A brief introduction to Deligne-Mumford Stacks Summer 2025

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Contents

1 What is Deligne-Mumford Stack?

2 Characterization of DM stacks

Theorem 1. Let $\mathcal{X}$ be an algebraic stack. The following are equivalent:
(1) the stack $\mathcal{X}$ is a Deligne-Mumford;

(2) the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified; and
(3) every point of $\mathcal{X}$ has a discrete and reduced stabilizer group.

If $\mathcal{X}$ has quasi-compact diagonal (e.g., $\mathcal{X}$ is quasi-separated), then (3) is equivalent to requiring that every stabilizer is finite and reduced.

Theorem 2. Let $G \to S$ be a smooth affine group scheme acting on an algebraic space U over S. Then (1) $[U/G]$ is Deligne-Mumford if and only if every point of U has a discrete and reduced stabilizer group, or equivalently if and only if the action map $G \times U \to U \times U$ is unramified.
(2) $[U/G]$ is an algebraic space if and only if every point of U has a trivial stabilizer group, or equivalently if and only if the action map $G \times U \to U \times U$ is a monomorphism.

3 Local Structure of Deligne-Mumford Stacks

Theorem 3. Let $\mathcal{X}$ be a quasi-separated Deligne-Mumford stack and $x \in |\mathcal{X}|$ be a finite type point with geometric stabilizer G_x. There exists an étale morphism

$$f : ([\operatorname{Spec} A/G_x], w) \to (\mathcal{X}, x)$$

where $w \in [\operatorname{Spec} A/G_x]$ such that f induces an isomorphism of geometric stabilizer groups at w. Moreover, if $\mathcal{X}$ has separated (resp., affine) diagonal, then it can be arranged that f is representable and separated (resp., affine).

4 Coarse Moduli

5 Keel-Mori Theorem

5.1 Quotient stack of affine scheme by finite group defines a coarser moduli

5.2 Luna's etale slice theorem

5.3 Descending etale morphism to geometric quotient

5.4 Keel-Mori Theorem

Theorem 4. Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian algebraic space S. Then there exists a coarse moduli space $\pi : \mathcal{X} \to X$ with $\mathcal{O}_X = \pi_* \mathcal{O}_{\mathcal{X}}$ such that
(1) X is separated and of finite type over S,
(2) π is a proper universal homeomorphism, and
(3) for every flat morphism $X' \to X$ of algebraic spaces, the base change $\mathcal{X} \times_X X' \to X'$ is a coarse moduli space. Moreover, X is proper if and only if $\mathcal{X}$ is.

6 Geometry of Deligne-Mumford Stack

6.1 Criterion for vector bundle on DM stack descent to coarse moduli

6.2 Deligne–Mumford stack has a finite cover by scheme

Theorem 5. (1) Let $\mathcal{X}$ be a Deligne-Mumford stack separated and of finite type over a noetherian scheme S. Then there exists a finite, generically étale, and surjective morphism $Z \to \mathcal{X}$ from a scheme Z.
(2) Let X be a normal algebraic space of finite type over a noetherian scheme S. Show that there is a normal scheme U with an action of a finite abstract group G such that X is the quotient of U by G, i.e., $[U/G] \to X$ is a coarse moduli space.

6.3 Zariski Main theorem

6.4 Chow Lemma

7 Examples

7.1 Moduli of smooth curves as a Deligne-Mumford Stack

References