Rational Curves on Kähler varieties Spring 2025

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The aim of this note are two:

(1) We will study the rational curves on Kähler variety ([HP16], [CH20]),

(2) We will give an introduction to the BDPP theorem for projective and Kahler manifold ([BDPP13],[Ou25]).

Some further discussion and applications can be found in Cone Theorem for Kähler MMP.

Contents

1 Transcendental cone

On a compact Kähler manifold, there may not have plently of divisors. To make sense varies positivities, it is necessary to introduce the transcendentla cones.

2 Duality between varies cones

The following theorem shows the duality between pseudo-effective cone and movable cone on the projective manifold.

Lemma 2.1. Let X be a projective manifold. Let $\gamma \in N_1(X)$ be a movable class. Then given any prime divisor E, there exist a representative γ_E such that γ_E intersect E properly and $\gamma \equiv \gamma_E$.

Remark 2.2. I am not pretty sure, if the result is also true for Kähler manifold?

Proof. □

Theorem 2.3. Let X be a projective manifold, then the pseudo-effective cone is dual to the cone of movable curves

$$\mathcal{E} = \overline{\mathrm{Mov}(X)}^{\vee}.$$

In other words, a divisor is pseudo-effective iff it has non-negative intersection with any movable curves.

Remark 2.4. David [**David19**] proved ...

Remark 2.5. Let us briefly sketch the idea of the proof.

Proof. Let C be a movable curve, By Lemma 2.1, we can choose some C' such that $C \equiv C'$ and C' meets the given pseudo-effective divisor properly. Hence

$$\mathcal{E} \subset \overline{\mathrm{Mov}(X)}^{\vee}.$$

Conversely, if the inclusion is strict then there exist some

$$\xi \in \partial(\mathcal{E}(X)), \quad \xi \in \mathrm{int}\left(\overline{\mathrm{Mov}}(X)^*\right).$$

We want to deduce contradition. Since X is projective, we can find some ample divisor H such that $\xi - \epsilon H$ still in the movable cone. So that

$$\frac{(\xi \cdot C)}{(H \cdot C)} \geq \varepsilon, \quad \forall\, C \in \overline{\mathrm{Mov}(X)}.$$

On the other hand we can apply Fujita approximation to the class $\xi + tH$ for the ample H. And gets

$$\mu_t : X_t \to X$$

such that

$$\mu_t^*(\xi + tH) = A_t + E_t$$

choose $C = \mu_* A_t^{n-1}$, then apply the Asymptotic orthogonality of Fujita approximation to $\xi \cdot C$ and Teissier-Hovanskii inequality to deduce an upper bound

$$\delta_t \geq \frac{\xi \cdot C}{H \cdot C} \geq \epsilon$$

with $\delta_t \to 0$ when $t \to 0$ (here δ_t is a constant depend on the volume of A_t, since $\mathrm{vol}(\xi) = 0$ by Fujita approximation $\mathrm{vol}(A_t) \to 0$ when $t \to 0$).

□

One can generalize the duality theorem to the normal Moishezon space using standard blow up arguement.

Theorem 2.6. Let X be a normal Moishezon space, then the pseudo-effective cone is dual to the movable cone of curves.

Proof. □

Using the duality theorem, we can show that cone of nef curves coinside with the movable cone of curves.

Theorem 2.7. Let X be a normal Moishezon space, then the Batyrev nef cone coinside with the movable cone of curves.

3 Characterization of the projective uniruled manifold

The projective uniruled manifold is characterized by the pseudo-effectiveness of the canonical bundle.

Lemma 3.1. Given a movable curve C, there exist a covering family $\bigcup_{t \in S} C_t$ contains C, which covers a dense open subset of X. To be more precise, we can find a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\phi"] \arrow[d, "f"'] & X \\ S & \end{tikzcd}$$

with f a fibration, with fibers C_t and ϕ is dominant generic finite morphism, with $\{C_t\}_{t \in S}$ lies the same numerical class.

Proof. □

Theorem 3.2 ([BDPP13, Corollary 0.3])**.** Let X be a projective manifold. Then X is uniruled iff K_X is not pseudo-effectiveness.

Remark 3.3. One direction of the proof is easy, and can be adopted to the Kähler manifold. The converse direction (say K_X is not pseudo-effective) implies uniruled of X is non-trivial, which requires the Mori bend and break technique and the duality between pseudo-effective cone and movable cone.

Remark 3.4. Miyaoka and Mori [**MM86**] proved that a projective manifold is uniruled iff there exist an open subset over which there exist a K_X-negative curve passing through it. For more discussion about Miyaokao-Mori theorem (and varies properties of uniruled manifold) see my Note 15.

Proof. It's sufficient to prove that if K_X is not pseudo-effective, then X is uniruled. By duality of pseudo-effective cone and movable cone, we know that there exists a movable curve such that

$$K_X \cdot C < 0.$$

By Lemma 3.1, we can produce a covering family of K_X-negative irreducible curves using the movable curve C. $\square$

We can generalize the BDPP theorem to the singular case.

Theorem 3.5. Let (X, B) be a $\mathbb{Q}$-factorial log pair. If $K_X + B$ is not pseudo-effective, then X is uniruled.

Remark 3.6. Rational curves on singular space is tricky. See more discussion on my notes note-9 Rational curves on Moishezon space, Kaehler varieties.

Proof. Taking the log resolution

$$f : X' \to X,$$

such that $f^*(K_X+B) = K_{X'}+B'$. Since being uniruled is birational invariant, if X is not uniruled, then so it is X'. Then by the BDPP theorem we just proved, $K_{X'}$ is pseudo-effective, thus K_X is pseudo-effective. Since B is effective, $K_X + B$ is pseudo-effective. $\square$

We can also show the converse direction for canonical singularity.

Theorem 3.7.

The following example indicate that BDPP theorem may fail for singular variety however.

We can characterize the uniruled variety using subsheaf of tangent sheaf

Theorem 3.8. Let X be a projective manifold, $\mathscr{F} \subset T_X$ be a coherent subsheaf such that $\det \mathscr{F}^* \subset T_X$ is not pseudo-effective, then X is uniruled.

Proof. $\square$

4 Proof of BDPP conjecture for Kähler manifold

Recently, [**Ou**] proved the BDPP conjecture for the compact Kähler manifold. In this section, we will briefly introduce the result that he proved.

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4.1 Algebraic integrability criteria under Kähler setting

4.2 Pseudo-effectiveness of the adjoint class

4.3 Relative Albanese reduction

4.4 Proof of BDPP conjecture for compact Kähler manifold

5 Several Applications of BDPP Theorem

5.1 Applications of duality of pseudo-effective cone and cone of movable curves

5.2 Producing rational curves using BDPP conjecture

5.3 Cone theorem using BDPP conjecture

6 Höring-Peternell's approach of Mori bend and break for Kähler 3-fold

In this section, we will briefly summarize the idea of Höring-Peternell on Mori bend and break for Kähler 3-fold [HP16].

Let us first define the working condition for this section.

Definition 6.1 (Condition (*))**.** Let X be a $\mathbb{Q}$-factorial normal compact Kähler threefold with at most terminal singularity, assume K_X is pseudo-effective, such that $S_1, \cdots, S_k$ are the components of the negative part of Zariski decomposition of K_X.

Similar as the standard Mori bend and break, if the anti-canonical degree is greater than a threshold b then the curves breaks and produce some rational curve. Let us first define the threshold for our theorem

$$b = \max\{1, -K_X \cdot Z \mid Z \in S_{j,\mathrm{Sing}} \text{ or } Z \in S_j \cap S_i\}.$$

Theorem 6.2. Under condition (*)

1. If C is a curve with large anti-canonical degree (say $-K_X \cdot C > b$), then it can break into

$$[C] = [C_1] + [C_2]$$

2. Let $\overline{\mathrm{NE}(X)}$ has a K_X-negative extremal ray $\mathbb{R}_+[\Gamma]$, such that the representative Γ is not very rigid. Then we can find a representable $C \in \mathbb{R}_+[\Gamma]$ such that $\dim_C \mathrm{Chow}(X) > 0$, and $\mathbb{R}_+[\Gamma]$ contains ratioal curves.

Remark 6.3. Let us first briefly sketch the idea of the proof: The general idea is if the anti-canonical degree is large, then the curve C is deformable in the sense that $\dim_C \mathrm{Chow}(X) > 0$.

We then prove that the deformation of the curve contains in a component S_i of negative par of Zariski decomposition $N(K_X)$. (Note that the surface S_i has K_{S_i} not pseudo-effective, thus it is a uniruled surface)

Therefore, we reduce the problem onto the surface S_i. We try to prove that K_{S_i}-negative curve on the uniruled surface breaks and produce a rational curve.

There are several issues in the idea above.

Before going into the proof, let us introduce an important result that repeatly use in the poof is that the minimal resolution on the surface will decrease the K_X-degree.

6.1 Mori bend and break for K_S-negative curve on smooth projective uniruled surface

Lemma 6.4. Let S be a smooth projective uniruled surface. Let C be a curve on S such that

1. (existence of rational curve) $K_S \cdot C < 0$, then the curve C is numerical equivalent to
$$[\sum c_i C_i] = [C]$$
2. (curve with anti-canonical degree breaks) If $K_S \cdot C < -4$ then the there exist a 1-cycle (with at least 2 components) $\sum_{i=1}^{m\geq 2} C_i$ such that
$$[C] = [\sum_{i=1}^{i\geq 2} C_i],$$
such that $K_S \cdot C_1 < 0$ and $K_S \cdot C_2 < 0$.

Proof. □

6.2 curve with large anti-canonical degree is deformable

This technical core of the proof is the following lemma, which shows that a curve with large anti-canonical degree is deformable in then sense that $\dim \mathrm{Chow}_C(X) > 0$.

Lemma 6.5. Under the setting of (*), assume that
$$-K_X \cdot C > b,$$
then $\dim \mathrm{Chow}_C(X) > 0$

Proof. □

6.3 Proof of main theorem

We can now prove the main theorem.

Proof of Theorem 6.2. □

7 Cao-Höring's approach produce rational curve for Kähler manifold

Cao-Höring found that the BDPP conjecture is useful to produce K_X-negative rational curves on a compact Kähler manifold when K_X is pseudo-effective but not nef. The major reference of this section is [CH20]. The technical core (and also the most difficult part) of the proof is the generalization of subadjunction formula to compact Kähler case.

Theorem 7.1. Let X be a compact Kähler manifold, such that K_X is pseudo-effective but not nef. Then there exist a rational curve C on X such that

7.1 Pseudo effectiveness of the adjoint class

The major technical tools that will be used in the proof of the Cao-Höring's theorem is the following pseudo-effectiveness theorem.

Theorem 7.2.

Remark 7.3.

7.2 Cao-Höring's subadjunction theorem

7.3 Proof of Cao-Höring's main theorem

7.4 Further discussion of Cao-Höring's theorem

We try to make slightly generalization of Cao-Höring result.

References

[BDPP13] S. Boucksom et al. "The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension". In: *J. Algebraic Geom.* 22.2 (2013), pp. 201–248.

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