General type locus and Moishezon locus Summer 2025

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1 Overview

The aim of this note is to study the general type locus, Moishezon locus and projective locus (see Definition 2.1). The motivation of this topic comes from the following observation on distribution of polarized (projective) K3 surfaces in the universal family of (marked) complex K3 surfaces.

Let $X \to D^{20}$ be a universal family of K3 surfaces. A smooth, compact surface is Moishezon iff it is projective. The projective fibers of $X \to D^{20}$ correspond to a countable union of hypersurfaces $H_{2g} \subset D^{20}$. As we can see from this example, the projective locus (which corresponds to projective K3 surfaces) is a countable union of the hypersurface in the moduli space D^{20}.

Is natural to ask, how the general type locus, Moishezon locus and projective locus distribute in a proper morphism between normal analytic spaces?

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We first give the definitions for the very big locus, Moishezon locus, general type locus, and the projetive locus.

Definition 2.1 (Very big locus, general type locus, Moishezon locus, [Kol22a, Definition 18])**.** Let $g : X \to S$ be a proper morphism of normal analytic spaces and L a line bundle on X. Set

1. $\mathrm{VB}_S(L) := \{s \in S : L_s \text{ is very big on } X_s\} \subset S$,
2. $\mathrm{GT}_S(X) := \{s \in S : X_s \text{ is of general type }\} \subset S$,

3. $\mathrm{MO}_S(X) := \{s \in S : X_s \text{ is Moishezon }\} \subset S$,
4. $\mathrm{PR}_S(X) := \{s \in S : X_s \text{ is projective }\} \subset S$.

Here, very big means the place $s \in S$ that $X_s \dashrightarrow \mathrm{Proj}_S(g_*L_s)$ is bimeromorphic onto its closure of the image.

Definition 2.2 (Locus V that satisfies alternating property)**.** Let $g : X \to S$ be a proper morphism of normal analytic spaces, we say the locus

$$V := \{s \in S \mid X_s \text{ admits property } P\},$$

satisfies *the alternating property over* S if $V \subset S$ is

(1) either nowhere dense (in the analytic Zariski topology),
(2) or it contains a dense open subset of S.

Remark 2.3. Note that the property that $V \subset S$ satisfies the alternating property does not care about the information on the special fibers (the alternating property is some generic property). Due to the following observation.

If there exists some non-empty Zariski open subset $S' \subset S$, such that V satisfies the alternating property on S'.

(a) If V is nowhere dense in S', then V is also nowhere dense in S (easy to check by definition),

(b) If V is dense in S', then V is also dense in S. (by the transitive property of dense subsets).

In other words, if V satisfies alternating property on some non-empty Zariski open subset S', then V also satisfies the alternating property on S.

We first show that the very big locus satisfies an alternating property; that is, it is either nowhere dense or contains a dense open subset.

Theorem 2.4 (Alternating property for very big locus, [Kol22a, Lemma 19])**.**
Let $g : X \to S$ be a proper morphism of normal irreducible analytic spaces(and therefore S is integral) and L a line bundle on X. Then $\mathrm{VB}_S(L) \subset S$ is

(1) either nowhere dense (in the analytic Zariski topology),
(2) or it contains a dense open subset of S, and $g : X \to S$ is Moishezon.

Proof. By passing to an open subset of S, we may assume that g is flat, g_*L is locally free and commutes with restriction to fibers. We get a meromorphic map $\phi : X \dashrightarrow \mathbb{P}_S(g_*L)$. There is thus a smooth, bimeromorphic model $\pi : X' \to X$ such that $\phi \circ \pi : X' \to \mathbb{P}_S(g_*L)$ is a morphism.

After replacing X by X' and again passing to an open subset of S, we may assume that g is flat, g_*L is locally free, commutes with restriction to fibers, and $\phi : X \to \mathbb{P}_S(g_*L)$ is a morphism. Let $Y \subset \mathbb{P}_S(g_*L)$ denote its image and $W \subset X$ the Zariski closed set of points where $\pi : X \to Y$ is not smooth. Set $Y^\circ := Y \backslash \phi(W)$ and $X^\circ := X \backslash \phi^{-1}(\phi(W))$. The restriction $\phi^\circ : X^\circ \to Y^\circ$ is then smooth and proper. We divide the problem into two cases:

Case 1. If we assume that the set of points

$$W = \{y \in Y \mid \phi^{-1}(y) \text{ is single points}\} \subset Y,$$

is not dense in Y. That is $\bar{W} \subset Y$ is a proper analytic subset. Then clearly this will imply that

$$S' = \{s \in S \mid Y_s \cap W \subset Y_s \text{ is dense in } Y_s\},$$

is a nowhere dense subset.

Case 2. If we assume that $\phi^{-1}(y)$ is a single point for a dense set in Y, then for a dense set in Y°. Since ϕ° is smooth and proper, it is then an isomorphism (since etale morphism in degree 1 is an isomorphism). Thus ϕ is bimeromorphic on every irreducible fiber that has a nonempty intersection with X°. Since image of dense subset is dense, this will imply that very big locus contains a dense open subset of S. And by the definition , $g : X \to S$ is actually a Moishezon morphism. □

As a direct consequence (combined with the classical birational boundedness result by Hacon and Mckernan) we have the general type locus which also admits alternating property.

Theorem 2.5 (The alternating property for general type locus, [Kol22a, Corollary 20]). Let $g : X \to S$ be a proper morphism of normal, irreducible analytic spaces. Then

$$\mathrm{GT}_S(X) = \{s \in S \mid X_s \text{ is of general type}\},$$

(1) either nowhere dense (in the analytic Zariski topology), (2) or it contains a dense open subset of S, and $g : X \to S$ is Moishezon

Proof. Using resolution of singularities, we may assume that X is smooth. By passing to an open subset of S, we may also assume that S and g are smooth. By [HM06] there is an m (depending only on $\dim X_s$) such that $|mK_{X_s}|$ is very big whenever X_s is of general type. Thus Lemma 2.4 applies to $L = mK_X$. □

3 The alternating property of the Moishezon locus

In this section, we will prove that Moishezon locus also admits alternating property. Before proving Theorem 21. Let us first recall the basic idea that being used in [RT22], which has being rephrased as the Theorem 3.2 of Kollár.

Theorem 3.1 (Uncoutnable many fibers are Moishezon with deformation invariance of Hodge number implies the morphism is Moishezon, see [RT22] Proposition 3.15).

Let $\pi : \mathcal{X} \to \Delta$ be a one-parameter degeneration. (1) Assume that there exists an uncountable subset B of Δ such that for each $t \in B$, the fiber X_t admits a line bundle L_t with the property that $c_1(L_t)$ comes from the restriction to X_t of some cohomology class in $H^2(\mathcal{X}, \mathbb{Z})$. (2) Assume further that the Hodge number $h^{0,2}(X_t) := h^1(X_t, \mathcal{O}_{X_t})$ is independent of $t \in \Delta$ (the original theorem require only Hodge (0,1) deformation invariance, we prove a weaker version).

Proof. Apply the sheaf exponential exact sequence so that

$$\begin{tikzcd}
{} \arrow[r] & H^1(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}}) \arrow[r] \arrow[d] & H^2(\mathcal{X}, \mathbb{Z}) \arrow[r, "e_2"] \arrow[d] & H^2(\mathcal{X}, \mathcal{O}_X) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H^1(X_s, \mathcal{O}^*_{X_s}) \arrow[r] & H^2(X_s, \mathbb{Z}) \arrow[r, "e_2"'] & H^2(X_s, \mathcal{O}_{X_s}) \arrow[r] & {}
\end{tikzcd}$$

Observe that

$$H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \cong R^2\pi_*\mathcal{O}_X(\Delta),\ H^2(X_s, \mathcal{O}_{X_s}) \cong R^2\pi_*\mathcal{O}_{\mathcal{X}}(s)$$

Indeed

(1) By Cartan B. we have

$$H^p(S, R^q\pi_*\mathcal{O}_X) = 0,\ p > 0$$

the Leray spectral sequence arguement thus implies the first isomorphism, (2) Since we assume the cohomological constant of $h^{0,2}$, by Grauert base change theorem it will imply the second isomorphism.

Thus the commutative diagram becomes

$$\begin{tikzcd}
 & & & H^0(\mathcal{X}, R^2\pi_*\mathcal{O}_{\mathcal{X}}) \arrow[d, "\cong"] & \\
\arrow[r] & H^1(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}}) \arrow[r] \arrow[d] & H^2(\mathcal{X}, \mathbb{Z}) \arrow[r] \arrow[d] & H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \arrow[r] \arrow[d] & {} \\
\arrow[r] & H^1(X_s, \mathcal{O}^*_{X_s}) \arrow[r] & H^2(X_s, \mathbb{Z}) \arrow[r] & H^2(X_s, \mathcal{O}_{X_s}) \arrow[r] \arrow[d, "\cong"] & {} \\
 & & & R^2\pi_*\mathcal{O}_{\mathcal{X}}(s) &
\end{tikzcd}$$

Where we have the evaluation $\mathrm{ev}_s : H^0(\mathcal{X}, R^2\pi_*\mathcal{O}_{\mathcal{X}}) \to R^2\pi_*\mathcal{O}_{\mathcal{X}}(s)$ in the diagram above.

The idea is to find a cohomology class $c \in H^2(X_s, \mathbb{Z})$ by the simply connectness of Δ it will lift it to $c \in H^2(\mathcal{X}, \mathbb{Z})$, if we can prove the vanishing of $e_2(c) \in H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ then by the exactness of the sequence we can find some global line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$.

Observe that the cohomology group $H^2(X_s, \mathbb{Z}) \cong H^2(\mathcal{X}, \mathbb{Z})$ by Ehresmann's theorem, and this $\mathbb{Z}$ coefficient cohomology group can only have counable many elements, taking uncountble many L_t, it must have some $c \in H$ such that there is uncountable many t such that $c_1(L_t) = c$.

Since this $c \in H^2(X_s, \mathbb{Z})$ coming from $\mathrm{Pic}(X_s)$, we have $e_2(c) = 0 \in R^2\pi_*\mathcal{O}_{\mathcal{X}}(s)$ and thus if we lift it to $c \in H^2(\mathcal{X}, \mathbb{Z})$ the global section $e_2(c) \in H^0(\mathcal{X}, R^2\pi_*\mathcal{O}_{\mathcal{X}})$ it will be zero on uncounable many points. Thus by the identity principle easy to see $e_2(c) = 0 \in H^2(\mathcal{X}, R^2\pi_*\mathcal{O}_{\mathcal{X}})$. Thus thus lift to some global line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ with the restriction $c_1(\mathcal{L}|_{X_s}) = c_1(L_s)$ and we can now apply the lemma about deformation density of Iitaka-Kodaira dimension below and conclude that $\mathcal{L}$ is indeed a big line bundle to finish the proof.

□

Now let's going into the proof of alternating property of Moishezon locus, the proof is motivated by the arguement of [RT22].

Theorem 3.2 ([Kol22a, Theorem 21])**.** Let $g : X \to S$ be a smooth, proper morphism of normal, irreducible analytic spaces. Then $\mathrm{MO}_S(X) \subset S$ is

(1) either contained in a countable union $\cup_i Z_i$, where $Z_i \subsetneq S$ are Zariski closed,
(2) or $\mathrm{MO}_S(X)$ contains a dense, open subset of S.

Furthermore, if $R^2 g_* \mathcal{O}_X$ is torsion free then (2) can be replaced by
(3) $\mathrm{MO}_S(X) = S$ and g is locally Moishezon.

Remark 3.3. The condition (1) is slightly differ from the nowhere dense condition compared with Lemma 2.4 and Theorem 2.5. Indeed the countable union of nowhere dense subset needs not to be nowhere dense (e.g. $\mathbb{Q}$ as countable union of nowhere dense subset is no longer nowhere dense). As we will see in the proof, this replacement is necessary.

Remark 3.4. Another difference compared with Lemma 2.4 and Proposition 2.5 is here we assume the morphism is smooth.

Remark 3.5. Compared with the original proof of [RT22], Kollár's proof do not need the base to be Δ. So that there are two changes, the direct image $R^2 g_* \mathcal{O}_X$ is only torsion free needs not to be locally free sheaf, (2)

Proof. Assume first that $R^2 g_* \mathcal{O}_X$ is torsion free, as coherent sheaf is torsion free on a (Zariski) open subset.

Consider the sheaf exponential sequence

$$0 \to \mathbb{Z}_X \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 1$$

gives

$$R^1 g_* \mathcal{O}_X^\times \to R^2 g_* \mathbb{Z}_X \xrightarrow{e_2} R^2 g_* \mathcal{O}_X.$$

We may pass to the universal cover of S and assume that $R^2 g_* \mathbb{Z}_X$ is a trivial $H^2(X_s, \mathbb{Z})$-bundle. (since the local system on the simply connected space is constant and therefore $R^2 g_* \mathbb{Z}_X \otimes \mathcal{O}_S$ is a trivial bundle). Let $\{\ell_i\}$ be those global sections of $R^2 g_* \mathbb{Z}_X$ such that $e_2(\ell_i) \in H^0(S,\ R^2 g_* \mathcal{O}_X)$ is identically 0, and $\left\{\ell'_j\right\}$ the other global sections (those $\{\ell_i, \ell'_i\}$ are countable since we consider the $\mathbb{Z}$-coefficient cohomology). The ℓ_i then lift back to global sections of $R^1 g_* \mathcal{O}_X^\times$, hence to line bundles L_i on X. We then divide the problem into two cases:

Case 1. If there is an L_i such that $\mathrm{VB}_S(L_i)$ contains a dense open subset of S, then $X \to S$ is Moishezon (by Proposition 2.4). Thus complete the proof.

Case 2. If any such line bundle L_i has nowhere dense very big locus $\mathrm{VB}_S(L_i)$. We claim

$$\mathrm{MO}_S(X) \subset \cup_i \mathrm{VB}_S(L_i) \bigcup \cup_j \left(e_2\left(\ell'_j\right) = 0\right).$$

For if $s \in \mathrm{MO}_S(X)$, and $s \notin \cup_j \left(e_2\left(\ell'_j\right) = 0\right)$. Then every line bundle on X_s is numerically equivalent to some $L_i|_{X_s}$. We then Since being big is preserved by numerical equivalence, we see that X_s has a big line bundle (since $s \in \mathrm{MO}_S(X)$) $\Leftrightarrow L_i|_{X_s}$ is big for some $i \Leftrightarrow L_i|_{X_s}$ is very big for some i (and therefore $s \in \cup_i \mathrm{VB}_S(L_i)$). This completes the case when $R^2 g_* \mathcal{O}_X$ is torsion free. $\square$

The following theorem is about the deformation density of Kodaira dimension.

Theorem 3.6 (Deformation density of Iitaka-Kodaira dimension, see [**LiebermanSernesi**; RT22])**.**

Let $\pi : \mathcal{X} \to Y$ be a flat family from a complex manifold over a one-dimensional connected complex manifold Y with possibly reducible fibers.

If there exists a holomorphic line bundle L on $\mathcal{X}$ such that the Kodaira-Iitaka dimension $\kappa\left(L_t\right) = \kappa$ for each t in an uncountable set B of Y, then any fiber X_t in π has at least one irreducible component C_t with $\kappa\left(L|_{C_t}\right) \geq \kappa$.

In particular, if any fiber X_t for $t \in Y$ is irreducible, then $\kappa\left(L_t\right) \geq \kappa$.

We next show that fiberwise Moishezon morphism is locally Moishezon if the morphism is smooth. Before proving the result, let us give an locally free criterion for direct image when the fibers satisfies the "DuBois property".

Theorem 3.7 (Locally freeness criterion for $R^i f_* \mathcal{O}_X$, [Kol22a, Theorem 24])**.** Let $f : X \to S$ be a smooth, proper morphism of analytic spaces. Assume that $H^i\left(X_s, \mathbb{C}\right) \to H^i\left(X_s, \mathcal{O}_{X_s}\right)$ is surjective for every i for some $s \in S$. Then $R^i g_* \mathcal{O}_X$ is locally free in a neighborhood of s for every i.

Remark 3.8. The reason that I call it DuBois property is because for proper algebraic variety with DuBois singularity, the map

$$H^i\left(X^{\mathrm{an}}, \mathbb{C}\right) \to H^i\left(X^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}}\right) \simeq H^i\left(X, \mathcal{O}_X\right)$$

is always surjective for all i. (see [Kol23]).

Proof. We begin our proof by noticing by the direct image theorem it's enough to show the surjectivity of the base change morphism

$$\phi_s^i : R^i f_* \mathcal{O}_X \to H^i\left(X_s, \mathcal{O}_{X_s}\right)$$

for every i.

Indeed the base change theorem shows that the surjectivity of the base change morphisms ϕ_s^i and ϕ_s^{i-1} implies the locally freeness of the direct image $R^i f_*(\mathcal{O}_X)$.

Next by the Theorem on Formal Functions, it is enough to prove this when S is replaced by any Artinian local scheme S_n, whose closed point is s.

By Cartan B easy to see the vanishing of $H^p(S_n, R^i f_* \mathcal{O}_X) = 0,\ \forall q, \forall p > 0$ then by the Leray spectral sequence arguement we get

$$H^0\left(S_n, R^i f_* \mathcal{O}_X\right) = H^i\left(X_n, \mathcal{O}_{X_n}\right)$$

On the affine base the fiber of the coherent sheaf is indeed the global section, as a consequence

$$R^i f_* \mathcal{O}_X(s) = H^0(S_n, R^i f_* \mathcal{O}_X) = H^i(X_n, \mathcal{O}_{X_n})$$

The base change morphism thus becomes

$$\psi^i : H^i\left(X_n, \mathcal{O}_{X_n}\right) \to H^i\left(X_s, \mathcal{O}_{X_s}\right).$$

Let $\mathbb{C}_{X_n}$ (resp. $\mathbb{C}_{X_s}$) denote the sheaf of locally constant functions on X_n (resp. X_s) and $j_n : \mathbb{C}_{X_n} \to \mathcal{O}_{X_n}$ (resp. $j_s : \mathbb{C}_{X_s} \to \mathcal{O}_{X_s}$) the natural inclusions. We have a commutative diagram

$$
\begin{tikzcd}
H^i(X_n, \mathbb{C}_{X_n}) \arrow[r, "\alpha^j"] \arrow[d, "j'_n"'] & H^i(X_s, \mathbb{C}_{X_s}) \arrow[d, "j'_s"] \\
H^i(X_n, \mathcal{O}_{X_n}) \arrow[r, "\psi^j"'] & H^i(X_s, \mathcal{O}_{X_s})
\end{tikzcd}
$$

Note that α^i is an isomorphism since the inclusion $X_s \hookrightarrow X_n$ is a homeomorphism, and j_s^i is surjective since X_s is Du Bois. Thus ψ^i is also surjective. □

Using this we can prove the theorem below

Theorem 3.9 (Fiberwise Moishezon morphism is locally Moishezon if it's smooth, see [Kol22a],Corollary 22)**.** Let $g : X \to S$ be a smooth, proper morphism of normal, irreducible analytic spaces whose fibers are Moishezon. Then g is locally Moishezon.

Proof. [Kol22a] Since we have proved the Moishezon manifolds admit strong Hodge decomposition (that we proved in the first time), the morphism

$$H^i(X_s, \mathbb{C}) \to H^i(X_s, \mathcal{O}_{X_s})$$

is surjective for every i.

The result then follows clearly by 3.7 and [Kol22a] Theorem 21. □

4 The alternating property of the projective locus

Theorem 4.1 (Alternating property of projective locus, see [Kol22b], Proposition 17)**.** Let $g : X \to S$ be a proper morphism of normal, irreducible analytic spaces. Then there is a dense, Zariski open subset $S^\circ \subset S$ such that

(1) either X is locally projective over S°,

(2) or $\mathrm{PR}_S(X) \cap S^\circ$ is locally contained in a countable union of Zariski closed, nowhere dense subsets.

Proof. We will prove it in the next time. □

References

[HM06] Christopher D. Hacon and James McKernan. "Boundedness of pluricanonical maps of varieties of general type". In: *Invent. Math.* 166.1 (2006), pp. 1–25.

[Kol23] János Kollár. *Families of varieties of general type*. Vol. 231. Cambridge Tracts in Mathematics. With the collaboration of Klaus Altmann and Sándor J. Kovács. Cambridge University Press, Cambridge, 2023, pp. xviii+471.

[Kol22a] János Kollár. "Moishezon morphisms". In: *Pure Appl. Math. Q.* 18.4 (2022), pp. 1661–1687.

[Kol22b] János Kollár. "Seshadri's criterion and openness of projectivity". In: *Proc. Indian Acad. Sci. Math. Sci.* 132.2 (2022), Paper No. 40, 12.

[RT22] Sheng Rao and I-Hsun Tsai. "Invariance of plurigenera and Chow-type lemma". In: *Asian J. Math.* 26.4 (2022), pp. 507–554.