Contraction in Kähler MMP reading notes Spring 2024

Note 3 — 28, 09, 2025 (draft version)

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Contents

1 Overview

The aim of this note is to introduce the contraction theorems in the Kähler minimal model program.

2 Fujiki's blowing down theoren and Grauert contraction theorem

Theorem 1. Let X be a reduced complex space, A an effective **Cartier** divisor. Assume $f : A \to A'$ be a proper surjective morphism to another complex space A'. Such that

[1] **WARNING:** (1) Round 1: sketch notes; (2) Round 2: more details but contains errors; (3) Round 3: correct version but not smooth to read; (4) Round 4: close to the published version.
To ensure a pleasant reading experience. Please read my notes from ROUND ≥ 4.

1. The restriction $\mathcal{O}_A(-A)$ is f-ample,
2. The higher direct image $R^1 f_* \mathcal{O}_X(-jA) = 0$ for all $j > 0$.

Then there exists a contraction morphism $F : X \to X'$ such that $F|_A = f$. Assume $\mathcal{S}_{X,A,f}$ is defined by the exact sequence

$$0 \to \mathcal{S}_{X,A,f} \to \mathcal{O}_X \to \mathcal{O}_A/\mathrm{im}(f^*\mathcal{O}_{A'} \to \mathcal{O}_A),$$

then $F_*(\mathcal{S}_{X,A,f}) = \mathcal{O}_X$.

3 Kollár-Mori's extension of contraction theorem

Theorem 2 ([KM92, Theorem 11.4])**.**

The proof of Kollár-Mori 11.4 is very likely motivated by Horikawa's paper, the proof constitutes two parts:

(1) First showing that obstruction of deformation of the contraction morphism lies in the $R^1 f_* \mathcal{O}_X$ (in particular the vanishing of 1st direct image implies unobstructness of the deformation),

(2) Second using the Douady space argument showing that extension from formal neighborhood to actual neighborhood is unconditional.

We will sketch the idea of the proof, and some issues for discussion are highlighted in red boxes.

3.1 Step 1. Infinitesimal level extension (Formal extension)

Kollar Mori did not prove this part. Our proof is motivated by [DH24, Theorem 5.8], while Horikawa's approach uses some Cech process.

First, we have the following fundamental exact sequence

$$0 \to \mathcal{O}_X(-jX_0) \to \mathcal{O}_X \to \mathcal{O}_{jX_0} \to 0, \tag{1}$$

which will then induce

$$\boxed{0 \to \mathcal{O}_{X_0}(-jX_0) \to \mathcal{O}_{(j+1)X_0} \to \mathcal{O}_{jX_0} \to 0.} \tag{2}$$

> Remark: For (2) to be exact, one requires X_0 to be Cartier correct (I mean $\mathbb{Q}$-Cartier condition is insufficiet)?
>
> On the other hand (1) holds as long as X_0 is a Weil divisor, and does not require any singularity assumption?

So that taking direct image on (2) will induce a long exact sequence

$$0 \to f_*(\mathcal{O}_{X_0}(-jX_0)) \to f_*\mathcal{O}_{(j+1)X_0} \to f_*\mathcal{O}_{jX_0} \to R^1 f_*\mathcal{O}_{X_0}(-jX_0) \to R^1 f_*\mathcal{O}_{(j+1)X_0} \to \cdots , \tag{3}$$

Similarly to the proof of [DH24, Theorem 5.8], we need to show that certain 1st direct image vanishes (for [DH24], the positivity and singularity conditions are needed in order to apply the KV vanishing theorem). For our setting, we need to prove

$$R^1 f_* \mathcal{O}_{X_0}(-jX_0) = 0,$$

(which is expected to follows from the assumption $R^1 f_* \mathcal{O}_{X_0} = 0$, to do this my idea is to apply projection formula for higher direct image

$$R^1 f_*(\mathcal{O}_{X_0}(-jX_0)) = R^1 f_*(\mathcal{O}_{X_0} \otimes \mathcal{O}_X(-jX_0)) = R^1 f_*(\mathcal{O}_{X_0} \otimes f^*\mathcal{L}) = R^1 f_* \mathcal{O}_{X_0} \otimes \mathcal{L} = 0.$$

Here is the gap that I cannot fix.

QUESTION 1: How to show that

$$\mathcal{O}_X(-jX_0) = f^*\mathcal{L}?$$

After that we can inductively define the $(j+1)$-th order infinitesimal extension from j-th order, by defining

$$Y_0^{(j+1)} = \mathrm{Specan}(f_* \mathcal{O}_{(j+1)X_0}),$$

which will induce the infinitesimal thickening of the morphism (since the construction here is functorial).

$$\begin{array}{ccc} X_0^{(n+1)} & \xrightarrow{f^{(n+1)}} & Y_0^{(n+1)} \\ \uparrow & & \uparrow \\ X_0^{(n)} & \xrightarrow{f^{(n)}} & Y_0^{(n)} \end{array}$$

3.2 Step 2. Douady space argument (Effectiveness)

From formal neighborhood to analytic neighborhood, typically do not need to impose additional conditions. Our proof in this section is motivated by [?], rewrite it in a more clear manner by the help of the book [BM19] (last chapter).

Remark 3. I am trying to give a new proof that replace Douady space by Chow-Barlet cycle space, as cycle space is more powerful when dealing with non-reduced structure. The proof below is still in the Douady space setting.

The idea is to use the correspondence between holomorphic morphism and graph of that morphism, so that it is possible to convert the deformation of holomorphic map problem into deformation of complex subspace problem. Thus, the Douady space can be used to "parameterize" the family of holomorphic maps. And the holomorphic section of the relative Douady space **glue** fiberwise holomorphic map together.

Let

$$\pi' : \mathcal{Y} \to \mathrm{Def}(Y_0)$$

be the Kuranishi family of deformation of Y_0 (here $\mathrm{Def}(Y_0)$ is some complex analytic space). It can be very singular, but there exists some holomorphic curve $\phi : U \to \mathrm{Def}(Y_0)$ passing through $0 \in \mathrm{Def}(Y_0)$, for some $U \subset S$. And therefore, we have the following pull back diagram.

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "h"] & \mathcal{Y} \arrow[d, "\pi'"] \\ U \arrow[r, "\phi"] & \mathrm{Def}(Y_0) \end{tikzcd}$$

Let $h \times g : Y \times X \to U \times U$ be the product family. We can define the relative Douady space $\mathcal{D} = \mathrm{Douady}(Y \times X/U \times U)$ that parameterizes the complex subspaces in the fibers of $h \times g$. Douady (and Pourcin for the relative version) proved that $\mathcal{D}$ admits some complex space structure, and there exists a holomorphic map $\Phi : \mathcal{D} \to U \times U$.

Furthermore, the same proof as the Hom scheme is Zariski open in the Hilbert scheme, there exists an open subset $\mathcal{D}' \subset \mathcal{D}$ that parametrizes the graph of the morphism (to be more precise, if we pick a point $[\Gamma_{s,t}] \in \mathcal{D}'$, then it represents a complex subspace $\Gamma_{s,t} \subset X_s \times Y_t$ such that the natural projection $q_1 : \Gamma_{s,t} \to X_s$ is an isomorphism.

$$\begin{tikzcd} & \Gamma_{s,t} \arrow[dl, "q_1"'] \arrow[d, hook] \arrow[dr, "q_2"] & \\ X_s & X_s \times Y_t \arrow[l] \arrow[r] & Y_t \end{tikzcd}$$

By definition, the representative $[\Gamma_f]$ of the graph of the morphism $f : X_0 \to Y_0$ lies on the fiber $\Phi^{-1}(0,0)$ and $[\Gamma_f] \in \mathcal{D}'$. The collection of infinitesimal thickenings of the morphism $\{f^{(n)}\}_{n\in\mathbb{N}}$ (we constructed in Step 1) defines a formal section around $(0,0) \in U \times U$, as the picture shows below.

$$
\begin{tikzcd}
\widetilde{y}_0 \in \widetilde{E}_0 \arrow[r, hook] \arrow[d, "p"] & \widetilde{S}_0 \arrow[r, hook] \arrow[d, "p"] & \widetilde{X} \arrow[d, "p"] \\
\widehat{y}_0 \in E_0 \arrow[r, hook] \arrow[d, "\pi"] & \widehat{S}_0 \arrow[r, hook] \arrow[d, "\pi"] & \widehat{X} \arrow[d, "\pi"] \\
y_0 \in C_0 \arrow[r, hook] & S_0 \arrow[r, hook] & X
\end{tikzcd}
$$

Figure 1: Formal section of the relative Doaudy space.

Thus, by Artin [Art70], if furthur shrink U, there exists a holomorphic section $\sigma : U \times U \to \mathcal{D}'$ around some analytic neighborhood $0 \in U \times U$, and we restrict the section on the diagonal $\sigma : U \to \mathcal{D}'$.

Next, we claim that the holomorphic section $\sigma : U \to \mathcal{D}'$ will define a holomorphic map $X \to Y/U$ which is an extension of $f : X_0 \to Y_0$.

To see this, note that the section $\sigma : U \to \mathcal{D}'$ gives an analytic family of subspace $(\Gamma_s)_{s\in U}$ in $X \times Y$. By [BM19, Theorem 4.3.3], the graph

$$G_U = \{(x, y, s) \in (X \times Y) \times U \mid (x, y) \in |\Gamma_s|\}.$$

admits a complex space structure (Barlet proves the result in Barlet-Chow Cycle space setting, it should be true for Douady space as well). (This complex space structure is important, as it glues the fiber together).

Therefore, we have the natural projection (in the category of complex spaces) shown below.

$$
\begin{tikzcd}
 & G_U = \bigcup_{s\in U} \Gamma_s \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[dd, "p_3"] & \\
X \arrow[dr] & & Y \arrow[dl] \\
 & U &
\end{tikzcd}
$$

Since, by the definition of $\Gamma_s \in \mathcal{D}'$, the restriction $p_1|_{\Gamma_s}$ is isomorphism for all $s \in U$, so that p_1 is isomorphism and therefore there exists a holomorphic map

$$f = p_2 \circ p_1^{-1} : X \to Y,$$

that commute the diagram, which completes the proof.

3.3 Step 3. Applications of Theorem 1 to MMP steps

We finally apply Theorem 1 to the MMP steps, the proof is motivated by [**?**, Theorem 4.1]).

We try to show that the following condition holds (which appears in my paper Corollary 2.12).

Theorem 4. Assume that we have a contraction morphism $f : X_0 \to Y_0$ from a projective variety X_0 with canonical singularity,

(1) If $R^1 f_* \mathcal{O}_X = 0$, then the extension $f : X \to Y$ is a contraction morphism (say $f_* \mathcal{O}_X = \mathcal{O}_Y$).

(2) If $f : X_0 \to Y_0$ is a $(K_{X_0} + \Delta_0 + \beta_0)$-divisorial or flipping contraction (in an MMP step), then there exists an extension $f : X \to Y$ which is a fiberwise bimeromorphic contraction morphism.

(3) If on the central fiber, $f : X_0 \to Y_0$ is a divisorial or fiber type contraction. So it is for the nearby fibers, however, the flipping contraction on the central fiber may restrict to identity for the nearby fibers (Totaro's example).

Proof of the contraction extend to contraction morphism on some neighborhood of X_0. First by construction at step 1, we know that on any n-th order infinitesimal thickening, we have

$$\mathcal{O}_{Y,y} \to (f_* \mathcal{O}_X)_y,$$

is isomorphism. Then we can apply the formal function theorem so that

QUESTION 3: Why the deformation of contraction morphism is still contraction morphism?

On the infinitesimal level, we define the structure sheaf on the target to be a direct image say

$$\mathcal{O}_{Y_0^{(j+1)}} := f_* \mathcal{O}_{(j+1)X_0},$$

however, it's not clear for me why from formal to actual extension is still a contraction morphism, thus I cannot actually prove Theorem 3 (1).

QUESTION 4: (Relative MMP v.s. Absolute MMP v.s. fiberwise MMP)
Could I ask, what is the difference between these 3 concepts.

QUESTION 5: (Analytic flip v.s. Algebraic flip)
Could I ask the new phenomenon that appears in the analytic flip? That is ananlytic flip may flip one component while algebra flip may flip several components.

Proof of fiberwise bimeromorphic. Take (1) as a grantee and prove (2). By Garf-Kirschner's decomposition theorem, on a projective variety with rational singularity, we can represent $-(K_{X_0} + \Delta_0 + \beta_0)$ by some ample divisor $-(K_{X_0} + \Delta_0 + B_0)$. Therefore, by the Kodaira vanishing theorem $R^1 f_* \mathcal{O}_{X_0} = 0$. And therefore by Theorem **??**, there exists some extension $f : X \to Y$. Since the deformation is flat morphism, we have $\dim X = \dim Y$. Since contraction in the MMP step preserve the normal condition, we have Y_0 is normal. Since being normal is an open condition, so that Y is still normal. Since we assune Theorem 3 (1), we have $f : X \to Y$ is a contraction morphism, therefore by Zariski main theorem (or Nakayama II.2.12) we have $f : X \to Y$ is actually a bimeromorphic morphism, in particular it's fiberwise bimeromorphic.

Proof of contraction type being preserved. The same as [**?**, Theorem 4.1], use semi-continuity. See also [KM92, Theorem 12.3.1]. I copy the statement of KM92 below.

Let $g : X \to S$ be a proper flat morphism of complex spaces. Assume that for some $0 \in S$ the fiber X_0 is a projective variety with only $\mathbb{Q}$ factorial rational singularities, $\dim X_0 \geq 3$. Let $f_0 : X_0 \to Y_0$ be the contraction of an extremal ray $C_0 \subset X_0$. By Theorem **??**, there is a proper flat morphism $Y \to S$ and a factorisation

$$g : X \xrightarrow{f} Y \to S$$

Then there is an open neighborhood $0 \in U \subset S$ such that if f_0 contracts a subset of codimension at least two (resp. contracts a divisor, resp. is a fiber space of generic relative dimension k) then f_s contracts a subset of codimension at least two (which may be empty) (resp. contracts a divisor; resp. is a fiberspace of generic relative dimension k) if $s \in U$.

Proof. □

4 Das-Hacon's contraction theorem for generalized Kähler PLT pairs

In this section we will introduce the contraction theorem of Das and Hacon.

Theorem 5 ([DH24, Theorem 5.8])**.** Let $(X, S+B+\boldsymbol{\beta})$ be a generalized PLT pair with $\lfloor S+B \rfloor = S$ irreducible, such that the following condition holds

1. S is a $\mathbb{Q}$-Cartier divisor,
2. There exist a contraction morphism $f : S \to T$ such that $-S|_S$ is f-ample,
3. the restriction of the canonical class $-(K_X + S + B + \boldsymbol{\beta})|_S$ is Kähler over T.

Then we can find a (bimeromorphic) contraction morphism $F : X \to Y$, with $F|_S = f$.

Remark 6. Let us briefly sketh the idea of the proof. Compared with the Fujiki blowing down theorem, we do not have Cartier condition on S and we do not have the vanishing of higher direct image (of conormal sheaf) condition in the statement.

Since S is $\mathbf{Q}$-Cartier, there exists a $r \in \mathbf{Z}$ such that rS is Cartier. We first show that there exists an extension on the infinitesimal thickending rS, with positivities preserved under thickening. Second we apply the Serre vanishing and change of index trick showing that the higher direct image of the conormal sheaf vanish. Then apply the Fujiki blowing down theorem yield the result. The major difficulty of the proof lies in showing the obstruction of infinitesimal extension vanish. To do this, we need adjunction for the generalized PLT pairs and Kawamata-Viehweg vanishing for the complex analytic spaces.

Proof. Assume $r \in \mathbb{Z}_+$ to be an integer, such that rS is a Cartier divisor. We have the following short exact sequence of sheaf of $\mathcal{O}_X$-module for thickening of S,

$$0 \to \mathcal{O}_S(-jS) \to \mathcal{O}_{(j+1)S} \to \mathcal{O}_{jS} \to 0.$$

On the other we have the quotient exact sequence

$$0 \to \mathcal{O}_X(-(j+1)S) \to \mathcal{O}_X(-jS) \to \mathcal{E}_j \to 0,$$

here $\mathcal{E}_j$ is a rank 1 reflexive sheaf supported on S.

Combine this 2 exact sequences, we get □

5 Applications of Analytic Contraction Theorems

5.1 Applications in Kähler minimal model program

Das-Hacon-Păun use Theorem **??** prove the semi-stable MMP for Kähler 4 folds over a curve.

Theorem 7 ([DHP24, Theorem 8.12])**.** Let $f : (X, B) \to T$ be a semi-stable klt pair of dimension 4 and $W \subset T$ a compact subset. If $(X/T; W)$ is $\mathbb{Q}$-factorial and $K_X + B$ is effective over W, then we can run the $(K_X + B)$-MMP over a neighborhood of W in T which ends with a minimal model over W.

5.2 Applications in algebraic approximation problems

Lin adopt Theorem **??** in the proof of the following result.

Theorem 8 ([**?**, Proposition 1.7])**.** Let X' be a normal compact complex variety and let X be a compact complex variety with at worst rational singularities that is bimeromorphic to X'. Assume that X' has a Y-locally trivial algebraic approximation for every subvariety $Y \subset X'$ satisfying $\dim Y \leq \dim X' - 2$. Then X has an algebraic approximation.

Remark 9. Without going too much into detail, the existence of the Y-locally trivial condition (for $\dim Y \leq \dim X' - 2$) in the theorem implies that an algebraic approximation of X' can induce an algebraic approximation of the higher model X'', provided that the bimeromorphic map

$$X'' \to X',$$

is isomorphism in codimension 1 on the target.

Proof. Let

$$\tau : X' \dashrightarrow X$$

be the bimeromorphic map in the assumption. The idea is to take the common resolution

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\nu"'] \arrow[dr, "\eta"] & \\ X' \arrow[rr, dashed, "\tau"] & & X \end{tikzcd}$$

We then try to apply the target stable result on ν, and source stable result on η (that is deformation of X induce deformation of ν and deformation of $\tilde{X}$ induce deformation of η).

By [?, Lemma 2.7], algebraic approximation on X' will induce an algebraic approximation on $\tilde{X}$.

Since the target X has rational singularity, the direct image $R^1\eta_*\mathcal{O}_{\tilde{X}} = 0$. And therefore, by Theorem ??, the algebraic approximation on $\tilde{X}$ descends to the algebraic approximation on X. $\square$

5.3 Applications in deformation of Calabi-Yau problem

Kollár use the Theorem ??, proved the following result (which says that one always has extension of contraction except that the target is uniruled or in some Abelian variety case).

Theorem 10 ([?, Theorem 33]). Let X be a projective variety with rational singularities, Y a normal variety, and $g : X \to Y$ a surjective morphism with connected fibers. Assume that Y is not uniruled. Then at least one of the following holds:

(1) Every small deformation of X gives a deformation of $(g : X \to Y)$.

(2) There is a quasi-étale cover $\tilde{Y} \to Y$, a normal variety Z, and positive dimensional Abelian varieties A_1, A_2 such that the lifted morphism $\tilde{g} : \tilde{X} := X \times_Y \tilde{Y} \to \tilde{Y}$ factors as

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d, "\tilde{g}"'] & Z \times A_1 \times A_2 \arrow[d] \\ \tilde{Y} \arrow[r, "\cong"'] & Z \times A_1 \end{tikzcd}$$

Kollár expects that Theorem ?? is useful in solving the following conjecture about deformation of Calabi-Yau fibration.

Conjecture 11 ([?, Theorem 52]). Let $g_0 : (X_0, \Delta_0) \to B_0$ be a relatively minimal log CalabiYau fiber space where (X_0, Δ_0) is a proper klt pair and $H^2(X_0, O_{X_0}) = 0$.

Let (X, Δ) be a klt pair and $h : (X, \Delta) \to (0 \in S)$ a flat proper morphism whose central fiber is (X_0, Δ_0).

Then, after passing to an analytic or étale neighborhood of $0 \in S$, there is a proper, flat morphism $B \to (0 \in S)$ whose central fiber is B_0 such that g_0 extends to a log Calabi-Yau fiber space $g : (X, \Delta) \to B$.

5.4 Applications in the invariance of plurigenera problems

Levine-Cao-Păun's invariance of plurigenera adopt a very similar strategy as Theorem ??. They first prove that infinitesimal extensions of the pluricanonical section exist when certain $\bar{\partial}$-equation holds and satisfies some L^2-estimate.

Theorem 12. Let $f : X \to \Delta$ be a smooth family of Kähler manifolds,

$$s \in H^0(X_0^{(k)}, \mathcal{L}|_{X_0^{(k)}})$$

with $\mathcal{L} = (m-1)K_X$, which admits a C^∞ extension s_k so that if we write $\bar{\partial} s_k = t^{k+1}\Lambda_k$, the integral

$$\int_X \left|\frac{\Lambda_k}{dt}\right|^2 e^{-(1-\varepsilon)\varphi_L} dV < \infty,$$

Converges for any positive $\varepsilon > 0$. Then there exists a section $\widehat{s}$ of $\mathcal{L}|_{X_0^{(k+1)}}$ such that $s = \pi_k(\widehat{s})$.

Remark 13. Compared with the 1st direct image vanishing condition in Theorem **??**, they imposed some L2 condition. (I am not pretty sure it's parallel condition or not).

Remark 14. Levine proved a similar result that a smooth pluricanonical section on the central fiber admits infinitesimal thikening. Cao-Păun's gave an alternative of Levine's statement.

The extension of the pluricanonical section from the formal neighborhood of the central fiber to an actual analytic neighborhood is unconditional (which is expected to be true compared to our proof of Theorem **??** in Step 2).

Theorem 15. Let $f : X \to \Delta$ be a smooth family of compact complex manifolds. Assume that the pluricanonical section on X_0 admits infinitesimal extension, then it will admit some extension on an analytic neighborhood of 0.

The next topic we will discuss is positivties in Käher families.

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