

**A Brief Survey on Kähler MMP Reading Notes**

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## 1 Overview

The aim of this series of notes is to summarize recent developments in the Kähler minimal model program. We will divide the discussion into several topics. This note is intended as an overview of what is currently known about the Kähler minimal model program, and it briefly sketches some important ideas that appear along the way.

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## 2 Cone Theorems

The original proof of cone theorem requires the Mori bend and break techniques. Unfortunately, Mori bend and break remains an open question for Kähler manifold. Using recent development of BDPP conjecture [Ou25], cone theorem for Kähler varieties indeed holds true.

**Theorem 1** ([HP24, Theorem 0.5]). Let  $X$  be a compact  $\mathbb{Q}$ -factorial Kähler variety of dimension  $n$  such that  $(X, B + \beta)$  is generalized klt (resp. and  $K_X + B + \beta_X$  is pseudo-effective), then the following holds

- (1) (Cone theorem) There are at most countably many rational curves  $\{\Gamma_i\}_{i \in I}$  such that

$$\overline{NA}(X) = \overline{NA}(X)_{K_X + B + \beta_X \geq 0} + \sum_{i \in I} \mathbb{R}^+ [\Gamma_i]$$

- (2) (Length of Extremal rays) that

$$0 < -(K_X + B + \beta_X) \cdot \Gamma_i \leq 2n$$

- (3) (Finiteness) if  $B + \beta_X$  or  $(K_X + B + \beta_X)$  is big, then  $I$  is finite.

## 3 Contraction Theorems

Currently, contraction theorem is only known for Kähler 3-folds. [HP16] proved for  $\mathbb{Q}$ -factorial compact Kähler 3-folds with terminal singularities. [DH25], [DH24a] completely proved the result for Kähler 3-folds  $\mathbb{Q}$ -factorial dlt pairs.

**Theorem 2** ([HP16, Theorem 1.3 + Corollary 7.9 + Theorem 7.12]). Let  $X$  be a normal  $\mathbb{Q}$ -factorial compact Kähler threefold with at most terminal singularities such that  $K_X$  is pseudoeffective. Let  $\mathbb{R}^+ [\Gamma_i]$  be a  $K_X$ -negative extremal ray in  $\overline{NA}(X)$ . Then the contraction of  $\mathbb{R}^+ [\Gamma_i]$  exists in the Kähler category.

- (1) (Divisorial Contraction when  $n(\alpha) = 0$ ) Suppose that the extremal ray  $R$  is divisorial and  $n(\alpha) = 0$ . Then the surface  $S$  can be blown down to a point: there exists a bimeromorphic morphism  $\varphi : X \rightarrow Y$  to a normal compact threefold  $Y$  with  $\dim \varphi(S) = 0$  such that  $\varphi|_{X \setminus S}$  is an isomorphism onto  $Y \setminus 0$ ;

- (2) (Divisorial Contraction when  $n(\alpha) = 1$ ) Suppose that the extremal ray  $R$  is divisorial and  $n(\alpha) = 1$ . Then there exists a fibration  $f : S \rightarrow B$  onto a curve  $B$  such that a curve  $C \subset S$  is contracted if and only if  $[C] \in R$ ;

- (3) (Flipping contraction case) Suppose that the extremal ray  $R = \mathbb{R}^+ [\Gamma_i]$  is small, and let  $C$  be the locus covered by curves in (cf. Notation 7.10). Then  $C$  is contractible.

**PROOF IDEA 3.** We will not give proof details here, but let us sketch the key point of the proof here.

Das-Hacon proved the contraction theorem for compact Kähler  $\mathbb{Q}$ -factorial dlt pairs.

**Theorem 4** ([DH24a, Theorem 1.2]). Let  $(X, B)$  be a  $\mathbb{Q}$ -factorial compact Kähler 3-fold dlt pair such that  $K_X + B$  is pseudo-effective.

Let  $\omega$  be a Kähler class such that  $\alpha = K_X + B + \omega$  is a nef and big class and  $\alpha^\perp \cap \overline{NA}(X) = R$  is an extremal ray.

Then there exists a projective bimeromorphic morphism  $f : X \rightarrow Z$  with connected fibers such that  $\alpha = f^*\alpha_Z$ , where  $\alpha_Z$  is a Kähler class on  $Z$ .

Moreover, if  $f$  is a divisorial contraction, then  $(Z, f_*B)$  has  $\mathbb{Q}$ -factorial klt singularities.

When flipping contraction exist, Das-Hacon proved the following result on existence of klt flips for generalized Kähler pairs in any dimension

**Theorem 5** ([DH24a, Theorem 5.12]). Let  $(X, B + \beta)$  be a  $\mathbb{Q}$ -factorial gdlt pair, where  $X$  is a compact analytic variety belonging to Fujiki's class  $\mathcal{C}$ , and  $f : X \rightarrow Z$  a flipping contraction. Then the flip of  $f$  exists.

## 4 Base Point Freeness

The transcendental base point freeness is widely open. For transcendental projective pair, the base point free theorem holds true.

**Theorem 6** ([DH24b, Theorem 3.5]). Let  $(X, B + \beta)$  be compact Kähler a generalized klt pair such that  $B + \beta$  is big. If  $K_X + B + \beta_X$  is nef then it is semiample.

## 5 Existence of minimal models, good minimal models, canonical models

The existence of minimal model is complete known in 3-folds case, and also for 4-folds effective pairs. For higher dimensional setting, the existence of good minimal model also true under maximal Albanese dimension setting.

### 5.1 Existence of minimal model for Kähler 3-folds

**Theorem 7** ([HP16, Theorem 1.1]). Theorem 1.1 Let  $X$  be a normal  $\mathbb{Q}$ -factorial compact Kähler threefold with at most terminal singularities. Suppose that  $K_X$  is pseudoeffective or, equivalently, that  $X$  is not uniruled. Then  $X$  has a minimal model, i.e. there exists a MMP

$$X \rightarrow X'$$

such that  $K_{X'}$  is nef.

**Theorem 8** ([DH24a, Theorem 1.4]). Let  $(X, B)$  be a dlt pair where  $X$  is a  $\mathbb{Q}$ -factorial compact Kähler 3-fold. If  $K_X + B$  is pseudo-effective, then there exists a finite sequence of flips and divisorial

contractions

$$\phi : X \dashrightarrow X_1 \dashrightarrow \dots \dashrightarrow X_n$$

such that  $K_{X_n} + \phi_* B$  is nef.

And later Das-Hacon-Yáñez proved the result for generalized Kähler 3-folds.

**Theorem 9** ([DHY23, Theorem 1.2]). Let  $(X, B + \beta)$  be a generalized compact klt Kähler 3-fold pair. (1) If  $K_X + B + \beta_X$  is big, then  $(X, B + \beta)$  has a log terminal model  $f : X \dashrightarrow X^m$  and a unique log canonical model  $g : X^m \rightarrow X^c$ .

(2) If  $K_X + B + \beta_X$  is pseudo-effective and  $\beta_X$  is big, then  $K_X + B + \beta_X$  has a log terminal model  $f : X \dashrightarrow X^m$  and there is a contraction  $g : X^m \rightarrow Z$  such that  $f_*(K_X + B + \beta_X) \equiv g^*\omega_Z$  where  $\omega_Z$  is a Kähler form on  $Z$ .

## 5.2 Existence of minimal model for Kähler 4-folds effective pairs

For Kähler folds effective dlt pairs, the existence of minimal model is proved by Das-Hacon-Păun.

**Theorem 10** ([DHP24, Theorem 1.1]). Let  $(X, B)$  be a  $\mathbb{Q}$ -factorial compact Kähler 4-fold dlt pair such that  $K_X + B \sim_{\mathbb{Q}} M \geq 0$ . Then  $(X, B)$  has a log minimal model.

**Theorem 11** ([DHP24, Corollary 7.12]). Let  $(X, B)$  be a  $\mathbb{Q}$ -factorial compact Kähler plt pair of dimension 4 such that  $\kappa(X, K_X + B) \geq 0$ . Then  $(X, B)$  has log terminal model.

## 5.3 Existence of good minimal model for Kähler variety with maximal Albanese dimension

Das-Hacon also prove the existence of good minimal model for Kahler klt pairs with maximal Albanese dimension.

**Theorem 12** ([DH24c, Theorem 1]). Let  $(X, B)$  be a compact Kähler klt pair of maximal Albanese dimension. Then  $(X, B)$  has a good minimal model.

## 5.4 Existence of good minimal model when general fibers admits good minimal models

Recently, Huang proved the existence of good minimal model for klt Kähler pair, when the Albanese map is projective and general fibers admits good minimal models.

**Theorem 13** ([Hua25, Theorem 1]). Let  $a_X : X \rightarrow A$  be the Albanese morphism of a compact Kähler variety  $X$  where  $(X, B)$  is a klt pair. Assume  $a_X$  is projective with connected fibers. Let  $F$  be the general fiber of  $a_X$ . Suppose  $(F, B_F)$  has a good minimal model, then  $(X, B)$  has a good minimal model.

## 6 Rational Curves on Kähler varieties and Ou's BDPP theorem for compact Kähler manifolds

The aim of this section is to give a brief survey of [Ou25] proof of BDPP conjecture for compact Kähler manifolds.

**Theorem 14** ([Ou25, Theorem 1]). Let  $X$  be a compact Kähler manifold, then  $K_X$  is not pseudo-effective iff  $X$  is uniruled.

When singularity is bad (say worse than klt singularities), the BDPP theorem is not true, by the following example of Totaro.

## 7 Projectivity Criterion for Kähler Varieties

Höring-Claudon developed a projectivity criterion for Kähler morphisms.

**Theorem 15** ([CH24, Theorem 1.1]). Let  $f : X \rightarrow Y$  be a fibration between compact Kähler manifolds. Assume one of the following:

- (1) The natural map  $f^* : H^0(Y, \Omega_Y^2) \rightarrow H^0(X, \Omega_X^2)$  is an isomorphism or
- (2) The morphism  $f$  is Moishezon, i.e. there exists a line bundle  $L \rightarrow X$  that is  $f$ -big.

Then  $f$  is a projective morphism.

**PROOF IDEA 16.**

## 8 Canonical Bundle Formulas

Canonical bundle formula for proper morphism between compact Kähler manifold is still an open question. Some partial results are known by [HP24].

**Theorem 17** ([HP24]).

## 9 Abundance Conjecture for Kahler varieties

Abundance for Kähler 3-folds lc pairs has been proved by [DO24].

**Theorem 18** ([DO24, Theorem 1.1] + [DO25, Corollary 1.3]). Let  $(X, \Delta)$  be a lc pair such that  $X$  is a compact Kähler threefold. If  $K_X + \Delta$  is nef, then it is semi-ample.

**PROOF IDEA 19.**

## References

[CH24] Benoît Claudon and Andreas Höring, *Projectivity criteria for kähler morphisms*, 2024.

- [DH24a] O. Das and C.D. Hacon, *On the minimal model program for Kähler 3-folds*, 2024.
- [DH24b] O. Das and C.D. Hacon, *Transcendental minimal model program for projective varieties*, 2024.
- [DH24c] Omprokash Das and Christopher Hacon, *Existence of good minimal models for Kähler varieties of maximal Albanese dimension*, C. R. Math. Acad. Sci. Paris **362** (2024), 83–91. MR 4762189
- [DH25] ———, *The log minimal model program for Kähler 3-folds*, J. Differential Geom. **130** (2025), no. 1, 151–207.
- [DHP24] O. Das, C.D. Hacon, and M. Păun, *On the 4-dimensional minimal model program for Kähler varieties*, Adv. Math. **443** (2024), Paper No. 109615, 68.
- [DHY23] O. Das, C.D. Hacon, and J.I. Yáñez, *Mmp for generalized pairs on Kähler 3-folds*, 2023.
- [DO24] O. Das and W.-H. Ou, *On the log abundance for compact Kähler threefolds*, Manuscripta Math. **173** (2024), no. 1-2, 341–404.
- [DO25] Omprokash Das and Wenhao Ou, *On the log abundance for compact Kähler threefolds ii*, 2025.
- [HP16] A. Höring and T. Peternell, *Minimal models for Kähler threefolds*, Invent. Math. **203** (2016), no. 1, 217–264.
- [HP24] C.D. Hacon and M. Păun, *On the canonical bundle formula and adjunction for generalized kähler pairs*, 2024.
- [Hua25] Yu-Ting Huang, *On the existence of good minimal models for kähler varieties with projective albanese map*, 2025.
- [Ou25] W.-H. Ou, *A characterization of uniruled compact kähler manifolds*, 2025.