Fiberwise Bimeromorphic Problems Summer 2025

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1 Overview

The aim of this note is to study the fiberwise bimeromorphic problems. To be more concrete, we consider the following two problems. The first one is:

Question 1.1. Let

$$\pi : X \to B \text{ and } \pi' : X' \to B,$$

be proper flat morphisms from a complex analytic space to a smooth connected curve B. Assume that the generic fibers of π and π' are bimeromorphic. Under what conditions, the special fibers between these two families also admit a certain bimeromorphic relation?

The second one focuses on the Moishezon morphisms, under which condition we can let a Moishezon morphism fiberwise bimeromorphic to a projective morphism:

Question 1.2. Let $g : X \to \mathbb{D}$ be a flat, proper, Moishezon morphism. Under which conditions, it is actually fiberwise bimeromorphic to a projective morphism $g^p : X^p \to \mathbb{D}$?

We will discuss the first question in Sections 2–4 and the second question in Section 5. The main references for this note are [Kol22] and [KT19].

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2 A Fiberwise Birational Criterion

We first recall the definition of the meromorphic S map that we introduced in the first time.

Definition 2.1 (Meromorphic S-map)**.** Let X, Y be reduced complex spaces. We call the S-map a *meromorphic S-map*

$$\begin{tikzcd} & \Gamma \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\alpha"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

if there exists a subvariety $\Gamma \subset X \times_S Y$ with the restriction of the first projection $p : \Gamma \to X$ be a proper bimeromorphic morphism. Moreover, if the restriction on the second projection $q : \Gamma \to Y$ is also a proper bimeromorphic morphism, then we call α *proper bimeromorphic S-map*.

Definition 2.2 (Fiberwise bimeromorphic map, [Kol22, Definition 26])**.** Let $g_i : X^i \to S$ be a proper morphisms. A bimeromorphic S-map $\phi : X^1 \dashrightarrow X^2/S$ is *fiberwise bimeromorphic* if ϕ induces a bimeromorphic map $\phi_s : X^1_s \dashrightarrow X^2_s$ for every $s \in S$.

Remark 2.3 (Fiberwise bimeromorphic $\neq$ fibers bimeromorphic equivalent, [CRT25, Example 2.15])**.** Let $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \to \mathbb{P}^1$ be the Hirzebruch surface of index n. By construction easy to see that all the Hirzebruch surface are birational equivalent to $\mathbb{P}^1 \times \mathbb{P}^1$. Let Z be any compact complex manifold. So that $\mathbb{F}_n \times Z \to \mathbb{P}^1$ is binational equivalent to $(\mathbb{P}^1 \times \mathbb{P}^1) \times Z \to \mathbb{P}^1$.

$$\begin{tikzcd} & \mathrm{Bl}_{p_1, \cdots, p_n}(\mathbb{P}^1 \times \mathbb{P}^1) \times Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}^1 \times \mathbb{P}^1 \times Z \arrow[rr, dashed, "q \circ p^{-1}"] \arrow[dr] & & \mathbb{F}_n \times Z \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd}$$

Note that fibers of these two families are birational equivalent (as both side have fiber $\mathbb{P}^1 \times Z$). However the restriction of the map $q \circ p^{-1}$ does not give the bimeromorphic map of the fiber (since the strict transform of the fibers of $\mathbb{P}^1 \times \mathbb{P}^1 \times Z$ via p^{-1} will be contracted by q).

Although the bimeromorphic map needs not to be fiberwise bimeromorphic in general, it is indeed fiberwise bimeromorphic on a dense open subset. As the following proposition shows.

Proposition 2.4 (Bimeromorphic on the generic fiber)**.** Let $f : X \dashrightarrow Y$ be a bimeromorphic S-map between two proper surjective family $g : X \to S,\ h : Y \to S$ over the base S. Then on the generic fiber, the map f will induce a bimeromorphic $f_s : X_s \dashrightarrow Y_s$.

Proof. Since f is bimeromorphic, by definition, the graph $\Gamma \subset X \times_S Y$ will induce two bimeromorphic morphisms

$$p : \Gamma \to X, \quad q : \Gamma \to Y,$$

such that there exists some non-empty analytic Zariski open subset $U_X \subset X$, $U_Y \subset Y$ with $p : p^{-1}(U_X) \to U_X$, $q : q^{-1}(U_Y) \to U_Y$ be isomorphisms. On the generic fiber, the dimension equalities hold

$$\dim X_s = \dim X - \dim S, \quad \dim Y_s = \dim Y - \dim S.$$

On the other hand, if we denote the analytic subset $E_X = X - U_X$ and $E_Y = Y - U_Y$, then the intersection with the generic fiber X_s (resp. Y_s), say

$$E_X \cap X_s = E_{X,s}, \text{ (resp. } E_Y \cap Y_s = E_{Y,s}),$$

are proper analytic subset in X_s (resp. Y_s). Indeed, only needs to show that $\dim E_{X,s} < \dim X_s$ (resp. $\dim E_{Y,s} < \dim Y_s$). As intersection of analytic subvariety is still analytic subvariety and dimension strict less, it's automatically proper analytic subset. Thus by definition $p_s : \Gamma_s \to X_s$ (resp. $q_s : \Gamma_s \to Y_s$) are bimeromorphic morphisms. To see that $\dim E_{X,s} < \dim X_s$, we divide it into two cases: (1) If $g(E_X) \subset S$ is proper analytic subset, then clearly the generic fiber has $\dim E_{X,s} = 0$. (2) If $g(E_X) = S$ then the generic fiber $\dim E_{X,s} = \dim E_X - \dim S$ and we know that $\dim E_X < \dim X$ and therefore

$$\dim E_{X,s} = \dim E_X - \dim S < \dim X - \dim S = \dim X_s.$$

Since the base change preserves the properness, we have $p_s : \Gamma_s \to X_s$, $q_s : \Gamma_s \to Y_s$ are still proper. Thus, complete the proof. □

We now prove the first main theorem of this note, which is about the specialization of the birational map when the pluricanonical system is non-empty.

Proposition 2.5 (Kollár's Specialization of birational map, [Kol23, Proposition 1.25])**.** Let $f_i : X^i \to B$ be two smooth families of projective varieties over a smooth curve B. Assume that the generic fibers X_b^1 and X_b^2 (for $b \neq 0$) are birational, and further assume that the pluricanonical system $\left|mK_{X_b^i}\right|$ is non empty for some $m > 0$. Then for every $b \in B$, the fibers X_b^1 and X_b^2 are birational.

Proof. Pick a birational map $\phi : X_b^1 \dashrightarrow X_b^2$ (for the generic fiber), and let $\Gamma \subset X^1 \times_B X^2$ be the closure of the graph of ϕ. Let $Y \to \Gamma$ be the resolution of the graph with projections $p_i : Y \to X^i$.

$$\begin{tikzcd}
& Y \arrow[d] \arrow[ddl, "p_1"'] \arrow[ddr, "p_2"] & \\
& \Gamma \arrow[dl] \arrow[dr] & \\
X^1 \arrow[dr, "f_1"'] & & X^2 \arrow[dl, "f_2"] \\
& B &
\end{tikzcd}$$

Note that by definition, both of the p_i are open embeddings on $Y \setminus (\mathrm{Ex}\,(p_1) \cup \mathrm{Ex}\,(p_2))$.

Thus if we prove that neither $p_1\,(\mathrm{Ex}\,(p_1) \cup \mathrm{Ex}\,(p_2))$ nor $p_2\,(\mathrm{Ex}\,(p_1) \cup \mathrm{Ex}\,(p_2))$ contains a fiber of f_1 or f_2, then $p_2 \circ p_1^{-1} : X^1 \dashrightarrow X^2$ (it needs not to be birational) restricts to a birational map $X^1_b \dashrightarrow X^2_b$ for every $b \in B$.

We may assume that B is affine (as we only care about the special fiber, thus we can focus on the affine base around b) and let $\mathrm{Bs}\,|mK_{X^i}|$ denote the set-theoretic base locus. Let $L_i = \mathcal{O}_{X^i}(mK_{X^i})$. The direct image $\mathcal{E}_i = f_{i*}L_i$ as a torsion free sheaf is locally free on the smooth curve B (so that the vanishing locus of a section of $\mathcal{E}_i$ is a subvariety).

By assumption $|mK_{X^i_b}| \neq \emptyset$ for generic $b \in B$, we claim that $|mK_{X^i}|$ is non-empty as well. Indeed, since the restriction map

$$H^0(X^i, mK_{X^i}) \to H^0(X^i_b, mK_{X^i_b}),$$

is surjective on the generic fibers. Thus, the pluricanonical system on the generic fiber satisfies $|mK_{X^i}|_{X^i_b} = |mK_{X^i_b}| \neq \emptyset$. In particular, this means that $|mK_{X^i}| \neq \emptyset$.

On the other hand, since $H^0(X, mK_{X^i}) \neq 0$ and the base is affine, for any point $s \in B$, there exists a non-zero section

$$0 \neq \sigma \in H^0(B, \mathcal{E}_i),$$

such that $\sigma(s) \neq 0$. Therefore consider the restriction commutative diagram (note that in general it's not clear the base change morphism $H^0(X^i_s, L_i|_{X^i_s}) \to \mathcal{E}_i(s)$ is isomorphism or not on the special fiber)

$$\begin{array}{ccc} H^0(X^i, L_i) & \xrightarrow{\text{res}} & H^0(X^i_s, L_i|_{X^i_s}) \\ \simeq\downarrow & & \downarrow \\ H^0(B, \mathcal{E}_i) & \xrightarrow[\text{res}]{} & \mathcal{E}_i(s) \end{array}$$

there exists a section $\sigma' \in H^0(X^i, L_i)$ which maps down to $\sigma \in H^0(B, \mathcal{E}_i)$ such that $\sigma(s) \neq 0$. So that $\sigma'|_{X^i_s} \neq 0$ in $H^0(X^i_s, L_i|_{X^i_s})$. And therefore the base locus $\mathrm{Bs}|mK_{X^i}|$ cannot contain the fiber.

Since X^i are smooth,

$$K_Y \sim p_i^* K_{X^i} + E_i, \quad \text{ where } E_i \geq 0 \text{ and } \mathrm{Supp}\, E_i = \mathrm{Ex}\,(p_i)\,.$$

So that every section of $\mathcal{O}_Y\,(mK_Y)$ pulls back from X^i, Thus

$$\mathrm{Bs}\,|mK_Y| = p_i^{-1}\,(\mathrm{Bs}\,|mK_{X^i}|) + \mathrm{Supp}\, E_i,$$

Comparing these for $i = 1, 2$, we conclude that

$$p_1^{-1}\,(\mathrm{Bs}\,|mK_{X^1}|) + \mathrm{Supp}\, E_1 = p_2^{-1}\,(\mathrm{Bs}\,|mK_{X^2}|) + \mathrm{Supp}\, E_2,$$

Therefore,

$$\boxed{p_1\,(\mathrm{Supp}\, E_2) \subset p_1\,(\mathrm{Supp}\, E_1) + \mathrm{Bs}\,|mK_{X^1}|}$$

Since E_1 is p_1-exceptional, $p_1\,(\mathrm{supp}\, E_1)$ has codimension ≥ 2 in X^1, hence it does not contain any of the fibers of f_1. Combined with $\mathrm{Bs}\,|mK_{X^1}|$ does not contain any of the fibers either.

Thus, $p_1(\mathrm{Ex}(p_1) \cup \mathrm{Ex}(p_2))$ does not contain any of the fibers, and the same argument shows for $p_2(\mathrm{Ex}(p_1) \cup \mathrm{Ex}(p_2))$. $\square$

As a remark by [Kol23], the result holds true even when the pluricanonical systems are empty. That is what we will prove in the next section.

3 Kontsevich-Tschinkel's Fiberwise Birational Theorem

Theorem 3.1 ([KT19, Theorem 1]). Let

$$\pi : X \to B \text{ and } \pi' : X' \to B$$

be smooth proper morphisms to a smooth connected curve B, over a field of characteristic zero. Assume that the generic fibers of π and π' are birational over the function field of B. Then, for every closed point $b \in B$, the fibers of π and π' over b are birational over the residue field at b.

We first introduce some new notions that needed in the proof.

Definition 3.2 (semi-ring). A *semi-ring* $(S, +, \times)$ consists of a set S equipped with two binary operations $+, \times$. Such that $+$ makes S a commutative monoid (which does not need to be an Abelian group compared to the definition of a ring).

Definition 3.3 (Burnside semi-ring over a field k, [KT19, Definition 2]). The *Burnside semi-ring* $\mathrm{Burn}_+(k)$ of a field k is the set of $\sim_k$ equivalence classes of smooth schemes of finite type over k endowed with a semi-ring structure where multiplication and addition are given by disjoint union and product over k. (here the $\sim_k$ equivalence of two schemes X, X' are defined as follows: $X/k \sim_k X'/k$ if and only if X and X' are k-birational). To be more precise, the addition and multiplication of semi-ring structure is defined as follows:

(a) Addition: Disjoint union $[X] + [Y] = [X \sqcup Y]$.
(b) Multiplication: Cartesian product $[X] \cdot [Y] = [X \times Y]$.

We then introduce the Grothendieck ring, and we denote $\mathrm{Burn}(k)$ the Grothendieck ring generated by $\mathrm{Burn}_+(k)$.

Definition 3.4 (The Grothendieck ring $\mathrm{Burn}(k)$). The *Grothendieck ring* $\mathrm{Burn}(k)$ thhat is associated to the Bunrside semi-ring $\mathrm{Burn}(k)^+$ is defined as the set of equivalence classes of pairs $([X], [Y])$, where $[X], [Y] \in \mathrm{Burn}(k)^+$. Intuitively, $([X], [Y])$ represents the "difference" $[X] - [Y]$. With the equivalence relation: We say $([X], [Y]) \sim ([X'], [Y'])$ if there exists $[Z] \in \mathrm{Burn}(k)^+$ such that:

$$[X] + \left[Y'\right] + [Z] = \left[X'\right] + [Y] + [Z].$$

The ring Operations is defined as follows

(a) Addition: $([X], [Y]) + ([X'], [Y']) = ([X] + [X'], [Y] + [Y'])$.
(b) Multiplication: $([X], [Y]) \cdot ([X'], [Y']) = ([X \times X'] + [Y \times Y'], [X \times Y'] + [Y \times X'])$.

Remark 3.5. The reason to introduce the Grothendieck ring over the Burnside semi-ring is that it allows one to implement formal subtraction, cut and paste operations.

Remark 3.6. Note that we can decompose the

$$\mathrm{Burn}(k) = \sqcup_{n\geq 0} \mathrm{Bir}_n(k),$$

where $\mathrm{Bir}_n(k)$ denotes k-birational equivalent class of smooth variety of dimension n. Each class can be denoted by $[L/k]$ with $L = k(X)$.

Proposition 3.7 (Existence of SNC model)**.** Let R be a complete dvr, let K be the fractional field (i.e. the generic point of $\mathrm{Spec}(R)$) and k the residue field (i.e. the special point of $\mathrm{Spec}(R)$). Let X/K be a geometric connected smooth proper variety defined over K, then there exists a regular flat separated R-scheme of finite type $\mathscr{X}$, endowed with an isomorphism of K-scheme $\mathscr{X}_K \to X$ such that the special fiber $\mathscr{X}_k$ is a divisor with strict normal crossing. We call $\mathscr{X}/R$ is a SNC model of X/K.

Proof. Let us first briefly sketch out the idea. We first reduce the problem to the projective case. □

Remark 3.8. The SNC model also plays an important role in the

Definition 3.9 (Specialization map, [KT19, (3.2)])**.** Let $\mathfrak{o}$ be a complete dvr, let K be the fractional field (i.e. the generic point of $\mathrm{Spec}(\mathfrak{o})$) and k the residue field (i.e. the special point of $\mathrm{Spec}(\mathfrak{o})$). We define

$$\rho_n : \mathrm{Bir}_n(K) \to \mathbb{Z}\left[\mathrm{Bir}_n(k)\right],$$

as follows: given a smooth projective family $X_K \to \mathrm{Spec}(K)$ (with the function field $L := K(X_K)$), choose one of family

$$\pi : X \to \mathrm{Spec}(\mathfrak{o}),$$

where π is proper, such that the generic fibers is X_K and special fiber

$$X_0 = \bigcup_{i\in I} d_i D_i,$$

is a SNC divisor, with the strata $D_J := \bigcap_{j\in J} D_j$. We then define the specialization map to be

$$\boxed{\rho_n([L/K]) := \sum_{\emptyset\neq J\subseteq I} (-1)^{\#J-1}\left[D_J \times \mathbb{A}^{\#J-1}/k\right]}$$

One of the main difficulties in the proof is verifying that the specialization map $\rho_n : \mathrm{Bir}_n(K) \to \mathbb{Z}[\mathrm{Bir}_n(k)]$ is well-defined (i.e., it does not depend on the choice of the family $X \to D$) or representative X in $\mathrm{Bir}_n(X)$. We omit the proof of this part; for details, see [KT19, Theorem 4]).

Proof of Theorem 3.1. We first reduce the problem onto Spec of a complete dvr. Let $\pi : X \to B$ be a smooth proper morphism to a smooth connected curve B over k with fiber X over the generic point of B. Let $K = k(B)$ be the function field of B. Let κ_b be the residue field at b, a finite extension of k. Let K_b be the completion of K at b. Then K_b is a local field with residue field κ_b. Let

$$\phi_b : K \to K_b,$$

be the canonical inclusion. By functoriality, it defines a homomorphism

$$\phi_{b,*} : \mathrm{Burn}(K) \to \mathrm{Burn}\left(K_b\right).$$

We then consider the specialization map over the complete dvr K_b. Note that we have the specialization homomorphism

$$\rho : \mathrm{Burn}\left(K_b\right) \to \mathrm{Burn}\left(\kappa_b\right),$$

and the following identity

$$[X_b/\kappa_b] = \rho\left(\phi_{b,*}([X/K])\right),$$

which follows immediately from the Definition 3.9 of ρ, since the special fiber is smooth and irreducible. This shows that the birational type of the special fiber is determined by the birational type of the generic fiber. $\square$

4 Fiberwise Bimeromorphic Criterion using Plurigenera

In this section, we will give a criterion for fiberwise bimeromorphic map using plurigenera. For readers who want to know more about this, please refer to [CRT25].

Lemma 4.1 ([GPR94, Theorem 1.19])**.** Let $f : X \to Y$ be a proper surjective holomorphic map, assume that X is reduced and irreducible. Then the set

$$\{y \in Y \mid \dim_x X_y > \dim X - \dim Y \text{ for some } x \in X_y\}.$$

is analytic in Y and of codimension at least 2.

Proposition 4.2 ([CRT25, Theorem 1.4])**.** Let X, Y and S be complex analytic spaces. Assume that X is reduced (not necessarily normal) and irreducible, Y is normal, and S is a smooth curve. Assume further that both $\pi_1 : X \to S$ and $\pi_2 : Y \to S$ are proper surjective holomorphic maps. Suppose that there is a bimeromorphic morphism $f : X \to Y$ over S. For some $t \in S$, if D_t is an irreducible component of Y_t that is of codimension 1 in Y, then there exists an irreducible component C_t (equipped with the reduced structure) of X_t that is bimeromorphic to D_t, induced by f.

In particular, if the fibers of $X \to S$ and $Y \to S$ are irreducibles then f is fiberwise bimeromorphic map.

Proof. Since X is reduced and irreducible, by lemma above, we have the set of points that $\dim X_y = 0$ is a big open subset in Y (with the complement an analytic subset V such that $\mathrm{codim}_V(X) \geq 2$). Since Y is normal, and $f : X - f^{-1}(V) \to Y - V$ is bijective. Thus $f : X - f^{-1}(V) \to Y - V$ is biholomorphic. Additionally, f is surjective by the definition of a bimeromorphic morphism. Consequently, there exists an irreducible component C_t of X_t such that $f\left(C_t\right) = D_t$ by the irreducibility of D_t.

In view of the codimensions of V and D_t, it follows that $D_t \nsubseteq V$, and consequently, $C_t \nsubseteq f^{-1}(V)$. Clearly, $D_t \cap V$ is a thin analytic subset of D_t, and $C_t \cap f^{-1}(V)$ is a thin analytic subset of C_t. Hence, one can easily check by definition that $f : C_t \to D_t$ is bimeromorphic. $\square$

We next prove a simplify version of the fiberwise bimeromorphic cirterion using plurigenera, for a much more general version, please refer to [CRT25].

Theorem 4.3 ([CRT25, Theorem 1.6])**.** Let

$$\pi_1 : X \to S,\ \ \pi_2 : Y \to S$$

be two (locally) Moishezon morphism with irreducible fibers that admits canonical singularities, such that $\kappa(X_0) \geq 0$. Then the bimeromorphic map that connects π_1 and π_2 is indeed fiberwise bimeromorphic.

Let us briefly sketch out the idea. We first take the resolution of indeterminacy, by further resolution we can guarantee the generic fibers of $W \to S$ being smooth.

$$\begin{tikzcd}
& W \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X \arrow[rr, dashed] \arrow[dr, "\pi_1"'] & & Y \arrow[dl, "\pi_2"] \\
& S &
\end{tikzcd}$$

We claim that the strict transform $\tilde{X}_0 = p_*^{-1}(X_0)$ and the strict transform $\tilde{Y}_0 = q_*^{-1}(Y_0)$ must coincide. For otherwise, since plurigenera is bimeromorphic invariant we have $P_m(X_0) = P_m(\tilde{X}_0)$, $P_m(Y_0) = P_m(\tilde{Y}_0)$ and $P_m(W_t) = P_m(X_t) = P_m(Y_t)$. On the other hand, since the family $W \to S$ is Moishezon, by the lower semi-continuity of the plurigenera (for Moishezon morphism), we have $P_m(\tilde{X}_0) + P_m(\tilde{Y}_0) \leq P_m(W_t)$. Since $Y \to S$ be a Moishezon morphism with fiberwise canonical singularities, the plurigenera remain constant i.e. $P_m(Y_t) = P_m(Y_0)$. Putting those together, we have

$$\boxed{P_m(X_0) + P_m(\tilde{Y}_0) = P_m(\tilde{X}_0) + P_m(\tilde{Y}_0) \leq P_m(W_t) = P_m(X_t) = P_m(Y_t) = P_m(Y_0) = P_m(\tilde{Y}_0)}$$

so that the plurigenera $P_m(X_0) = 0$ which contradicts $\kappa(X_0) \geq 0$.

5 The Fiberwise Bimeromorphic Conjecture for Moishezon Morphisms

In the last section, we will prove the following conjecture under the additional assumption that the center fiber is KLT and not uniruled.

Conjecture 5.1 (Fiberwise bimeromorphic conjecture for Moishezon morphism, [Kol22, Conjecture 5])**.** Let $g : X \to \mathbb{D}$ be a flat, proper, Moishezon morphism. Assume that X_0 has canonical (resp. log terminal) singularities.

Then g is fiberwise birational to a flat, projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to \mathbb{D}$ such that

(1) X_0^{p} has canonical (resp. log terminal) singularities,
(2) X_s^{p} has terminal singularities for $s \neq 0$, and
(3) $K_{X^{\mathrm{p}}}$ is $\mathbb{Q}$-Cartier.

Remark 5.2. Before continuing our discussion of this conjecture, let us first look closely at what this conjecture is about. The conjecture shows that the flat Moishezon morphism is not only bimeromorphic to some projective model but it is indeed fiberwise bimeromorphic to some projective model, as long as the singularity on the central fiber is nice enough.

Kollár verifies the conjecture when the central fiber is KLT and not uniruled. Before proving the theorem, let us list some intermediate results that will be used.

Proposition 5.3 (Inversion of adjunction, [Kol22, Proposition 30])**.** Let X be a normal complex analytic space, $X_0 \subset X$ a Cartier divisor, and Δ an effective $\mathbb{R}$-divisor such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. Then $(X, X_0 + \Delta)$ is PLT in a neighborhood of X_0 iff $\left(X_0, \Delta|_{X_0}\right)$ is KLT.

Proposition 5.4 (Existence of canonical modification, [Kol22, Corollary 38])**.** Let $f : X \to \mathbb{D}$ be a flat, proper, Moishezon morphism. Assume that X_0 is log terminal. Then X has a canonical modification $\pi : X^{\mathrm{c}} \to X$, such that
(a) X_0^{c} is log terminal and,
(b) π is fiberwise birational.

Proof. The proof uses some algebraic approximation technique, see [Kol22]. □

Lemma 5.5 ([Kol22, Lemma 31.1])**.** Let $X \to S$ be a proper, Moishezon morphism, D an $\mathbb{R}$-divisor on X, and A a big $\mathbb{R}$-divisor on X such that $\mathbf{B}^{\mathrm{div}}(A) = \emptyset$. Then, for every prime divisor $F \subset X$,

$$\operatorname{coeff}_F \mathbf{B}_{-}^{\mathrm{div}}(D) = \lim_{\epsilon \to 0} \operatorname{coeff}_F \mathbf{B}_{-}^{\mathrm{div}}(D + \epsilon A)$$

Lemma 5.6 ([Kol22, Lemma 31.2])**.** Let $X_i \to S$ be proper, Moishezon morphisms, $h : X_1 \to X_2$ a proper, bimeromorphic morphism, D_2 a pseudo-effective, $\mathbb{R}$-Cartier divisor on X_2, and E an effective, h-exceptional divisor. Then

$$\mathbf{B}_{-}^{\mathrm{div}}\left(E + h^* D_2\right) \geq E.$$

The following proposition is useful in the proof.

Proposition 5.7. Let $f : X \to U$ be a proper morphism between complex varieties, (X, Δ) a DLT pair and $\phi : X \dashrightarrow X_M$ be a minimal model for $K_X + \Delta$ over U. Then the set of ϕ-exceptional divisors coincides with the set of divisors contained in $\mathbf{B}_{-}\left(K_X + \Delta / U\right)$.

Proof. Let $p : Y \to X$ and $q : Y \to X_M$ be a common resolution. Since ϕ is $(K_X + \Delta)$-negative, we have that $p^*\left(K_X + \Delta\right) = q^*\left(K_{X_M} + \phi_* \Delta\right) + E$ where E is effective, q-exceptional and the support of $p_* E$ is the set of ϕ-exceptional divisors. Since the minimal model assumption, we have $N_\sigma\left(p^*\left(K_X + \Delta\right)/U\right) = E$. we get

$$p_* E = N_\sigma(K_X + \Delta).$$

□

Lemma 5.8. Let $b_0 = 1, b_1, \ldots, b_n$ be real numbers which are linearly independent over $\mathbb{Q}$, and suppose that the divisor $\sum_{i=0}^{n} b_i B_i$ is $\mathbb{R}$-Cartier. Then each of the divisors B_i is $\mathbb{Q}$-Cartier.

Having introduced a bunch of lemma will be used in the proof. We can now dive into the proof of the last main theorem of this note.

Theorem 5.9 (A flat Moishezon morphism with KLT and non-uniruled central fiber will be fiberwise bimeromorphic to a projective morphism, [Kol22], Theorem 28)**.** Let $g : X \to \mathbb{D}$ be a flat, proper, Moishezon morphism. Assume that

1. X_0 has log terminal singularities and
2. X_0 is not uniruled

Then

(a) g is fiberwise birational to a flat, projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to \mathbb{D}$ (possibly over a smaller disc),

(b) X_0^{p} has log terminal singularities,

(c) X_s^{p} is not uniruled and has terminal singularities for $s \neq 0$,

(d) $K_{X^{\mathrm{p}}}$ is $\mathbb{Q}$-Cartier

Proof. We take a resolution of singularities $Y \to X$ such that $Y \to \mathbb{D}$ is projective, and then take a relative minimal model of $Y \to \mathbb{D}$. We hope that it gives what we want. There are, however, several obstacles.

Step 1. Take the canonical modification. We need to control the singularities of X. First for a flat proper Moishezon morphism with KLT central fiber, there exist a canonical modifiction (Theorem 5.4) which is fiberwise birational and the central fiber is KLT. Thus we in the case that K_X is $\mathbb{Q}$-Cartier.

Indeed by the canonical modificaiton we can find some canonical modification $X^c \to X$ such that X^c is canonical singularity and the the morphism $X^c \to X$ is the fiberwise birational. Thus, if we can prove the result for $X^c \to \mathbb{D}$ then it will also be true for the $X \to \mathbb{D}$ (since composition of fiberwise birational map is again fiberwise birational).

We assume this from now on. Then the inversion of adjunction for PLT pair implies that the pair (X, X_0) is PLT. by setting $\Delta = 0$ in the inversion of adjunction. (To apply the inversion of adjunction here we require K_X to be $\mathbb{Q}$-Cartier)

Step 2. Take the semi-stable reduction. After a base change $z \mapsto z^r$ we get $g^r : X^r \to \mathbb{D}$. For suitable r, there is a semi-stable, projective resolution $h : Y \to \mathbb{D}$; we may also choose it to be equivariant for the action of the cyclic group $G \cong \mathbb{Z}_r$. All subsequent steps will be G-equivariant. We denote by X_0^Y the birational transform of X_0 and by E_i the other irreducible components of Y_0.

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & X^r \arrow[r] \arrow[d] & X \arrow[d] \\ & \mathbb{D} \arrow[r] & \mathbb{D} \end{tikzcd}$$

Such that the following conditions hold:
(a) Y is non-singular,

(b) generic fibers are non-singular,
(c) The special fiber is a reduced divisor with SNC support,
(d) Denote that $Y_0 = X_0^Y + \sum c_i E_i$ (with X_0^Y be the strict transform on X_0), note that the strict transform X_0^Y will dominant X_0.

Step 3. Prove the generic fibers Y_s are not uniruled (for $s \neq 0$). We will prove it by contradiction, if the generic fibers Y_s are uniruled. Then, by Matsusaka's theorem (see [Kol96, Theorem VI.1.7]), all the irreducible components of Y_0 are uniruled. On the other hand, since X_0^Y dominant X_0, X_0 must be uniruled, a contradiction.

And finally by the BDPP theorem. easy to see K_{Y_s} is pseudo-effective. (Since we assume that generic fibers are smooth).

Step 4. Run the MMP. We require the condition that the general fibers are of log general type. To achieve this, let H be an ample, G-equivariant divisor such that $Y_0 + H$ is snc (note that this is possible by taking $H' = \otimes_{m=1}^n g \cdot H$, since G is finite group this is well defined ample line bundle). For $\epsilon > 0$ we get a pair $(Y, \epsilon H)$ whose general fibers $(Y_s, \epsilon H_s)$ are of log general type since K_{Y_s} is pseudoeffective by previous step. For such algebraic families, relative minimal models are known to exist by BCHM. We also know that $(Y, Y_0 + \epsilon H)$ is dlt for $0 < \epsilon \ll 1$ (since Y is smooth and $Y_0 + H$ is snc).

Thus we get the $(K_X + Y_0 + \epsilon H)$-relative MMP on the disc $\mathbb{D}$, (Note that the base is an analytic disc, thus the MMP is in the sense of Fujino [Fuj22] or Kollár-Nicaise-Xu [KNX18]).

$$\begin{tikzcd}
(Y, \epsilon H) \arrow[rr, dashed, "\phi"] \arrow[dr] & & (Y^m, \epsilon H^m) \arrow[dl] \\
& \mathbb{D} &
\end{tikzcd}$$

We claim $(Y^{\mathrm{m}}, Y_0^{\mathrm{m}} + \epsilon H^{\mathrm{m}})$ is DLT, and H^{m} is $\mathbb{Q}$-Cartier for general choice of ϵ and also thus $(Y^{\mathrm{m}}, Y_0^{\mathrm{m}})$ is also dlt.

Indeed, Step of MMP will preserve DLT condition (see [**BCHM**] Lemma 3.10.10.) easy to see $(Y^{\mathrm{m}}, Y_0^{\mathrm{m}} + \epsilon H^{\mathrm{m}})$ is DLT. On the other hand, by Lemma 5.8, easy to see if ϵ is sufficient general the $\mathbb{Q}$-linear independent condition satisfies and therefore H^m is indeed a $\mathbb{Q}$-Cartier divisor. And finally by [KM98, Corollary 2.39] the $(Y^{\mathrm{m}}, Y_0^{\mathrm{m}})$ is also DLT.

Recall that we have
$$\mathbf{B}_-^{\mathrm{div}}(K_Y + Y_0) \geq (1 + a(E_i, X^r, X_0))E_i,$$
since the discrepancy of a PLT pair $a(E_i, X^r, X_0) > -1$ thus all the exceptional divisors E_i contains in the divisorial part of the restricted base locus $\mathbf{B}_-^{\mathrm{div}}(K_Y + Y_0)$. On the other hand

$$\operatorname{coeff}_F \mathbf{B}_-^{\mathrm{div}}(D) = \lim_{\epsilon \to 0} \operatorname{coeff}_F \mathbf{B}_-^{\mathrm{div}}(D + \epsilon A),$$

for any prime divisor F. Thus, for sufficiently small ϵ, E_i also contains in the restricted base locus of $K_Y + Y_0 + \epsilon H$ (since the coefficients of E_i in $\mathbf{B}_-^{\mathrm{div}}(K_Y + Y_0 + \epsilon H)$ is also positive if $\operatorname{coeff}_{E_i}\mathbf{B}_-^{\mathrm{div}}(K_Y + Y_0) > 0$). Then, by Proposition 5.7, any MMP will contract those E_i.

Step 5. Prove fiberwise bimeromorphic. By the Cone theorem, those divisors contracted will be covered by rational curves. However, we assume that X_0^Y is not uniruled (thus, it is not contracted by the MMP). By Theorem 2.4 the generic fiber of $X \dashrightarrow Y^m$ is bimeromorphic, so one only needs to prove that the central fiber X_0 is bimeromorphic to Y_0^m. In fact, since the only component on Y_0^m is the strict transform of X_0^Y, X_0 is bimeromorphic to Y_0^m.

Step 6. Check the singularity assumptions. Note that the fibers Y_s of the family $h : Y \to \mathbb{D}$ is smooth away from Y_0 (by the semi-stable assumption) thus $(Y_s, \epsilon H_s)$ is terminal for $s \neq 0$ and $0 \leq \epsilon \ll 1$ (see [KM98, Corollary 2.35. (2)])

Since H_s is ample, by negativity lemma the MMP above will not contract H_s. Note that $(Y_s^{\mathrm{m}}, \epsilon H_s^{\mathrm{m}})$ is still terminal (by [KM98, Corollary 3.43]). Thus, Y_s^{m} also admits the terminal singularity (see [KM98, Corollary 2.35]). Since $(Y^{\mathrm{m}}, Y_0^{\mathrm{m}})$ is DLT, it's also PLT thanks to the irreducible of Y_0^{m} ([KM98, Proposition 5.51]). And therefore Y_0^{m} is KLT by the easy direction of inversion of adjunction (see Theorem 5.3). ☐

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