Fiberwise Bimeromorphic map Summer 2025

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Yi Li

1 Overview

The aim of this note is to study the fiberwise bimeromorphic map in birational geometry. The topic is around the fiberwise bimeromorphic conjecture given in the following. The main references of this note are [Kol22] and [CRT25], [KT19].

Conjecture 1.1 ([Kol22, Conjecture 5])**.** Let $g : X \to \mathbb{D}$ be a flat, proper, Moishezon morphism. Assume that X_0 has canonical (resp. log terminal) singularities. Then g is fiberwise birational to a flat, projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to \mathbb{D}$ such that
(1) X_0^{p} has canonical (resp. log terminal) singularities,
(2) X_s^{p} has terminal singularities for $s \neq 0$, and
(3) $K_{X^{\mathrm{p}}}$ is $\mathbb{Q}$-Cartier.

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2 A fiberwise birational criterion

Definition 2.1 (Meromorphic S-map)**.** Let X, Y be reduced complex space. We call the S-map a meromorphic S-map (not necessary morphism) if

$$\begin{tikzcd} & \Gamma \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\alpha"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

with $\Gamma \subset X \times_S Y$, and $p : \Gamma \to X$ is a proper bimeromosphic morphism. Moreover if q is also proper bimeromorphic morphism, then we all α proper bimeromorphic S-map.

Definition 2.2 (Fiberwise bimeromorphic map, [Kol22, Definition 26])**.** Let $g_i : X^i \to S$ be a proper morphisms. A bimeromorphic map $\phi : X^1 \dashrightarrow X^2$ is fiberwise bimeromorphic if ϕ induces a bimeromorphic map $\phi_s : X^1_s \dashrightarrow X^2_s$ for every $s \in S$.

Remark 2.3. In general, the bimeromorphic S-map does not need to be fiberwise bimeromorphic. Since the graph $\Gamma \subset X \times Y$ needs not to contains in $X \times_S Y$.

Remark 2.4. Fiberwise bimeromorphic is different from having bimeromorphic equivalent fibers ([CRT25, Example 2.15]). Let $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \to \mathbb{P}^1$ be the Hirzebruch surface of index n. By construction easy to see that all the Hirzebruch surface are birational equivalent to $\mathbb{P}^1 \times \mathbb{P}^1$. Let Z be any compact complex manifold. In particular, we see that $\mathbb{F}_n \times Z \to \mathbb{P}^1$ has an explicit birational map between $(\mathbb{P}^1 \times \mathbb{P}^1) \times Z \to \mathbb{P}^1$.

$$\begin{tikzcd}
& \mathrm{Bl}_{p_1,\cdots,p_n}(\mathbb{P}^1 \times \mathbb{P}^1) \times Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\
\mathbb{P}^1 \times \mathbb{P}^1 \times Z \arrow[rr, dashed, "q\circ p^{-1}"] \arrow[dr] & & \mathbb{F}_n \times Z \arrow[dl] \\
& \mathbb{P}^1 &
\end{tikzcd}$$

Note that fibers of these two families are birational equivalent (as both side have fiber $\mathbb{P}^1 \times Z$). However the restriction of the map $q \circ p^{-1}$ does not give the bimeromorphic map of the fiber (since the strict transform of the fibers of $\mathbb{P}^1 \times \mathbb{P}^1 \times Z$ on the blow up being contracted by q).

Although the bimeromorphic map is not fiberwise bimeromorphic in general, it is indeed fiberwise bimeromorphic on a dense open subset.

Proposition 2.5 (Bimeromorphic S-map is generic fiberwise bimeromorphic)**.** Let $f : X \dashrightarrow Y$ be a bimeromorphic S-map between complex varieties over the base S, prove that on the generic fiber the morphism induces a bimeromorphic map on the fiber.

Proof. Since f is bimeromorphic there exist some open dense subset such that $f|_V : V \xrightarrow{\sim} U$ then I claim the morphism induce bimeromorphic map on the fibers X_s such that $X_s \cap V \neq \emptyset$.

Indeed since $X_s \cap V \subset X_s$ is dense in X_s indeed we have

$$\overline{X_s \cap V} \subset X_s \cap \overline{V} = X_s \cap X = X_s$$

thus we have $X_s \cap V$ dense in X_s.

we have that $X_s \cap V$ is dense open subset of X_s, and therefore it induce an bimeromorphism on the fiber

$$X_s \dashrightarrow Y_s$$

Finally note that the set

$$\{s \in S \mid X_s \cap V \neq \emptyset\} = f(V) = \{s \in S \mid X_s \dashrightarrow Y_s \text{ is bimeromorphism}\}$$

and image of dense subset under a continuous map is dense, thus we find the bimeromorphic map induce bimeromorphic map on the generic fiber of the morphism.

□

Under some additional assumption, the bimeromorphic S-map is indeed fiberwise bimeromorphic map.

Proposition 2.6 (Fiberwise birational criterion under non-vanishing condition, [Kol23, Proposition 1.25])**.** Let $f_i : X^i \to B$ be two smooth families of projective varieties over a smooth curve B. Assume that the generic fibers $X^1_{k(B)}$ and $X^2_{k(B)}$ are birational, and further assume that the pluri-canonical system $\left|mK_{X^i_{k(B)}}\right|$ is nonempty for some $m > 0$. Then we have fiberwise bimeromorphic condition.

Proof. Pick a birational map $\phi : X^1_{k(B)} \dashrightarrow X^2_{k(B)}$ (for the generic fiber), and let $\Gamma \subset X^1 \times_B X^2$ be the closure of the graph of ϕ. Let $Y \to \Gamma$ be the normalization with projections $p_i : Y \to X^i$.

$$\begin{tikzcd}
 & Y \arrow[dl, "p_1"'] \arrow[d] \arrow[dr, "p_2"] & \\
 & \Gamma \arrow[l] \arrow[r] & \\
X^1 \arrow[dr] & & X^2 \arrow[dl] \\
 & S &
\end{tikzcd}$$

Note that both of the p_i are open embeddings on $Y \setminus (\mathrm{Ex}\,(p_1) \cup \ \mathrm{Ex}\,(p_2))$.

Thus if we prove that neither $p_1\,(\mathrm{Ex}\,(p_1) \cup \mathrm{Ex}\,(p_2))$ nor $p_2\,(\mathrm{Ex}\,(p_1) \cup \ \mathrm{Ex}\,(p_2))$ contains a fiber of f_1 or f_2, then $p_2 \circ p_1^{-1} : X^1 \dashrightarrow X^2$ (it needs not to be birational) restricts to a birational map $X^1_b \dashrightarrow X^2_b$ for every $b \in B$.

We may assume that B is affine (as we only care about the special fiber, thus we can focus on the affine base around b) and let $\mathrm{Bs}\,|mK_{X^i}|$ denote the set-theoretic base locus. By assumption, we cliam $|mK_{X^i}|$ is not empty.

Since

$$\mathrm{Bs}(|mK_{X^i(b)}|) \subset \mathrm{Bs}\,(|mK_{X^i}|)|_{X^i(b)},$$

if the LHS is non-empty, then so will the right hand side.

We know that direct image of torsion free sheaf is torsion free if the morphism is dominant and torsion free is locally free on a smooth curve. If we denote $L = \mathcal{O}_{X^i}(mK_{X^i})$, then

$$f_{i*}L = \mathcal{E}$$

is a locally free sheaf. On the other hand, since $H^0(X, mK_{X^i}) \neq 0$ thus for any point $s \in B$, there exist a section

$$0 \neq \sigma \in H^0(B, \mathcal{E})$$

such that $\sigma(s) \neq 0$. Therefore consider the restriction map

$$\begin{tikzcd} H^0(X^i, L) \arrow[r, "\mathrm{res}"] \arrow[d, "\simeq"'] & H^0(X^i_s, L|_{X^i_s}) \arrow[d] \\ H^0(B, f_*L) \arrow[r, "\mathrm{res}_s"'] & f_*L(s) \end{tikzcd}$$

such that there exist a section $s \in H^0(X^i, L)$ which maps down to σ such that $\sigma(s) \neq 0$. So that $s|_{X^i_s} \neq 0$. And therefore the base locus can not contains the fiber.

Since the X^i are smooth,

$$K_Y \sim p_i^* K_{X^i} + E_i, \quad \text{ where } E_i \geq 0 \text{ and } \operatorname{Supp} E_i = \operatorname{Ex}(p_i)$$

So that every section of $\mathcal{O}_Y(mK_Y)$ pulls back from X^i, Thus

$$\operatorname{Bs}|mK_Y| = p_i^{-1}(\operatorname{Bs}|mK_{X^i}|) + \operatorname{Supp} E_i$$

Comparing these for $i = 1, 2$, we conclude that

$$p_1^{-1}(\operatorname{Bs}|mK_{X^1}|) + \operatorname{Supp} E_1 = p_2^{-1}(\operatorname{Bs}|mK_{X^2}|) + \operatorname{Supp} E_2$$

Therefore,

$$\boxed{p_1(\operatorname{Supp} E_2) \subset p_1(\operatorname{Supp} E_1) + \operatorname{Bs}|mK_{X^1}|}$$

Since E_1 is p_1-exceptional, $p_1(\operatorname{supp} E_1)$ has codimension ≥ 2 in X^1, hence it does not contain any of the fibers of f_1. We saw that $\operatorname{Bs}|mK_{X^1}|$ does not contain any of the fibers either. Thus $p_1(\operatorname{Ex}(p_1) \cup \operatorname{Ex}(p_2))$ does not contain any of the fibers, and similarly for $p_2(\operatorname{Ex}(p_1) \cup \operatorname{Ex}(p_2))$. $\square$

As a remark in [Kol23], the result holds true even when the pluricanonical systems are empty. That is what we will prove in the next section.

3 Kontsevich-Tschinkel's fiberwise birational theorem

Theorem 3.1 ([KT19, Theorem 1])**.** Let

$$\pi : \mathcal{X} \to B \text{ and } \pi' : \mathcal{X}' \to B$$

be smooth proper morphisms to a smooth connected curve B, over a field of characteristic zero. Assume that the generic fibers of π and π' are birational over the function field of B. Then, for every closed point $b \in B$, the fibers of π and π' over b are birational over the residue field at b.

The proof of Kontsevich-Tschinkel's theorem involve some new notions, let us first introduce it.

Definition 3.2 (semi-ring)**.** A semiring $(S, +, \times)$ consists of a set S equipped with two binary operations $+, \times$. Such that $+$ makes S an commutative monoid (which needs not to be Abelian group compared with Ring).

Definition 3.3 (Burnside ring over a field k)**.** The Burnside semi-ring $\mathrm{Burn}_+(k)$ of a field k is the set of $\sim_k$- equivalence classes of smooth schemes of finite type over k endowed with a semi-ring structure where multiplication and addition are given by disjoint union and product over k. (where wo such schemes X, X' we have $[X/k] = [X'/k]$ (i.e. $\sim_k$) if and only if X and X' are k-birational).

We denote by $\mathrm{Burn}(k)$ the Grothendieck ring generated by $\mathrm{Burn}_+(k)$.

Remark 3.4. The reason to introduce the Grothendieck ring is that it allow to do formal subtraction. And it allows some cut and paste operation on the geometric level.

Remark 3.5. Note that we can decompose the

$$\mathrm{Burn}(k) = \sqcup_{n\geq 0} \mathrm{Bir}_n(k),$$

where $\mathrm{Bir}_n(k)$ denotes k-birational equivalent class of smooth variety of dimension n. Each class can be denoted by $[L/k]$ with $L = k(X)$.

Definition 3.6 (Specialization map)**.** We define

$$\rho_n : \mathrm{Bir}_n(K) \to \mathbb{Z}\left[\mathrm{Bir}_n(k)\right],$$

as follows: given a smooth projective variety X/K, choose one of family

$$\pi : \mathcal{X} \to D,$$

where π is proper, such that generic fibers are X and special fiber

$$X_0 = \bigcup_{i\in I} d_i D_i,$$

is a SNC divisor, with the strata $D_J := \bigcap_{j\in J} D_j$. We then define the specialization map to be

$$\rho_n([L/K]) := \sum_{\emptyset\neq J\subseteq I} (-1)^{\#J-1}\left[D_J \times \mathbb{A}^{\#J-1}/k\right].$$

One of the main difficulities of the proof is to check the specialization map is well defined (i.e. it does not depend on the choose the family $\mathcal{X} \to D$) and representative X in $\mathrm{Bir}_n(X)$. We omit the proof of this part, for more detail of the proof see [KT19, Theorem 4]).

Remark 3.7 (Relation with the dual complex)**.**

Proof of the main theorem. Let $\pi : \mathcal{X} \to B$ be a smooth proper morphism to a smooth connected curve B over k with fiber X over the generic point of B. Let $K = k(B)$ be the function field of B. Let κ_b be the residue field at b, a finite extension of k. Let K_b be the completion of K at b. It is a local field with residue field κ_b, isomorphic to $\kappa_b((t))$, where t is a formal local coordinate. Let

$$\phi_b : K \to K_b$$

be the canonical inclusion. By functoriality (see Remark 3), it defines a homomorphism

$$\phi_{b,*} : \mathrm{Burn}(K) \to \mathrm{Burn}\left(K_b\right).$$

We have the specialization homomorphism

$$\rho : \mathrm{Burn}\left(K_b\right) \to \mathrm{Burn}\left(\kappa_b\right)$$

and the following identity

$$\left[X_b/\kappa_b\right] = \rho\left(\phi_{b,*}([X/K])\right),$$

which follows immediately from the definition of ρ, since the special fiber is smooth and irreducible. This shows that the birational type of the special fiber is determined by the birational type of the generic fiber. □

4 Fiberwise bimeromorphic criterion and plurigenera (Chen-Rao-Tsai)

In this section, we will give a criterion for fiberwise bimeromorphic map using plurigenera. For more detailed discussion, see [CRT25].

Proposition 4.1 ([CRT25, Theorem 1.4])**.** Let X, Y and S be complex analytic spaces. Assume that X is reduced (not necessarily normal) and irreducible, Y is normal, and S is a smooth curve. Assume further that both $\pi_1 : X \to S$ and $\pi_2 : Y \to S$ are proper surjective holomorphic maps. Suppose that there is a bimeromorphic morphism $f : X \to Y$ over S. For some $t \in S$, if D_t is an irreducible component of Y_t that is of codimension 1 in Y, then there exists an irreducible component C_t (equipped with the reduced structure) of X_t that is bimeromorphic to D_t, induced by f.

In particular, if the fibers of $X \to S$ and $Y \to S$ are irreducibles then f is fiberwise bimeromorphic map.

Proof. Since both X, Y are irreducible and reduced and S is smooth curve, both π_1, π_2 are flat with fiber of codimension 1. Since X is reduced and irreducible, by [GPR94, Theorem 1.19], we have the set of points that $\dim X_y = 0$ is a big open subset in Y (with the complement an analytic subset V such that $\mathrm{codim}_V(X) \geq 2$). Since Y is normal, and $f : X - f^{-1}(V) \to Y - V$ is bijective. Thus $f : X - f^{-1}(V) \to Y - V$ is biholomorphic. Additionally, f is surjective by the definition of a bimeromorphic morphism. Consequently, there exists an irreducible component C_t of X_t such that $f\left(C_t\right) = D_t$ by the irreducibility of D_t.

In view of the codimensions of V and D_t, it follows that $D_t \nsubseteq V$, and consequently, $C_t \nsubseteq f^{-1}(V)$. Clearly, $D_t \cap V$ is a thin analytic subset of D_t, and $C_t \cap f^{-1}(V)$ is a thin analytic subset of C_t. Hence, one can easily check by definition that $f : C_t \to D_t$ is bimeromorphic. □

Theorem 4.2 ([CRT25, Theorem 1.6])**.** Let

$$\pi_1 : X \to S, \pi_2 : Y \to S$$

be two (locally) Moishezon morphism with irreducible fibers that admits canonical singularities, such that $\kappa(X_0) \geq 0$. Then the bimeromorphic map that connects π_1 and π_2 is indeed fiberwise bimeromorphic.

Let us briefly sketch the idea out. Take the resolution of indetermancy.

$$\begin{tikzcd}
& W \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X \arrow[rr, dashed] \arrow[dr, "\pi_1"'] & & Y \arrow[dl, "\pi_2"] \\
& S &
\end{tikzcd}$$

We claim that strict transform $\tilde{X_0}$ must coinside with $\tilde{Y_0}$. For otherwise, by the lower semi-continuity of the plurigenera (for Moishezon morphism), we know that

$$P_m(X_0) + P_m(\tilde{Y_0}) = P_m(\tilde{X_0}) + P_m(\tilde{Y_0}) \leq P_m(W_t) = P_m(X_t) = P_m(Y_t) = P_m(Y_0) = P_m(\tilde{Y_0}),$$

so that the plurigenera $P_m(X_0) = 0$ contradict to $\kappa(X_0) \geq 0$.

5 Proof of fiberwise bimeromorphic conjecture when X_0 is KLT and not uniruled

In the last section, we will prove the following conjecture in this section under the assumption that center fiber is KLT and not uniruled.

Conjecture 5.1 (Fiberwise bimeromorphic conjecture for Moishezon morphism, see [Kol22, Conjecture 5])**.** Let $g : X \to \mathbb{D}$ be a flat, proper, Moishezon morphism. Assume that X_0 has canonical (resp. log terminal) singularities.

Then g is fiberwise birational to a flat, projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to \mathbb{D}$ such that

(1) X_0^{p} has canonical (resp. log terminal) singularities,
(2) X_s^{p} has terminal singularities for $s \neq 0$, and
(3) $K_{X^{\mathrm{p}}}$ is $\mathbb{Q}$-Cartier.

Remark 5.2. Before continuing the discussion about this conjecture, let us first look closely at what this conjecture is about. The conjecture shows that flat Moishezon morphim is not only bimeromorphic to some projective model it's indeed fiberwise bimeromorphic to some projective model, if we assume the singularity on the central fiber is nice.

Kollár verifies the conjecture when the central fiber is KLT and not uniruled. Before proving the theorem, let us list some intermediate results that will be used.

Theorem 5.3 (Inversion of adjunction,[Kol22, Proposition 30])**.** Let X be a normal, complex analytic space, $X_0 \subset X$ a Cartier divisor and Δ an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then $(X, X_0 + \Delta)$ is PLT in a neighborhood of X_0 iff $\left(X_0, \Delta|_{X_0}\right)$ is KLT.

Proof. The proof uses some algebraic approximation technique, and it's omit here. ☐

Theorem 5.4 (Existence of canonical modification for flat Moishezon family with KLT central fiber, [Kol22, Collary 38])**.** Let $f : X \to \mathbb{D}$ be a flat, proper, Moishezon morphism. Assume that X_0 is log terminal. Then X has a canonical modification $\pi : X^{\mathrm{c}} \to X$, such that
(a) X_0^{c} is log terminal and,
(b) π is fiberwise birational.

Proof. The proof use some algebraic approximation technique. See more details in the paper. ☐

Lemma 5.5 (A limiting expression for restricted base locus, see [Kol22], (31.1))**.** Let $X \to S$ be a proper, Moishezon morphism, D an $\mathbb{R}$-divisor on X, and A a big $\mathbb{R}$-divisor on X such that $\mathbf{B}^{\mathrm{div}}(A) = \emptyset$. Then, for every prime divisor $F \subset X$,

$$\operatorname{coeff}_F \mathbf{B}_{-}^{\mathrm{div}}(D) = \lim_{\epsilon \to 0} \operatorname{coeff}_F \mathbf{B}_{-}^{\mathrm{div}}(D + \epsilon A)$$

Lemma 5.6 (An estimate for restricted base locus, see [Kol22], (31.2))**.** Let $X_i \to S$ be proper, Moishezon morphisms, $h : X_1 \to X_2$ a proper, bimeromorhic morphism, D_2 a pseudo-effective, $\mathbb{R}$-Cartier divisor on X_2, and E an effective, h-exceptional divisor. Then

$$\mathbf{B}_{-}^{\mathrm{div}}\left(E + h^* D_2\right) \geq E$$

The following proposition is useful in the proof.

Proposition 5.7. Let $f : X \to U$ be a projective morphism, (X, Δ) a dlt pair and $\phi : X \dashrightarrow X_M$ be a minimal model for $K_X + \Delta$ over U. Then the set of ϕ-exceptional divisors coincides with the set of divisors contained in $\mathbf{B}_{-}\left(K_X + \Delta / U\right)$ and

Proof. Let $p : Y \to X$ and $q : Y \to X_M$ be a common resolution. Since ϕ is $(K_X + \Delta)$-negative, we have that $p^*\left(K_X + \Delta\right) = q^*\left(K_{X_M} + \phi_* \Delta\right) + E$ where E is effective, q-exceptional and the support of $p_* E$ is the set of ϕ-exceptional divisors. Since the minimal model assumption, we have $N_\sigma\left(p^*\left(K_X + \Delta\right) / U\right) = E$. we get

$$p_* E = N_\sigma(K_X + \Delta).$$

☐

The following lemma is useful in the proof of the main result.

Lemma 5.8. Let $b_0 = 1, b_1, \ldots, b_n$ be real numbers which are linearly independent over $\mathbb{Q}$, and suppose that the divisor $\sum_{i=0}^n b_i B_i$ is $\mathbb{R}$-Cartier. Then each of the divisors B_i is $\mathbb{Q}$-Cartier.

Now we can dive into the proof of the main theorem.

Theorem 5.9 (A flat Moishezon morphism with KLT and non-uniruled central fiber will be fiberwise bimeromorphic to a projective morphism, [Kol22], Theorem 28)**.** Let $g : X \to \mathbb{D}$ be a flat, proper, Moishezon morphism. Assume that

1. X_0 has log terminal singularities and
2. X_0 is not uniruled

Then

(a) g is fiberwise birational to a flat, projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to \mathbb{D}$ (possibly over a smaller disc),

(b) X_0^{p} has log terminal singularities,

(c) X_s^{p} is not uniruled and has terminal singularities for $s \neq 0$,

(d) $K_{X^{\mathrm{p}}}$ is $\mathbb{Q}$-Cartier

Proof. We take a resolution of singularities $Y \to X$ such that $Y \to \mathbb{D}$ is projective, and then take a relative minimal model of $Y \to \mathbb{D}$. We hope that it gives what we want. There are, however, several obstacles.

Step 1. Take canonical modification.

We need to control the singularities of X. First for a flat proper Moishezon morphism with KLT central fiber, there exist a canonical modifiction (Theorem 5.4) which is fiberwise birational and the central fiber is KLT. Thus we in the case that K_X is $\mathbb{Q}$-Cartier.

Indeed by the canonical modificaiton we can find some canonical modification $X^c \to X$ such that X^c is canonical singularity and the the morphism $X^c \to X$ is the fiberwise birational map, thus if we can prove the result for $X^c \to \mathbb{D}$ then it will also be true for the $X \to \mathbb{D}$ (since composition of fiberwise birational map is again fiberwise birational)

We assume this from now on. Then the inversion of adjunction for PLT pair implies that the pair (X, X_0) is PLT. by setting $\Delta = 0$ in the inversion of adjunction. (To apply the inversion of adjunction here we require K_X to be $\mathbb{Q}$-Cartier)

Step 2. Doing semi-stable reduction

After a base change $z \mapsto z^r$ we get $g^r : X^r \to \mathbb{D}$. For suitable r, there is a semi-stable, projective resolution $h : Y \to \mathbb{D}$; we may also choose it to be equivariant for the action of the cyclic group $G \cong \mathbb{Z}_r$. All subsequent steps will be G-equivariant. We denote by X_0^Y the birational transform of X_0 and by E_i the other irreducible components of Y_0.

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & X^r \arrow[r] \arrow[d] & X \arrow[d] \\ & \mathbb{D} \arrow[r] & \mathbb{D} \end{tikzcd}$$

Such that the following condition holds:
(a) Y is non-singular,
(b) generic fibers are non-singular,
(c) The special fiber is a reduced divisor with SNC support,

(d) Denote that $Y_0 = X_0^Y + \sum c_i E_i$ (with X_0^Y be the strict transform on X_0), note that the strict transform X_0^Y will dominant X_0.

Step 3. The generic fibers Y_s are not uniruled (for $s \neq 0$).

We will prove it by contradiction, if the generic fibers Y_s are uniruled. Then, by the Matsusaka's theorem (see [Kol96, Theorem VI.1.7]), all the irreducible components of Y_0 are uniruled. On the other hand, since X_0^Y dominant X_0, X_0 must be uniruled, a contradiction.

And finally by the BDPP theorem. easy to see K_{Y_s} is pseudo-effective. (Since we assume that generic fibers are smooth).

Step 4. Run the MMP using BCHM

We require the condition that the general fibers are of log general type. To achieve this, let H be an ample, G-equivariant divisor such that $Y_0 + H$ is snc (note that this is possible by taking $H' = \otimes_{m=1}^{n} g \cdot H$, since G is finite group this is well defined ample line bundle). For $\epsilon > 0$ we get a pair $(Y, \epsilon H)$ whose general fibers $(Y_s, \epsilon H_s)$ are of log general type since K_{Y_s} is pseudoeffective by previous step. For such algebraic families, relative minimal models are known to exist by BCHM. We also know that $(Y, Y_0 + \epsilon H)$ is dlt for $0 < \epsilon \ll 1$ (since Y is smooth and $Y_0 + H$ is snc).

Thus we get the relative MMP over the disc $\mathbb{D}$, (note here the base is an analytic disc, thus it require the [Fuj22]).

$$\begin{tikzcd}
(Y, \epsilon H) \arrow[rr, dashed, "\phi"] \arrow[dr] & & (Y^m, \epsilon H^m) \arrow[dl] \\
& \mathbb{D} &
\end{tikzcd}$$

We claim $(Y^{\mathrm{m}}, Y_0^{\mathrm{m}} + \epsilon H^{\mathrm{m}})$ is DLT, and H^{m} is $\mathbb{Q}$-Cartier for general choice of ϵ and also thus $(Y^{\mathrm{m}}, Y_0^{\mathrm{m}})$ is also dlt.

Indeed, Step of MMP will preserve DLT condition (see [**BCHM**] Lemma 3.10.10.) easy to see $(Y^{\mathrm{m}}, Y_0^{\mathrm{m}} + \epsilon H^{\mathrm{m}})$ is DLT. On the other hand by Lemma 5.8, easy to see if ϵ is sufficient general the $\mathbb{Q}$-linear independent condition satisfies and therefore H^m is indeed a $\mathbb{Q}$-Cartier divisor. And finally by [KM98, Corollary 2.39] the $(Y^{\mathrm{m}}, Y_0^{\mathrm{m}})$ is also DLT.

Recall that we have
$$\mathbf{B}_{-}^{\mathrm{div}}(K_Y + Y_0) \geq (1 + a(E_i, X^r, X_0))E_i,$$
On the other hand
$$\operatorname{coeff}_F \mathbf{B}_{-}^{\mathrm{div}}(D) = \lim_{\epsilon \to 0} \operatorname{coeff}_F \mathbf{B}_{-}^{\mathrm{div}}(D + \epsilon A),$$
for any prime divisor F. Thus for sufficient small ϵ, E_i also contains in the restricted base locus of $K_Y + Y_0 + \epsilon H$ then by Proposition 5.7, any MMP will contract those E_i.

Step 5. The morphism $X \dashrightarrow Y^m$ is fiberwise birational morphism. Since Cone theorem, those divisor being contracted will be covered by rational curves. But we assume that X_0^Y is not uniruled (thus it does not being contracted by the MMP). By Theorem 2.5 the generic fiber of $X \dashrightarrow Y^m$ are bimeromorphic, so that one only needs to prove that the central fiber X_0 is bimeromorphic to Y_0^m. Indeed, since the only component that left on Y_0^m is the strict transform of X_0^Y, thus X_0 is bimeromorphic to Y_0^m.

Step 6. Check the singularity assumption. Note that $h : Y \to \mathbb{D}$ is smooth away from Y_0 (by the semi-stable family) thus $(Y_s, \epsilon H_s)$ is terminal for $s \neq 0$ and $0 \leq \epsilon \ll 1$ (see [KM98] Corollary 2.35. (2))

Since H_s is ample, by negativity lemma we do not contract it. Note that $(Y_s^{\mathrm{m}}, \epsilon H_s^{\mathrm{m}})$ is still terminal (see [KM98, Corollary 3.43], note that the divisorial contraction preserve the terminal singularity require the exceptional set does not contains in the support of H_s). Thus Y_s^{m} also admits terminal singularity (see [KM98, Corollary 2.35]). Since $(Y^{\mathrm{m}}, Y_0^{\mathrm{m}})$ is DLT, it's also plt thanks to the irreducible of Y_0^{m} ([KM98, Proposition 5.51]). And therefore Y_0^{m} is KLT by the easy direction of inversion of adjunction (see Theorem 5.3). □

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