General type locus and Moishezon locus — Summer 2025

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1 Overview

The aim of this note is to study the Moishezon locus and general type locus (i.e. the place that have general type fibers or Moishezon fibers).

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2 Moishezon locus, fiberwise Moishezon morphism is locally Moishezon under smoothness assumption

We begin this section by defining very big locus, general type locus and Moishezon locus.

Definition 2.1 (Very big locus, general type locus, Moishezon locus, see [**Moishezonmorphism**] Definition 18)**.**
Let $g : X \to S$ be a proper morphism of normal analytic spaces and L a line bundle on X. Set

1. $\mathrm{VB}_S(L) := \{s \in S : L_s \text{ is very big on } X_s\} \subset S$,
2. $\mathrm{GT}_S(X) := \{s \in S : X_s \text{ is of general type }\} \subset S$,
3. $\mathrm{MO}_S(X) := \{s \in S : X_s \text{ is Moishezon }\} \subset S$,
4. $\mathrm{PR}_S(X) := \{s \in S : X_s \text{ is projective }\} \subset S$.

here very big means the place $s \in S$ that

$$X_s \dashrightarrow \mathrm{Proj}_S(g_*L_s) = (\mathrm{Proj}_S(g_*L))_s$$

is birational onto its closure of the image.

We first show that very big locus admits alternating property i.e. it's either nowhere dense or contains some dense open subset

Theorem 2.2 (Alternating property for very big locus, see [**Moishezonmorphism**] Lemma 19)**.**
Let $g : X \to S$ be a proper morphism of normal irreducible analytic spaces(and therefore S is integral) and L a line bundle on X. Then $\mathrm{VB}_S(L) \subset S$ is

(1) either nowhere dense (in the analytic Zariski topology),
(2) or it contains a dense open subset of S, and $g : X \to S$ is Moishezon.

Proof. By passing to an open subset of S, we may assume that g is flat, g_*L is locally free and commutes with restriction to fibers. We get a meromorphic map $\phi : X \dashrightarrow \mathbb{P}_S(g_*L)$. There is thus a smooth, bimeromorphic model $\pi : X' \to X$ such that $\phi \circ \pi : X' \to \mathbb{P}_S(g_*L)$ is a morphism.

After replacing X by X' and again passing to an open subset of S, we may assume that g is flat, g_*L is locally free, commutes with restriction to fibers, and $\phi : X \to \mathbb{P}_S(g_*L)$ is a morphism. Let $Y \subset \mathbb{P}_S(g_*L)$ denote its image and $W \subset X$ the Zariski closed set of points where $\pi : X \to Y$ is not smooth. Set $Y^\circ := Y \backslash \phi(W)$ and $X^\circ := X \backslash \phi^{-1}(\phi(W))$. The restriction $\phi^\circ : X^\circ \to Y^\circ$ is then smooth and proper.

We assume that $\phi^{-1}(y)$ is a single point for a dense set in Y, hence for a dense set in Y°. Since ϕ° is smooth and proper, it is then an isomorphism. Thus ϕ is bimeromorphic on every irreducible fiber that has a nonempty intersection with X°. □

As a direct consequence (combined with the classical result by Hacon and Mckernan [**HaconMckernan**]) we have the general type locus also admits alternating property.

Theorem 2.3 (Alternating property for general type locus, see [**Moishezonmorphism**] Corollary 20)**.**
Let $g : X \to S$ be a proper morphism of normal, irreducible analytic spaces. Then

$$\mathrm{GT}_S(X) = \{s \in S \mid X_s \text{ is of general type}\}$$

(1) either nowhere dense (in the analytic Zariski topology), (2) or it contains a dense open subset of S, and $g : X \to S$ is Moishezon

Proof. Proof. Using resolution of singularities, we may assume that X is smooth. By passing to an open subset of S, we may also assume that $\underline{S}$ and g are smooth. By [**HaconMckernan**] there is an m (depending only on $\dim X_s$) such that $|mK_{X_s}|$ is very big whenever X_s is of general type. Thus Lemma 2.2 applies to $L = mK_X$. □

Before proving Theorem 21. Let us first recall the basic idea that being used in [**RaoTsai**]

Theorem 2.4 (Uncoutnable many fibers are Moishezon with deformation invariance of Hodge number implies the morphism is Moishezon, see [**RaoTsai**] Proposition 3.15)**.**

Let $\pi : \mathcal{X} \to \Delta$ be a one-parameter degeneration. (1) Assume that there exists an uncountable subset B of Δ such that for each $t \in B$, the fiber X_t admits a line bundle L_t with the property that $c_1(L_t)$ comes from the restriction to X_t of some cohomology class in $H^2(\mathcal{X}, \mathbb{Z})$. (2) Assume further that the Hodge number $h^{0,2}(X_t) := h^1(X_t, \mathcal{O}_{X_t})$ is independent of $t \in \Delta$ (the original theorem require only Hodge (0,1) deformation invariance)

Proof. Apply the sheaf exponential exact sequence so that

$$\begin{tikzcd}
{} \arrow[r] & H^1(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}}) \arrow[r] \arrow[d] & H^2(\mathcal{X}, \mathbb{Z}) \arrow[r, "e_2"] \arrow[d] & H^2(\mathcal{X}, \mathcal{O}_X) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H^1(X_s, \mathcal{O}^*_{X_s}) \arrow[r] & H^2(X_s, \mathbb{Z}) \arrow[r, "e_2"'] & H^2(X_s, \mathcal{O}_{X_s}) \arrow[r] & {}
\end{tikzcd}$$

Observe that

$$H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \cong R^2\pi_*\mathcal{O}_X(\Delta),\ H^2(X_s, \mathcal{O}_{X_s}) \cong R^2\pi_*\mathcal{O}_{\mathcal{X}}(s)$$

Indeed

(1) By Cartan B. we have

$$H^p(S, R^q\pi_*\mathcal{O}_X) = 0,\ p > 0$$

the Leray spectral sequence arguement thus implies the first isomorphism, (2) Since we assume the cohomological constant of $h^{0,2}$, by Grauert base change theorem it will imply the second isomorphism.

Thus the commutative diagram becomes

$$\begin{tikzcd}
 & & & H^0(\mathcal{X}, R^2\pi_*\mathcal{O}_{\mathcal{X}}) \arrow[d, "\cong"] & \\
{} \arrow[r] & H^1(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}}) \arrow[r] \arrow[d] & H^2(\mathcal{X}, \mathbb{Z}) \arrow[r] \arrow[d] & H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & H^1(X_s, \mathcal{O}^*_{X_s}) \arrow[r] & H^2(X_s, \mathbb{Z}) \arrow[r] & H^2(X_s, \mathcal{O}_{X_s}) \arrow[r] \arrow[d, "\cong"] & {} \\
 & & & R^2\pi_*\mathcal{O}_{\mathcal{X}}(s) &
\end{tikzcd}$$

Where we have the evaluation $\mathrm{ev}_s : H^0(\mathcal{X}, R^2\pi_*\mathcal{O}_{\mathcal{X}}) \to R^2\pi_*\mathcal{O}_{\mathcal{X}}(s)$ in the diagram above.

The idea is to find a cohomology class $c \in H^2(X_s, \mathbb{Z})$ by the simply connectness of Δ it will lift it to $c \in H^2(\mathcal{X}, \mathbb{Z})$, if we can prove the vanishing of $e_2(c) \in H^2(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ then by the exactness of the sequence we can find some global line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$.

Observe that the cohomology group $H^2(X_s, \mathbb{Z}) \cong H^2(\mathcal{X}, \mathbb{Z})$ by Ehresmann's theorem, and this $\mathbb{Z}$ coefficient cohomology group can only have counable many elements, taking uncountble many L_t, it must have some $c \in H$ such that there is uncountable many t such that $c_1(L_t) = c$.

Since this $c \in H^2(X_s, \mathbb{Z})$ coming from $\mathrm{Pic}(X_s)$, we have $e_2(c) = 0 \in R^2\pi_*\mathcal{O}_{\mathcal{X}}(s)$ and thus if we lift it to $c \in H^2(\mathcal{X}, \mathbb{Z})$ the global section $e_2(c) \in H^0(\mathcal{X}, R^2\pi_*\mathcal{O}_{\mathcal{X}})$ it will be zero on uncounable many points. Thus by the identity principle easy to see $e_2(c) = 0 \in H^2(\mathcal{X}, R^2\pi_*\mathcal{O}_{\mathcal{X}})$. Thus thus lift to some global line bundle $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ with the restriction $c_1(\mathcal{L}|_{X_s}) = c_1(L_s)$ and we can now apply the lemma about deformation density of Iitaka-Kodaira dimension below and conclude that $\mathcal{L}$ is indeed a big line bundle to finish the proof.

$\square$

Theorem 2.5 (Deformation density of Iitaka-Kodaira dimension, see [**RaoTsai**; **LiebermanSernesi**]).

Let $\pi : \mathcal{X} \to Y$ be a flat family from a complex manifold over a one-dimensional connected complex manifold Y with possibly reducible fibers.

If there exists a holomorphic line bundle L on $\mathcal{X}$ such that the Kodaira-Iitaka dimension $\kappa\left(L_t\right) = \kappa$ for each t in an uncountable set B of Y, then any fiber X_t in π has at least one irreducible component C_t with $\kappa\left(\left.L\right|_{C_t}\right) \geq \kappa$.

In particular, if any fiber X_t for $t \in Y$ is irreducible, then $\kappa\left(L_t\right) \geq \kappa$.

Today we will continue our discussion on the paper Moishezon morphism. We will first finish our discussion on the Moishezon locus, we will prove a interesting locally freeness result about the direct image sheaves. Then we will delve into today's main topic, the proof of the Conjecture 5 with additional assumptions that the central fiber is KLT and not uniruled.

3 The Moishezon locus

We first prove an interesting locally freeness criterion for direct image sheaves.

Theorem 3.1 (locally freeness criterion for $R^i f_* \mathcal{O}_X$, see [**Moishezonmorphism**], Theorem 24)**.** Let $f : X \to S$ be a smooth, proper morphism of analytic spaces. Assume that $H^i\left(X_s, \mathbb{C}\right) \to H^i\left(X_s, \mathcal{O}_{X_s}\right)$ is surjective for every i for some $s \in S$. Then $R^i g_* \mathcal{O}_X$ is locally free in a neighborhood of s for every i.

Proof. We begin our proof by noticing by the direct image theorem it's enough to show the surjectivity of the base change morphism

$$\phi_s^i : R^i f_* \mathcal{O}_X \to H^i\left(X_s, \mathcal{O}_{X_s}\right)$$

for every i.

Indeed the base change theorem shows that the surjectivity of the base change morphisms ϕ_s^i and ϕ_s^{i-1} implies the locally freeness of the direct image $R^i f_*(\mathcal{O}_X)$.

Next by the Theorem on Formal Functions, it is enough to prove this when S is replaced by any Artinian local scheme S_n, whose closed point is s.

By Cartan B easy to see the vanishing of $H^p(S_n, R^i f_* \mathcal{O}_X) = 0,\ \forall q, \forall p > 0$ then by the Leray spectral sequence arguement we get

$$H^0\left(S_n, R^i f_* \mathcal{O}_X\right) = H^i\left(X_n, \mathcal{O}_{X_n}\right)$$

On the affine base the fiber of the coherent sheaf is indeed the global section, as a consequence

$$R^i f_* \mathcal{O}_X(s) = H^0(S_n, R^i f_* \mathcal{O}_X) = H^i(X_n, \mathcal{O}_{X_n})$$

The base change morphism thus becomes

$$\psi^i : H^i\left(X_n, \mathcal{O}_{X_n}\right) \to H^i\left(X_s, \mathcal{O}_{X_s}\right).$$

Let $\mathbb{C}_{X_n}$ (resp. $\mathbb{C}_{X_s}$) denote the sheaf of locally constant functions on X_n (resp. X_s) and $j_n : \mathbb{C}_{X_n} \to \mathcal{O}_{X_n}$ (resp. $j_s : \mathbb{C}_{X_s} \to \mathcal{O}_{X_s}$) the natural inclusions. We have a commutative diagram

$$\begin{tikzcd}
H^i(X_n, \mathbb{C}_{X_n}) \arrow[r, "\alpha^j"] \arrow[d, "j'_n"'] & H^i(X_s, \mathbb{C}_{X_s}) \arrow[d, "j'_s"] \\
H^i(X_n, \mathcal{O}_{X_n}) \arrow[r, "\psi^j"'] & H^i(X_s, \mathcal{O}_{X_s})
\end{tikzcd}$$

Note that α^i is an isomorphism since the inclusion $X_s \hookrightarrow X_n$ is a homeomorphism, and j_s^i is surjective since X_s is Du Bois. Thus ψ^i is also surjective. □

Using this we can prove the theorem below

Theorem 3.2 (Fiberwise Moishezon morphism is locally Moishezon if it's smooth, see [**Moishezonmorphism**],Co
22)**.** Let $g : X \to S$ be a smooth, proper morphism of normal, irreducible analytic spaces whose fibers are Moishezon. Then g is locally Moishezon.

Proof. Since we have proved the Moishezon manifolds admit strong Hodge decomposition, the morphism

$$H^i\left(X_s, \mathbb{C}\right) \to H^i\left(X_s, \mathcal{O}_{X_s}\right)$$

is surjective for every i.

The result then follows clearly by 3.1 and [**Moishezonmorphism**] Theorem 21. □