Hacon-Mckernan-Xu's canonical bundle formula Summer 2025

Note 8 — 06, 06, 2025 (draft version)

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The aim of this note is to prove the following Hacon-Mckernan-Xu's subadjunction theorem and show varies applications. The major references are [Bir19] and [HMX14].

Theorem 1 ([HMX14, Theorem 4.2])**.** Assume the following condition holds:
(1) Let (X, B) be a projective klt pair;

(2) $G \subset X$ is a subvariety with normalisation F, (we want to do subadjunction on this F),

(3) X is $\mathbb{Q}$-factorial near the generic point of G,

(4) $\Delta \geq 0$ is an $\mathbb{R}$-Cartier divisor on X; and $(X, B + \Delta)$ is lc near the generic point of G, (the LC condition allow us to find some DLT model near G),

(5) there is a unique non-klt place of this pair whose centre is G.

(6) Let $d \in \mathbb{N}$, and let Φ be a subset of $[0, 1]$ that contains 1, and $\dim X = d$ and $B \in \Phi$.

We then use MMP arguement (see more detail below) find some DLT model $\psi : Y \to X$, such that there exists unique component S with coefficient 1, and a contraction $h : S \to F$. We then apply the canonical bundle formula

$$K_S + \Xi_S \sim_{\mathbb{R}} h^*(K_F + \Theta_F + P_F),$$

where Ξ_S is some axillary boundary divisor (which will be contruced below) and Θ_F is the discriminant divisor and P_F is the moduli divisor in the canonical bundle formula.

Then Hacon-Mckernan-Xu subadjunction theorem says that:

(a) (coefficient control on discriminant part) The coefficient of the discriminant divisor Θ_F belong to $\Psi := \{a \mid 1 - a \in \mathrm{LCT}_{d-1}(D(\Phi))\} \cup \{1\}$, thus only depends on B not Δ.

(b) (positivity on moduli part) P_F is pseudo-effective.

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1.5 subadjunction for ϵ-LC

2 Applications of Hacon-Mckernan-Xu's subadjunction

References

[Bir19] Caucher Birkar, *Anti-pluricanonical systems on fano varieties*, Annals of Mathematics **190** (2019), no. 2, 345–463.

[HMX14] Christopher D. Hacon, James McKernan, and Chenyang Xu, *ACC for log canonical thresholds*, Ann. of Math. (2) **180** (2014), no. 2, 523–571. MR 3224718