective variety and the sum of the strict transform of Θ and every exceptional divisor has global normal crossings.

As (S, Θ) is KLT, we may write

$$K_X + \Delta = \pi^* (K_S + \Theta) + E,$$

where $E \geq 0$ is exceptional, $\pi_* \Delta = \Theta$ and $\lfloor \Delta \rfloor = 0$. In particular (X, Δ) is KLT.

Adding a small multiple of the sum of the exceptional divisors to both sides, we may assume that both the support of Δ and the support of E contains every exceptional divisor.

In particular Δ is big. (Since we assume that Θ is big and thus so it's the strict transform and Δ contains some effective divisor in it, thus Δ is a big divisor).

Suppose first that $K_X + \Delta$ is not pseudo-effective. Then K_X is not pseudo-effective either,

By BDPP for projective variety implies that X is covered by curves on which K_X is negative and X is uniruled. Either way, S is uniruled. In this case there is a rational curve through every point of S and the result is clear.

Therefore we may assume that $K_X + \Delta$ is pseudo-effective. By BCHM (X, Δ) has a log terminal model, $f : X \dashrightarrow Y$. As Γ is big, it follows that $K_Y + \Gamma$ is semiample. And if $\phi : S \dashrightarrow Y$ is the induced birational map then ϕ is a log terminal model of $K_S + \Theta$.

$$\begin{tikzcd} X \arrow[r, dashed] \arrow[d] & Y \\ S \arrow[ur, dashed, "\phi"'] & \end{tikzcd}$$

Let Z be the indeterminancy locus of $\phi^{-1} : Y \dashrightarrow S$.

If $K_S + \Theta$ is already semi-ample, then there is nothing to prove. Otherwise, by the Theorem 5.4 (Zariski-Fujita theorem). There exist a curve C contains in the stable base locus $\mathbf{B}(K_S + \Theta)$. Let $p : W \to S$ and $q : W \to Y$ resolve the indetermancy $X \dashrightarrow Y$. Since $S \dashrightarrow Y$ is $(K_S + \Theta)$-negative and $K_Y + \Gamma$ is semi-ample, by Proposition 5.3, thus $C \subset p(q^{-1}(Z))$.

Since (Y, Γ) is KLT, the indetermancy locus Z is uniruled, thus $p(q^{-1}(Z))$ is uniruled. And therefore there exists a rational curve passing through $c \in C$. □

Theorem 5.2 ([McK17, Corollary 0.2])**.** Let S be a $\mathbb{Q}$-factorial proper algebraic space and let (S, Θ) be a kawamata log terminal pair.

If $C \subset S$ is a curve such that $(K_S + \Theta) \cdot C < 0$ then for every point $c \in C$ there is a rational curve $c \in M \subset S$.

Proof. As S is birational to a projective variety there is a big divisor $B \geq 0$ (a priopr it's not clear this is Cartier or not it's only a rank 1 reflexive sheaf) on S. As S is $\mathbb{Q}$-factorial, B is $\mathbb{Q}$-Cartier.

Possibly replacing B by a small multiple, we may assume that $(S, \Theta + B)$ is KLT and $(K_S + \Theta + B) \cdot C < 0$.

Then $\Theta + B$ is big and C belongs to the stable base locus of $K_S + \Theta + B$. Thus, apply the Theorem 5.1, shows that there exists a rational curve. □

The following result is useful in the proof of the theorem above.

Proposition 5.3. Let S and Y be proper normal algebraic spaces, let D be an $\mathbb{R}$-Cartier divisor on S and let $C \subset S$ be a curve. Let $\phi : S \dashrightarrow Y$ be a D-negative birational map.

Let $p : W \longrightarrow S$ and $q : W \longrightarrow Y$ be proper birational morphisms which resolve ϕ. Let Z be the indeterminancy locus of ϕ^{-1}.

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ S \arrow[rr, dashed, "\phi"] & & Y \end{tikzcd}$$

Then one of the following three conditions hold.

(1) $C \subset p\left(q^{-1}(Z)\right)$, or

(2) Every curve $\Sigma \subset Y$ such that $C \subset p\left(q^{-1}(\Sigma)\right)$ belongs to the stable base locus of $\phi_* D$, or

(3) C does not belong to the stable base locus of D.

The proof is ad hoc.

Proof. Proof. Suppose that (1) and (2) do not hold. As (2) doesn't hold, we may find a curve Σ such that $C \subset p\left(q^{-1}(\Sigma)\right)$ and Σ does not belong to the stable base locus of $\phi_* D$. Then we may find a divisor

$$0 \leq D' \sim_{\mathbb{R}} \phi_* D$$

which does not contain Σ. As ϕ is D-negative,

$$B = \phi^* D' + p_* E \geq 0$$

On the other hand

$$\begin{aligned} B = \phi^* D' + p_* E &= p_*\left(q^* D' + E\right) \\ &\sim_{\mathbb{R}} p_*\left(q^* \phi_* D + E\right) \\ &= p_* p^* D = D \end{aligned}$$

Let $U = Y - Z, \Sigma_0 = \Sigma \cap U, D'_0 = D'|_U$ and $\pi_0 = \phi^{-1}\big|_U : U \longrightarrow S$. Then D'_0 does not contain $\Sigma_0, D'_0 \sim_{\mathbb{R}} \pi_0^* D$ and moreover Σ_0 dominates C, since (1) does not hold. It follows that D'_0 does not contain $\pi_0^{-1}(C)$. But then $\pi_0\left(D'\right)$ does not contain C. On the other hand, $p_* E$ does not contain C as (1) does not hold. Therefore $B = \phi^* D' + p_* E$ does not contain C and so (3) holds. ☐

Theorem 5.4 (Zariski-Fujita theorem, [Laz04, Remark 2.1.31])**.** Let L be a line bundle on a projective variety X with the property that the base locus $\mathrm{Bs}(|L|)$ is a finite set. Then L is semiample

References

[Kol22] János Kollár. "Moishezon morphisms". In: *Pure Appl. Math. Q.* 18.4 (2022), pp. 1661–1687.

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I.* Vol. 48. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Classical setting: line bundles and linear series. Springer-Verlag, Berlin, 2004, pp. xviii+387.

[McK17] James McKernan. "Rational curves on algebraic spaces". In: *Higher dimensional algebraic geometry—in honour of Professor Yujiro Kawamata's sixtieth birthday.* Vol. 74. Adv. Stud. Pure Math. Math. Soc. Japan, Tokyo, 2017, pp. 313–319.

[Ou25] Wenhao Ou. *A characterization of uniruled compact Kähler manifolds.* 2025. arXiv: 2501.18088 [math.AG]. URL: https://arxiv.org/abs/2501.18088.

[VP21] David Villalobos-Paz. *Moishezon Spaces and Projectivity Criteria.* 2021. arXiv: 2105.14630 [math.AG]. URL: https://arxiv.org/abs/2105.14630.