Contraction in Kähler MMP reading notes Fall 2025

Note 2 — 28, 09, 2025 (draft version)

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The aim of these notes is to give an elementary introduction to positivity in the Kähler MMP. We will organize the notes as follows: first, we introduce the notion of positivity for line bundles and cohomology classes; then, we discuss some special properties that hold for the canonical divisor or adjoint classes.

Contents

1 Intersection Theory

2 Numerical trivial, linearly trivial relation

A complex analytic variety may not contain any subvariety.

Example 1.

[1] **WARNING:** (1) Round 1: sketch notes; (2) Round 2: more details but contains errors; (3) Round 3: correct version but not smooth to read; (4) Round 4: close to the published version.
To ensure a pleasant reading experience. Please read my notes from ROUND ≥ 4.

One of the central proposition that we will repeatly apply is the following.

Theorem 2. Let $f : X \to Y$ be a proper morphism with connected fibers between normal compact complex spaces with rational singularities. Assume that one of the following two conditions hold:
(1) X and Y are in Fujiki's class $\mathcal{C}$ and f is bimeromorphic, or,
(2) there is an effective $\mathbb{Q}$-divisor $B \geq 0$ such that (X, B) is klt, $-(K_X + B)$ is f-nef-big and f is projective.

Pull back $f^* : H^{1,1}_{\mathrm{BC}}(Y) = H^1(Y, \mathcal{H}_Y) \to H^{1,1}_{\mathrm{BC}}(X) = H^1(X, \mathcal{H}_X)$ and a $f^* : H^2(Y, \mathbb{R}) \to H^2(X, \mathbb{R})$ are both injective, and

So that the relative Picard number is

$$\rho(X/Y) = \dim H^{1,1}_{\mathrm{BC}}(X) - \dim H^{1,1}_{\mathrm{BC}}(Y) = \rho(X) - \dim \operatorname{im}(f^*) = \rho(X) - (\rho(Y) - \dim(\ker(f)^*))$$

and

$$\operatorname{Im}(f^*) = \left\{\alpha \in H^{1,1}_{\mathrm{BC}}(X) \mid \alpha \cdot C = 0 \text{ for all curves } C \subset X \ s.t.\ \ f_*(C) = \mathrm{pt}\right\}$$

and

$$\operatorname{Im}(f^*) = \left\{\alpha \in H^2(X) \mid \alpha \cdot C = 0 \text{ for all curves } C \subset X \ s.t.\ \ f_*(C) = \mathrm{pt}\right\}$$

Proof. □

3 Pseudo-effectiveness

4 Nefness

5 Bigness

One of central result in birational geometry is the following Kawamata-Viehweg vanishing theorem.

Definition 3 (Kawamata–Viehweg's vanishing theorem, [DH24, Theorem 5.1])**.** Let $(X, B + \boldsymbol{\beta})$ be a gklt pair, $g : X \to T$ a proper morphism of analytic varieties, and D a $\mathbb{Q}$-Cartier $\mathbb{Z}$-divisor such that $D - (K_X + B + \boldsymbol{\beta}_X)$ is nef and big over T. Then $R^i g_* \mathcal{O}_X(D) = 0$ for all $i > 0$.

Proof. The idea is simple, we try to construct a new klt pair (X', B^*) over the bimeromorphic model $\nu : X' \to X$. And showing that □

6 Semi-positive

7 Semi-ampleness

8 Positivities in families

We first prove the classical result about ampleness in families.

Theorem 4 ([Laz04, Theorem 1.2.17] and [**?**, Proposition 3.7])**.** Let $f : X \to Y/S$ be a proper surjective contraction morphism of complex analytic varieties over a smooth connected curve S. For each $t \in S$, denote by

$$f_t : X_t \to Y_t$$

the induced proper contraction on the fiber over t. If L is a $\mathbb{Q}$-line bundle on X such that $L|_{X_0}$ is f_0-ample, then there exists a Zariski open subset $U \subset S$ around 0 such that $L|_{X_t}$ is f_t-ample for all $t \in U$.

PROOF IDEA 5.

References

[DH24] O. Das and C.D. Hacon, *On the minimal model program for Kähler 3-folds*, 2024.

[Laz04] Robert Lazarsfeld, *Positivity in algebraic geometry. I*, vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series.