Hacon-Popa-Schnell's Readings Notes Fall 2025

Note III.3 — 2025 09 12 (draft version 0)

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The aim of this note is to prove the Iitaka conjecture when the base has maximal Albanese dimension.

Theorem 0.1 ([HPS18, Theorem 1.1])**.** Let $f : X \to Y$ be an algebraic fiber space with general fiber F. Assume that Y has maximal Albanese dimension, then $\kappa(X) \geq \kappa(F) + \kappa(Y)$.

There are three essential ingredients appears in the proof.

(a) Green-Lazarsfeld-Simpson's structure theorem on cohomological support loci and generic vanishing theorem;
(b) Hacon-Popa-Schnell's construction of singular Hermitian metric on $f_*\omega_X^{\otimes n}$;
(c) Kawamata's Iitaka conjecture when the base is of general type and structure theorem for finite cover of Abelian varieties.

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1.1 Basic Properties of GV sheaves

Definition 1.1 (GV sheaves)**.**

1.2 Green-Lazarsfeld-Simpson's generic vanishing theorem

Theorem 1.2.

2 Hacon-Popa-Schnell's construction of semi-positive singular Hermitian metric on $f_*\omega_X^{\otimes n}$

We will divide the construction into 3 steps. First we try to find some semi-positive singular Hermitian metric on $\mathscr{F} = f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)$ as long as (L, h) is semi-positive using the optimal Ohsawa-Takegoshi extension theorem. In the second step, we try to construct the m-th Narasimhan-Simha metric on the $\omega_{X/Y}$ and hence a semi-positive line bundle $(\omega_{X/Y}^{\otimes(m-1)}, h)$. And finally, using the elimination of multiplier ideal lemma, we can transport the semi-positivity from $\mathscr{F} = f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)$ to $\mathscr{F}_m = f_*(\omega_{X/Y}^{\otimes m})$. It's worth mentioning that up to now there is no algebraic proof of such semi-positivity theorem.

2.1 Construction of semi-positive metric on $f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h))$

We first prove there exists a semi-positive singular Hermitian metric on $\mathscr{F} = f_*\left(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\right)$ as long as (L, h) is semi-positive.

Proposition 2.1 ([HPS18, Theorem 21.1])**.** Let $f : X \to Y$ be a projective surjective morphism between two connected complex manifolds. If (L, h) is a holomorphic line bundle with a singular hermitian metric of semi-positive curvature on X, then the pushforward sheaf

$$\mathscr{F} = f_*\left(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\right)$$

(1) has a canonical singular hermitian metric H;
(2) This metric has semi-positive curvature and;
(3) satisfies the minimal extension property.

PROOF IDEA 2.2. The construction of the metric is a bit involved. Let us briefly sketch the idea of the proof first. We try to first construct some semi-positive singular Hermitian metric on the "nice" locus where the direct image is a vector bundle. And then we try to apply the Riemann extension theorem to extend such semi-positive metric to semi-positive metric for torsion free sheaf $f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h))$.

2.2 Construction of Narasimhan-Simha metric on $\omega_{X/Y}$

Our next goal is to construct the m-th Narasimhan-Simha metric on $\omega_{X/Y}$.

Proposition 2.3. Let $f : X \to Y$ be a surjective projective morphism with connected fibers between two complex manifolds. Suppose that $f_*\omega_{X/Y}^{\otimes m} \neq 0$ for some $m \geq 2$.

The line bundle $\omega_{X/Y}$ has a canonical singular hermitian metric with semipositive curvature, called the m-th Narasimhan-Simha metric. This metric is continuous on the preimage of the smooth locus of f.

PROOF IDEA 2.4.

2.3 Construction of semi-positive metric on $f_*\omega_{X/Y}^{\otimes m}$

In order to finish the construction of semi-positive metric on $f_*\omega_{X/Y}^{\otimes m}$ we need the following lemma to eliminate the multiplier ideal sheaf $\mathcal{I}(h)$.

Proposition 2.5. Let $f : X \to Y$ be a surjective projective morphism with connected fibers between two complex manifolds. Suppose that $f_*\omega_{X/Y}^{\otimes m} \neq 0$ for some $m \geq 2$. (b) If h denotes the singular hermitian metric on $L = \omega_{X/Y}^{\otimes(m-1)}$ induced by the Narasimhan-Simha metric we just constructed, then

$$f_*\left(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h)\right) \hookrightarrow f_*\omega_{X/Y}^{\otimes m}$$

is an isomorphism over the smooth locus of f.

Hence we can transport the semi-positivity from the $f_*(\omega_{X/Y} \otimes L \otimes \mathcal{I}(h))$ to the direct image of relative pluricanonical sheaves using the following lemma.

Lemma 2.6. Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism between two torsion-free coherent sheaves that is generically an isomorphism. If $\mathscr{F}$ has a singular hermitian metric with semi-positive curvature, then so does $\mathscr{G}$.

Theorem 2.7 ([HPS18, Theorem 4.6])**.** Let $f : X \to Y$ be an algebraic fiber space. Suppose Y is compact.

(a) For any $m \in \mathbb{N}$, the torsion-free sheaf $f_*\omega_{X/Y}^{\otimes m}$ has a canonical singular hermitian metric with semi-positive curvature;
(b) If $c_1\left(\det f_*\omega_{X/Y}^{\otimes m}\right) = 0$ in $H^2(Y, \mathbb{R})$, $f_*\omega_{X/Y}^{\otimes m}$ is locally free, and the singular hermitian metric on it is smooth and flat;
(c) Every nonzero morphism $f_*\omega_{X/Y}^{\otimes m} \to \mathcal{O}_Y$ is split surjective.

Proof. □

3 A brief review of Kawamata's work on Iitaka conjecture when base is of general type

3.1 Several Critera for Albanese map to be fibration

Theorem 3.1 (Albanese map as fibration when Kodaira dimension 0,[Kaw81, Theorem 1])**.** Let X be a non-singular and projective algebraic variety and assume that $\kappa(X) = 0$. Then the Albanese map $\alpha : X \to A(X)$ is an algebraic fiber space.

3.2 Structure Theorem for Finite Covers of Abelian Varieties

Theorem 3.2 ([Kaw81, Theorem 13])**.**

3.3 Kawamata's proof of Iitaka conjecture when base is of general type

Theorem 3.3.

4 Proof of the Iitaka conjecture when the base has maximal Albanese dimension

Let us finish this note by the proof of Hacon-Popa-Schnell's result.

Theorem 4.1. Let $f : X \to Y$ be an algebraic fiber space with general fiber F. Assume that Y has maximal Albanese dimension, then $\kappa(X) \geq \kappa(F) + \kappa(Y)$.

4.1 Reduce the problem to the case when the base is Abelian Varieties and $\kappa(X) = 0$

Proposition 4.2. To prove Theorem 4.1, it's sufficient to prove the case when $\kappa(X) = 0$ and Y is an Abelian variety.

PROOF IDEA 4.3. We will divide the proof into 3 cases.
Case 1. We will prove the result in the case when $\kappa(X) = -\infty$. Green-Lazarsfeld-Simpson's generic vanishing theorem is used.
Case 2. We assume that $\kappa(Y) = 0$. Using Theorem 3.1, we can reduce the base to an Abelian variety Y. We then take a Iitaka fibration on the total space

$$h : X \to Z,$$

with the general fiber G. We then restrict the fibration $f : X \to Y$ to a new fibration

$$G \to B',$$

with $\kappa(G) = 0$ and B' is an Abelian variety and H be the general fiber. which is exactly the situation of Proposition 4.1, that is

$$\kappa(G) \geq \kappa(H) + \kappa(B'),$$

thus $\kappa(H) = 0$.

$$\begin{tikzcd}
H = F \cap G \arrow[r, hook] \arrow[d, hook] & F \arrow[r] \arrow[d, hook] & h(F) \arrow[d, hook] \\
G \arrow[r, hook] \arrow[d] & X \arrow[r, "h"] \arrow[d, "f"] & Z \\
B \arrow[r, hook] & Y &
\end{tikzcd}$$

We then apply the easy addition formula to $F \to h(F)$, so that

$$\kappa(F) \leq \kappa(H) + \dim h(F) = \dim h(F),$$

which complete the proof of this case.
Case 3. When the base Y has maximal Albanese dimension. Then by the Kawamata's structure theorem for finite cover of Abelian varieties, we can decompose the base Y into a product of Abelian variety K and a variety of general type z.

$$\begin{tikzcd}
F \arrow[r, hook] \arrow[dr, hook] & E \arrow[d, hook] \arrow[dr] & \\
& X \arrow[r] \arrow[d] \arrow[dr, "f"] & K \\
& Z & Y = Z \times K \arrow[l] \arrow[u]
\end{tikzcd}$$

The key point is to observe that the general fiber F of $f : X \to Y$ is also the general fiber of the induced $E \to K$. We then apply the Iitaka conjecture when base Z is of general type so that

$$\kappa(X) = \kappa(E) + \kappa(Z)$$

on the other hand since K is Abelian variety, so that it's possible to prove the Iitaka conjecture for $E \to K$ that is

$$\kappa(E) \geq \kappa(F).$$

This complete the proof of the 3rd case.

We can apply the result of Iitaka conjecture when the base is of general type to finish the proof.

4.2 Analytic Proof of Iitaka conjecture when base is Abelian Varieties and $\kappa(X) = 0$

We will finish the proof by showing that

Proposition 4.4. If $f : X \to A$ is an algebraic fiber space over an abelian variety, with $\kappa(X) = 0$, then we have

$$\mathscr{F}_m = f_* \omega_X^{\otimes m} \simeq \mathscr{O}_A$$

for every $m \in \mathbb{N}$ such that $H^0\left(X, \omega_X^{\otimes m}\right) \neq 0$.

PROOF IDEA 4.5. To prove this result, we need to apply the Theorem 2.7, so that there the

References

[HPS18] Christopher Hacon, Mihnea Popa, and Christian Schnell. "Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Păun". In: *Local and global methods in algebraic geometry*. Vol. 712. Contemp. Math. Amer. Math. Soc., [Providence], RI, [2018] ©2018, pp. 143–195. ISBN: 978-1-4704-3488-5. DOI: 10.1090/conm/712/14346. URL: https://doi.org/10.1090/conm/712/14346.

[Kaw81] Yujiro Kawamata. "Characterization of abelian varieties". In: *Compositio mathematica* 43.2 (1981), pp. 253–276.