Applications of OT Extension Theorems Readings Notes Fall 2025

Note I.1 — 2025 09 12 (draft version 0)

Yi Li

1 Introduction

The aim of this note is to give an overview of applications of the Ohsawa–Takegoshi extension theorem in birational geometry. A more thorough discussion can be found in the series of notes that follow this one. In this note, we will only highlight the key ideas that appear in the proofs.

Contents

2 A Brief summary of Ohsawa-Takegoshi extension theorems

2.1 Local Version of Ohsawa-Takegoshi Extension Theorem

Theorem 2.1 (Local Version Ohsawa-Takegoshi Extension Theorem)**.** Assume $X = \mathbb{D}^n$ be a polydisc in $\mathbb{C}^n$ assume $Y = X \cap H$ (non empty) for some hyperplane.

(1) φ is a plurisubharmonic function on X, and

(2) $f \in H^0(Y, \mathcal{O}_Y)$ is a holomorphic funtion on Y (i.e. the trivial line bundle on Y) such that

$$\int_Y |f|^2 e^{-\varphi} < +\infty$$

Then

(a) There exist a holomorphic function $F \in H^0(X, \mathcal{O}_X)$ such that $F|_Y = f$,

(b) The norm estimate

$$\int_X |F|^2 e^{-\varphi} dV_X \leq C \int_Y |f|^2 e^{-\varphi} dV_Y$$

with constant C depends only on X and Y (independent of f and φ.)

Remark 2.2. As is well known to experts, to invoke Ohsawa-Takegoshi Extension theorem, one needs to check 2 conditions: (1) The positivity condition, (2) The L^2-condition on the section of the hyperplane.

The nontrivial part of the theorem lies in (b), the L^2-norm estimate, while (a) is a rather mild which holds unconditionally.

PROOF IDEA 2.3. The result is based on Hörmander's L^2-technique. To be more precise.

2.2 Global Version Ohsawa-Takegoshi Extension

Theorem 2.4 (Global Version Ohsawa-Takegoshi Extension Theorem)**.** Let $\pi : X \to \Delta$ be as before and let (L, h) be a (singular) Hermitian line bundle.

Let ω be a global Kähler metric on X, and dV_X, dV_{X_0} the respective induced volume elements on X_0 and X. Assume that h_{X_0} is well defined.

Then a section $u \in H^0(X_0, (K_X + L)|_{X_0})$ extends to X if the following 2 conditions hold

(a) The curvature current $\Theta_{L,h} \geqslant 0$ on X,

(b) The L2-estimate

$$\int_{X_0} \|u\|^2_{h|_{X_0}} < +\infty$$

(i.e. u vanish along the multiplier ideal $\mathcal{J}(h|_{X_0})$)

Moreover we have the following L^2 estimate

$$\int_X \|\widetilde{u}\|^2_{\omega\otimes h} dV_X \leqslant C_0 \int_{X_0} \|u\|^2_{\omega\otimes h} dV_{X_0},$$

where $C_0 \geqslant 0$ is some universal constant (independent of $X, L, \ldots$).

PROOF IDEA 2.5. Let us briefly sketch the idea of the proof.

2.3 Ohsawa-Takegoshi Extension for Vector Bundles

Theorem 2.6 (Ohsawa-Takegoshi Extension for Vector Bundles)**.**

2.4 Optimal Ohsawa-Takegoshi Extension

Theorem 2.7 (Optimal Ohsawa-Takegoshi Extension Theorem)**.** Let $p : X \to \Delta$ be a fibration from a Kähler manifold to the unit disc $\Delta \in \mathbb{C}^n$. and let L be a line bundle endowed with a possible singular metric h_L such that $\sqrt{-1}\Theta_{h_L}(L) \geq 0$ in the sense of current. Let $m \in \mathbb{N}$. We suppose that the center fiber X_0 is smooth and let $f \in H^0\left(X_0, mK_{X_0} + L\right)$ such that

$$\int_{X_0} |f|_{h_L}^{\frac{2}{m}} < +\infty$$

Then there exists a $F \in H^0\left(X, mK_{X/Y} + L\right)$ such that

(1) $F|_{X_0} = f$

(2) The following $L^{\frac{2}{m}}$ bound holds

$$\int_X |F|_{h_L}^{\frac{2}{m}} \leq C_0 \int_{X_0} |f|_{h_L}^{\frac{2}{m}}$$

where C_0 is the volume of the unit disc Δ.

2.5 Other Variants of Ohsawa-Takegoshi Extension

3 Ohsawa-Takegoshi extension theorem in extension of pluricanonical sections and Invariance of Plurigenera

In this section, we will see one of the major applications of Ohsawa-Takegoshi Extension says extension of pluricanonical sections.

Our first result is Siu's analytic proof of invariance of plurigenera. We will discuss this result in much more details in the following series of notes, for now we just highlight some key points of this result.

Theorem 3.1. Let $X \to S$ be a proper holomorphic map defining a family of smooth projective varieties of general type on an irreducible base S. Then the plurigenus $p_m\left(X_t\right) = h^0\left(X_t, mK_{X_t}\right)$ of fibers is independent of t for all $m \geqslant 0$.

PROOF IDEA 3.2.

Demailly-Hacon-Păun use variant version of Ohsawa-Takegoshi prove the following DLT extension theorem.

Theorem 3.3.

PROOF IDEA 3.4.

4 Ohsawa-Takegoshi extension in positivity results

Ohsawa-Takegoshi extension theorem are used in the construction of Bergman type metric. Bergman type metric plays an important role in positivity results in algebraic geometry.

Recently, [PaT18] and [HPS18] found an application of Optimal Ohsawa-Takegoshi extension in the proof of positivity of $f_*\omega_{X/Y}^{\otimes m}$. In this section, we briefly introduce the result that can be proof using Optimal Ohsawa-Takegoshi extension, further discussion can found found in

5 Ohsawa-Takegoshi extension theorem in extension of metric across the singular locus

There are two classical approach to extend a metric across singular locus: (1) Hartog's extension theorem, (2) Riemann extension theorem.

References

[HPS18] Christopher Hacon, Mihnea Popa, and Christian Schnell. "Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Păun". In: *Local and global methods in algebraic geometry*. Vol. 712. Contemp. Math. Amer. Math. Soc., 2018, pp. 143–195.

[PaT18] Mihai P˘ aun and Shigeharu Takayama. "Positivity of twisted relative pluricanonical bundles and their direct images". In: *J. Algebraic Geom.* 27.2 (2018), pp. 211–272.