Hacon-Popa-Schnell's Readings Notes Fall 2025

Note III.3 — 2025 09 12 (draft version 0)

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The aim of this note is to prove the Iitaka conjecture when the base has maximal Albanese dimension.

Theorem 0.1 ([HPS18, Theorem 1.1])**.** Let $f : X \to Y$ be an algebraic fiber space with general fiber F. Assume that Y has maximal Albanese dimension, then $\kappa(X) \geq \kappa(F) + \kappa(Y)$.

There are three essential ingredients that appear in the proof:

(a) The Green–Lazarsfeld–Simpson structure theorem on cohomological support loci and the generic vanishing theorem;
(b) Hacon–Popa–Schnell's construction of singular Hermitian metrics on $f_*\omega_X^{\otimes n}$;
(c) Kawamata's proof of the Iitaka conjecture when the base is of general type, together with his structure theorem for finite covers of Abelian varieties.

Contents

1 Green-Lazarsfeld's Generic Vanishing theorem

In this section, we will introduce Green-Lazarsfeld's generic vanishing theorem. The structure is organized as follows, we will first introduce the Fourier-Mukai Transform. And then a complete proof of Hacon-Pareschi-Popa's criterion of GV sheaf is given. Using this, we will prove the Green-Lazarsfeld generic vanishing theorem and then

1.1 Theorem of Hacon-Pareschi-Popa and GV sheaves

Definition 1.1 (GV (generic vanishing) sheaves)**.** Given a coherent $\mathscr{O}_T$-module $\mathscr{F}$ on a compact complex torus T, define

$$S^i(T, \mathscr{F}) = \left\{L \in \operatorname{Pic}^0(T) \mid H^i(T, \mathscr{F} \otimes L) \neq 0\right\}$$

We say that $\mathscr{F}$ is a GV-sheaf if $\operatorname{codim} S^i(T, \mathscr{F}) \geq i$ for every $i \geq 0$; we say that $\mathscr{F}$ is M-regular if codim $S^i(T, \mathscr{F}) \geq i+1$ for every $i \geq 1$.

Definition 1.2 (Cohomological support loci)**.** Recall that for any coherent sheaf $\mathscr{F}$ on an abelian variety A, we consider for all $k \geq 0$ the cohomological support loci

$$V^k(\mathscr{F}) = \left\{P \in \operatorname{Pic}^0(A) \mid H^k(X, \mathscr{F} \otimes P) \neq 0\right\}$$

They are closed subsets of $\operatorname{Pic}^0(A)$, by the semi-continuity theorem for cohomology.

Definition 1.3 (Unipotent Vector Bundle)**.** A vector bundle on A is called unipotent if it has a filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_n = U$$

such that $U_i/U_{i-1} \simeq \mathscr{O}_A$ for all $i = 1, \ldots, n$. Note in particular that $\det U \simeq \mathscr{O}_A$.

That is we have a succesive extension

$$0 \to U_{i-1} \to U_i \to E_i \to 0$$

with

$$U_i/U_{i-1} \cong \mathcal{O}_A = E_i$$

Before going into the proof of theorem of Hacon-Pareschi-Popa, let us briefly introduce the Fourier-Mukai Transform.

Definition 1.4 (Fourier-Mukai Transform)**.** Let A be an Abelian variety. We will identify $\operatorname{Pic}^0(A)$ with the dual abelian variety $\widehat{A}$, and denote by P the normalized Poincaré bundle on $A \times \widehat{A}$. It induces the integral transforms

$$\mathbf{R}\Phi_P : D^b_{coh}\left(\mathscr{O}_A\right) \longrightarrow D^b_{coh}\left(\mathscr{O}_{\widehat{A}}\right), \quad \mathbf{R}\Phi_P \mathscr{F} = \mathbf{R}p_{2*}\left(p_1^* \mathscr{F} \otimes P\right).$$

and

$$\mathbf{R}\Psi_P : D^b_{coh}\left(\mathscr{O}_{\widehat{A}}\right) \longrightarrow D^b_{coh}\left(\mathscr{O}_A\right), \quad \mathbf{R}\Psi_P \mathscr{G} = \mathbf{R}p_{1*}\left(p_2^* \mathscr{G} \otimes P\right),$$

where $p_1 : A \times \hat{A} \to A$ and $p_2 : A \times \hat{A} \to \hat{A}$. (Note that in some of the references, the Fourier-Mukai transform $R\Psi_P = R\hat{S}$ and $R\Phi_P = RS$).

The following result of Mukai is fundamental.

Theorem 1.5. The Fourier-Mukai functors $(\mathbf{R}\Psi_P, \mathbf{R}\Phi_P)$ are e equivalences of derived categories, usually called the Fourier-Mukai transforms; moreover,

$$\mathbf{R}\Psi_P \circ \mathbf{R}\Phi_P = (-1_A)^* [-g] \quad \text{and} \quad \mathbf{R}\Phi_P \circ \mathbf{R}\Psi_P = \left(-1_{\widehat{A}}\right)^* [-g]$$

where $[-g]$ denotes shifting g places to the right.

We will give the proof of Mukai's theorem in the Series of notes A brief introduction to Fourier-Mukai Transform..

One can prove the following commutation relation of Fourier-Mukai Transform and Grothendieck Dual.

Proposition 1.6 (Commutation relation of Grothendieck dual and Fourier-Mukai Transform)**.**

$$D_A \circ RS = \left((-1_A)^* \circ RS \circ D_{\hat{A}}\right)[g].$$

Using the Fourier-Mukai, the following non-vanishing theorem comes for free. (Which shows to get non-vanihing of $h^0(\mathscr{F})$ has needs a lot of vanishing on h^i for $i > 0$).

Proposition 1.7. If $\mathcal{F} \in \mathrm{Coh}(A)$ is a non-zero coherent sheaf such that $h^i(\mathcal{F} \otimes P) = 0$ for all $i > 0$ and $P \in \hat{A}$, then $h^0(\mathcal{F}) \neq 0$.

PROOF IDEA 1.8. The proof is an immediate application of Mukai's theorem. Assume by contradiction that $h^0(\mathscr{F}) = 0$. Since Euler characteristic is independent of topological trivial line bundle, this means that for any $P, Q \in \mathrm{Pic}^0(A)$, the Euler characteristic remain the same. That is

$$\chi(\mathscr{F} \otimes P) = \chi(\mathscr{F}) = h^0(\mathscr{F}) = 0.$$

And consequently, we get $h^0(\mathscr{F} \otimes P) = 0$ as well.

We then claim that $R\hat{S}(\mathscr{F}) = 0$ using Grauert base change and definition of Poincare line bundle as

$$R^i p_{\hat{A}*}(p_A^* \mathscr{F} \otimes \mathcal{P})(\hat{x}) = H^i(A, (p_A^* \mathscr{F} \otimes \mathcal{P})|_{A \times \hat{x}}) = H^i(A, (\mathscr{F} \otimes P_{\hat{x}})) = 0, \ \forall i \geq 0, \hat{x} \in \mathrm{Pic}^0(A).$$

(Note that Grauert base change theorem holds since the cohomological dimension is constant 0).

Then the result easily follows by Mukai's theorem $RS \circ R\hat{S}(\mathscr{F}) = (-1_A)^* \mathscr{F}[-g]$, that is $\mathscr{F} = 0$ a contradiction.

Hacon-Pareschi-Popa proved the following equivalence characterization of GV sheaves.

Theorem 1.9 (Theorem of Hacon-Pareschi-Popa, [HPS18, Lemma 7.3 + Theorem 7.7])**.** Let A be an Ableian variety over $\mathbb{C}$, $\mathscr{F}$ be a coherent sheaf on X. Then the following are equivalent.

(1) (Vanishing condition) $h^i(A, \mathscr{F} \otimes \hat{L}^\vee) = 0$ for any $i > 0$. Here L is a sufficient ample line bundle on $\hat{A}$, and $\hat{L}$ is the vector bundle on A that you get by Fourier-Mukai transform $\hat{L} = R^0 S(L)$;

(2) (The Fourier-Mukai condition)

$$R\hat{S}(D_A(\mathscr{F})) = R^g \hat{S}(D_A(\mathscr{F})),$$

where D_A is the Grothendieck derived dual and $R\hat{S}$ denotes the Fourier-Mukai transform.

(3) (Codimension condition on cohomological support loci)

$$\operatorname{codim}_A R^i\hat{S}(\mathscr{F}) \geq i, \quad \forall i > 0$$

Proof. We first prove that (1) $\implies$ (2). By Grothendeick duality

$$D_k\left(R\Gamma\left(\mathcal{F}\otimes\hat{L}^\vee\right)\right) \cong R\Gamma\left(D_A\left(\mathcal{F}\otimes\hat{L}^\vee\right)\right).$$

By [Muk81], $\hat{L}$ is locally free, and therefore (using some property of RHom see e.g. Vakil's FOAG 2022.14.2.F)

$$R\Gamma\Big(D_A\Big(\mathcal{F}\otimes\hat{L}^\vee\Big)\Big) \cong R\Gamma\Big(D_A(\mathcal{F})\otimes\hat{L}\Big).$$

Then by definition of $\hat{L}$, there is an isomorphism

$$R\Gamma\left(D_A(\mathcal{F})\otimes\hat{L}\right) \cong R\Gamma\left(D_A(\mathcal{F})\otimes p_{A,*}\left(\mathcal{P}\otimes p_{\hat{A}}^*L\right)\right).$$

Therfore by projection formula

$$R\Gamma\left(D_A(\mathcal{F})\otimes p_{A,*}\left(\mathcal{P}\otimes p_{\hat{A}}^*L\right)\right) \cong R\Gamma\Big(\Big(p_A^*D_A(\mathcal{F})\otimes\mathcal{P}\otimes p_{\hat{A}}^*L\Big)\Big) \cong R\Gamma\left(R\hat{S}\left(D_A(\mathcal{F})\right)\otimes L\right).$$

In summary, we proved

$$D_k\left(R\Gamma\left(\mathcal{F}\otimes\hat{L}^\vee\right)\right) \cong R\Gamma\left(R\hat{S}\left(D_A(\mathcal{F})\right)\otimes L\right).$$

Easy to see $D_k\left(R\Gamma\left(\mathcal{F}\otimes\hat{L}^\vee\right)\right)$ is a sheaf in degree 0 if and only if $H^i\left(A,\mathcal{F}\otimes\hat{L}^\vee\right) = 0$ for all $i > 0$.

Since L is sufficient ample, we may assume that each $R^j\hat{S}\left(D_A(\mathcal{F})\right)\otimes L$ are globally generated with vanishing higher cohomologies. And therefore by Leray spectral sequence arguement

$$R^j\Gamma\left(R\hat{S}\left(D_A(\mathcal{F})\right)\otimes L\right) \cong H^0\left(R^j\hat{S}\left(D_A(\mathcal{F})\right)\otimes L\right).$$

So that $R\Gamma\Big(R\hat{S}(D_A(\mathcal{F}))\otimes L\Big)$ is a sheaf in degree 0 if and only if $H^0\left(R^j\hat{S}\left(D_A(\mathcal{F})\right)\otimes L\right) = 0$ for all $j \neq 0$. Since $R^j\hat{S}\left(D_A(\mathcal{F})\right)\otimes L$ is global generated, this is equivalent to $R^j\hat{S}\left(D_A(\mathcal{F})\right) = 0$ for all $j \neq 0$.

(2) $\implies$ (3). Let

$$\mathcal{G} = R^g\hat{S}\left(D_A(\mathcal{F})\right)$$

then by our assumption then

$$\mathcal{G}[g] = R\hat{S}(D_A(\mathcal{F})).$$

Then

$$R\hat{S}(\mathcal{F}) = D_{\hat{A}}D_{\hat{A}}(R\hat{S}(\mathcal{F})) \quad \cong D_A\Big((-1_{\hat{A}})^*R\hat{S}(D_A(\mathcal{F}[g]))\Big) = (-1_{\hat{A}})^*D_A(\mathcal{G}).$$

So that

$$R^i\hat{S}(\mathcal{F}) = \left(-1_{\hat{A}}\right)^*\mathcal{E}xt^i\left(\mathcal{G},\omega_X^\bullet\right).$$

Then by [HL97, Proposition 1.1.6], the result follows easily.

And finally (3) $\implies$ (1). Suppose that $\operatorname{codim}\operatorname{Supp} R^i\hat{S}(\mathcal{F}) \geq i$ for all $i > 0$, then $H^j\left(R^i\hat{S}(\mathcal{F}) \otimes L^\vee\right) = 0$ for all $i + j > g$ and any line bundle L.From the spectral sequence

$$H^j\left(R^i\hat{S}(\mathcal{F}) \otimes L^\vee\right) \implies R^{i+j}\Gamma\left(R\hat{S}(\mathcal{F}) \otimes L^\vee\right)$$

it follows that $R^l\Gamma\left(R\hat{S}(\mathcal{F}) \otimes L^\vee\right) = 0$ for $l > g$. We have

$$R^l\Gamma\left(R\hat{S}(\mathcal{F}) \otimes L^\vee\right) = R^l\Gamma\left(Rp_{\hat{A},*}\left(p_A^*\mathcal{F} \otimes \mathcal{P}\right) \otimes L^\vee\right) = R^l\Gamma\left(p_A^*\mathcal{F} \otimes \mathcal{P} \otimes p_{\hat{A}}^*L^\vee\right) = R^l\Gamma\left(Rp_{A,*}\left(p_A^*\mathcal{F} \otimes \mathcal{P} \otimes p_{\hat{A}}^*L^\vee\right)\right.$$

It follows that $H^l\left(\mathcal{F} \otimes \hat{L}^\vee\right) = R^l\Gamma\left(\mathcal{F} \otimes \hat{L}^\vee\right) = R^{l+g}\Gamma\left(p_A^*\mathcal{F} \otimes \mathcal{P} \otimes p_{\hat{A}}^*L^\vee\right) = 0, \forall l > 0.$ ☐

For GV sheaves, the cohomological support loci satisfies certain order relation.

Theorem 1.10. Let $\mathscr{F}$ be a coherent sheaf on A. Then if $\mathscr{F}$ is a GV-sheaf, then

$$V^g(\mathscr{F}) \subseteq \cdots \subseteq V^1(\mathscr{F}) \subseteq V^0(\mathscr{F}).$$

PROOF IDEA 1.11.

As an immediate consequence, we can get a non-vanishing criterion for GV sheaves.

Theorem 1.12 ([HPS18, Lemma 7.4])**.** Let A be an Abelian variety, $\mathscr{F}$ be a GV sheaf on A. Then

$$\mathscr{F} = 0 \iff V^0(\mathscr{F}) = \emptyset.$$

Proof. ☐

Using the theorem of Hacon-Pareschi-Popa we can prove the direct image of pluricanonical sheaves are GV sheaves.

Proposition 1.13. Let $f : X \to A$ be a morphism from a smooth projective variety to an abelian variety. For every $m \in \mathbb{N}$, the sheaf $\mathscr{F}_m = f_*\omega_X^{\otimes m}$ is a GV-sheaf on A.

PROOF IDEA 1.14. The idea is to use the vanishing condition we just proved, and then the result will follows from the Kollár's vanishing.

1.2 Green-Lazarsfeld generic vanishing theorem

Let us first recall the Green–Lazarsfeld–Simpson generic vanishing theorem. We will begin by introducing Green and Lazarsfeld's original proof, and then present Hacon's proof of the Green–Lazarsfeld theorem using the Fourier–Mukai transform.

Theorem 1.15. If $f : X \to A$ is a morphism from a smooth projective variety to an abelian variety, then for any $j, k \geq 0$ we have
(1) $\operatorname{codim}_{\operatorname{Pic}^0(A)} V^k\left(R^jf_*\omega_X\right) \geq k$;
(2) Every irreducible component of $V^k\left(R^jf_*\omega_X\right)$ is a translate of an abelian subvariety of A by a point of finite order.

PROOF IDEA 1.16. Let us first briefly sketch the idea of Green-Lazarsfeld's original proof. There are two essential ingredients that appear in the proof (1) The $\partial\bar{\partial}$-Lemma, (2)

For our interest, we need the generic vanishing for direct image of pluricanonical sheaves.

Theorem 1.17 ([HPS18, Theorem 4.1])**.** Let X be a smooth projective variety. For each $m \in \mathbb{N}$, the locus

$$\left\{P \in \operatorname{Pic}^0(X) \mid H^0\left(X, \omega_X^{\otimes m} \otimes P\right) \neq 0\right\} \subseteq \operatorname{Pic}^0(X)$$

is a finite union of abelian subvarieties translated by points of finite order.

This theorem implies that $V^0\left(A, \mathscr{F}_m\right)$ (for $\mathscr{F}_m = f_*\omega_X^{\otimes m}$) is also a finite union of abelian subvarieties translated by points of finite order. where

$$\begin{aligned} V^0\left(A, \mathscr{F}_m\right) &= \left\{P \in \operatorname{Pic}^0(A) \mid H^0\left(A, \mathscr{F}_m \otimes P\right) \neq 0\right\} \\ &= \left\{P \in \operatorname{Pic}^0(A) \mid H^0\left(X, \omega_X^{\otimes m} \otimes f^*P\right) \neq 0\right\} \subseteq \operatorname{Pic}^0(A). \end{aligned}$$

PROOF IDEA 1.18.

Using the result we just developed above, we can prove the following criterion for direct image of pluricanonical sheaves to be a indecomposable unipotent vector bundle.

Definition 1.19. Let E be a vector bundle on complex manifold M, we say it's unipotent if

Theorem 1.20 ([HPS18, Corollary 4.3])**.** Let X be a smooth projective variety with $\kappa(X) = 0$, and let $f : X \to A$ be an algebraic fiber space over an abelian variety.

If $H^0\left(X, \omega_X^{\otimes m}\right) \neq 0$ for some $m \in \mathbb{N}$, then the coherent sheaf $\mathscr{F}_m$ is an indecomposable unipotent vector bundle.

PROOF IDEA 1.21.

2 Hacon-Popa-Schnell's construction of semi-positive singular Hermitian metric on $f_*\omega_X^{\otimes n}$

We will divide the construction into 3 steps. First we try to find some semi-positive singular Hermitian metric on $\mathscr{F} = f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)$ as long as (L, h) is semi-positive using the optimal Ohsawa-Takegoshi extension theorem. In the second step, we try to construct the m-th Narasimhan-Simha metric on the $\omega_{X/Y}$ and hence a semi-positive line bundle $(\omega_{X/Y}^{\otimes(m-1)}, h)$. And finally, using the elimination of multiplier ideal lemma, we can transport the semi-positivity from $\mathscr{F} = f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)$ to $\mathscr{F}_m = f_*(\omega_{X/Y}^{\otimes m})$. It's worth mentioning that up to now there is no algebraic proof of such semi-positivity theorem.

2.1 Construction of semi-positive metric on $f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h))$

We first prove there exists a semi-positive singular Hermitian metric on $\mathscr{F} = f_*\left(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\right)$ as long as (L, h) is semi-positive.

Proposition 2.1 ([HPS18, Theorem 21.1])**.** Let $f : X \to Y$ be a projective surjective morphism between two connected complex manifolds. If (L, h) is a holomorphic line bundle with a singular hermitian metric of semi-positive curvature on X, then the pushforward sheaf

$$\mathscr{F} = f_*\left(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\right)$$

(1) has a canonical singular hermitian metric H;
(2) This metric has semi-positive curvature and;
(3) satisfies the minimal extension property.

PROOF IDEA 2.2. The construction of the metric is a bit involved. Let us briefly sketch the idea of the proof first. We try to first construct some semi-positive singular Hermitian metric on the "nice" locus where the direct image is a vector bundle. And then we try to apply the Riemann extension theorem to extend such semi-positive metric to semi-positive metric for torsion free sheaf $f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h))$. Now let's explain the idea of the proof step by step.

Step 1. (Find the nice locus that on which $\mathscr{F}$ is a locally free sheaf E). We first find a nice Zariski open subset $U = Y \setminus Z$, such that: (a) f is submersion on U (so that Eheresmann theorem can apply), (b) the direct image $\mathscr{F} = f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h))$ and $f_*(\omega_{X/Y} \otimes L)$ are locally free on U; (c) the base change property holds for $f_*(\omega_{X/Y} \otimes L)$.

Step 2. (Construction of the Singular metric H on E). Let $s \in E(y) \subset H^0(X_y, K_{X_y} \otimes L_y)$ for some $y \in U$, we define the singular Hermitian inner product on E_y as

$$\|s\| = \int_{X_y} |s|_h^2.$$

We will check that this is finite and positive almost everywhere. Hence defines a singular Hermitian metric on E.

Step 3. (Prove the semi-positivity of the metric H using optimal Ohsawa-Takegoshi extension). By definition, in order to prove the semi-positive of the metric on E we just defined. We need to show the psh of the function $\psi(y) = \log |g|_{H^*}$ for local section $g \in H^0(U, E^*)$. To do this, we need to verify the upper semi-continuity and mean value inequality on the nice locus. Let us briefly sketch the idea for the proof of mean-value inequality. To do this we first simplify the problem onto the unit ball $i : B \hookrightarrow Y$ (which sends $0 \mapsto y$). By optimal Ohsawa-Takegoshi extension, for any $\alpha \in E_y$ with finding some section $s \in H^0(B, i_*\mathscr{F})$ such that

$$s(0) = \alpha, \frac{1}{\mu(B)} \int_B |s|_H^2 d\mu \leq 1.$$

And therefore, by the extremal characterization of dual metric H^* we have

$$|g|_{H^*} = \sup_{s \in E_y, \|s\|_{H,y} \leq 1} \|\langle g(y), s\rangle\| \geq |\langle g(y), s/\|s\|_{H,y}\rangle|.$$

And therefore taking log, we get

$$2\psi(y) \geq \log |\langle g(y), s\rangle|^2 - \log \|s\|_{H,y}^2$$

And therefore, by integration

$$\frac{1}{\pi} \int_\Delta 2\psi d\mu \geq \frac{1}{\pi} \int_\Delta \log |g(s)|^2 d\mu - \frac{1}{\pi} \int_\Delta \log |s|_H^2 d\mu.$$

For the 2nd term on the RHS, we apply the Jensen's inequality, and optimal L2 estimate. For the 1st term on the RHS, since $g(s)$ defines a holomorphic function and therefore we can apply the mean value inequality which complete the proof of the mean value inequality for ψ.

As a side remark, the optimal L^2 estimate plays a crucial role in the argument above.

Step 4. (Extend the semi-positive metric onto the whole torsion free sheaves) In order to extend the metric onto $\mathscr{F}$, we use the Riemann extension theorem, thus need to check the uniform boundedness of the psh function $\psi(y)$ on Y.

2.2 Construction of Narasimhan-Simha metric on $\omega_{X/Y}$

Our next goal is to construct the m-th Narasimhan-Simha metric on $\omega_{X/Y}$.

Proposition 2.3. Let $f : X \to Y$ be a surjective projective morphism with connected fibers between two complex manifolds. Suppose that $f_*\omega_{X/Y}^{\otimes m} \neq 0$ for some $m \geq 2$.

The line bundle $\omega_{X/Y}$ has a canonical singular hermitian metric with semipositive curvature, called the m-th Narasimhan-Simha metric. This metric is continuous on the preimage of the smooth locus of f.

PROOF IDEA 2.4. Let us first briefly sketch the idea of the proof, the idea is very similar to the construction of the metric on $f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h))$, we first construction relative Bergman-type metric on the smooth locus. We then compute the local weight function, and try to prove it's psh function. And we reduce the problem to the mean value inequality so that one can apply the Optimal L2-estimate. To be more concrete.

(Step 1, Construction of absolute version NS metric and showing its semi-positivity). We first define a length function, given a pluricanoncial section $v \in H^0(X, \omega_X^{\otimes m}) = V_m$, we define the length function

$$\ell(v) = \left(\int_X (c_n^m v \wedge \bar{v})^{1/m}\right)^{m/2},$$

where $c_n = 2^{-n}(-1)^{n^2/2}$. We then define the singular Hermitian metric on ω_X fiberwise, that is given an element $\xi \in \omega_X(x)$ (where $\omega_X(x)$ is the fiber of ω_X at $x \in X$). We set

$$|\xi|_{h_m,x} = \inf\left\{\ell(v)^{1/m} \mid v \in V_m \text{ satisfies } v(x) = \xi^{\otimes m}\right\} \in [0, +\infty]$$

This will induce a singular Hermitian metric on ω_X.

We then construct the relative version NS metric. First given $v \in V_m = H^0\left(X, \omega_X^{\otimes m}\right)$ and a point $y \in Y$, we have

$$v|_{X_y} = v_y \otimes (dt_1 \wedge \cdots \wedge dt_r)^{\otimes m}$$

for a unique m-canonical form $v_y \in H^0\left(X_y, \omega_{X_y}^{\otimes m}\right) = V_{m,y}$. We then define the length of v_y as

$$\ell_y(v_y) = \left(\int_{X_y} (c_n^m v_y \wedge \bar{v_y})^{1/m}\right)^{m/2}$$

Given an element $\xi \in \omega_X(x)$ (with $f(x) = y$), we set the norm

$$|\xi|_{h_m,x} = \inf\left\{\ell_y(v_y)^{1/m} \mid v \in V_m \text{ satisfies } v(x) = \xi^{\otimes m}\right\}.$$

One can check that this will define a singular Hermitian metric on $\omega_{X/Y}$.

One of the key point in the construction is the extremal characterization of the local weight $\varphi_m(x)$.

(Step 2. Prove the continuity of the metric).

(Step 3. Prove the semi-positivity of the NS metric). We first try to find some local coordinate, under which the line bundle ω_X can be trivialized (under the local frame s_0). Therefore the metric weight can be written as

$$\varphi_m = -\log |s_0|^2_{h_m} : U \to [-\infty, +\infty).$$

Any global section $v \in V_m = H^0(X, \omega_X^{\otimes m})$, under the local frame can be expressed as

$$v|_U = g_v s_0^{\otimes m}.$$

Then, we show that

$$\varphi_m(x) = \frac{2}{m} \sup \{\log |g_v(x)| \mid v \in V_m \text{ satisfies } \ell(v) \leq 1\}.$$

Since $\log |g_v(x)|$ are psh functions, and therefore using upper envelope of an equicontinuous family of plurisubharmonic functions we show that φ_m is psh and hence h_m is a semi-positive metric on ω_X.

(Step 4. Proof of semi-positive of the relative version NS metric on smooth locus). Only need to prove the local weight function of the metric is pluri-subharmonic. And therefore, it's sufficient to prove continuous of the metric and mean value inequality for the metric weight function. And we already proved the continuity of the NS metric (in Step 2). Thus only need to prove the mean value inequality. To do this we first prove the optimal L2-estimate for the length. For each $u \in H^0\left(X_0, \omega_{X_0}^{\otimes m}\right)$, there is some $v \in H^0\left(X, \omega_X^{\otimes m}\right)$ with

$$\ell(v) \leq \mu(B)^{m/2} \cdot \ell_0(u)$$

We then using the extremal definition we prove

$$\varphi_m(x) \geq \frac{2}{m} \log |g_v(x)| - \log \ell_y\left(v_y\right)^{2/m}$$

and then taking integration we get

$$\frac{1}{\pi} \int_B \varphi_m(i(y)) d\mu \geq \frac{1}{\pi} \int_B \frac{2}{m} \log |g_v(i(y))| \, d\mu - \frac{1}{\pi} \int_B \log \ell_y\left(v_y\right)^{2/m} d\mu,$$

and therefore apply the mean value inequality to the 1st term on the RHS and Jensen inequality with optimal estimate to the 2nd term on the RHS, which shows the mean value inequality for φ_m.

(Step 5. Extension of the semi-positive relative NS metric across the singular locus). To prove that it's possible to extend across the singular locus, we will apply the Riemann extension theorem. And therefore, only need to prove the locally uniformly boundedness of the local weight φ_m. To be more concrete, by extremal characterization of $\varphi_m(x)$. For every $x \in X$, there exists some $v \in H^0(X, \omega_X^{\otimes m})$ with $\ell(v) \leq C$ such that

$$\varphi_m(x) = \frac{2}{m} \log |g_v(x)|$$

with $\ell(v) \sim \int |g_v|^{2/m} d\mathrm{vol}$. We then try to use mean value inequality combined with Jensen's inequality prove

$$\varphi_m(x_0) \leq \frac{1}{\mathrm{vol}(B_R)} \int_{B_R} \frac{2}{m} \log |g_v(x)| \lesssim_R \ell(v) \leq C.$$

Thus φ_m is locally uniformly bounded and only depends on the constant R, C (independent of choice of $\varphi_m(x_0)$).

2.3 Construction of semi-positive metric on $f_* \omega_{X/Y}^{\otimes m}$

In order to finish the construction of semi-positive metric on $f_* \omega_{X/Y}^{\otimes m}$ we need the following lemma to eliminate the multiplier ideal sheaf $\mathcal{I}(h)$.

Proposition 2.5. Let $f : X \to Y$ be a surjective projective morphism with connected fibers between two complex manifolds. Suppose that $f_* \omega_{X/Y}^{\otimes m} \neq 0$ for some $m \geq 2$. (b) If h denotes the singular hermitian metric on $L = \omega_{X/Y}^{\otimes(m-1)}$ induced by the Narasimhan-Simha metric we just constructed, then

$$f_* \left(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h) \right) \hookrightarrow f_* \omega_{X/Y}^{\otimes m}$$

is an isomorphism over the smooth locus of f.

PROOF IDEA 2.6. We try to prove that the natrual inclusion is isomorphism on $Y \setminus Z$ using Ohsawa-Takegoshi theorem.

Hence we can transport the semi-positivity from the $f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h))$ to the direct image of relative pluricanonical sheaves using the following lemma.

Lemma 2.7. Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism between two torsion-free coherent sheaves that is generically an isomorphism. If $\mathscr{F}$ has a singular hermitian metric with semi-positive curvature, then so does $\mathscr{G}$.

Theorem 2.8 ([HPS18, Theorem 4.6])**.** Let $f : X \to Y$ be an algebraic fiber space. Suppose Y is compact.

(a) For any $m \in \mathbb{N}$, the torsion-free sheaf $f_* \omega_{X/Y}^{\otimes m}$ has a canonical singular hermitian metric with semi-positive curvature;
(b) If $c_1 \left(\det f_* \omega_{X/Y}^{\otimes m} \right) = 0$ in $H^2(Y, \mathbb{R})$, $f_* \omega_{X/Y}^{\otimes m}$ is locally free, and the singular hermitian metric on it is smooth and flat;
(c) Every nonzero morphism $f_* \omega_{X/Y}^{\otimes m} \to \mathcal{O}_Y$ is split surjective.

To finish the proof of (b) we need to following two propositions.

Proposition 2.9 (Theorem of Katz and André)**.** Any coherent sheaf of $\mathcal{O}_X$ module, with integrable connection(flat connection) is locally free.

PROOF IDEA 2.10. The proof of the result requires the theory of D-module. To be more precise.

3 A brief review of Kawamata's work on Iitaka conjecture when base is of general type

3.1 Several Critera for Albanese map to be fibration

Theorem 3.1 (Albanese map as fibration when Kodaira dimension 0,[Kaw81, Theorem 1])**.** Let X be a non-singular and projective algebraic variety and assume that $\kappa(X) = 0$. Then the Albanese map $\alpha : X \to A(X)$ is an algebraic fiber space. In addition, if X has maximal Albanese dimension then the Albanese map is a birational contraction morphism.

3.2 Structure Theorem for Finite Covers of Abelian Varieties

Theorem 3.2 ([Kaw81, Theorem 13])**.** Let $f : X \to A$ be a finite morphism from a complete normal algebraic variety to an abelian variety.

Then $\kappa(X) \geq 0$ and there are an abelian subvariety B of A, etale covers $\tilde{X}$ and $\tilde{B}$ of X and B, respectively, and a complete normal algebraic variety $\tilde{Y}$ such that
(1) $\tilde{Y}$ is finite over A/B;
(2) $\tilde{X}$ is isomorphic to $\tilde{B} \times \tilde{Y}$;
(3) $\kappa(\tilde{Y}) = \dim \tilde{Y} = \kappa(X)$.

3.3 Kawamata's proof of Iitaka conjecture when base is of general type

Theorem 3.3 ([Kaw81, Theorem])**.** If $f : X \to Y$ is algebraic fibre space with general fibre F and $\kappa(Y) = \dim Y$ then $\kappa(X) = \kappa(Y) + \kappa(F)$.

PROOF IDEA 3.4.

4 Proof of the Iitaka conjecture when the base has maximal Albanese dimension

Let us finish this note by the proof of Hacon-Popa-Schnell's result.

Theorem 4.1. Let $f : X \to Y$ be an algebraic fiber space with general fiber F. Assume that Y has maximal Albanese dimension, then $\kappa(X) \geq \kappa(F) + \kappa(Y)$.

4.1 Reduce the problem to the case when the base is an Abelian Variety and $\kappa(X) = 0$

Proposition 4.2. To prove Theorem 4.1, it's sufficient to prove the case when $\kappa(X) = 0$ and Y is an Abelian variety.

PROOF IDEA 4.3. We will divide the proof into 3 cases.
Case 1. We will prove the result in the case when $\kappa(X) = -\infty$. Green-Lazarsfeld-Simpson's

generic vanishing theorem is used. we assume by contradiction that $P_m(F) > 0$ then this will imply that $f_*\omega_X^{\otimes m} \neq 0$. And consequently, we can apply the generic vanishing theorem we developed in Section 1, find some torsion point $P \in \mathrm{Pic}^0(A)$ with $h^0\left(X, \omega_X^{\otimes m} \otimes g^*P\right) = h^0\left(A, g_*\omega_X^{\otimes m} \otimes P\right) \neq 0$ and then

$$h^0\left(X, \omega_X^{\otimes km}\right) = h^0\left(X, \left(\omega_X^{\otimes m} \otimes P\right)^{\otimes k}\right) \neq 0,$$

which contradicts to our assumption that $\kappa(X) = -\infty$;

Case 2. We assume that $\kappa(Y) = 0$. Using Theorem 3.1, we can reduce the base to an Abelian variety Y. We then take a Iitaka fibration on the total space

$$h : X \to Z,$$

with the general fiber G. We then restrict the fibration $f : X \to Y$ to a new fibration

$$G \to B',$$

with $\kappa(G) = 0$ and B' is an Abelian variety and H be the general fiber. which is exactly the situation of Proposition 4.1, that is

$$\kappa(G) \geq \kappa(H) + \kappa(B'),$$

thus $\kappa(H) = 0$.

$$\begin{tikzcd}
H = F \cap G \arrow[r, hook] \arrow[d, hook] & F \arrow[r] \arrow[d, hook] & h(F) \arrow[d, hook] \\
G \arrow[r, hook] \arrow[d] & X \arrow[r, "h"] \arrow[d, "f"] & Z \\
B \arrow[r, hook] & Y &
\end{tikzcd}$$

We then apply the easy addition formula to $F \to h(F)$, so that

$$\kappa(F) \leq \kappa(H) + \dim h(F) = \dim h(F),$$

which complete the proof of this case.

Case 3. When the base Y has maximal Albanese dimension. Then by the Kawamata's structure theorem for finite cover of Abelian varieties, we can decompose the base Y into a product of Abelian variety K and a variety of general type Z.

$$\begin{tikzcd}
F = E \cap H \arrow[r, hook] \arrow[d, hook] \arrow[dr] & E \arrow[r] \arrow[d, hook] & K \arrow[d] \\
H \arrow[r, hook] \arrow[d] & X \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, "f"] & K \\
Z \arrow[r] & Z & Y = Z \times K \arrow[l] \arrow[u]
\end{tikzcd}$$

The key point is to observe that the general fiber F of $f : X \to Y$ is also the general fiber of the induced $E \to K$. We then apply the Iitaka conjecture when base Z is of general type so that

$$\kappa(X) = \kappa(E) + \kappa(Z)$$

on the other hand since K is Abelian variety, so that it's possible to prove the Iitaka conjecture for $E \to K$ that is

$$\kappa(E) \geq \kappa(F).$$

This complete the proof of the 3rd case. We can then apply the result of Iitaka conjecture when the base is of general type to finish the proof.

4.2 Proof of the Iitaka Conjecture when the Base is an Abelian Variety and $\kappa(X) = 0$

We will finish the proof by showing that

Proposition 4.4. If $f : X \to A$ is an algebraic fiber space over an abelian variety, with $\kappa(X) = 0$, then we have

$$\mathscr{F}_m = f_* \omega_X^{\otimes m} \simeq \mathscr{O}_A$$

for every $m \in \mathbb{N}$ such that $H^0\left(X, \omega_X^{\otimes m}\right) \neq 0$.

PROOF IDEA 4.5. By Theorem 1.20 and Theorem 2.8, the direct image $\mathscr{F}_m$ is a flat unipotent vector bundle. Then using a bit representation theory, we can prove that the vector bundle is actually $\mathscr{F}_m = \bigoplus_{i=1}^{r} \mathcal{O}_A$. Using the assumption that $\kappa(X) = 0$, we have $r = 1$ and hene finish our proof.

Proof of Iitaka conjecture when base is Abelian variety and $\kappa(X) = 0$. By assumption, it's sufficient to prove that $\kappa(F) = 0$. Then, the result easily follows by Proposition 4.4. □

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