BDPP reading notes Spring 2025

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1 Overview

This aim of this note is to introduce the BDPP theorem for projective [BDPP13] and Kahler manifold [Ou25]. Varies applications of the BDPP theorem are shown.

2 Transcendentla cone

On a compact Kähler manifold, there may not have plently of divisors. To make sense varies positivities, it is necessary to introduce the transcendentla cones.

3 Duality between varies cones

The following theorem shows the duality between pseudo-effective cone and movable cone on the projective manifold.

Theorem 1. Let X be a projective manifold, then the pseudo-effective cone is dual to the cone of movable curves

$$\mathcal{E} = \overline{\mathrm{Mov}(X)}^{\vee}.$$

In other words, a divisor is pseudo-effective iff it has non-negative intersection with any movable curves.

Remark 2. David [WN19] proved ...

4 Characterization of the projective uniruled manifold

The projective uniruled manifold is characterized by the pseudo-effectiveness of the canonical bundle.

Theorem 3 ([BDPP13, Corollary 0.3])**.** Let X be a projective manifold. Then X is uniruled iff K_X is not pseudo-effectiveness.

Remark 4. One direction of the proof is easy, and can be adopted to the Kähler manifold. The converse direction (say K_X is not pseudo-effective) implies uniruled of X is non-trivial, which requires the Mori bend and break technique and the duality between pseudo-effective cone and movable cone.

Remark 5. Miyaoka and Mori [MM86] proved that a projective manifold is uniruled iff there exist an open subset over which there exist a K_X-negative curve passing through it. For more discussion about Miyaokao-Mori theorem (and varies properties of uniruled manifold) see my Note 15.

We can generalize the BDPP theorem to the singular case.

Theorem 6. Let (X, B) be a $\mathbb{Q}$-factorial log pair. If $K_X + B$ is not pseudo-effective, then X is uniruled.

Remark 7. Rational curves on singular space is tricky. See more discussion on my notes note-9 Rational curves on Moishezon space, Kaehler varieties.

Proof. Taking the log resolution

$$f : X' \to X,$$

such that $f^*(K_X + B) = K_{X'} + B'$. Since being uniruled is birational invariant, if X is not uniruled, then so it is X'. Then by the BDPP theorem we just proved, $K_{X'}$ is pseudo-effective, thus K_X is pseudo-effective. Since B is effective, $K_X + B$ is pseudo-effective. □

5 Proof of BDPP conjecture for Kähler manifold

Recently, [Ou25] proved the BDPP conjecture for the compact Kähler manifold. In this section, we will briefly introduce the result that he proved.

5.1 Algebraic integrability criteria under Kähler setting

5.2 Proof of BDPP conjecture for compact Kähler manifold

6 Varies applications

6.1 Applications of duality of pseudo-effective cone and cone of movable curves

6.2 Producing rational curves using BDPP conjecture

6.3 Cone theorem using BDPP conjecture

References

[BDPP13] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Păun, and Thomas Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), no. 2, 201–248.

[MM86] Yoichi Miyaoka and Shigefumi Mori, *A numerical criterion for uniruledness*, Ann. of Math. (2) **124** (1986), no. 1, 65–69.

[Ou25] Wenhao Ou, *A characterization of uniruled compact kähler manifolds*, 2025.

[WN19] David Witt Nyström, *Duality between the pseudoeffective and the movable cone on a projective manifold*, J. Amer. Math. Soc. **32** (2019), no. 3, 675–689, With an appendix by Sébastien Boucksom.