Flipping contraction in Kähler MMP reading notes Spring 2025

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1 Overview

2 Das-Hacon's approach to flipping contraction for Kähler 3-fold MMP

In this section, we will prove the following theorem.

Theorem 1 ([DH24, Theorem 6.9])**.** Let (X, B) be a strong $\mathbb{Q}$-factorial Kähler 3-fold KLT pair. With the following condition holds

1. $K_X + B$ is pseudo-effective
2. $\alpha = [K_X + B + \beta]$ is nef and big class such that β is Kähler,
3. The negative extremal ray $R = \overline{\mathrm{NA}}(X) \cap \alpha^{\perp}$ is flipping type.

Then there exists an α-trivial flipping contraction

$$f : X \to Z,$$

such that there exist some Kähler form α_Z on Z such that $\phi^*(\alpha_Z) = \alpha$. And the flip exist.

Remark 2. Before proving the theorem, let us briefly sketch the idea of the proof.

2.1 Apply the DLT modification

2.2 Find a negative extremal ray on one of the component $S' \subset \lfloor \Delta' \rfloor$

2.3 Apply the PLT contraction theorem

In this step, we try to extend to contraction from $S' \to T'$ to $X' \to Z'$.

2.4 Prove the contraction morphism $X' \to Z'$ is divisorial or flipping contraction

We try to prove the contraction we just contructed is a divisorial or flipping contraction, thus it's an MMP step.

2.5 Control the MMP

Similar to the divisorial contraction case, the MMP we just contructed has some good properties, we will show that

2.6 Find the contraction $f : X \to Z$

In this step, we take the normalization on the graph of the map $\psi : X \dashrightarrow X^+$. Let $\eta := \alpha + \delta\,(K_X + B)$, we try to show that

$$E = p^*\eta - q^*\eta^+,$$

satisfies the condition that $-E|_E$ is ample. So that we can apply the Grauert contraction theorem, and get a contraction $g : W \to Z$.

We then try to show that this contraction will induce $f : X \to Z$ and $f^+ : X^+ \to Z$ as the diagram below shows.

$$\begin{tikzcd}
X' \arrow[r, dashed, "\phi"] \arrow[d, "\nu"'] & X^+ \arrow[ddr, "f^+"] & \\
X \arrow[ur, dashed, "\psi"] \arrow[drr, "f"'] & W \arrow[l, "p"'] \arrow[u, "q"'] \arrow[dr, "g"] & \\
& & Z
\end{tikzcd}$$

We then show that $f : X \to Z$ will contract every curves in the negative extremal ray R and it's a $(K_X + B)$-negative contraction.

2.7 Prove base point freeness

Just as what we did in the divisorial contraction case, we try to use that fact that proper bimeromorphic morphism $f : X \to Z$ between Kähler varieties with rational singularity satisfies the condition

$$\mathrm{im}(f^*) = \{\alpha \in H^{1,1}_{\mathrm{BC}}(X) \mid \alpha \cdot C = 0,\ \forall\, C \in N_1(X/Z)\}.$$

Then apply singular version Demailly-Păun Kahlerness criterion to the big and nef class α_Z.

2.8 Prove the existence of flips in any dimension

3 Höring-Peternell's approach for flipping contraction

References

[DH24] Omprokash Das and Christopher Hacon, *On the minimal model program for kähler 3-folds*, 2024.