Divisorial contraction in Kähler MMP reading notes Spring 2025

Lecture 4 — 25, 02, 2025 (draft version)

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1 Overview

2 Das-Hacon's approach to divisorial contraction for Kähler 3-fold MMP

In this section, we will prove the following theorem.

Theorem 1 ([DH24, Theorem 6.9])**.** Let (X, B) be a strong $\mathbb{Q}$-factorial Kähler 3-fold KLT pair. With the following condition holds

1. $K_X + B$ is pseudo-effective
2. $\alpha = [K_X + B + \beta]$ is nef and big class such that β is Kähler,
3. The negative extremal ray $R = \overline{\mathrm{NA}}(X) \cap \alpha^{\perp}$ is divisorial.

Then there exists an α-trivial divisorial contraction

$$f : X \to Z,$$

such that there exist some Kähler form α_Z on Z such that $\phi^*(\alpha_Z) = \alpha$.

Before going to the proof let us briefly sketch the idea. We first try to prove that the null locus $\mathrm{Null}(\alpha)$ is the Moishezon surface whose smooth model is projective uniruled. We then take a DLT modification

$$\varphi : (X', \Delta') \to (X, \Delta)$$

of the pair $(X, \Delta = B + (1-b)S)$ (note that this pair (X, Δ) differs from the original pair (X, B) and it is not a KLT pair).

We show that the DLT modification φ preserve the geometry outside the null locus $\mathrm{Null}(\alpha)$. We then run the relative Kähler MMP for (X', Δ') over (X, Δ), which becomes the core of the proof. Since it's Kähler 3-fold MMP, the termination is known. So that it's possible to produce positivity (say $K_{X^m} + \Delta^m$ is nef over (X, Δ)) by the termination theorem.

We need to control the divisors being contracted in the MMP.

So that the induced bimeromorphic map $f : X \dashrightarrow X^m$ is a morphism, and this is the divisorial contraction we want.

In the final step, we will show that the base point freeness holds for the divisorial contraction, say α as pull back of some Kähler form α_Z down stairs.

2.1 The null locus is a Moishezon surface whose smooth model is projective uniruled

In this subsection, we will proof the following lemma.

Lemma 2. In the same setting as Theorem 1. The null locus $\mathrm{Null}(\alpha)$ is a irreducible Moishezon surface, whose smooth model is projective uniruled. Such that the curves in the negative extremal ray R covers the surface S with

$$R \cdot S < 0.$$

Remark 3. Let us breifly sketch the idea. The class $\alpha|_S, \alpha|_{S^\nu}, \alpha|_{S'}$ play important role in this lemma (for simplicity let us assume for now that S is a smooth surface). The idea is to try to use that if a smooth surface is not pseudo-effective, then it's uniruled projective surface. The non-pseudo-effectiveness comes from some intersection number analysis. To be more precise, we will use that $S = \mathrm{Null}(\alpha)$, so that volume $\mathrm{vol}(\alpha|_S) = (\alpha|_S)^2 = 0$ (by definition of null locus). In particular, the restriction $\alpha|_S$ can not be a big class. On the other hand, we can apply adjunction to

$$\alpha|_S = (K_X + B + \beta)|_S.$$

If the coefficient of S in B is 1, then everything is nice and we get

$$\alpha|_S = (K_X + B' + S + \beta)|_S = K_S + B'|_S + \beta|_S.$$

Since $B'|_S \geq 0$ and $\beta|_S$ Kahler, this will imply that K_S can not be pseudo-effective.

However, the coefficient of S in B is not 1, so that we need to take some scaling,

What nice on the projective uniruled surface is that the (0,2)-Hodge number is 0, so that the Bott-Chern class can be realized as a $\mathbf{R}$-divisor (which is also a $\mathbb{R}$-curve on the surface).

Finally, we need to prove that $R \cdot S < 0$. To do this, Batyrev cone theorem for mobile curve is applied.

Proof. □

2.2 Take DLT modification

2.3 Run the relative MMP

2.4 Control the set of divisors being contracted

2.5 Proof of the base point freeness

3 Höring-Peternell's approach for Kähler 3-fold MMP

References

[DH24] Omprokash Das and Christopher Hacon, *On the minimal model program for kähler 3-folds*, 2024.