Generic Vanishing Readings Notes Fall 2025

Note I.1 — 2025 09 12 (draft version 0)

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The aim of this note is to introduce the following generic vanishing theorem.

Contents

1 Green-Lazarsfeld's Generic Vanishing theorem

In this section, we will introduce Green-Lazarsfeld's generic vanishing theorem. The structure is organized as follows, we will first introduce the Fourier-Mukai Transform. And then a complete proof of Hacon-Pareschi-Popa's criterion of GV sheaf is given. Using this, we will prove the Green-Lazarsfeld generic vanishing theorem and then

1.1 Theorem of Hacon-Pareschi-Popa and GV sheaves

Definition 1.1 (GV (generic vanishing) sheaves)**.** Given a coherent $\mathscr{O}_T$-module $\mathscr{F}$ on a compact complex torus T, define

$$S^i(T,\mathscr{F}) = \left\{L \in \operatorname{Pic}^0(T) \mid H^i(T,\mathscr{F}\otimes L) \neq 0\right\}$$

We say that $\mathscr{F}$ is a GV-sheaf if $\operatorname{codim} S^i(T,\mathscr{F}) \geq i$ for every $i \geq 0$; we say that $\mathscr{F}$ is M-regular if codim $S^i(T,\mathscr{F}) \geq i+1$ for every $i \geq 1$.

Definition 1.2 (Cohomological support loci)**.** Recall that for any coherent sheaf $\mathscr{F}$ on an abelian variety A, we consider for all $k \geq 0$ the cohomological support loci

$$V^k(\mathscr{F}) = \left\{P \in \operatorname{Pic}^0(A) \mid H^k(X,\mathscr{F}\otimes P) \neq 0\right\}$$

They are closed subsets of $\operatorname{Pic}^0(A)$, by the semi-continuity theorem for cohomology.

Definition 1.3 (Unipotent Vector Bundle)**.** A vector bundle on A is called unipotent if it has a filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_n = U$$

such that $U_i/U_{i-1} \simeq \mathscr{O}_A$ for all $i = 1,\ldots,n$. Note in particular that $\det U \simeq \mathscr{O}_A$.

That is we have a succesive extension

$$0 \to U_{i-1} \to U_i \to E_i \to 0$$

with

$$U_i/U_{i-1} \cong \mathcal{O}_A = E_i$$

Before going into the proof of theorem of Hacon-Pareschi-Popa, let us briefly introduce the Fourier-Mukai Transform.

Definition 1.4 (Fourier-Mukai Transform)**.** Let A be an Abelian variety. We will identify $\mathrm{Pic}^0(A)$ with the dual abelian variety $\widehat{A}$, and denote by P the normalized Poincaré bundle on $A \times \widehat{A}$. It induces the integral transforms

$$RS : D^b_{coh}(\mathscr{O}_A) \longrightarrow D^b_{coh}\left(\mathscr{O}_{\widehat{A}}\right), \quad RS\mathscr{F} = \mathbf{R}p_{2_*}\left(p_1^*\mathscr{F} \otimes P\right).$$

and

$$R\hat{S} : D^b_{coh}\left(\mathscr{O}_{\widehat{A}}\right) \longrightarrow D^b_{coh}(\mathscr{O}_A), \quad R\hat{S}\mathscr{G} = \mathbf{R}p_{1_*}\left(p_2^*\mathscr{G} \otimes P\right),$$

where $p_1 : A \times \hat{A} \to A$ and $p_2 : A \times \hat{A} \to \hat{A}$. (Note that in some of the references, the Fourier-Mukai transform $R\Psi_P = R\hat{S}$ and $R\Phi_P = RS$).

The following result of Mukai is fundamental.

Theorem 1.5. The Fourier-Mukai functors $(\mathbf{R}S, \mathbf{R}\hat{S})$ are e equivalences of derived categories, usually called the Fourier-Mukai transforms; moreover,

$$\mathbf{R}\hat{S} \circ \mathbf{R}S = (-1_A)^*[-g] \quad \text{and} \quad \mathbf{R}S \circ \mathbf{R}\hat{S} = \left(-1_{\widehat{A}}\right)^*[-g]$$

where $[-g]$ denotes shifting g places to the right.

We will give the proof of Mukai's theorem in the Series of notes A brief introduction to Fourier-Mukai Transform.

One can prove the following commutation relation of Fourier-Mukai Transform and Grothendieck Dual.

Proposition 1.6 (Commutation relation of Grothendieck dual and Fourier-Mukai Transform)**.**

$$D_A \circ RS = \left((-1_A)^* \circ RS \circ D_{\hat{A}}\right)[g].$$

Using the Fourier-Mukai, the following non-vanishing theorem comes for free. (Which shows to get non-vanihing of $h^0(\mathscr{F})$ has needs a lot of vanishing on h^i for $i > 0$).

Proposition 1.7. If $\mathcal{F} \in \mathrm{Coh}(A)$ is a non-zero coherent sheaf such that $h^i(\mathcal{F} \otimes P) = 0$ for all $i > 0$ and $P \in \hat{A}$, then $h^0(\mathcal{F}) \neq 0$.

PROOF IDEA 1.8. The proof is an immediate application of Mukai's theorem. Assume by contradiction that $h^0(\mathscr{F}) = 0$. Since Euler characteristic is independent of topological trivial line bundle, this means that for any $P, Q \in \mathrm{Pic}^0(A)$, the Euler characteristic remain the same. That is

$$\chi(\mathscr{F} \otimes P) = \chi(\mathscr{F}) = h^0(\mathscr{F}) = 0.$$

And consequently, we get $h^0(\mathscr{F}\otimes P)=0$ as well.

We then claim that $R\hat{S}(\mathscr{F})=0$ using Grauert base change and definition of Poincare line bundle as

$$R^i p_{\hat{A}*}(p_A^*\mathscr{F}\otimes\mathcal{P})(\hat{x})=H^i(A,(p_A^*\mathscr{F}\otimes\mathcal{P})|_{A\times\hat{x}})=H^i(A,(\mathscr{F}\otimes P_{\hat{x}}))=0,\ \forall i\geq 0,\hat{x}\in\mathrm{Pic}^0(A).$$

(Note that Grauert base change theorem holds since the cohomological dimension is constant 0).

Then the result easily follows by Mukai's theorem $RS\circ R\hat{S}(\mathscr{F})=(-1_A)^*\mathscr{F}[-g]$, that is $\mathscr{F}=0$ a contradiction.

Hacon-Pareschi-Popa proved the following equivalence characterization of GV sheaves.

Theorem 1.9 (Theorem of Hacon-Pareschi-Popa, [HPS18, Lemma 7.3 + Theorem 7.7])**.** Let A be an Ableian variety of dimension g over $\mathbb{C}$, $\mathscr{F}$ be a coherent sheaf on A. Then the following are equivalent.

(1) (Vanishing condition) $h^i(A,\mathscr{F}\otimes\hat{L}^\vee)=0$ for any $i>0$. Here L is a sufficient ample line bundle on $\hat{A}$, and $\hat{L}$ is the vector bundle on A that you get by Fourier-Mukai transform $\hat{L}=R^0S(L)$;

(2) (The Fourier-Mukai condition)

$$R\hat{S}(D_A(\mathscr{F}))=R^g\hat{S}(D_A(\mathscr{F})),$$

where D_A is the Grothendieck derived dual and $R\hat{S}$ denotes the Fourier-Mukai transform.

(3) (Codimension condition on cohomological support loci)

$$\mathrm{codim}_A R^i\hat{S}(\mathscr{F})\geq i,\quad \forall i>0$$

Proof. We first prove that (1) $\implies$ (2). By Grothendeick duality

$$D_k\left(R\Gamma\left(\mathcal{F}\otimes\hat{L}^\vee\right)\right)\cong R\Gamma\left(D_A\left(\mathcal{F}\otimes\hat{L}^\vee\right)\right).$$

By [Muk81], $\hat{L}$ is locally free, and therefore (using some property of RHom see e.g. Vakil's FOAG 2022.14.2.F)

$$R\Gamma\Big(D_A\Big(\mathcal{F}\otimes\hat{L}^\vee\Big)\Big)\cong R\Gamma\Big(D_A(\mathcal{F})\otimes\hat{L}\Big).$$

Then by definition of $\hat{L}$, there is an isomorphism

$$R\Gamma\left(D_A(\mathcal{F})\otimes\hat{L}\right)\cong R\Gamma\left(D_A(\mathcal{F})\otimes p_{A,*}\left(\mathcal{P}\otimes p_{\hat{A}}^*L\right)\right).$$

Therfore by projection formula

$$R\Gamma\left(D_A(\mathcal{F})\otimes p_{A,*}\left(\mathcal{P}\otimes p_{\hat{A}}^*L\right)\right)\cong R\Gamma\Big(\Big(p_A^*D_A(\mathcal{F})\otimes\mathcal{P}\otimes p_{\hat{A}}^*L\Big)\Big)\cong R\Gamma\left(R\hat{S}\left(D_A(\mathcal{F})\right)\otimes L\right).$$

In summary, we proved

$$D_k\left(R\Gamma\left(\mathcal{F}\otimes\hat{L}^\vee\right)\right)\cong R\Gamma\left(R\hat{S}\left(D_A(\mathcal{F})\right)\otimes L\right).$$

Easy to see $D_k\left(R\Gamma\left(\mathcal{F}\otimes\hat{L}^\vee\right)\right)$ is a sheaf in degree 0 if and only if $H^i\left(A,\mathcal{F}\otimes\hat{L}^\vee\right)=0$ for all $i>0$.

Since L is sufficient ample, we may assume that each $R^j\hat{S}\left(D_A(\mathcal{F})\right)\otimes L$ are globally generated with vanishing higher cohomologies. And therefore by Leray spectral sequence arguement

$$R^j\Gamma\left(R\hat{S}\left(D_A(\mathcal{F})\right)\otimes L\right)\cong H^0\left(R^j\hat{S}\left(D_A(\mathcal{F})\right)\otimes L\right).$$

So that $R\Gamma\Big(R\hat{S}(D_A(\mathcal{F}))\otimes L\Big)$ is a sheaf in degree 0 if and only if $H^0\left(R^j\hat{S}\left(D_A(\mathcal{F})\right)\otimes L\right)=0$ for all $j\neq 0$. Since $R^j\hat{S}\left(D_A(\mathcal{F})\right)\otimes L$ is global generated, this is equivalent to $R^j\hat{S}\left(D_A(\mathcal{F})\right)=0$ for all $j\neq 0$.

(2) $\implies$ (3). Let

$$\mathcal{G}=R^g\hat{S}\left(D_A(\mathcal{F})\right)$$

then by our assumption then

$$\mathcal{G}[g]=R\hat{S}(D_A(\mathcal{F})).$$

Then

$$R\hat{S}(\mathcal{F})=D_{\hat{A}}D_{\hat{A}}(R\hat{S}(\mathcal{F}))\quad\cong D_A\Big((-1_{\hat{A}})^*R\hat{S}(D_A(\mathcal{F}[g]))\Big)=(-1_{\hat{A}})^*D_A(\mathcal{G}).$$

So that

$$R^i\hat{S}(\mathcal{F})=\left(-1_{\hat{A}}\right)^*\mathcal{E}xt^i\left(\mathcal{G},\omega_X^\bullet\right).$$

Then by [HL97, Proposition 1.1.6], the result follows easily.

And finally (3) $\implies$ (1). Suppose that $\operatorname{codim}\operatorname{Supp}R^i\hat{S}(\mathcal{F})\geq i$ for all $i>0$, then $H^j\left(R^i\hat{S}(\mathcal{F})\otimes L^\vee\right)=0$ for all $i+j>g$ and any line bundle L. From the spectral sequence

$$H^j\left(R^i\hat{S}(\mathcal{F})\otimes L^\vee\right)\implies R^{i+j}\Gamma\left(R\hat{S}(\mathcal{F})\otimes L^\vee\right)$$

It follows that $R^l\Gamma\left(R\hat{S}(\mathcal{F})\otimes L^\vee\right)=0$ for $l>g$. We have

$$R^l\Gamma\left(R\hat{S}(\mathcal{F})\otimes L^\vee\right)=R^l\Gamma\left(Rp_{\hat{A},*}\left(p_A^*\mathcal{F}\otimes\mathcal{P}\right)\otimes L^\vee\right)=R^l\Gamma\left(p_A^*\mathcal{F}\otimes\mathcal{P}\otimes p_{\hat{A}}^*L^\vee\right)=R^l\Gamma\left(Rp_{A,*}\left(p_A^*\mathcal{F}\otimes\mathcal{P}\otimes p_{\hat{A}}^*L^\vee\right.\right.$$

It follows that $H^l\left(\mathcal{F}\otimes\hat{L}^\vee\right)=R^l\Gamma\left(\mathcal{F}\otimes\hat{L}^\vee\right)=R^{l+g}\Gamma\left(p_A^*\mathcal{F}\otimes\mathcal{P}\otimes p_{\hat{A}}^*L^\vee\right)=0,\forall l>0.$ □

For GV sheaves, the cohomological support loci satisfies certain order relation.

Theorem 1.10. Let $\mathscr{F}$ be a coherent sheaf on A. Then if $\mathscr{F}$ is a GV-sheaf, then

$$V^g(\mathscr{F})\subseteq\cdots\subseteq V^1(\mathscr{F})\subseteq V^0(\mathscr{F}).$$

PROOF IDEA 1.11. The idea is to prove that if $P\notin V^i(\mathcal{F})$ then $P\notin V^{i+1}(\mathcal{F})$ as well.

Suppose that $P\notin V^i(\mathcal{F})$, then $0=H^i(A,\mathcal{F}\otimes P)^\vee\cong H^{-i}\left(D_kR\Gamma(\mathcal{F}\otimes P)\right)\cong R^{-i}\Gamma\left(D_A(\mathcal{F})\otimes P^\vee\right)$. (where we assume that $i\geq 0$). Since for $-i$ the surjective automatically hold, by Grauert base change we know that

$$\varphi^{-i-1}(x_{P^\vee}):R^{-i-1}p_{A*}\Big(p_{\hat{A}}^*D_A(\mathcal{F})\otimes\mathcal{P}\Big)(x_{P^\vee})\to R^{-i-1}\Gamma\Big(D_A(\mathcal{F})\otimes P^\vee\Big)$$

is surjective. On the other hand, by condition (2) in Theorem 1.9, we know that for $i\geq 0$, $R^{-i-1}\hat{S}\left(D_A(\mathcal{F})\right)=0$. Therefore $H^{i+1}(A,\mathcal{F}\otimes P)^\vee\cong R^{i+1}\Gamma\left(D_A(\mathcal{F})\otimes P^\vee\right)=0$. In particular, $P\notin V^{i+1}(\mathcal{F})$. Thus $V^i(\mathcal{F})\supset V^{i+1}(\mathcal{F})$.

As an immediate consequence, we can get a non-vanishing criterion for GV sheaves.

Corollary 1.12 ([HPS18, Lemma 7.4])**.** Let A be an Abelian variety, $\mathscr{F}$ be a GV sheaf on A. Then

$$\mathscr{F} = 0 \iff V^0(\mathscr{F}) = \emptyset.$$

Proof. If $V^0(\mathscr{F}) = \emptyset$, then by the theorem we just proved, all the cohomological support loci $V^i(\mathscr{F}) = \emptyset$. Then $\mathscr{F} = 0$ has to be zero. Since by Grauert base change we have $RS(\mathscr{F}) = 0$. And therefore very similar as the proof of Lemma 1.7, using Mukai's theorem this will imply $\mathscr{F} = 0$. $\square$

Using the theorem of Hacon-Pareschi-Popa we can prove the direct image of pluricanonical sheaves are GV sheaves.

Proposition 1.13. Let $f : X \to A$ be a morphism from a smooth projective variety to an abelian variety. For every $m \in \mathbb{N}$ and $i \geq 1$, the sheaf $\mathscr{F}_m = R^i f_* \omega_X^{\otimes m}$ is a GV-sheaf on A.

PROOF IDEA 1.14. The idea is to use the vanishing condition we have just proved, and then the result will follow easily from Kollár's vanishing via the base change under isogeny $\varphi_L : \hat{A} \to A$. To be more concrete, To prove that $\mathscr{F}_m = R^i f_* \omega_X^{\otimes m}$ is a GV sheaf, we apply the 1st criterion in the Theorem 1.9, that is we need to prove

$$h^j\left(R^i f_* \omega_X^{\otimes m} \otimes \hat{L}^\vee\right) = 0$$

for $j > 0$ and L sufficient ample. And we try to prove it using Kollár's vanishing. And to do this, we will take a base change via the isogeny induced by L.

$$\begin{tikzcd} \hat{X} \arrow[r, "\psi"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ \hat{A} \arrow[r, "\varphi_L"'] & A \end{tikzcd}$$

Then we try to prove the vanishing of

$$H^j\left(\widehat{A}, R^i h_* \omega_{\hat{X}}^{\otimes m} \otimes \varphi_L^* \hat{L}^\vee\right) = 0,$$

Since then $H^j\left(\widehat{A}, R^i f_* \omega_X^{\otimes m} \otimes \hat{L}^\vee\right)$ is a direct sum of $H^j\left(\widehat{A}, h_* \omega_{\hat{X}}^{\otimes m} \otimes \varphi_L^* \hat{L}^\vee\right)$. So that it admits vanishing as well. Since by Mukai's lemma, we have

$$H^j\left(\widehat{A}, R^i h_* \omega_{\hat{X}}^{\otimes m} \otimes \varphi_L^* \hat{L}^\vee\right) = \bigoplus_{h^0(L)} H^j\left(\widehat{A}, R^i h_* \omega_{\hat{X}}^{\otimes m} \otimes L\right)$$

and each copy satisfies the condition of (generalized) Kollár's vanishing (by [PS14, Corollary 2.9.]), which complete the proof of the result.

1.2 Green-Lazarsfeld generic vanishing theorem

Let us first recall the Green–Lazarsfeld–Simpson generic vanishing theorem. We will begin by introducing Green and Lazarsfeld's original proof, and then present Hacon's proof of the Green–Lazarsfeld theorem using the Fourier–Mukai transform.

Theorem 1.15. If $f : X \to A$ is a morphism from a smooth projective variety to an abelian variety, then for any $j, k \geq 0$ we have
(1) $\operatorname{codim}_{\operatorname{Pic}^0(A)} V^k \left(R^j f_* \omega_X\right) \geq k$;
(2) Every irreducible component of $V^k \left(R^j f_* \omega_X\right)$ is a translate of an abelian subvariety of A by a point of finite order.

For our interest, we need the generic vanishing for direct image of pluricanonical sheaves.

Theorem 1.16 ([HPS18, Theorem 4.1])**.** Let X be a smooth projective variety. For each $m \in \mathbb{N}$, the locus

$$\left\{P \in \operatorname{Pic}^0(X) \mid H^0\left(X, \omega_X^{\otimes m} \otimes P\right) \neq 0\right\} \subseteq \operatorname{Pic}^0(X)$$

is a finite union of abelian subvarieties translated by points of finite order.

This theorem implies that $V^0\left(A, \mathscr{F}_m\right)$ (for $\mathscr{F}_m = f_* \omega_X^{\otimes m}$) is also a finite union of abelian subvarieties translated by points of finite order. where

$$\begin{aligned} V^0\left(A, \mathscr{F}_m\right) &= \left\{P \in \operatorname{Pic}^0(A) \mid H^0\left(A, \mathscr{F}_m \otimes P\right) \neq 0\right\} \\ &= \left\{P \in \operatorname{Pic}^0(A) \mid H^0\left(X, \omega_X^{\otimes m} \otimes f^* P\right) \neq 0\right\} \subseteq \operatorname{Pic}^0(A). \end{aligned}$$

PROOF IDEA 1.17.

Using the result we just developed above, we can prove the following criterion for direct image of pluricanonical sheaves to be a indecomposable unipotent vector bundle.

Definition 1.18. Let E be a vector bundle on complex manifold M, we say it's unipotent if

Theorem 1.19 ([HPS18, Corollary 4.3])**.** Let X be a smooth projective variety with $\kappa(X) = 0$, and let $f : X \to A$ be an algebraic fiber space over an abelian variety.

If $H^0\left(X, \omega_X^{\otimes m}\right) \neq 0$ for some $m \in \mathbb{N}$, then the coherent sheaf $\mathscr{F}_m$ is an indecomposable unipotent vector bundle.

PROOF IDEA 1.20.

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