Projectivity Criterira and it's deformation behavior Summer 2025

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1 Overview

The aim of this note is to give some projectivity critera for Moishezon/Kähler morphism and study the deformation bahavior of projectivity. The ultimate goal is to finish the proof of [Kol22, Theorem 2].

Theorem 1.1 ([Kol23, Theorem 2])**.** Let $g : X \to S$ be a proper morphism of complex analytic spaces and $S^* \subset S$ a dense, Zariski open subset such that g is flat over S^*. Assume that

(1) X_0 is projective for some $0 \in S$,
(2) the fibers X_s have rational singularities for $s \in S^*$, and
(3) g is bimeromorphic to a projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to S$.

Then there is a Zariski open neighborhood $0 \in U \subset S$ and a locally closed, Zariski stratification $U \cap S^* = \cup_i S_i$ such that each $g|_{X_i} : X_i := g^{-1}(S_i) \to S_i$ is projective.

Contents

2 Several projectivity critera

In this section, we try to summarize the projectivity critera related to Moishezon varieties (we will use the Moishezon variety or algebraic space interchangeably).

2.1 Kodaira's projectivity criterion

Proposition 2.1. Let X be a compact Kähler variety with rational singularities such that $H^2(X, \mathcal{O}_X) = 0$, then X is projective.

Proof. Take the resolution $\nu : X' \to X$, where X' is Kähler manifold. Since X has rational singularity, $R^i\nu_*\mathcal{O}_{X'} = 0$ with $i > 0$. Thus, by the Leray spectral sequence arguement, $H^2(X, \mathcal{O}_X) = H^2(X', \mathcal{O}_{X'}) = 0$ and therefore by Kodaira's projectivity criterion for smooth manifold, X' is projective. And therefore X is a Kähler Moishezon variety with rational singularity. By the result we proved in the 1st time, X is a projective variety. □

2.2 Nakai-Moishezon ampleness critera

Proposition 2.2 ([Kol90]). Let X be a proper algebraic space and let H be a line bundle on X. Then H is ample on X if an only if for every irreducible closed subspace $Z \subset X$, the intersection product $H^{\dim(Z)} \cdot Z$ is positive.

2.3 Seshadri criterion of projectivity, line bundle version

Seshadri constant was first introduced by Demailly in the early 90s, when he tried to prove Fujita's conjecture.

Conjecture 2.3. Let X be a smooth projective variety of dimension n, with L being ample. Then

(a) $K_X + (n+1)L$ is global generated,

(b) $K_X + (n+2)L$ is very ample.

Demailly define the Seshadri constant as

Definition 2.4. Given a proper analytic space X and a line bundle L, the Seshari constant is defined to be

$$\epsilon(L, x) := \inf_{C \ni x} \frac{L \cdot C}{\operatorname{mult}_x C}.$$

Then he showed the following result holds true.

Theorem 2.5.
(a) If $\epsilon(L, x) > \frac{n}{n+1}$ then $K_X + (n+1)L$ is global generated,

(b) If $\epsilon(L, x) > \frac{2n}{n+2}$ then $K_X + (n+2)L$ is very ample.

For the reader want to know more about this, please read the [Dem82].

Proposition 2.6. Let X be a proper algebraic space, and D a divisor on X (the same also true for $\mathbb{Q}$, $\mathbb{R}$ dvisor). Then D is ample if and only if there exists a positive number $\varepsilon > 0$ such that

$$\frac{(D \cdot C)}{\text{mult}_{x} C} \geq \varepsilon$$

for every point $x \in X$ and every irreducible curve $C \subseteq X$ passing through x.

Proof. The proof is standard, see [Laz04]. Only needs to show how to generalize the result to the Moishezon setting. That is if thre exist a divisor satisfies the Seshadri inequality, then it's ample and X is projective.

□

2.4 Seshadri criterion of projectivity, cohomology class version

For cohomology class, the Seshadri criterion is also true.

Lemma 2.7. Let X be a normal compact complex space with rational singularities of dimension n in Fujiki class $\mathcal{C}$. Then the canonical map

$$\Phi : N^1(X) \to N_1(X)^\vee, \quad [\omega] \mapsto \lambda_\omega$$

is an isomorphism. Here we define

$$\lambda_\omega : N_1(X) \to \mathbb{R}, \quad [T] \mapsto T(\omega)$$

Proposition 2.8. Let X be a proper algebraic space over $\mathbb{C}$ with 1-rational singularities (see Definition 11). Then X is projective iff there is a cohomology class $\Theta \in H^2(X(\mathbb{C}), \mathbb{Q})$ and an $\epsilon > 0$ such that

$$\Theta \cap [C] \geq \epsilon \cdot \text{mult}_p C$$

for every integral curve $C \subset X$ and every $p \in C$.

Proof. Note that the cup product induce a $\mathbb{Q}$-bilinear form

$$(-) \cap (-) : H^2(X, \mathbb{Q}) \times H_2(X, \mathbb{Q}) \to \mathbb{Q},$$

which will induce a $\mathbb{Q}$-linear functional on $H_2(X, \mathbb{Q})$. If $C \mapsto [C]$ gives an injection $N_1(X, \mathbb{Q}) \hookrightarrow H_2(X(\mathbb{C}), \mathbb{Q})$, then we can view $C \mapsto \Theta \cap [C]$ as a linear map

$$\Theta \cap N_1(X, \mathbb{Q}) \to \mathbb{Q}.$$

By the previous lemma, line bundles span the dual space of $N_1(X, \mathbb{Q})$, so there is a line bundle L on X and an $m > 0$ such that $\deg(L|_C) = m \cdot \Theta \cap [C]$ for every integral curve $C \subset X$. Thus

$$\deg(L|_C) = m \cdot \Theta \cap [C] \geq m\epsilon \cdot \text{mult}_p C$$

for every integral curve $C \subset X$ and every $p \in C$. Then L is ample by the line bundle version Seshadri criterion, so X is projective.

It remains to show that $C \mapsto [C]$ gives an injection $N_1(X, \mathbb{Q}) \hookrightarrow H_2(X(\mathbb{C}), \mathbb{Q})$ if X has 1-rational singularities. (see Note-1). □

The relative version is also true.

Proposition 2.9. Let $f : X \to S$ be a proper Moishezon morphism, such that fibers have rational singularity for $s \neq 0$. Then f is projective morphism iff there is a cohomology class $\Theta \in H^2(X, \mathbb{Q})$ and $\epsilon > 0$ such that on each fiber

$$\Theta|_{X_s} \cap [C] \geq \epsilon \cdot \mathrm{mult}_p C$$

for every integral curve $C \subset X_s$, and $p \in C$.

Proof. □

2.5 Kollár-Paz Klieman's ampleness criterion for Moishezon spaces

Proposition 2.10 ([VP21])**.** Suppose that Y is a Moishezon space with $\mathbb{Q}$-factorial, log terminal singularities and that L is a Cartier divisor on Y. Then L is ample if and only if L has positive degree on every irreducible curve on Y and L induces a strictly positive function on $\overline{\mathrm{NE}}(Y)$.

Remark 2.11. It reamins open if the result is still true without the $\mathbb{Q}$-factorial KLT assumption.

Proof. The proof is bit involved and it will be given in the next note. □

3 Approximate the Chow-Barlet cycle space by countable many projective morphism

In this Section, we will study the main technical tools: Chow-Barlet cycle space, which allows to approximate the Chow-Barlet cycle space using countable many projective morphism. The tool in this section is crucial for the proof of Theorem 6.1.

Theorem 3.1 (Approximation Chow-Barlet cycle space using countable many projective morphisms)**.** Let $g : X \to S$ be a proper morphism of complex analytic spaces that is bimeromorphic to a projective morphism. Fix $m \in \mathbb{N}$. Then there are countably many diagrams of complex analytic spaces over S,

$$\begin{tikzcd} C_i \arrow[r, hook] \arrow[d, shift right, "w_i"'] & W_i \times_S X \\ W_i \arrow[u, shift right, "\sigma_i"'] & \end{tikzcd}$$

indexed by $i \in I$, such that

(1) the $w_i : C_i \to W_i$ are proper, of pure relative dimension 1 and flat over a dense, Zariski open subset $W_i^\circ \subset W_i$,

(2) the fiber of w_i over any $p \in W_i^\circ$ has multiplicity m at $\sigma_i(p)$,

(3) the W_i are irreducible, the structure maps $\pi_i : W_i \to S$ are projective, and

(4) the fibers over all the W_i° give all irreducible curves that have multiplicity m at the marked point.

Proof. We will divide the proof into several steps.

□

4 Kollár Projectivity on very general fiber

The following upper semi-continuity result is needed in the proof.

Lemma 4.1 (upper semi-continuity of the multiplicities, [BM19, Proposition 4.3.10])**.** Let $(X_s)_{s\in S}$ be an analytic family of n-cycles of a complex space M. Then the function

$$S \times M \longrightarrow \mathbb{N}, (s, z) \mapsto \mathrm{mult}_z\,(X_s)$$

is upper semicontinuous in the Zariski topology of $S \times M$.

Proof. The proof of the lemma is a big complicated and we omit it here. □

Remark 4.2. In particular, let $f : X \to S$ be a proper flat morphism of relative dimension 1, assume that there is a holomorphic section $\sigma : S \to X$. Then the multiplicity

$$\mathrm{mult} : S \to \mathbb{Z}, \quad s \mapsto \mathrm{mult}_{\sigma(s)} X_s$$

is Zariski upper-semicontinuous.

Proof. Since the fibers $\{X_s\}$ clearly forms an anlytic family of cycles in X. Since the section map $\sigma : S \to X$ is holomorphic, thus

$$S \to S \times X \to \mathbb{N}, \quad s \mapsto (s, \sigma(s)) \mapsto \mathrm{mult}_{\sigma(s)} X_s,$$

as composition of continuous function and upper semi-continuous function is upper semi-continuous. □

We can now prove the following theorem, which is the key step to deduce the main theorem.

Theorem 4.3 (Projectivity on very general fiber, [Kol22, Proposition 14])**.** Let $g : X \to S$ be a proper morphism of complex analytic spaces and $S^* \subset S$ a dense, Zariski open subset such that g is flat over S^*. Assume that
(1) X_0 is projective for some $0 \in S$,
(2) the fibers X_s have rational singularities for $s \in S^*$, and
(3) g is bimeromorphic to a projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to S$.

Then there is a Euclidean open neighborhood $0 \in U \subset S$ and countably many nowhere dense, closed, analytic subsets $\{H_j \subset U : j \in J\}$, such that X_s is projective for every $s \in U \setminus \cup_j H_j$.

Proof. □

5 The alternating property of projective locus

In this section, we will finish the proof of the alternating property about the projective locus.

Proposition 5.1 ([Kol22, Proposition 17])**.** Let $g : X \to S$ be a proper morphism of normal, irreducible analytic spaces. Then there is a dense, Zariski open subset $S^\circ \subset S$ such that
(1) either X is locally projective over S°,
(2) or $\mathrm{PR}_S(X) \cap S^\circ$ is locally contained in a countable union of Zariski closed, nowhere dense subsets.

If g is bimeromorphic to projective morphism, then X is projective over S^o.

Proof. ☐

Sometimes we need the global projective condition, which is guranteed if the Moishezon condition holds. Before proving the proposition, we need to the following statement on the local system of Neron-Sever group.

Proposition 5.2.

Proof. ☐

Proposition 5.3. Assume that $g : X \to S$ be a proper Moishezon morphism of normal irreducible analytic spaces. Assume that there exist a dense Zariski open subset $S^o \subset S$ such that X is locally projective over S^o then it's actually global projective.

Proof. ☐

Beginer provided a similar statement about the distribution of projective locus.

6 Kollár's stratification of Moishezon morphism to projective morphisms

Now we can prove the main theorem of this note.

Theorem 6.1 (Openess of projectivity, [Kol22, Theorem 2])**.** Let $g : X \to S$ be a proper morphism of complex analytic spaces and $S^* \subset S$ a dense, Zariski open subset such that g is flat over S^*. Assume that

(1) X_0 is projective for some $0 \in S$,

(2) the fibers X_s have rational singularities for $s \in S^*$, and

(3) g is bimeromorphic to a projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to S$.

Then there is a Zariski open neighborhood $0 \in U \subset S$ and a locally closed, Zariski stratification $U \cap S^* = \cup_i S_i$ such that each

$$g|_{X_i} : X_i := g^{-1}(S_i) \to S_i \text{ is projective.}$$

Proof. □

7 Claudon-Höring's projectivity criterion for Kähler morphism

In this section, we try to prove the following projectivity criterion for Kähler morphism (due to Claudon-Höring).

Theorem 7.1 ([CH24, Theorem 3.1])**.** Let $f : X \to Y$ be a fibration between normal compact Kähler spaces. Assume that X has strongly $\mathbb{Q}$-factorial klt singularities. Assume one of the following:
(a.1) The normal space Y has klt singularities and the natural map

$$f^* : H^0\left(Y, \Omega_Y^{[2]}\right) \longrightarrow H^0\left(X, \Omega_X^{[2]}\right)$$

is an isomorphism.
(a.2) The morphism f is Moishezon.

Then f is a projective morphism.

Proof. If time permit, we will dicuss it in the next time. □

Final words, there are other projectivity critera developed by Kollár in [Kol90] (which we does not cover in this note).

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