Projectivity Criterira Reading Seminars Fall 2024

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1 Overview

This time we will finish the proof of the main theorem of the paper [Kol22], Theorem 2. Before proving this openess of projectivity theorem, we need another result about alternating property of projective locus of a proper (Moishezon) morphism. Combined with the theorem about projectivity of very general fibers that we proved last time, the proof of the main theorem is immediate.

2 Alternating property of projective locus

In this section, we will focus on the proof of the following theorem

Theorem 2.1 (Alternating property of projective locus, see [Kol22], Proposition 17)**.**

Let $g : X \to S$ be a proper morphism of normal, irreducible analytic spaces. Then there is a dense, Zariski open subset $S^\circ \subset S$ such that

(1) either X is locally projective over S°,

(2) or $\mathrm{PR}_S(X) \cap S^\circ$ is locally contained in a countable union of Zariski closed, nowhere dense subsets.

Proof. The main technical tool is the sheaf exponential sequence with its restriction on the fiber.

□

3 Proof of the main theorem

Theorem 3.1 (Openess of projectivity, see [Kol22], Theorem 2)**.**

Let $g : X \to S$ be a proper morphism of complex analytic spaces and $S^* \subset S$ a dense, Zariski open subset such that g is flat over S^*. Assume that

(1) X_0 is projective for some $0 \in S$,

(2) the fibers X_s have rational singularities for $s \in S^*$, and

(3) g is bimeromorphic to a projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to S$.

Then there is a Zariski open neighborhood $0 \in U \subset S$ and a locally closed, Zariski stratification $U \cap S^* = \cup_i S_i$ such that each

$$g|_{X_i} : X_i := g^{-1}(S_i) \to S_i \text{ is projective.}$$

Proof. □

References

[Kol22] János Kollár. "Seshadri's criterion and openness of projectivity". In: *Proc. Indian Acad. Sci. Math. Sci.* 132.2 (2022), Paper No. 40, 12. ISSN: 0253-4142,0973-7685. DOI: 10.1007/s12044-022-00680-9. URL: https://doi.org/10.1007/s12044-022-00680-9.