Hacon-Popa-Schnell's Readings Notes Fall 2025

Note III.3 — 2025 09 12 (draft version 0)

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The aim of this note is to prove the Iitaka conjecture when the base has maximal Albanese dimension.

Theorem 0.1 ([HPS18, Theorem 1.1])**.** Let $f : X \to Y$ be an algebraic fiber space with general fiber F. Assume that Y has maximal Albanese dimension, then $\kappa(X) \geq \kappa(F) + \kappa(Y)$.

There are three essential ingredients appears in the proof.

(a) Green-Lazarsfeld-Simpson's structure theorem on cohomological support loci and generic vanishing theorem;
(b) Hacon-Popa-Schnell's construction of singular Hermitian metric on $f_*\omega_X^{\otimes n}$;
(c) Kawamata's Iitaka conjecture when the base is of general type and structure theorem for finite cover of Abelian varieties.

Contents

1 Green-Lazarsfeld-Simpson's Generic Vanishing theorem

1.1 Basic Properties of GV sheaves

Definition 1.1 (GV sheaves)**.** Given a coherent $\mathscr{O}_T$-module $\mathscr{F}$ on a compact complex torus T, define

$$S^i(T,\mathscr{F}) = \{L \in \operatorname{Pic}^0(T) \mid H^i(T, \mathscr{F} \otimes L) \neq 0\}$$

We say that $\mathscr{F}$ is a GV-sheaf if $\operatorname{codim} S^i(T,\mathscr{F}) \geq i$ for every $i \geq 0$; we say that $\mathscr{F}$ is M-regular if $\operatorname{codim} S^i(T,\mathscr{F}) \geq i+1$ for every $i \geq 1$.

Definition 1.2 (Cohomological support loci)**.** Recall that for any coherent sheaf $\mathscr{F}$ on an abelian variety A, we consider for all $k \geq 0$ the cohomological support loci

$$V^k(\mathscr{F}) = \left\{P \in \operatorname{Pic}^0(A) \mid H^k(X, \mathscr{F} \otimes P) \neq 0\right\}$$

They are closed subsets of $\operatorname{Pic}^0(A)$, by the semi-continuity theorem for cohomology.

Definition 1.3 (Unipotent Vector Bundle)**.** A vector bundle on A is called unipotent if it has a filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_n = U$$

such that $U_i/U_{i-1} \simeq \mathscr{O}_A$ for all $i = 1, \ldots, n$. Note in particular that $\det U \simeq \mathscr{O}_A$.

That is we have a succesive extension

$$0 \to U_{i-1} \to U_i \to E_i \to 0$$

with

$$U_i/U_{i-1} \cong \mathcal{O}_A = E_i$$

One of the most important example of GV that interests us is the direct image of pluricanonical sheaf.

Proposition 1.4. Let $f : X \to A$ be a morphism from a smooth projective variety to an abelian variety. For every $m \in \mathbb{N}$, the sheaf $\mathscr{F}_m = f_*\omega_X^{\otimes m}$ is a GV-sheaf on A.

1.2 Green-Lazarsfeld generic vanishing theorem

Let us first introduce the original version Green-Lazarsfeld-Simpson's generic vanishing theorem. We will first introduce Grenn-Lazarsfeld's original proof, and then provide Hacon's proof of Green-Lazarsfeld theorem using Fourier-Mukai transform.

Theorem 1.5. If $f : X \to A$ is a morphism from a smooth projective variety to an abelian variety, then for any $j, k \geq 0$ we have
(1) $\operatorname{codim}_{\operatorname{Pic}^0(A)} V^k\left(R^j f_*\omega_X\right) \geq k$;
(2) Every irreducible component of $V^k\left(R^j f_*\omega_X\right)$ is a translate of an abelian subvariety of A by a point of finite order.

PROOF IDEA 1.6.

For our interest, we need the generic vanishing for direct image of pluricanonical sheaves.

Theorem 1.7 ([HPS18, Theorem 4.1])**.** Let X be a smooth projective variety. For each $m \in \mathbb{N}$, the locus

$$\left\{P \in \operatorname{Pic}^0(X) \mid H^0\left(X, \omega_X^{\otimes m} \otimes P\right) \neq 0\right\} \subseteq \operatorname{Pic}^0(X)$$

is a finite union of abelian subvarieties translated by points of finite order.

This theorem implies that $V^0\left(A, \mathscr{F}_m\right)$ (for $\mathscr{F}_m = f_*\omega_X^{\otimes m}$) is also a finite union of abelian subvarieties translated by points of finite order. where

$$\begin{aligned} V^0\left(A, \mathscr{F}_m\right) &= \left\{P \in \operatorname{Pic}^0(A) \mid H^0\left(A, \mathscr{F}_m \otimes P\right) \neq 0\right\} \\ &= \left\{P \in \operatorname{Pic}^0(A) \mid H^0\left(X, \omega_X^{\otimes m} \otimes f^*P\right) \neq 0\right\} \subseteq \operatorname{Pic}^0(A). \end{aligned}$$

Proof. □

For GV sheaves, we have the following non-zero criterion.

Theorem 1.8 ([HPS18, Lemma 7.4])**.** If $\mathscr{F}$ is a GV-sheaf on A, then $\mathscr{F} = 0$ if and only if $V^0(\mathscr{F}) = \emptyset$.

Proof. □

Using the result we just developed above, we can prove the following criterion for direct image of pluricanonical sheaves to be a indecomposable unipotent vector bundle.

Definition 1.9. Let E be a vector bundle on complex manifold M, we say it's unipotent if

Theorem 1.10 ([HPS18, Corollary 4.3])**.** Let X be a smooth projective variety with $\kappa(X) = 0$, and let $f : X \to A$ be an algebraic fiber space over an abelian variety.

If $H^0\left(X, \omega_X^{\otimes m}\right) \neq 0$ for some $m \in \mathbb{N}$, then the coherent sheaf $\mathscr{F}_m$ is an indecomposable unipotent vector bundle.

2 Hacon-Popa-Schnell's construction of semi-positive singular Hermitian metric on $f_*\omega_X^{\otimes n}$

We will divide the construction into 3 steps. First we try to find some semi-positive singular Hermitian metric on $\mathscr{F} = f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)$ as long as (L, h) is semi-positive using the optimal Ohsawa-Takegoshi extension theorem. In the second step, we try to construct the m-th Narasimhan-Simha metric on the $\omega_{X/Y}$ and hence a semi-positive line bundle $(\omega_{X/Y}^{\otimes(m-1)}, h)$. And finally, using the elimination of multiplier ideal lemma, we can transport the semi-positivity from $\mathscr{F} = f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)$ to $\mathscr{F}_m = f_*(\omega_{X/Y}^{\otimes m})$. It's worth mentioning that up to now there is no algebraic proof of such semi-positivity theorem.

2.1 Construction of semi-positive metric on $f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h))$

We first prove there exists a semi-positive singular Hermitian metric on $\mathscr{F} = f_* \left(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\right)$ as long as (L, h) is semi-positive.

Proposition 2.1 ([HPS18, Theorem 21.1])**.** Let $f : X \to Y$ be a projective surjective morphism between two connected complex manifolds. If (L, h) is a holomorphic line bundle with a singular hermitian metric of semi-positive curvature on X, then the pushforward sheaf

$$\mathscr{F} = f_* \left(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\right)$$

(1) has a canonical singular hermitian metric H;
(2) This metric has semi-positive curvature and;
(3) satisfies the minimal extension property.

PROOF IDEA 2.2. The construction of the metric is a bit involved. Let us briefly sketch the idea of the proof first. We try to first construct some semi-positive singular Hermitian metric on the "nice" locus where the direct image is a vector bundle. And then we try to apply the Riemann extension theorem to extend such semi-positive metric to semi-positive metric for torsion free sheaf $f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h))$. Now let's explain the idea of the proof step by step.

Step 1. (Find the nice locus that on which $\mathscr{F}$ is a locally free sheaf E). We first find a nice Zariski open subset $U = Y \setminus Z$, such that: (a) f is submersion on U (so that Eheresmann theorem can apply), (b) the direct image $\mathscr{F} = f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h))$ and $f_*(\omega_{X/Y} \otimes L)$ are locally free on U; (c) the base change property holds for $f_*(\omega_{X/Y} \otimes L)$.

Step 2. (Construction of the Singular metric H on E). Let $s \in E(y) \subset H^0(X_y, K_{X_y} \otimes L_y)$ for some $y \in U$, we define the singular Hermitian inner product on E_y as

$$\|s\| = \int_{X_y} |s|_h^2.$$

We will check that this is finite and positive almost everywhere. Hence defines a singular Hermitian metric on E.

Step 3. (Prove the semi-positivity of the metric H using optimal Ohsawa-Takegoshi extension). By definition, in order to prove the semi-positive of the metric on E we just defined. We need to show the psh of the function $\psi(y) = \log |g|_{H^*}$ for local section $g \in H^0(U, E^*)$. To do this, we need to verify the upper semi-continuity and mean value inequality on the nice locus. Let us briefly sketch the idea for the proof of mean-value inequality. To do this we first simplify the problem onto the unit ball $i : B \hookrightarrow Y$ (which sends $0 \mapsto y$). By optimal Ohsawa-Takegoshi extension, for any $\alpha \in E_y$ with finding some section $s \in H^0(B, i_*\mathscr{F})$ such that

$$s(0) = \alpha, \frac{1}{\mu(B)} \int_B |s|_H^2 d\mu \le 1.$$

And therefore, by the extremal characterization of dual metric H^* we have

$$|g|_{H^*} = \sup_{s \in E_y, \|s\|_{H,y} \le 1} \|\langle g(y), s\rangle\| \ge |\langle g(y), s/\|s\|_{H,y}\rangle|.$$

And therefore, taking log, we get

$$2\psi(y) \geq \log|\langle g(y), s\rangle|^2 - \log\|s\|^2_{H,y}$$

And therefore, by integration

$$\frac{1}{\pi}\int_\Delta 2\psi d\mu \geq \frac{1}{\pi}\int_\Delta \log|g(s)|^2 d\mu - \frac{1}{\pi}\int_\Delta \log|s|^2_H d\mu.$$

For the 2nd term on the RHS, we apply the Jensen's inequality, and optimal L2 estimate, which will gives the mean value inequality.

Step 4. (Extend the semi-positive metric onto the whole torsion free sheaves) In order to extend the metric onto $\mathscr{F}$, we use the Riemann extension theorem, thus need to check the uniform boundedness of the psh function $\psi(y)$ on Y.

2.2 Construction of Narasimhan-Simha metric on $\omega_{X/Y}$

Our next goal is to construct the m-th Narasimhan-Simha metric on $\omega_{X/Y}$.

Proposition 2.3. Let $f : X \to Y$ be a surjective projective morphism with connected fibers between two complex manifolds. Suppose that $f_*\omega_{X/Y}^{\otimes m} \neq 0$ for some $m \geq 2$.

The line bundle $\omega_{X/Y}$ has a canonical singular hermitian metric with semipositive curvature, called the m-th Narasimhan-Simha metric. This metric is continuous on the preimage of the smooth locus of f.

PROOF IDEA 2.4.

2.3 Construction of semi-positive metric on $f_*\omega_{X/Y}^{\otimes m}$

In order to finish the construction of semi-positive metric on $f_*\omega_{X/Y}^{\otimes m}$ we need the following lemma to eliminate the multiplier ideal sheaf $\mathcal{I}(h)$.

Proposition 2.5. Let $f : X \to Y$ be a surjective projective morphism with connected fibers between two complex manifolds. Suppose that $f_*\omega_{X/Y}^{\otimes m} \neq 0$ for some $m \geq 2$. (b) If h denotes the singular hermitian metric on $L = \omega_{X/Y}^{\otimes(m-1)}$ induced by the Narasimhan-Simha metric we just constructed, then

$$f_*\left(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\right) \hookrightarrow f_*\omega_{X/Y}^{\otimes m}$$

is an isomorphism over the smooth locus of f.

Hence we can transport the semi-positivity from the $f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h))$ to the direct image of relative pluricanonical sheaves using the following lemma.

Lemma 2.6. Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism between two torsion-free coherent sheaves that is generically an isomorphism. If $\mathscr{F}$ has a singular hermitian metric with semi-positive curvature, then so does $\mathscr{G}$.

Theorem 2.7 ([HPS18, Theorem 4.6])**.** Let $f : X \to Y$ be an algebraic fiber space. Suppose Y is compact.

(a) For any $m \in \mathbb{N}$, the torsion-free sheaf $f_*\omega_{X/Y}^{\otimes m}$ has a canonical singular hermitian metric with semi-positive curvature;
(b) If $c_1\left(\det f_*\omega_{X/Y}^{\otimes m}\right) = 0$ in $H^2(Y, \mathbb{R})$, $f_*\omega_{X/Y}^{\otimes m}$ is locally free, and the singular hermitian metric on it is smooth and flat;
(c) Every nonzero morphism $f_*\omega_{X/Y}^{\otimes m} \to \mathcal{O}_Y$ is split surjective.

Proof of (a). □

To finish the proof of (b) we need to following two propositions.

Proposition 2.8 (Theorem of Katz and Andre)**.**

PROOF IDEA 2.9. The proof of the result requires the theory of D-module.

Proposition 2.10.

Proof. □

3 A brief review of Kawamata's work on Iitaka conjecture when base is of general type

3.1 Several Critera for Albanese map to be fibration

Theorem 3.1 (Albanese map as fibration when Kodaira dimension 0,[Kaw81, Theorem 1])**.** Let X be a non-singular and projective algebraic variety and assume that $\kappa(X) = 0$. Then the Albanese map $\alpha : X \to A(X)$ is an algebraic fiber space. In addition, if X has maximal Albanese dimension then the Albanese map is a birational contraction morphism.

3.2 Structure Theorem for Finite Covers of Abelian Varieties

Theorem 3.2 ([Kaw81, Theorem 13])**.** Let $f : X \to A$ be a finite morphism from a complete normal algebraic variety to an abelian variety.

Then $\kappa(X) \geq 0$ and there are an abelian subvariety B of A, etale covers $\tilde{X}$ and $\tilde{B}$ of X and B, respectively, and a complete normal algebraic variety $\tilde{Y}$ such that
(1) $\tilde{Y}$ is finite over A/B;
(2) $\tilde{X}$ is isomorphic to $\tilde{B} \times \tilde{Y}$;
(3) $\kappa(\tilde{Y}) = \dim \tilde{Y} = \kappa(X)$.

3.3 Kawamata's proof of Iitaka conjecture when base is of general type

Theorem 3.3 ([Kaw81, Theorem])**.** If $f : X \to Y$ is algebraic fibre space with general fibre F and $\kappa(Y) = \dim Y$ then $\kappa(X) = \kappa(Y) + \kappa(F)$.

PROOF IDEA 3.4.

4 Proof of the Iitaka conjecture when the base has maximal Albanese dimension

Let us finish this note by the proof of Hacon-Popa-Schnell's result.

Theorem 4.1. Let $f : X \to Y$ be an algebraic fiber space with general fiber F. Assume that Y has maximal Albanese dimension, then $\kappa(X) \geq \kappa(F) + \kappa(Y)$.

4.1 Reduce the problem to the case when the base is Abelian Varieties and $\kappa(X) = 0$

Proposition 4.2. To prove Theorem 4.1, it's sufficient to prove the case when $\kappa(X) = 0$ and Y is an Abelian variety.

PROOF IDEA 4.3. We will divide the proof into 3 cases.
Case 1. We will prove the result in the case when $\kappa(X) = -\infty$. Green-Lazarsfeld-Simpson's generic vanishing theorem is used. we assume by contradiction that $P_m(F) > 0$ then this will imply that $f_*\omega_X^{\otimes m} \neq 0$. And consequently, we can apply the generic vanishing theorem we developed in Section 1, find some torsion point $P \in \mathrm{Pic}^0(A)$ with $h^0\left(X, \omega_X^{\otimes m} \otimes g^*P\right) = h^0\left(A, g_*\omega_X^{\otimes m} \otimes P\right) \neq 0$ and then

$$h^0\left(X, \omega_X^{\otimes km}\right) = h^0\left(X, \left(\omega_X^{\otimes m} \otimes P\right)^{\otimes k}\right) \neq 0,$$

which contradicts to our assumption that $\kappa(X) = -\infty$;
Case 2. We assume that $\kappa(Y) = 0$. Using Theorem 3.1, we can reduce the base to an Abelian variety Y. We then take a Iitaka fibration on the total space

$$h : X \to Z,$$

with the general fiber G. We then restrict the fibration $f : X \to Y$ to a new fibration

$$G \to B',$$

with $\kappa(G) = 0$ and B' is an Abelian variety and H be the general fiber. which is exactly the situation of Proposition 4.1, that is

$$\kappa(G) \geq \kappa(H) + \kappa(B'),$$

thus $\kappa(H) = 0$.

$$\begin{tikzcd}
H = F \cap G \arrow[r, hook] \arrow[d, hook] & F \arrow[r] \arrow[d, hook] & h(F) \arrow[d, hook] \\
G \arrow[r, hook] \arrow[d] & X \arrow[r, "h"] \arrow[d, "f"] & Z \\
B \arrow[r, hook] & Y &
\end{tikzcd}$$

We then apply the easy addition formula to $F \to h(F)$, so that

$$\kappa(F) \leq \kappa(H) + \dim h(F) = \dim h(F),$$

which complete the proof of this case.
Case 3. When the base Y has maximal Albanese dimension. Then by the Kawamata's structure theorem for finite cover of Abelian varieties, we can decompose the base Y into a product of Abelian variety K and a variety of general type z.

$$\begin{tikzcd}
F = E \cap H \arrow[r, hook] \arrow[d, hook] \arrow[dr] & E \arrow[r] \arrow[d, hook] & K \arrow[d] \\
H \arrow[r, hook] \arrow[d] & X \arrow[r, "p"] \arrow[d, "q"] \arrow[dr, "f"] & K \\
Z \arrow[r] & Z & Y = Z \times K \arrow[l] \arrow[u]
\end{tikzcd}$$

The key point is to observe that the general fiber F of $f : X \to Y$ is also the general fiber of the induced $E \to K$. We then apply the Iitaka conjecture when base Z is of general type so that

$$\kappa(X) = \kappa(E) + \kappa(Z)$$

on the other hand since K is Abelian variety, so that it's possible to prove the Iitaka conjecture for $E \to K$ that is

$$\kappa(E) \geq \kappa(F).$$

This complete the proof of the 3rd case. We can then apply the result of Iitaka conjecture when the base is of general type to finish the proof.

4.2 Analytic Proof of Iitaka conjecture when base is Abelian Varieties and $\kappa(X) = 0$

We will finish the proof by showing that

Proposition 4.4. If $f : X \to A$ is an algebraic fiber space over an abelian variety, with $\kappa(X) = 0$, then we have

$$\mathscr{F}_m = f_* \omega_X^{\otimes m} \simeq \mathscr{O}_A$$

for every $m \in \mathbb{N}$ such that $H^0\left(X, \omega_X^{\otimes m}\right) \neq 0$.

PROOF IDEA 4.5. By Theorem 1.10 and Theorem 2.7, the direct image $\mathscr{F}_m$ is a flat unipotent vector bundle. Then using a bit representation theory, we can prove that the vector bundle is actually $\mathscr{F}_m = \bigoplus_{i=1}^r \mathcal{O}_A$. Using the assumption that $\kappa(X) = 0$, we have $r = 1$ and hene finish our proof.

Proof of Proposition 4.4. □

Proof of Iitaka conjecture when base is Abelian variety and $\kappa(X) = 0$. By assumption, it's sufficient to prove that $\kappa(F) = 0$. Using Proposition 4.4, easy to see this is the case. □

References

[HPS18] Christopher Hacon, Mihnea Popa, and Christian Schnell. "Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Păun". In: *Local and global methods in algebraic geometry*. Vol. 712. Contemp. Math. Amer. Math. Soc., 2018, pp. 143–195.

[Kaw81] Yujiro Kawamata. "Characterization of abelian varieties". In: *Compositio mathematica* 43.2 (1981), pp. 253–276.