Projectivity Criterira and it's deformation behavior Summer 2025

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1 Overview

The aim of this note is to give some projectivity critera for Moishezon/Kähler morphism and study the deformation bahavior of projectivity. The ultimate goal is to finish the proof of [Kol22, Theorem 2].

Contents

2 Projectivity critera of Nakai-Moishezon and Seshadri

2.1 Seshadri criterion of projectivity, line bundle version

2.2 Seshadri criterion of projectivity, cohomology class version

Numerical equivalent relation and Homological equivalent relation

Now we can prove the cohomological version Seshadri criterion.

Theorem 2.1 (see [Kol22], Proposition).

Last time we proved the Seshadri projectivity criterion of Moishezon variety.

This time we first introduce the Chow-Barlet cycle space and prove a technical result which shows that we can approximate the Chow-Barlet cycle space by countable many projective morphism. Using this result, we can prove the main theorem of today's seminar that is some lower semi-continuity result on Seshadri constant.

Let us brief sketch the idea of the proof.

3 Chow-Barlet cycle space

In this Section, we will study the main technical tools

We can now state the result which allows to approximate the Chow-Barlet cycle space using countable many projective morphism

Theorem 3.1 (Approximation Chow-Barlet cycle space using countable many projective morphisms)**.** Let $g : X \to S$ be a proper morphism of complex analytic spaces that is bimeromorphic to a projective morphism. Fix $m \in \mathbb{N}$. Then there are countably many diagrams of complex analytic spaces over S,

$$\begin{tikzcd} C_i \arrow[r, hook] \arrow[d, shift right, "w_i"'] & W_i \times_S X \\ W_i \arrow[u, shift right, "\sigma_i"'] & \end{tikzcd}$$

indexed by $i \in I$, such that

(1) the $w_i : C_i \to W_i$ are proper, of pure relative dimension 1 and flat over a dense, Zariski open subset $W_i^\circ \subset W_i$,

(2) the fiber of w_i over any $p \in W_i^\circ$ has multiplicity m at $\sigma_i(p)$,

(3) the W_i are irreducible, the structure maps $\pi_i : W_i \to S$ are projective, and

(4) the fibers over all the W_i° give all irreducible curves that have multiplicity m at the marked point.

Proof. We will divide the proof into several steps.

□

4 Kollár Openness of projectivity

Theorem 4.1 (Openness of projectivity over Euclidean open subset, see [Kol22], Proposition 14.)**.**

Proof. □

Theorem 4.2 (Openess of projectivity, [Kol22, Theorem 2])**.**

Let $g : X \to S$ be a proper morphism of complex analytic spaces and $S^* \subset S$ a dense, Zariski open subset such that g is flat over S^*. Assume that

(1) X_0 is projective for some $0 \in S$,

(2) the fibers X_s have rational singularities for $s \in S^*$, and

(3) g is bimeromorphic to a projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to S$.

Then there is a Zariski open neighborhood $0 \in U \subset S$ and a locally closed, Zariski stratification $U \cap S^* = \cup_i S_i$ such that each

$$g|_{X_i} : X_i := g^{-1}(S_i) \to S_i \text{ is projective.}$$

Proof. □

5 Claudon-Höring's projectivity criterion for Kähler morphism

6 Kollár's projectivity criterion for moduli space

References

[Kol22] János Kollár. "Seshadri's criterion and openness of projectivity". In: *Proc. Indian Acad. Sci. Math. Sci.* 132.2 (2022), Paper No. 40, 12. ISSN: 0253-4142,0973-7685. DOI: 10.1007/s12044-022-00680-9. URL: https://doi.org/10.1007/s12044-022-00680-9.