

## 1 Overview

The aim of this note is to study the fiberwise bimeromorphic map in birational geometry. The major reference of this note are [Kol22].

## Contents

1	Overview	1
2	Criteria for fiberwise bimeromorphic map	1
3	Kontsevich-Tschinkel's fiberwise birational theorem (without non-vanishing condition)	4
4	Fiberwise bimeromorphic criterion and plurigenera	6
5	Proof of fiberwise bimeromorphic conjecture when $X_0$ is KLT and not uniruled	6

## 2 Criteria for fiberwise bimeromorphic map

**Definition 2.1** (Meromorphic  $S$ -map). Let  $X, Y$  be reduced complex space. We call the  $S$ -map a meromorphic  $S$ -map (not necessary morphism) if

$$\begin{array}{ccc}
 & \Gamma & \\
 p \swarrow & & \searrow q \\
 X & \overset{\alpha}{\dashrightarrow} & Y \\
 f \searrow & & \swarrow g \\
 & S &
 \end{array}$$

with  $\Gamma \subset X \times_S Y$ , and  $p : \Gamma \rightarrow X$  is a proper bimeromorphic morphism. Moreover if  $q$  is also proper bimeromorphic morphism, then we call  $\alpha$  proper bimeromorphic  $S$ -map.

**Definition 2.2** (Fiberwise bimeromorphic map, [Kol22, Definition 26]). Let  $g_i : X^i \rightarrow S$  be a proper morphisms. A bimeromorphic map  $\phi : X^1 \dashrightarrow X^2$  is fiberwise bimeromorphic if  $\phi$  induces a bimeromorphic map  $\phi_s : X_s^1 \dashrightarrow X_s^2$  for every  $s \in S$ .

**Remark 2.3.** In general, the bimeromorphic  $S$ -map does not need to be fiberwise bimeromorphic.

**Remark 2.4.** The fiberwise bimeromorphic phenomenon is closely related to the separatedness property in the moduli theory. To be more precise ...

Although the bimeromorphic map is not fiberwise bimeromorphic in general, it is indeed fiberwise bimeromorphic on a dense open subset.

**Proposition 2.5** (Bimeromorphic  $S$ -map is generic fiberwise bimeromorphic). Let  $f : X \dashrightarrow Y$  be a bimeromorphic  $S$ -map between complex varieties over the base  $S$ , prove that on the generic fiber the morphism induces a bimeromorphic map on the fiber.

*Proof.* Since  $f$  is bimeromorphic there exist some open dense subset such that  $f|_V : V \xrightarrow{\sim} U$  then I claim the morphism induce bimeromorphic map on the fibers  $X_s$  such that  $X_s \cap V \neq \emptyset$ .

Indeed since  $X_s \cap V \subset X_s$  is dense in  $X_s$  indeed we have

$$\overline{X_s \cap V} \subset X_s \cap \overline{V} = X_s \cap X = X_s$$

thus we have  $X_s \cap V$  dense in  $X_s$ .

we have that  $X_s \cap V$  is dense open subset of  $X_s$ , and therefore it induce an bimeromorphism on the fiber

$$X_s \dashrightarrow Y_s$$

Finally note that the set

$$\{s \in S \mid X_s \cap V \neq \emptyset\} = f(V) = \{s \in S \mid X_s \dashrightarrow Y_s \text{ is bimeromorphism}\}$$

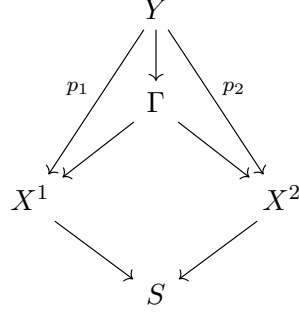
and image of dense subset under a continuous map is dense, thus we find the bimeromorphic map induce bimeromorphic map on the generic fiber of the morphism.

□

Under some additional assumption, the bimeromorphic  $S$ -map is indeed fiberwise bimeromorphic map.

**Proposition 2.6** (Fiberwise birational criterion under non-vanishing condition, [Kol23, Proposition 1.25]). Let  $f_i : X^i \rightarrow B$  be two smooth families of projective varieties over a smooth curve  $B$ . Assume that the generic fibers  $X_{k(B)}^1$  and  $X_{k(B)}^2$  are birational, and further assume that the pluricanonical system  $|mK_{X_{k(B)}^i}|$  is nonempty for some  $m > 0$ . Then we have fiberwise bimeromorphic condition.

*Proof.* Pick a birational map  $\phi : X_{k(B)}^1 \dashrightarrow X_{k(B)}^2$  (for the generic fiber), and let  $\Gamma \subset X^1 \times_B X^2$  be the closure of the graph of  $\phi$ . Let  $Y \rightarrow \Gamma$  be the normalization with projections  $p_i : Y \rightarrow X^i$ .



Note that both of the  $p_i$  are open embeddings on  $Y \setminus (\text{Ex}(p_1) \cup \text{Ex}(p_2))$ .

Thus if we prove that neither  $p_1(\text{Ex}(p_1) \cup \text{Ex}(p_2))$  nor  $p_2(\text{Ex}(p_1) \cup \text{Ex}(p_2))$  contains a fiber of  $f_1$  or  $f_2$ , then  $p_2 \circ p_1^{-1} : X^1 \dashrightarrow X^2$  (it needs not to be birational) restricts to a birational map  $X_b^1 \dashrightarrow X_b^2$  for every  $b \in B$ .

We may assume that  $B$  is affine (as we only care about the special fiber, thus we can focus on the affine base around  $b$ ) and let  $\text{Bs}|mK_{X^i}|$  denote the set-theoretic base locus. By assumption, we claim  $|mK_{X^i}|$  is not empty.

Since

$$\text{Bs}(|mK_{X^{i(b)}}|) \subset \text{Bs}(|mK_{X^i}|)|_{X^{i(b)}}$$

thus if the LHS is non-empty, then so will the right hand side.

Using that fact, torsion free = locally free for coherent sheaf on one dimensional non singular variety (smooth curve), and on the other hand we know that Direct image of locally free sheaf needs not to be locally free, direct image of torsion free sheaf is torsion free if the morphism is dominant. If we denote  $L = \mathcal{O}_{X^i}(mK_{X^i})$ , then

$$f_{i*}L = \mathcal{E}$$

is a locally free sheaf. On the other hand, since  $H^0(X, mK_{X^i}) \neq 0$  thus for any point  $s \in B$ , there exist a section

$$0 \neq \sigma \in H^0(B, \mathcal{E})$$

such that  $\sigma(s) \neq 0$ .

Therefore consider the restriction map

$$\begin{array}{ccc} H^0(X^i, L) & \xrightarrow{\text{res}} & H^0(X_s^i, L|_{X_s^i}) \\ \cong \downarrow & & \downarrow \\ H^0(B, f_*L) & \xrightarrow{\text{res}_s} & f_*L(s) \end{array}$$

such that there exist a section  $s \in H^0(X^i, L)$  which maps down to  $\sigma$  such that  $\sigma(s) \neq 0$ . So that  $s|_{X^i_s} \neq 0$ . And therefore the base locus can not contains the fiber.

Since the  $X^i$  are smooth,

$$K_Y \sim p_i^* K_{X^i} + E_i, \quad \text{where } E_i \geq 0 \text{ and } \text{Supp } E_i = \text{Ex}(p_i)$$

So that every section of  $\mathcal{O}_Y(mK_Y)$  pulls back from  $X^i$ , Thus

$$\text{Bs } |mK_Y| = p_i^{-1}(\text{Bs } |mK_{X^i}|) + \text{Supp } E_i$$

Comparing these for  $i = 1, 2$ , we conclude that

$$p_1^{-1}(\text{Bs } |mK_{X^1}|) + \text{Supp } E_1 = p_2^{-1}(\text{Bs } |mK_{X^2}|) + \text{Supp } E_2$$

Therefore,

$$\boxed{p_1(\text{Supp } E_2) \subset p_1(\text{Supp } E_1) + \text{Bs } |mK_{X^1}|}$$

Since  $E_1$  is  $p_1$ -exceptional,  $p_1(\text{supp } E_1)$  has codimension  $\geq 2$  in  $X^1$ , hence it does not contain any of the fibers of  $f_1$ . We saw that  $\text{Bs } |mK_{X^1}|$  does not contain any of the fibers either. Thus  $p_1(\text{Ex}(p_1) \cup \text{Ex}(p_2))$  does not contain any of the fibers, and similarly for  $p_2(\text{Ex}(p_1) \cup \text{Ex}(p_2))$ .  $\square$

As a remark in [Kol23], the result holds true even when the pluricanonical system are empty. That's what we will prove in the next section.

### 3 Kontsevich-Tschinkel's fiberwise birational theorem (without non-vanishing condition)

**Theorem 3.1** ([KT19, Theorem 1]). Let

$$\pi : \mathcal{X} \rightarrow B \text{ and } \pi' : \mathcal{X}' \rightarrow B$$

be smooth proper morphisms to a smooth connected curve  $B$ , over a field of characteristic zero. Assume that the generic fibers of  $\pi$  and  $\pi'$  are birational over the function field of  $B$ . Then, for every closed point  $b \in B$ , the fibers of  $\pi$  and  $\pi'$  over  $b$  are birational over the residue field at  $b$ .

The proof of Kontsevich-Tschinkel's theorem involve some new notions, let us first introduce it.

**Definition 3.2** (semi-ring). A semiring  $(S, +, \times)$  consists of a set  $S$  equipped with two binary operations  $+$ ,  $\times$ . Such that  $+$  makes  $S$  an commutative monoid (which needs not to be Abelian group compared with Ring).

**Definition 3.3** (Burnside ring over a field  $k$ ). The Burnside semi-ring  $\text{Burn}_+(k)$  of a field  $k$  is the set of  $\sim_k$ - equivalence classes of smooth schemes of finite type over  $k$  endowed with a semi-ring structure where multiplication and addition are given by disjoint union and product over  $k$ . (where wo such schemes  $X, X'$  we have  $[X/k] = [X'/k]$  (i.e.  $\sim_k$ ) if and only if  $X$  and  $X'$  are  $k$ -birational).

We denote by  $\text{Burn}(k)$  the Grothendieck ring generated by  $\text{Burn}_+(k)$ .

**Remark 3.4.** The motivation to introduce the Grothendieck ring is that it allow to do formal subtraction. And it allows some cut and paste operation on the geometric level.

**Remark 3.5.** Note that we can decompose the

$$\text{Burn}(k) = \sqcup_{n \geq 0} \text{Bir}_n(k),$$

where  $\text{Bir}_n(k)$  denotes  $k$ -birational equivalent class of smooth variety of dimension  $n$ . Each class can be denoted by  $[L/k]$  with  $L = k(X)$ .

**Definition 3.6** (Specialization map). We define

$$\rho_n : \text{Bir}_n(K) \rightarrow \mathbb{Z}[\text{Bir}_n(k)],$$

as follows: given a smooth projective variety  $X/K$ , choose one of family

$$\pi : \mathcal{X} \rightarrow D,$$

where  $\pi$  is proper, such that generic fibers are  $X$  and special fiber

$$X_0 = \bigcup_{i \in I} d_i D_i,$$

is a SNC divisor, with the strata  $D_J := \bigcap_{j \in J} D_j$ . We then define the specialization map to be

$$\rho_n([L/K]) := \sum_{\emptyset \neq J \subseteq I} (-1)^{\#J-1} [D_J \times \mathbb{A}^{\#J-1}/k].$$

One of the main difficulties of the proof is to check the specialization map is well defined (i.e. it does not depend on the choose the family  $\mathcal{X} \rightarrow D$ ) and representative  $X$  in  $\text{Bir}_n(X)$ . We omit the proof of this part, for more detail of the proof see [KT19, Theorem 4]).

*Proof.* Let  $\pi : \mathcal{X} \rightarrow B$  be a smooth proper morphism to a smooth connected curve  $B$  over  $k$  with fiber  $X$  over the generic point of  $B$ . Let  $K = k(B)$  be the function field of  $B$ . Let  $\kappa_b$  be the residue field at  $b$ , a finite extension of  $k$ . Let  $K_b$  be the completion of  $K$  at  $b$ . It is a local field with residue field  $\kappa_b$ , isomorphic to  $\kappa_b((t))$ , where  $t$  is a formal local coordinate. Let

$$\phi_b : K \rightarrow K_b$$

be the canonical inclusion. By functoriality (see Remark 3), it defines a homomorphism

$$\phi_{b,*} : \text{Burn}(K) \rightarrow \text{Burn}(K_b).$$

We have the specialization homomorphism

$$\rho : \text{Burn}(K_b) \rightarrow \text{Burn}(\kappa_b)$$

and the following identity

$$[X_b/\kappa_b] = \rho(\phi_{b,*}([X/K])),$$

which follows immediately from the definition of  $\rho$ , since the special fiber is smooth and irreducible. This shows that the birational type of the special fiber is determined by the birational type of the generic fiber.  $\square$

## 4 Fiberwise bimeromorphic criterion and plurigenera

In this section, we will give a criterion for fiberwise bimeromorphic map using the plurigenera as detector. The major reference of this part of the note is the paper [CRT25].

**Theorem 4.1** ([CRT25]). Let

$$\pi_1 : \mathcal{X} \rightarrow S, \pi_2 : \mathcal{Y} \rightarrow S$$

be two Moishezon morphism, such that all the fibers of  $\pi_1$  has  $\kappa(X_t) \geq 0$ . Then the bimeromorphic map that connect  $\pi_1$  and  $\pi_2$  is indeed fiberwise bimeromorphic.

*Proof.* □

## 5 Proof of fiberwise bimeromorphic conjecture when $X_0$ is KLT and not uniruled

In the last section, we will prove the following conjecture in this section under the assumption that center fiber is KLT and not uniruled.

**Conjecture 5.1** (Fiberwise bimeromorphic conjecture for Moishezon morphism, see [Kol22, Conjecture 5]). Let  $g : X \rightarrow \mathbb{D}$  be a flat, proper, Moishezon morphism. Assume that  $X_0$  has canonical (resp. log terminal) singularities.

Then  $g$  is fiberwise birational to a flat, projective morphism  $g^p : X^p \rightarrow \mathbb{D}$  such that

- (1)  $X_0^p$  has canonical (resp. log terminal) singularities,
- (2)  $X_s^p$  has terminal singularities for  $s \neq 0$ , and
- (3)  $K_{X^p}$  is  $\mathbb{Q}$ -Cartier.

**Remark 5.2.** Before continuing the discussion about this conjecture, let us first look closely at what this conjecture is about. The conjecture shows that flat Moishezon morphism is not only bimeromorphic to some projective model it's indeed fiberwise bimeromorphic to some projective model, if we assume the singularity on the central fiber is nice.

Kollár verifies the conjecture when the central fiber is KLT and not uniruled. Before proving the theorem, let us list some intermediate results that will be used.

**Theorem 5.3** (Inversion of adjunction, [Kol22, Proposition 30]). Let  $X$  be a normal, complex analytic space,  $X_0 \subset X$  a Cartier divisor and  $\Delta$  an effective  $\mathbb{R}$ -divisor such that  $K_X + \Delta$  is  $\mathbb{R}$ -Cartier. Then  $(X, X_0 + \Delta)$  is PLT in a neighborhood of  $X_0$  iff  $(X_0, \Delta|_{X_0})$  is KLT.

*Proof.* The proof is omit here. □

**Theorem 5.4** (Canonical modification theorem, see [Kol22, colloary 30]). Let  $f : X \rightarrow \mathbb{D}$  be a flat, proper, Moishezon morphism. Assume that  $X_0$  is log terminal. Then  $X$  has a canonical modification  $\pi : X^c \rightarrow X$ , such that

- (a)  $X_0^c$  is log terminal and,
- (b)  $\pi$  is fiberwise birational.

*Proof.* □

**Lemma 5.5** (A limiting expression for restricted base locus, see [Kol22], (31.1)). Let  $X \rightarrow S$  be a proper, Moishezon morphism,  $D$  an  $\mathbb{R}$ -divisor on  $X$ , and  $A$  a big  $\mathbb{R}$ -divisor on  $X$  such that  $\mathbf{B}^{\text{div}}(A) = \emptyset$ . Then, for every prime divisor  $F \subset X$ ,

$$\text{coeff}_F \mathbf{B}_-^{\text{div}}(D) = \lim_{\epsilon \rightarrow 0} \text{coeff}_F \mathbf{B}_-^{\text{div}}(D + \epsilon A)$$

*Proof.* □

**Lemma 5.6** (An estimate for restricted base locus, see [Kol22], (31.2)). Let  $X_i \rightarrow S$  be proper, Moishezon morphisms,  $h : X_1 \rightarrow X_2$  a proper, bimeromorphic morphism,  $D_2$  a pseudo-effective,  $\mathbb{R}$ -Cartier divisor on  $X_2$ , and  $E$  an effective,  $h$ -exceptional divisor. Then

$$\mathbf{B}_-^{\text{div}}(E + h^* D_2) \geq E$$

*Proof.* □

Finally, let me make a remark on why restricted base locus is useful here, indeed the restricted base locus contains precisely the divisors that will be contracted by the minimal model program:

**Theorem 5.7** (Restricted base locus contains the divisors that will be contracted by the MMP).

Now we can go into the proof of the theorem

**Theorem 5.8** (A flat Moishezon morphism with KLT and non-uniruled central fiber will be fiberwise bimeromorphic to a projective morphism, [Kol22], Theorem 28). Let  $g : X \rightarrow \mathbb{D}$  be a flat, proper, Moishezon morphism. Assume that

1.  $X_0$  has log terminal singularities and
2.  $X_0$  is not uniruled

Then

- (a)  $g$  is fiberwise birational to a flat, projective morphism  $g^p : X^p \rightarrow \mathbb{D}$  (possibly over a smaller disc),
- (b)  $X_0^p$  has log terminal singularities,
- (c)  $X_s^p$  is not uniruled and has terminal singularities for  $s \neq 0$ ,
- (d)  $K_{X^p}$  is  $\mathbb{Q}$ -Cartier

*Proof.* We take a resolution of singularities  $Y \rightarrow X$  such that  $Y \rightarrow \mathbb{D}$  is projective, and then take a relative minimal model of  $Y \rightarrow \mathbb{D}$ . We hope that it gives what we want. There are, however, several obstacles. Next we discuss these, and their solutions, but for all technical details we refer to later sections.

**Step 1. Reduce the variety to the one that has  $\mathbb{Q}$ -Cartier canonical divisor.**

We need to control the singularities of  $X$ . First for a flat proper Moishezon morphism with KLT central fiber, there exist a canonical modification which is fiberwise birational and the central fiber is KLT reduces us to the case when  $K_X$  is  $\mathbb{Q}$ -Cartier.

Indeed by the canonical modification we can find some canonical modification  $X^c \rightarrow X$  such that  $X^c$  is canonical singularity and the morphism  $X^c \rightarrow X$  is the fiberwise birational map, thus if we can prove the result for  $X^c \rightarrow \mathbb{D}$  then it will also be true for the  $X \rightarrow \mathbb{D}$  (since composition of fiberwise birational map is again fiberwise birational)

We assume this from now on. Then the inversion of adjunction for PLT pair implies that the pair  $(X, X_0)$  is PLT. by setting  $\Delta = 0$  in the inversion of adjunction. (To apply the inversion of adjunction here we require  $K_X$  to be  $\mathbb{Q}$ -Cartier)

**Step 2. Take base change morphism require the projective model to a semistable one.**

After a base change  $z \mapsto z^r$  we get  $g^r : X^r \rightarrow \mathbb{D}$ . For suitable  $r$ , there is a semi-stable, projective resolution  $h : Y \rightarrow \mathbb{D}$ ; we may also choose it to be equivariant for the action of the cyclic group  $G \cong \mathbb{Z}_r$ . All subsequent steps will be  $G$ -equivariant. We denote by  $X_0^Y$  the birational transform of  $X_0$  and by  $E_i$  the other irreducible components of  $Y_0$ .

**Step 3. The generic fibers are not uniruled.**

We will prove it by contradiction, note that for a dominant morphism if the source is uniruled then so is the target (see [Rationalcurve] IV. 1.2 Lemma). On the other hand, since the deformation limit of uniruled variety is uniruled on each irreducible and reduced components (see [Rationalcurve] IV 1.7) We have  $X_0^Y$  being uniruled but then  $X_0$  will also be uniruled which contradicts to the assumption.

And finally by [BDPP] Corollar 0.3. easy to see  $K_{Y_s}$  is pseudo-effective.

**Step 4. Run the MMP using BCHM**

We require the condition that the general fibers are of log general type. To achieve this, let  $H$  be an ample,

$G$ -equivariant divisor such that  $Y_0 + H$  is snc. For  $\epsilon > 0$  we get a pair  $(Y, \epsilon H)$  whose general fibers  $(Y_s, \epsilon H_s)$  are of log general type since  $K_{Y_s}$  is pseudoeffective. For such algebraic families, relative minimal models are known to exist by BCHM.

We also know that  $(Y, Y_0 + \epsilon H)$  is dlt for  $0 < \epsilon \ll 1$ .

Thus we get the MMP

$$\phi : (Y, \epsilon H) \dashrightarrow (Y^m, \epsilon H^m),$$

**Step 5. Singularity of the output minimal model**

We claim  $(Y^m, Y_0^m + \epsilon H^m)$  is DLT, and  $H^m$  is  $\mathbb{Q}$ -Cartier for general choice of  $\epsilon$  and also thus  $(Y^m, Y_0^m)$  is also dlt.

Indeed Step of MMP will preserve DLT condition (see [BCHM] Lemma 3.10.10.) easy to see  $(Y^m, Y_0^m + \epsilon H^m)$  is DLT. On the other hand by Lemma 1.5.1. of [Alex], easy to see if  $\epsilon$  is sufficient general the  $\mathbb{Q}$ -linear independent condition satisfies and therefore  $H^m$  is indeed a  $\mathbb{Q}$ -Cartier divisor. And finally by [KM98] Corollary 2.39. the  $(Y^m, Y_0^m)$  is also DLT (note that we really need  $\mathbb{Q}$ -Cartier condition).



**Step 6. The minimal model will contract precisely the divisors  $E_i$ .** Recall that we have

$$\mathbf{B}_-^{\text{div}}(K_Y + Y_0) \geq (a_i + 1)E_i$$

On the other hand

$$\text{coeff}_F \mathbf{B}_-^{\text{div}}(D) = \lim_{\epsilon \rightarrow 0} \text{coeff}_F \mathbf{B}_-^{\text{div}}(D + \epsilon A)$$

for any prime divisor  $F$ . Thus for sufficient small  $\epsilon$   $E_i$  also contains in the restricted base locus of  $K_Y + Y_0 + \epsilon H$  then by Theorem ?? the MMP will contract those  $E_i$ .

**Step 7. The morphism  $X \dashrightarrow Y^m$  is fiberwise birational morphism.**

Since Cone theorem, those divisor being contracted will be covered by rational curves. But we assume that  $X_0$  is not uniruled. By Theorem 2.5 the generic fiber of  $X \dashrightarrow Y^m$  are bimeromorphic, that is we know for  $s \neq 0$  there is bimeromorphic mapping between the fibers.

On needs to prove that the central fiber  $X_0$  is bimeromorphic to the central fiber  $Y_0^m$ . Indeed by the definition of strict transform, we pick the defining domain of the birational map  $Y \rightarrow X$  so that  $V \xrightarrow{\sim} U$  and we pick  $X_0 \cap U \xrightarrow{\sim} X_0^Y \cap V$ , observe that  $X_0 \cap U \subset X_0$  dense (since  $\overline{X_0 \cap U} \subset \overline{X_0} \cap \overline{U} = X_0 \cap X = X_0$ ) and  $X_0^Y \cap V \subset X_0^Y$  dense. We get that  $X_0$  and  $X_0^Y$  are birational.

**Step 8. The pair  $(Y_s, \epsilon H_s)$  is terminal, and also the pair  $(Y_s^m, \epsilon H_s^m)$  and also  $Y_s^m$ .**

Note that  $h : Y \rightarrow \mathbb{D}$  is smooth away from  $Y_0$  (by the semi-stable family) thus  $(Y_s, \epsilon H_s)$  is terminal for  $s \neq 0$  and  $0 \leq \epsilon \ll 1$  (see [KM98] Corollary 2.35. (2))

Since  $H_s$  is ample, by negativity lemma we do not contract it.  $(Y_s^m, \epsilon H_s^m)$  is still terminal (since minimal model program preserve the terminal singularity indeed we have flip diagram and divisorial contraction preserve KLT (DLT, LC, terminal) singularity (see [KM98] Corollary 3.43) note that the divisorial contraction preserve the terminal singularity require the exceptional set does not contains in the support of  $H_s$ . Hence so is  $Y_s^m$  (see [KM98] Corollary 2.35.)

**Step 9. The central fiber has KLT singularity.**

$(Y^m, Y_0^m)$  is dlt (since DLT), hence it's also plt thanks to the irreducible of  $Y_0^m$  (see [KM98] Proposition 5.51.). And therefore  $Y_0^m$  is KLT by the easy direction of inversion of adjunction (see Theorem 5.3).  $\square$

**Remark 5.9.** Finally, let us say a few words about the subtle differences between the different statements above.

## References

- [CRT25] Jian Chen, Sheng Rao, and I-Hsun Tsai. *Characterization of fiberwise bimeromorphism and specialization of bimeromorphic types I: the non-negative Kodaira dimension case*. 2025. arXiv: 2506.12670 [math.AG]. URL: <https://arxiv.org/abs/2506.12670>.
- [Kol23] János Kollár. *Families of varieties of general type*. Vol. 231. Cambridge Tracts in Mathematics. With the collaboration of Klaus Altmann and Sándor J. Kovács. Cambridge University Press, Cambridge, 2023, pp. xviii+471. ISBN: 978-1-009-34610-8.
- [Kol22] János Kollár. “Moishezon morphisms”. In: *Pure Appl. Math. Q.* 18.4 (2022), pp. 1661–1687.

- 
- [KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*. Vol. 134. Cambridge Tracts in Mathematics. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. Cambridge University Press, Cambridge, 1998, pp. viii+254. ISBN: 0-521-63277-3. DOI: [10.1017/CB09780511662560](https://doi.org/10.1017/CB09780511662560). URL: <https://doi.org/10.1017/CB09780511662560>.
- [KT19] Maxim Kontsevich and Yuri Tschinkel. “Specialization of birational types”. In: *Invent. Math.* 217.2 (2019), pp. 415–432. ISSN: 0020-9910,1432-1297. DOI: [10.1007/s00222-019-00870-9](https://doi.org/10.1007/s00222-019-00870-9). URL: <https://doi.org/10.1007/s00222-019-00870-9>.