Nakayama's Extension Theorems Summer 2025

Note I.1 — 08, 17, 2025 (draft version)

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The aim of this note is to prove the following two extension results appear in Nakayama's book [Nak04].

Theorem 1 ([Nak04, Theorem VI.3.7])**.**
(1) Let L be a π-pseudo-effective $\mathbb{Z}$-divisor of V such that $L-(K_V+X+\Delta)$ is π-nef. (2) Let Λ be a π-nef and π-big $\mathbb{Q}$-divisor of V such that $\Delta \geq \langle\Lambda\rangle$ and $k\Lambda \in \mathbb{E}_{\text{big}}$ is for some $k \in \mathbb{N}$. Then (A) The homomorphism

$$\pi_* \mathcal{O}_V\left(lL+\lfloor\Lambda\rfloor\right) \rightarrow \pi_* \mathcal{O}_X\left(lL+\lfloor\Lambda\rfloor\right)$$

is surjective for $l \gg 0$.

(B) If $\left.L\right|_X$ is $\left(\left.\pi\right|_X\right)$-pseudo-effective, then the homomorphism above is surjective for any $l>0$.

Theorem 2 ([Nak04, Theorem VI.3.16])**.** Let L be a π-abundant divisor of V. Suppose that (1) $\pi\left(X_i\right)$ is a prime divisor of S for any X_i, (2) $L-(K_V+X+\Delta)$ is π-nef and π-abundant, (3) $\kappa\left(\left.L\right|_{X_i} ; X_i / \pi\left(X_i\right)\right) \geq \kappa(L ; V / S)$ for any i.

Then the restriction homomorphism $\pi_* \mathcal{O}_V(lL) \rightarrow \pi_* \mathcal{O}_X(lL)$ is surjective for any $l \geq 1$.

Our presentation uses slightly different notation, and several simplifications are made. For example, we may assume the divisor X in the book to be the central fiber of the family, and several assumptions becomes vacus then. We also adopt the notations in [BCHM10] and [HM10] and also [Kaw24], for example the multiplier ideal sheaves are denoted as

Contents

1 Base ideal sheaves and Multiplier ideal sheaves

The majore tools that will be used is the multiplier ideal sheaves.

2 Nakayama's blueprint theorem

The following result will be the blueprint for the extension theorem.

Theorem 3. Let L and L' be $\mathbb{Q}$-divisors of V with $\langle L\rangle \leq \Delta$ such that (L, L') satisfies one of the three conditions (VI-3), (VI-4), and (VI-5). Suppose that there exist - a rational number $0<\beta<1$, positive integers m, m', and an integer b, - $\mathbb{Z}$-divisors A and D of V, and - a bimeromorphic morphism $\rho: W \rightarrow V$ from a non-singular variety satisfying the following conditions:

(1) $mL, m'L$, and bL' are $\mathbb{Z}$-divisors with $mL+A \in \mathbb{E}_V, m'L+bL' \in \mathbb{E}_V$;

(2) $m\beta \leq m'+b\beta$ and $L'-\beta L$ is π-semi-ample;

(3) $\mathcal{I}[mL] \subset \mathcal{J}[mL+A]$;

(4) D is an effective divisor containing no components of X and ($V \& X, \Delta+(1/m)D)$ is log-terminal along X;

(5) ρ satisfies the conditions $\mathbf{E}$ for $mL+A$ and $\mathbf{E}$ for $m'L+bL'$ in which the inequality

$$-E(mL+A) \leq \rho^* D-E\left(m'L+bL'\right)$$

holds. Then $\pi_* \mathcal{O}_V\left(L_L L_{\perp}\right) \rightarrow \pi_* \mathcal{O}_X\left(L_L L_{\perp}\right)$ is surjective.

3 Nakayama's extension theorem under nef and big conditions

4 Nakayama's extension theorem under nef and abundant conditions

In this section we will give a proof of the following result.

References

[BCHM10] C. Birkar, P. Cascini, C.D. Hacon, and J. McKernan, *Existence of minimal models for varieties of log general type*, Journal of the American Mathematical Society **23** (2010), no. 2, 405–468.

[HM10] C.D. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type. II*, J. Amer. Math. Soc. **23** (2010), no. 2, 469–490.

[Kaw24] Yujiro Kawamata, *Algebraic varieties: minimal models and finite generation*, Cambridge Studies in Advanced Mathematics, vol. 214, Cambridge University Press, Cambridge, 2024, Translated by Chen Jiang.

[Nak04] N. Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004.