

Divisorial contraction in Kähler MMP reading notes

Spring 2025

Lecture 4 — 25, 02, 2025 (draft version)*Yi Li***Contents**

1 Overview	1
2 Das-Hacon's approach to divisorial contraction for Kähler 3-fold MMP	1
2.1 The null locus is a Moishezon surface whose smooth model is projective uniruled . . .	2
2.2 Take minimal DLT modification	5
2.3 Run the relative MMP	6
2.4 Control the set of divisors being contracted	6
2.5 Proof of the base pointness	8
3 Höring-Peternell's approach for Kähler 3-fold MMP	9
3.1 Existence of supporting nef class α	9
3.2 The irreducible surface S being contracted	10
3.3 Case 1. When the nef dimension $n(\alpha) = 1$	10
3.4 Case 2. When the nef dimension $n(\alpha) = 0$	10
3.5 Prove the contraction preserve the Kählerness condition	10
3.6 Höring-Peternell's approach to the Mori fibre space case	10

1 Overview

The aim of this note is to summarize the construction of contraction morphism for Kähler 3 folds.

2 Das-Hacon's approach to divisorial contraction for Kähler 3-fold MMP

In this section, we will prove the following theorem.

Theorem 1 ([DH24, Theorem 6.9]). Let (X, B) be a strong \mathbf{Q} -factorial Kähler 3-fold KLT pair. With the following condition holds

1. $K_X + B$ is pseudo-effective
2. $\alpha = [K_X + B + \beta]$ is nef and big class such that β is Kähler,
3. The negative extremal ray $R = \overline{\text{NA}}(X) \cap \alpha^\perp$ is divisorial.

Then there exists an α -trivial divisorial contraction

$$f : X \rightarrow Z,$$

such that there exist some Kähler form α_Z on Z such that $\phi^*(\alpha_Z) = \alpha$.

Before going to the proof let us briefly sketch the idea. We first try to prove that the null locus $\text{Null}(\alpha)$ is the Moishezon surface whose smooth model is projective uniruled. We then take a DLT modification

$$\varphi : (X', \Delta') \rightarrow (X, \Delta)$$

of the pair $(X, \Delta = B + (1 - b)S)$ (note that this pair (X, Δ) differs from the original pair (X, B) and it is not a KLT pair).

We show that the DLT modification φ preserve the geometry outside the null locus $\text{Null}(\alpha)$. We then run the relative Kähler MMP for (X', Δ') over (X, Δ) , which becomes the core of the proof. Since it's Kähler 3-fold MMP, the termination is known. So that it's possible to produce positivity (say $K_{X^m} + \Delta^m$ is nef over (X, Δ)) by the termination theorem.

We need to control the divisors being contracted in the MMP.

So that the induced bimeromorphic map $f : X \dashrightarrow X^m$ is a morphism, and this is the divisorial contraction we want.

In the final step, we will show that the base point freeness holds for the divisorial contraction, say α as pull back of some Kähler form α_Z down stairs.

2.1 The null locus is a Moishezon surface whose smooth model is projective uniruled

In this subsection, we will proof the following lemma.

Lemma 2. In the same setting as Theorem 1. The null locus $\text{Null}(\alpha)$ is a irreducible Moishezon surface, whose smooth model is projective uniruled. Such that the curves in the negative extremal ray R covers the surface S with

$$R \cdot S < 0.$$

Remark 3. Let us briefly sketch the idea. The class $\alpha|_S, \alpha|_{S^\vee}, \alpha|_{S'}$ play important role in this lemma (for simplicity let us assume for now that S is a smooth surface). The idea is to try to use that if a smooth surface is not pseudo-effective, then it's uniruled projective surface. The non-pseudo-effectiveness comes from some intersection number analysis. To be more precise, we will use that $S = \text{Null}(\alpha)$, so that volume $\text{vol}(\alpha|_S) = (\alpha|_S)^2 = 0$ (by definition of null locus). In particular, the restriction $\alpha|_S$ can not be a big class. On the other hand, we can apply adjunction to

$$\alpha|_S = (K_X + B + \beta)|_S.$$

If the coefficient of S in B is 1, then everything is nice and we get

$$\alpha|_S = (K_X + B' + S + \beta)|_S = K_S + B'|_S + \beta|_S.$$

Since $B'|_S \geq 0$ and $\beta|_S$ Kahler, this will imply that K_S can not be pseudo-effective.

However, the coefficient of S in B is not 1, we need some tricky argument. Let ϵ small enough, so that $\alpha - \epsilon\omega$ is still big. Then apply the divisorial Zariski decomposition to

$$\alpha - \epsilon\omega = \sum c_i S_i + P.$$

We restrict the class on the surface S . If $S_i \neq S$ for all i , then

$$\alpha|_S = \epsilon\omega|_S + \sum c_i (S_i \cap S) + P|_S.$$

The right hand side is big, which contradict to the fact S is null locus of α . Thus there exists some component say $S_1 = S$. We try to make the coefficient of S in α is 1. So that we take the **scaling** that

$$(1 + \frac{1-b}{c_1})\alpha|_S = (K_X + B + \beta + \frac{1-b}{s_1}(\sum s_i S_i + P))|_S.$$

Note that in this case, the coefficient of S in α is 1. So that we can apply the adjunction

$$(K_X + B' + \beta + S + \sum_{j \geq 2} s_j S_j + P)|_S = K_S + B'_S + \beta|_S + \sum_{j \geq 2} s_j S_j \cap S + P|_S,$$

which will imply that K_S is not pseudo-effective. And by the classification theorem of complex surfaces, we know that S is uniruled (and projective as S is assumed to be smooth).

What nice on the projective uniruled surface is that the $(0,2)$ -Hodge number is 0, so that the Bott-Chern class $\alpha|_S$ can be realized as a \mathbf{R} -divisor (which is also a \mathbf{R} -curve on the surface). On the other hand since $\alpha|_{S'}$ is a nef class, this will consequently define a movable curve, thus we can apply the Beytev cone theorem, and write

$$\alpha|_{S'} = C_\epsilon + \sum c_i M_i$$

with M_i being a finite set of movable curves. Since M_i movable, if we can prove that there exist some M_i that is α' -trivial. Then we find a α' -trivial covering family of S .

Finally, we need to prove that $R \cdot S < 0$. To do this, Batyrev cone theorem for movable curve is applied. So that $\alpha|_S = C_\epsilon + \sum a_j H_j$. We try to prove that there exist a movable curve H_k in the component such that $\alpha \cdot H_k = 0$ and it generates the negative extremal ray R . So that apply it to the Zariski decomposition of $\alpha - \epsilon\omega = s_1 S + \sum_{j \geq 2} s_j S_j + P$, we get

$$s_1 H_i \cdot S = (\alpha - \epsilon\omega) \cdot H_k - (\sum s_j S_j \cdot H_k + P|_S \cdot H_k) < 0,$$

using that ω is Kähler, H_k meets S_j properly, and $P|_S$ is pseudo-effective and thus intersection with movable curve is non-negative.

Proof. We know give a complete proof based on the idea above. Since we assume that R is divisorial, thus the null locus $\text{Null}(\alpha)$ by definition is a surface, it may have multiple components, pick one of

the component $S \subset \text{Null}(\alpha)$. Since α is big and nef, thus if ϵ sufficient small, then $\alpha - \epsilon\omega$ is still big. Take the Zariski decomposition of the the class

$$\alpha - \epsilon\omega = P + \sum s_i S_i.$$

We claim that the component S is actually some irreducible component of the Zariski decomposition. The proof idea is very similar to the Horing-Peternell. We restrict the Zariski decomposition on the surface S . If we assume that $S \neq S_i$ for any i . Then the restriction becomes

$$(\alpha - \epsilon\omega)|_S = P|_S + \sum s_i S_i \cap S,$$

since P is modified nef, the restriction on a irreducible divisor becomes nef. On the other hand, since $s_j \geq 0$ this means the RHS is a pseudo-effective divisor. Since $\epsilon > 0$ this means that

$$\alpha|_S = \epsilon\omega|_S + P|_S + \sum s_i S_i \cap S$$

which is a big divisor.

On the other hand, since we assume that $S \subset \text{Null}(\alpha)$, which means that the volume

$$\int_S (\alpha|_S)^2 = 0$$

(as by definition the null locus

$$\text{Null}(\alpha) = \bigcup_{V \subset X, \int (\alpha|_V)^{\dim V} = 0} V$$

,

Therefore α is not big. Which gives the contradiction.

Now we can apply the adjunction, let $b = \text{mult}_S(B)$ and therefore

$$S = \frac{1}{s}(\alpha - \epsilon\omega - P - \sum_{j \geq 2} s_j S_j),$$

therefore if we if we scale α with $1 + \frac{1-b}{s}$ then the coefficient of S will becomes 1. And we consider the

$$(1 + \frac{1-b}{s})\alpha|_S = (K_X + B + \beta + \frac{1-b}{s}(\epsilon\omega + P + sS + \sum_{j \geq 2} s_j S_j))|_S = (K_X + B' + S + \beta + c'\omega + \sum_{j \geq 2} s_j S_j)|_S$$

then it will implies that $K_X S$ is not pseudo-effective, for otherwise. the restriction of $\alpha|_S$ will be a big class.

Therefore, by the classification result of surfaces, we know that S is a uniruled surface, such that the plurigenera will vanish. In particular, it's a smooth projective uniruled surface.

Our next goal is to prove that the surface S is covered by the α -trivial rational curves. The idea is try to find a movable curve M_i on S' that is α -trivial. Then since push forward of movable curve under birational morphism is still movable. And by the projection formula, this will define a movable curve downstairs. \square

2.2 Take minimal DLT modification

The idea is try to add the null locus S into the NKLT part of the boundary divisor $(X, B + (1-b)S)$ with $b = \text{mult}_B(S)$.

Lemma 4. There exist a projective resolution $\mu : X' \rightarrow X$ such that

1. (X', Δ') is a \mathbb{Q} -factorial DLT pair, with $\Delta' = \mu_*^{-1}(\Delta) + \text{Ex}(\mu)$,
2. $K_{X'} + \Delta'$ is nef over X , and $\mu^*(K_X + \Delta) - (K_{X'} + \Delta') \geq 0$,
3. The following relation holds

$$K_{X'} + \Delta' = \mu^*(K_X + B) + (1-b)\mu_*^{-1}S + \sum a_j E_j,$$

then $a_j > 0$ for all j .

Remark 5. One may ask why we need to take a DLT modification? There are several reasons.

First, we will apply the Das-Hacon PLT contraction theorem in the later step, which needs some control on the singularity.

Second, we need to control the place being contracted (which is the core of the proof).

We first take $b = \text{mult}_S(B)$ and define a new divisor $\Delta = B + (1-b)S$, we then take the log resolution so that

$$\mu : X' \rightarrow X$$

with

$$\Delta' = \mu_*^{-1}(\Delta) + \text{Ex}(\mu).$$

Since log smooth pair with SNC divisor is DLT, under the log resolution, we have a \mathbb{Q} -factorial DLT pair.

So that we can run the analytic BCHM, which will terminate

$$\tilde{X} \dashrightarrow X',$$

such that $K_{X'} + \Delta'$ is relative nef over X . Therefore we have

$$\mu^*(K_X + \Delta) - (K_{X'} + \Delta') \geq 0$$

by negativity lemma.

On the other hand, we have (X, B) being a KLT pair.

Proof.

□

2.3 Run the relative MMP

The core of the proof of Theorem 1 lies in the existence of a relative MMP in this section.

Theorem 6. Let (X', Δ') be the output of the Lemma 4. We can run the $(K_{X'} + \Delta')$ -MMP

$$(X', \Delta') \dashrightarrow (X^1, \Delta^1) \dashrightarrow \cdots \dashrightarrow (X^m, \Delta^m),$$

such that the following conditions hold

1. the birational contraction $\phi^i : X \dashrightarrow X^i$ is α' -trivial,
2. ϕ^i is isomorphism over U , and $\lfloor \Delta^i \rfloor = \text{supp}(\Theta^i)$,
3. $K_{X^i} + \Delta^i \equiv_{\alpha^i} \Theta^i$ such that the NKLT locus $\lfloor \Delta^i \rfloor \subset W^i := X^i - X_U^i$,
4. The MMP terminate at the pair (X^m, Δ^m) such that any prime divisor S_i in the $\lfloor \Delta^m \rfloor$ is

As usual, let us briefly sketch out the idea of the proof first.

Proof.

□

2.4 Control the set of divisors being contracted

What is interesting in the MMP process above is that we can have some control of the divisors being contracted.

We first apply the nef reduction on the normalization of the surface $S^\nu \rightarrow S$. Denote it $\nu : S^\nu \rightarrow T$, then given a component $P \subset \mu^*S$, it's easy to see that

We claim

Proposition 7. • If $n(\alpha|_{S^\nu}) = 0$, then the MMP $\phi^m : X' \dashrightarrow X^m$ will contract μ^*S and no other divisors

- If $n(\alpha|_{S^\nu}) = 1$, then the MMP $\phi^m : X' \dashrightarrow X^m$ will contract S' and those components E in $S' + \sum E_i$ such that $n(\alpha|_E) = 1$.

Remark 8. In particular, we can prove that when descend to $f : X \dashrightarrow X^m$,

- If $n(\alpha|_{S^\nu}) = 0$, then
- If $n(\alpha|_{S^\nu}) = 1$,

Since we have shown that $\lfloor \Delta^m \rfloor = 0$, the next proposition shows that the MMP $X' \dashrightarrow X^m$ actually descends to a morphism $f : X \rightarrow Z$.

Proposition 9. The induced bimeromorphic map $f : X \rightarrow Z := X^m$ is actually a morphism.

PROOF IDEA 10. The idea is simple, take the resolution of the graph of the bimeromorphic map $f : X \dashrightarrow X^m$, with $p : W \rightarrow X$ and $q : W \rightarrow X^m$. So that by rigidity lemma, if all the p -exceptional curve is also q -exceptional, then the bimeromorphic map is actually a morphism.

We prove this by contradiction, if there exist some curve $C \subset W$ such that $p_*(C) = 0$ but $q_*(C) \neq 0$, we pick a Kähler class ω^m on X^m . If we can prove that $q^*(\omega^m)$ is the pull back, say

$$p^*(\omega + rS) = q^*\omega^m.$$

Then this will deduce the contradiction, as

$$0 < q_*(C) \cdot \omega^m = p^*(\omega + rS) \cdot C = 0.$$

To see this, apply that $N^1(W/X)$ is generated by some p -exceptional divisors E_1, \dots, E_k . Thus the divisor

$$q^*\omega^m + \sum c_i E_i \equiv_X 0,$$

and therefore there exist some $\omega \in H_{\text{BC}}^{1,1}(X)$ such that

$$q^*\omega^m + \sum c_i E_i = p^*\omega.$$

Note that $S \cdot R < 0$, there exist some $r \in \mathbb{R}$ such that $(\omega + rS) \cdot R = 0$. We claim

$$Q = \sum c_i E_i + rp^*S = 0$$

which complete the proof of the proposition. To do this, we will use the negativity lemma. We try to show

(1) The Q is q -exceptional. Since we know that $\Theta^m = 0$, therefore all the μ -exceptional part being contracted by $X' \dashrightarrow X^m$. And consequently, the birational map $X^m \dashrightarrow X$ is small. In particular, the map $f : X \dashrightarrow Z$ does not extract any divisor (so that any p -exceptional **divisor** is also q -exceptional **divisor**). In particular we know that the first terms $\sum c_i E_i$ is q -exceptional. On the other hand, since $f : X \dashrightarrow Z$ contract S (i.e. $f_*S = 0$), so that the second term p^*S satisfies

$$f_*S = q_*(p^*S) = 0,$$

i.e. the second term is also q -exceptional.

(2) The $Q + q^*\omega^m$ is q -numerical trivial. That is for any q -exceptional curve C , we have

$$(Q + q^*\omega^m) \cdot C = 0.$$

This is clear, since by definition $Q + q^*\omega^m = p^*(\omega + rS)$ and therefore: if C is also p -exceptional, then the intersection (*) clearly holds. If C is not p -exceptional (i.e. $p_*(C) \neq 0$). Since

$$\alpha \cdot p_*(C) = C \cdot p^*(\alpha) = C \cdot q^*(\alpha^m) = 0.$$

(I am not pretty sure at this step, maybe $C \cdot p^*(\alpha) = 0$ use the α trivial of the contraction).

Therefore this will imply that

$$(\omega + rS) \cdot p_*C = 0.$$

As a consequence,

$$p^*(\omega + rS) \cdot C = (q^*(\omega^m) + Q) \cdot C = 0.$$

which is what we want.

2.5 Proof of the base pointness

To prove the base point freeness result, we need the following lemma, the first one says that image of pull back of Bott-Chern is those classes that curves being contracted are trivial on it (assume the singularity is nice and).

Lemma 11. Let $f : X \rightarrow Y$ be a morphism between normal compact complex spaces with rational singularity. If in addition one of the following two conditions hold,

1. f is a proper bimeromorphic morphism between Fujiki varieties,
2. f is surjective, there exist some boundary divisor B such that (X, B) is KLT. Moreover $(K_X + B)$ is f -big and nef.

Then the pull back

$$f^* : H_{\text{BC}}^{1,1}(Y) \rightarrow H_{\text{BC}}^{1,1}(X),$$

is injective, and the image

$$\text{im}(f^*) = \{\alpha \in H^{1,1}(Y) \mid \alpha \cdot C = 0, \forall C \in N_1(X/Z)\}.$$

Thus if the contraction morphism $f : X \rightarrow Z$ is α -trivial, then there exists some $\alpha_Z \in H_{\text{BC}}^{1,1}(Z)$ with $f^*\alpha_Z = \alpha$.

Remark 12. Before proving the lemma, let us compare this result with the projective contraction theorem. Recall that in the projective setting, if D is a Cartier divisor supporting some negative extremal ray R , then D comes from the pull back (i.e. $D \in \text{im}(f^* : \text{NS}(Y) \rightarrow \text{NS}(X))$). The proof requires the base point free theorem to show that mD is base point free. Thus, mD is the pull back of Serre twisted line bundle via map associated to mD . Finally, using rigidity lemma to show that the Kodaira map coincides with the contraction $f : X \rightarrow Z$. Thus the divisor mD is also pull back via $f : X \rightarrow Z$.

On the other hand, the transcendental case is relatively easier. Since ...

Proof. □

To check α_Z is Kähler, we need the following (singular version) Demailly-Păun Kählerness criterion.

Lemma 13. Let X be a compact normal complex variety. Let $\{\alpha\} \in H_{\text{BC}}^{1,1}(X)$ be a big and nef class. Then $\{\alpha\}$ is Kähler iff for any positive dimensional subvariety (or reduced analytic subset) W , the following holds true

$$\int_W (\alpha|_W)^{\dim W} > 0.$$

Proof. □

Now we can prove the base freeness for the divisorial contraction $f : X \rightarrow Z$, using Lemma 11 and Lemma 13

Proof of base point freeness. Compared with the flipping contraction case, this case is a bit harder. The reason is, for flipping contraction, the flipping curve being contracted to points, and thus there is no positive dimensional subvariety contains in it. In the divisorial contraction case, we need to consider two scenarios,

Case 1. When the positive dimensional subvariety $W' \subset f(S)$. We need to use the condition that curve C not being contracted must have positive intersection with α . Since we proved that S is irreducible. Thus only needs to consider that $W' = f(S)$ and it's a irreducible curve. Therefore, we may find some curve $W \subset S$ upstairs dominating W' .

Now use the condition that the curve W' not being contracted by f , it means

$$W' \cdot \alpha > 0.$$

And therefore, the Kählerness of α_Z is proved in this case.

Case 2. When the positive dimensional subvariety $W' \not\subset f(S)$. In this case, we can consider the strict transform of W' under f . And by the projection formula, we have

$$\alpha_Z^{\dim W'} \cdot W' = (f^* \alpha_Z)^{\dim W} \cdot W = \alpha^{\dim W} \cdot \dim W > 0,$$

the positive due to W does contains in the null locus.

□

3 Höring-Peternell's approach for Kähler 3-fold MMP

In this section, we will introduce Höring-Peternell's approach in [HP16]. In the paper, they proved that

Theorem 14. Let X be a compact normal \mathbb{Q} -factorial Kähler 3-fold with terminal singularity.

Let $R \subset \overline{\text{NA}}(X)$ be a divisorial type extremal ray. Then the divisorial contraction $f : X \rightarrow Z$ exist, moreover the divisorial contraction will preserve the Kählerness condition such that there exists some Kähler class α_Z such that

$$\alpha = f^* \alpha_Z.$$

For simplicity, we denote that following condition as condition (*).

Definition 15 ((Condition (*))). Let X be a compact normal \mathbb{Q} -factorial Kähler 3-fold with terminal singularity.

3.1 Existence of supporting nef class α

We first prove that for a negative extremal ray α , there always exist some supporting nef class.

Lemma 16. Let X be Kähler 3-fold satisfy the condition (*), let $R \subset \overline{\text{NA}}(X)$ be a negative extremal ray. Then there exist a nef class $\alpha \in H_{\text{BC}}^{1,1}(X)$ such that

$$R = \alpha^\perp \cap \overline{\text{NA}}(X),$$

moreover the class α is positive on the

$$\overline{\text{NA}}(X)_{K_X \geq 0} + \sum_{i \neq i_0} \mathbb{R}_+[\Gamma_i].$$

Proof.

□

3.2 The irreducible surface S being contracted

thus we find a surface that $\text{Null}(\alpha) = S$, we may take the normalization in \hat{S} (in order to apply the nef reduction).

3.3 Case 1. When the nef dimension $n(\alpha) = 1$

Recall that the nef dimension is always less than the geometric dimension. So that $n(\nu^*\alpha) \leq 2$. We need to show that $n(\nu^*\alpha) = 2$ cannot happen. This is because for the surface case if the nef dimension is 2, then for all but countable many curves on S satisfies $\alpha \cdot C > 0$. On the other hand, we know that S as null locus of α ,

The original proof of [HP16] contains some error. However, Höring fix the error in his note [?] later.

We first try to prove that the general fiber of nef reduction map is \mathbb{P}^1 . To be more precise

Theorem 17.

3.4 Case 2. When the nef dimension $n(\alpha) = 0$

In this section, we prove the existence of divisorial contraction when nef dimension is 0. Before proving it, let us first briefly sketch out the idea. We try to apply the Grauert contraction theorem to a point (see [DH24, Lemma 4.3]). The point is try to show that the for the surface spanned

3.5 Prove the contraction preserve the Kählerness condition

We finally prove that Kählerness condition is preserved when contracting the divisorial negative extremal ray in the generalized Mori cone. The tool that we need is the following

Lemma 18.

3.6 Höring-Peternell's approach to the Mori fibre space case

Combined with the flipping contraction case appear in another my note. This will give a complete contraction theorem for Kähler 3-folds.

References

- [DH24] Omprokash Das and Christopher Hacon, *On the minimal model program for kähler 3-folds*, 2024.
- [HP16] Andreas Höring and Thomas Peternell, *Minimal models for Kähler threefolds*, Invent. Math. **203** (2016), no. 1, 217–264.