Albanese varieties and Albanese mappings Fall 2025

Note I.2 — 22, 09, 2025 (draft version)

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The aim of this note is try to give an introduction of Albanese varieties and Albanese mapping with varies applications.

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1 Construction of Albanese Variety

Let us first construct the Albanese variety. To do this, we need the following proposition.

Proposition 1. Let X be a compact Kähler manifold, with the Kähler form ω.
(1) We have the following well defined map

$$\varphi : H_1(X,\mathbb{Z}) \to (H^0(X,\Omega^1_X))^\vee, \quad [\gamma] \mapsto (\alpha \mapsto \int_\gamma \alpha).$$

(2) The image of $H_1(X,\mathbb{Z})$ forms a lattice in $(H^0(X,\Omega^1_X))^\vee$, hence the quotient is a complex torus of dimension equals to the $\dim H^0(X,\Omega^1_X)$.

Proof of (1). Let (X,ω) be a compact Kähler manifold, by Lemma 2, given a holomorphic p-forms α, it is always closed. On the other hand, if two class $[\gamma] = [\gamma'] \in H_1(X,\mathbb{Z})$, then there exists a singular 2-chain such that

$$\gamma - \gamma' = \partial S.$$

In particular, by the Stoke's formula,

$$\int_{\gamma-\gamma'} \alpha = \int_{\partial S} \alpha = \int_S d\alpha = 0.$$

Thus it's independent of choice of representative. □

Proof of (2). □

Lemma 2. Let X be a compact Kähler manifold with a Kähler metric. Let $\alpha \in H^0(X, \Omega_X^p)$ be a holomorphic p-form, then α is always closed.

Proof. by definition we have $\bar{\partial}\alpha = 0$, on the other hand, by type reason, we know that $\bar{\partial}^*\alpha = 0$ as well, thus

$$\Delta_{\bar{\partial}}\alpha = 0 \implies \Delta_{\partial}\alpha = 0.(\text{by Kähler identity}).$$

And consequently,

$$(\partial\partial^*\alpha + \partial^*\partial\alpha, \alpha) = \|\partial\alpha\|^2 + \|\partial^*\alpha\|^2 = 0 \implies \partial\alpha = 0.$$

□

Definition 3 (Albanese variety)**.** Let X be a compact Kähler manifold (or more generally a compact complex manifold). We define the complex torus

$$\mathrm{Alb}(X) = (H^0(X, \Omega_X^1))^\vee / H_1(X, \mathbb{Z})$$

to be the Albanese variety associated to X, which is a complex torus.

When X is projective, we can show that Albanese variety is an Abelian variety.

Theorem 4. Let X be a projective manifold, the Albanese variety is an Abelian variety.

Proof. □

2 Construction of Albanese mapping

Definition 5 (Albanese mapping)**.** Let $[\omega_1], \cdots, [\omega_k]$ be the basis of $H^0(X, \Omega_X^1)$. Then the representative ω_i are closed $(1,0)$-forms. We then define the Albanese mapping as

$$\mathrm{alb}_X : X \to \mathrm{Alb}(X)., \quad z \mapsto (\int_{z_0}^{z} \omega_1, \cdots, \int_{z_0}^{z} \omega_k).$$

Proposition 6. The Albanese mapping is well defined.

Proof. First, by Lemma 2, the integration does not depend on the real path that we choose. Second, we need to check the linear functional does not depend on the □

Albanese mapping is also defined for complex manifold.

We next show how to deal with the Albanese mapping for singular varieties.

3 Some Basic Properties for Albanese mapping

We first introduce the universal property of Albanese mapping.

Proposition 7.

Proof. □

Note that when X is Moishezon manifold, the Albanese variety is an Abelian variety. In general however it's only a complex torus.

Proposition 8.

4 Conditions for Albanese mapping to be fibration

Note that in general Albanese mapping is not fibration.

5 Applications of Albanese Fibration

5.1 Albanese mapping in Iitaka conjecture

5.2 Albanese mapping in the classification problems

References