

1 Overview

The aim of this note is to give a brief introduction to Moishezon variety and Moishezon morphism. The major references of this note are [Kol22], [Fuj83], and [Uen75].

Why we study the Moishezon variety/morphism? First, Moishezon variety has more functorial behavior (compared with projective variety), as we will see in Section 2. Secondly, from almost any projective variety we can construct some Moishezon variety via bimeromorphic modification so that Moishezon varieties are versatile in birational geometry. Thirdly, by Artin's fundamental theorem, the category of Moishezon varieties appear naturally in the moduli theory. And finally let us cite a sentence by Shigefumi Mori: Projective variety like the classical paintings, after modern painting are introduced we can enjoy the classical paintings more. To me non-projective complete variety (maybe he mean Moishezon variety) seems like the modern paintings.

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2 Moishezon varieties

Definition 2.1 (Meromorphic map). Let X, Y be reduced complex space. We call the S -map a meromorphic S -map (not necessary morphism) if

$$\begin{array}{ccc}
& \Gamma & \\
p \swarrow & & \searrow q \\
X & \overset{\alpha}{\dashrightarrow} & Y \\
f \searrow & & \swarrow g \\
& S &
\end{array}$$

with $\Gamma \subset X \times_S Y$, and $p : \Gamma \rightarrow X$ is a proper bimeromorphic morphism. Moreover if q is also proper bimeromorphic morphism, then we call α proper bimeromorphic S -map.

Remark 2.2 (Comparison between meromorphic map and S -meromorphic map). Recall that by definition, we know that

$$X \times_S Y \hookrightarrow X \times Y$$

be an inclusion. Therefore easy to see by definition that

$$S\text{-meromorphic map} \implies \text{meromorphic map}$$

Conversely, the graph of meromorphic map $\Gamma \subset X \times Y$ needs not to contain in $X \times_S Y$, so that meromorphic map needs not to be S -meromorphic map.

Remark 2.3 (Comparison between S -meromorphic map with fiberwise meromorphic map). Note that S -bimeromorphic map needs not to be fiberwise bimeromorphic map. Since the restriction of the bimeromorphic map on subvariety (the fibers) need not to be bimeromorphic.

Definition 2.4 (Moishezon variety, 1st definition). A proper, irreducible, reduced analytic space X is Moishezon if it is bimeromorphic to a projective variety X^P .

Definition 2.5 (Moishezon variety, 2nd definition). A proper, irreducible, reduced analytic space X is Moishezon if

$$a(X) := \text{tr deg}_{\mathbb{C}} M(X) = \dim(X)$$

that is it has $\dim X$ number of algebraic dependent meromorphic function.

Definition 2.6 (Moishezon variety, 3rd version). A proper irreducible, reduced analytic space X is Moishezon if and only if it carries a big rank 1 reflexive sheaf \mathcal{F} .

Proposition 2.7. Three different definitions for Moishezon varieties are equivalent.

Proof. □

The first important property for Moishezon variety is that it locally looks like quasi-projective scheme up to étale cover.

Proposition 2.8 ([Kol22, Proposition 8.2]). Let X be a Moishezon variety. For every $x \in X$ there is a pointed quasi-projective scheme (x', X') and an étale morphism $(x', X') \rightarrow (x, X)$.

Proof. □

Proposition 2.9 ([Kol22, Proposition 8.3]). Let X be a Moishezon variety. If X is normal then there is a proper variety Y and a finite group G acting on Y such that $X \cong Y/G$. (Note that usually Y can not be chosen projective.)

Proof. First, by Proposition 2.9, there exists some etale cover of X (indeed, since the etale morphism is finite, we can find an open cover of X by the etale morphism). Since X is proper, we can find some finite cover of it. We try to \square

Artin [Art70] shows the following equivalence of category, which shows the importance of the category of Moishezon varieties in the moduli theory.

Proposition 2.10 ([Art70, Theorem 7.3]). There is a "natural" functor

$$\text{an} : (\text{algebraic space of finite type over } \mathbb{C}) \rightarrow (\text{complex spaces})$$

extending the functor an on the category (schemes of finite type $/\mathbb{C}$). This functor induces an equivalence of categories

$$(\text{complex algebraic schemes of finite type}/\mathbb{C}) \rightarrow (\text{Moishezon spaces})$$

In other words, every Moishezon spaces "is" in an unique way an algebraic space.

Proposition 2.11 ([Nam02]). Let X be a Moishezon variety with 1-rational singularities. If X is Kähler, then X is projective.

Before proving the theorem, let us first state two results that will be used in the proof.

Lemma 2.12. Let X be a compact Moishezon variety with 1-rational singularities, that is, X is normal and has a resolution $\pi : Y \rightarrow X$ such that $R^1\pi_*\mathcal{O}_Y = 0$. Then an analytic homology class $b \in A_2(X, \mathbb{Q})$ is zero if it is numerically equivalent to 0. In particular,

$$A_2(X, \mathbb{Q}) = N_1(X)_{\mathbb{Q}}$$

Lemma 2.13 (Nakai-Moishezon criterion for \mathbb{Q} -line bundles over Kähler Moishezon variety). Let X be a Kähler Moishezon variety with a Kähler form ω . Assume that an element $L \in \text{Pic}(X)_{\mathbb{Q}}$ satisfies the equality for any curve $C \subset X$:

$$(C.L) = \int_C \omega.$$

Then L is ample.

Proof of the Proposition 2.11. Since the numerical equivalence and the homological equivalence coincide for (analytic) 1-cycle by Lemma 2.12, we have a natural map

$$\alpha : N^1(X)_{\mathbb{Q}} \rightarrow (A_2(X, \mathbb{Q}))^*, \quad d \mapsto (- \cdot d),$$

and α is an isomorphism.

On the other hand, we have the morphism

$$\alpha : N^1(X)_{\mathbb{Q}} \rightarrow (A_2(X, \mathbb{Q}))^*$$

and the theorem shows this is actually an isomorphism.

Note that $\omega \in H^2(X, \mathbb{R})$ Kähler form as an element of $(A_2(X, \mathbb{R}))^*$. By simply define

$$\alpha_\omega : A_2(X, \mathbb{R}) \rightarrow \mathbb{R}, \quad C \mapsto \omega \cdot C = \int_C \omega$$

Since $\alpha_\mathbb{R}$ is surjective, there is an element $d \in N^1(X)_\mathbb{R}$ such that

$$(C \cdot d) = \int_C \omega,$$

for every curve $C \subset X$.

Approximate $d \in N^1(X)_\mathbb{R}$ by \mathbb{Q} -coefficient Approximate $d \in N^1(X)_\mathbb{R}$ by a convergent sequence $\{d_m\}$ of rational elements $d_m \in N^1(X)_\mathbb{Q}$.

Let us fix the basis b_1, \dots, b_l of the vector space $N^1(X)_\mathbb{Q}$. Each b_i is represented by an element $B_i \in \text{Pic}(X)$. Here

$$\text{Pic}(X)_\mathbb{Q} \rightarrow N^1(X)_\mathbb{Q}, \quad B_i \mapsto b_i,$$

Now d (respectively d_m) is represented by an element in $\text{Pic}(X)_\mathbb{R}$. Respectively $\text{Pic}(X)_\mathbb{Q}$,

$$D := \sum x_i B_i,$$

(respectively $D_m := \sum x_i^{(m)} B_i$) such that $\lim x_i^{(m)} = x_i$. Put $E_m := D_m - D$. Then there are d closed $(1, 1)$ -forms α_m corresponding to E_m such that $\{\alpha_m\}$ uniformly converge to 0.

If m is chosen sufficiently large, then $\omega_m := \omega + \alpha_m$ is a Kähler form. Since

$$(C \cdot D_m) = \int_C \omega_m,$$

for every curve $C \subset X$. We see that D_m is ample by Lemma 2.13 (Note that we have D_m being a \mathbb{Q} -divisor, so that it's possible to apply the Nakai-Moishezon criterion). In particular, X is projective. \square

Remark 2.14. There exists some Kähler Moishezon variety with bad singularity that is not projective.

Proposition 2.15 ([Kol22, Proposition 8]).

- (1) Let X be a proper Moishezon space, if $Z \rightarrow X$ be finite then Z is Moishezon.
- (2) If M be a Moishezon variety, and $f : M \rightarrow N$ be a surjective morphism of complex varieties. Then N is also Moishezon.
- (3) Subvarieties of a Moishezon variety are Moishezon varieties.

This is sometimes called functorial property of Moishezon variety, note that (2) needs not always be true for projective variety.

Proof of (1). By definition

$$\text{trdeg}_\mathbb{C} K(X) = \dim X,$$

and if Z is finite map then

$$K(X) \hookrightarrow K(Z),$$

is a finite field extension and therefore by additive property for a tower of field extension, we have

$$\mathrm{trdeg}_{\mathbb{C}}(K(Z)) = \mathrm{trdeg}_{\mathbb{C}}(K(X)) + \mathrm{trdeg}_{K(X)}K(Z) = \mathrm{trdeg}_{\mathbb{C}}(K(X)).$$

□

Proof of (2). We will give a proof of the result in the relative version in Proposition 3.14. □

Proof of (3). Consider the following pull back diagram.

$$\begin{array}{ccc} Z^p = f^{-1}(Z) & \longrightarrow & X^p \\ f_Z \downarrow & & \downarrow f \\ Z & \hookrightarrow & X \end{array}$$

Clearly Z^p is projective (as subvariety of X^p), and f_Z is surjective. And therefore apply (2), we know that Z is again Moishezon. □

The following proposition shows that Moishezon manifold admits strong Hodge decomposition.

Proposition 2.16. If X is a Moishezon manifold, then the Hodge decomposition holds indeed it admits strong Hodge decomposition.

Before proving the theorem, let us first define what is strong Hodge decomposition. We say that a compact manifold admits a strong Hodge decomposition if the natural maps

$$H_{\mathrm{BC}}^{p,q}(X, \mathbb{C}) \longrightarrow H^{p,q}(X, \mathbb{C}), [\alpha^{p,q}]_{\mathrm{BC}} \mapsto [\alpha^{p,q}]_{\bar{\partial}} \quad \bigoplus_{p+q=k} H_{\mathrm{BC}}^{p,q}(X, \mathbb{C}) \longrightarrow H^k(X, \mathbb{C}), \quad \sum [\alpha^{p,q}] \mapsto \sum \alpha^{p,q},$$

are isomorphisms.

Proof. The idea of the proof comes from [Dem97, Proposition (12.3)]. We first take the projective modification

$$\mu : \tilde{X} \rightarrow X,$$

such that \tilde{X} is a projective manifold. And therefore \tilde{X} admits a strong Hodge decomposition. On the other hand

We first observe that $\mu_*\mu^*\beta = \beta$ for every smooth form β on X .

In fact, this property is equivalent to the equality

$$\int_Y (\mu_*\mu^*\beta) \wedge \alpha = \int_X \mu^*(\beta \wedge \alpha) = \int_Y \beta \wedge \alpha.$$

for every smooth form α on X , and this equality is clear because μ is a biholomorphism outside sets of Lebesgue measure 0.

Consequently, the induced cohomology morphism μ_* is surjective and μ^* is injective (but these maps need not be isomorphisms).

$$\begin{array}{ccc}
H_{\text{BC}}^{p,q}(\tilde{X}, \mathbb{C}) \longrightarrow H^{p,q}(\tilde{X}, \mathbb{C}), & \bigoplus_{p+q=k} H_{\text{BC}}^{p,q}(\tilde{X}, \mathbb{C}) \longrightarrow H^k(\tilde{X}, \mathbb{C}) \\
\mu_* \downarrow \uparrow \mu^* & \mu_* \downarrow \uparrow \mu^* & \mu_* \downarrow \uparrow \mu^* & \mu_* \downarrow \uparrow \mu^* \\
H_{\text{BC}}^{p,q}(X, \mathbb{C}) \longrightarrow H^{p,q}(X, \mathbb{C}), & \bigoplus_{p+q=k} H_{\text{BC}}^{p,q}(X, \mathbb{C}) \longrightarrow H^k(X, \mathbb{C})
\end{array}$$

Now, we have commutative diagrams with either upward or downward vertical arrows. Hence the surjectivity or injectivity of the top horizontal arrows implies that of the bottom horizontal arrows. \square

We next introduce Campana's Moishezon criterion.

Proposition 2.17. Let X be a compact complex variety in the Fujiki class \mathcal{C} . Then X is Moishezon if and only if X is algebraically connected.

Proof. \square

Proposition 2.18. Let $f : X \rightarrow B$ be a fibration over an algebraically connected variety (e.g. a projective curve). Assume that X is in the Fujiki class \mathcal{C} and the general fiber of f is algebraically connected, then X is Moishezon if and only if f has a multi-section.

Proof. \square

3 Moishezon morphisms

Let us first recall the definition of projective morphism.

Definition 3.1 (Projective morphism, first definition). Let $X \rightarrow S$ be a proper morphism between complex spaces. f is projective if there exist an f -ample line bundle \mathcal{L} and a positive integer $m \in \mathbb{Z}_{>0}$ such that there exist a closed S -immersion $X \hookrightarrow \mathbb{P}_S(f_*\mathcal{L}^{\otimes m})$, in short we have the following diagram:

$$\begin{array}{ccc}
X & \xrightarrow{\quad} & \mathbb{P}_S(f_*\mathcal{L}^{\otimes m}) \\
& \searrow & \swarrow \\
& S &
\end{array}$$

Definition 3.2 (Projective morphism, second definition). Let $X \rightarrow S$ be a proper morphism between complex spaces. f is projective if X can be embedded in $\mathbb{P}^N \times S$ for some N , with the following the diagram commute.

$$\begin{array}{ccc}
X & \xrightarrow{\quad} & \mathbb{P}^N \times S \\
& \searrow & \swarrow \\
& S &
\end{array}$$

Definition 3.3 (Locally projective morphism). Let $f : X \rightarrow S$ be a proper morphism of complex spaces. We call f locally projective if for every relatively compact open subset Q of S the restriction $f_Q : X_Q \rightarrow Q$ is a projective morphism.

Remark 3.4. Easy to see the second definition will immediate implies the first definition. Converse direction also holds when the base is Stein.

Proof. Let $\mathcal{E} = f_*\mathcal{L}^{\otimes m}$, $f : X \rightarrow S$ and $g : Y = \mathbb{P}_S(f_*\mathcal{L}^{\otimes m}) \rightarrow S$. Let A be an g -ample line bundle. And, therefore by Serre vanishing theorem over some Stein compact subset $W \subset S$, for some sufficient large $n \gg 0$, we have

$$g^*g_*(\mathcal{E} \otimes A^{\otimes n}) \rightarrow \mathcal{E} \otimes A^{\otimes n},$$

is global generated. Since the base S is Stein, by Cartan A theorem, $g_*(\mathcal{E} \otimes A^{\otimes n})$ is global generated. And therefore so it's the pull back $g^*g_*(\mathcal{E} \otimes A^{\otimes n})$. Since the surjective map sends global generated coherent sheaf to global generated coherent sheaf. This means that $\mathcal{E} \otimes A^{\otimes n}$ is global generated.

By coherence of $\mathcal{E} \otimes A^{\otimes n}$, the cohomology group $V = H^0(Y, \mathcal{E} \otimes A^{\otimes n})$ is finite dimensional. And there is a surjection

$$V \otimes \mathcal{O}_Y \rightarrow \mathcal{E} \otimes A^{\otimes n}.$$

And therefore it will induce an embedding

$$X \hookrightarrow \mathbb{P}_B(\mathcal{E}) = \mathbb{P}_B(\mathcal{E} \otimes A^{\otimes m}) \hookrightarrow \mathbb{P}(V) \times B,$$

after shrink the base $B \subset S$. □

Remark 3.5. When the total space has only finite number of irreducible components then locally projective morphism is bimeromorphic to a projective morphism. (see [Fuj83, Lemma 1.3.1]).

In what follows, we may assume that the base S is reduced. However, in general, we do not require the total space X being reduced or not.

Definition 3.6 (Moishezon morphism, 1st definition). A proper morphism of analytic spaces $g : X \rightarrow S$ is Moishezon if $g : X \rightarrow S$ is bimeromorphic to a projective morphism $g^p : X^p \rightarrow S$.

That is, there is a closed subspace $Y \subset X \times_S X^p$ such that the coordinate projections $Y \rightarrow X$ and $Y \rightarrow X^p$ are bimeromorphic.

$$\begin{array}{ccccc}
& & Y & & \\
& \swarrow & & \searrow & \\
X & & & & X^p \\
& \swarrow & \text{---} & \searrow & \\
& & S & &
\end{array}$$

Definition 3.7 (Moishezon morphism, 2nd definition). A proper morphism of analytic spaces $g : X \rightarrow S$ is Moishezon if There is a projective morphism of algebraic varieties $G : \mathbb{X} \rightarrow \mathbb{S}$ and a meromorphic $\phi_S : S \dashrightarrow \mathbb{S}$ such that X is bimeromorphic to $\mathbb{X} \times_{\mathbb{S}} S$, the fiber product of rational maps is defined where the maps are defined, so on a dense open set.

Remark 3.8. Let us say few words about the fiber product for a rational map $\phi_S : S \dashrightarrow \mathbb{S}$, the fiber product is defined on the place that ϕ_S is holomorphic map.

Definition 3.9 (Moishezon morphism, 3rd definition). A proper morphism of analytic spaces $g : X \rightarrow S$ is Moishezon if there is a rank 1, reflexive sheaf L on X such that the natural map $X \dashrightarrow \text{Proj}_S(g_*L)$ is bimeromorphic onto the closure of its image.

Proposition 3.10. Three definitions of Moishezon morphism are equivalent.

Proof. **Definition 3.7** implies **Definition 3.6** is clear. □

Moishezon morphism satisfies the following Chow type lemma (which can be viewed as the deterministic property of Moishezon morphism). Before proving the result, let us first recall the relative Iitaka fibration theorem (which appears implicitly in some proof in [Naka04] but not explicitly stated in the book).

Theorem 3.11 (Relative Iitaka fibration).

Theorem 3.12 ([DH20, Lemma 2.18]). Let $f : X \rightarrow S$ be a proper surjective morphism of analytic varieties, and let L be a f -big line bundle on X (more slightly weaker, f is a Moishezon morphism) and D a \mathbb{Q} -divisor.

Then over any relatively compact open subset $V \subset S$, there exists a proper bimeromorphic morphism $\alpha : W \rightarrow f^{-1}V$ from a smooth analytic variety W such that $\beta = f|_{f^{-1}V} \circ \alpha : W \rightarrow V$ is a projective morphism and $(W, \alpha_*^{-1}(D|_{f^{-1}V}) + \text{Ex}(\alpha))$ is a log smooth pair.

First, let us compare the theorem above with the Definition 3.6, in the Definition, we only assume the existence of some bimeromorphic S -map, the Chow lemma allow us to choose some bimeromorphic morphism (which actually can be choosen to be a projective morphism).

Proof. Let $\phi : X \rightarrow Y$ be the relative Iitaka fibration of L over S and $g : Y \rightarrow S$ the induced projective morphism. Since L is f -big, $\phi : X \rightarrow Y$ is bimeromorphic. Let $p : \Gamma \rightarrow X$ and $q : \Gamma \rightarrow Y$ be the resolution of indeterminacy of ϕ so that p is proper.

Now fix a relatively compact open subset $V \subset S$. Choose another relatively compact open set $U \subset S$ containing V such that $\bar{V} \subset U$. Note that U is σ -compact, since it is relatively compact. Since f and g are both proper morphisms, it follows that $X_U := f^{-1}U$ and $Y_U := g^{-1}U$ are both σ -compact. Let $\Gamma_U := q^{-1}(g^{-1}U) = p^{-1}(f^{-1}U)$. Then from the commutative diagram above it follows that $q|_{\Gamma_U} : \Gamma_U \rightarrow g^{-1}U$ is a proper morphism. In particular, Γ_U is σ -compact. Note that $q|_{\Gamma_U}$ is bimeromorphic. Therefore by Theorem 2.17 there is a projective bimeromorphic morphism $h : Z \rightarrow \Gamma_U$ from an analytic variety Z such that $q|_{\Gamma_U} \circ h : Z \rightarrow Y_U$ is a projective bimeromorphic morphism. Since g is projective, so is $Z \rightarrow U$. Now we replace U by our previously fixed open

set V . Then $Z_V := (g \circ q \circ h)^{-1}V$ is a relatively compact open subset of Z . Let $r : W \rightarrow Z_V$ be the log resolution of $\left(Z_V, (p \circ h)_*^{-1} \left(D|_{f^{-1}V}\right)\right)$ as in Theorem 2.16. Let $\alpha := p|_{\Gamma_V} \circ h|_{h^{-1}\Gamma_V} \circ r$ and $\beta := g|_{g^{-1}V} \circ q|_{\Gamma_V} \circ h|_{h^{-1}\Gamma_V} \circ r$, where $\Gamma_V := p^{-1}(f^{-1}V) = q^{-1}(g^{-1}V)$. Note that β is a projective morphism, since it is a composition of projective morphisms over relatively compact bases.

□

Proposition 3.13 ([Fuj83, Proposition 1.5.(4)]). Suppose that there exist a locally projective morphism $g : Y \rightarrow S$ and a generically finite meromorphic S -map $h : X \dashrightarrow Y$. Then f is Moishezon.

$$\begin{array}{ccc} X & \overset{h}{\dashrightarrow} & Y \\ & \searrow f & \swarrow g \\ & S & \end{array}$$

Proof. First since being Moishezon is stable under bimeromorphic change, without loss of generality we can assume that h is a Moishezon. And since Moishezon morphism and locally projective morphism are proper. So that h is proper. Apply the Stein factorization theorem, such that h_2 is projective and h_1 is proper. And thus the composition $g \circ h_2$ is locally projective. And thus by definition $X \rightarrow S$ is a Moishezon morphism.

$$\begin{array}{ccccc} & & X^* & & \\ & \nearrow h_1 & & \searrow h_2 & \\ X & & \xrightarrow{h} & & Y \\ & \searrow & & \swarrow g & \\ & & S & & \end{array}$$

□

Proposition 3.14 ([Fuj83, Proposition 1.7]). Let $f : X \rightarrow S$ be a Moishezon morphism, and $g : Y \rightarrow S$ a proper morphism, of reduced complex spaces. Suppose that there is a generically surjective meromorphic S -map $h : X \dashrightarrow Y$. Then g also is Moishezon.

Proof. This Proposition can be viewed as a generalization of the Proposition 2.15. The proof is a bit involving, and we omit it here.

□

Proposition 3.15 ([Fuj83, Proposition 1.5]).

(1) The morphism $f : X \rightarrow S$ is Moishezon if and only if for each irreducible component X_i of X the restriction $f = f|_{X_i} : X_i \rightarrow S$ is Moishezon.

(2) Let $f : X \rightarrow S$ be a Moishezon morphism. Then: For every reduced analytic subspace $X' \subseteq X$ the induced morphism $f' = f|_{X'} : X' \rightarrow S$ is Moishezon.

Proof. For (1), let's take the normalization

$$\nu : X^\nu \rightarrow X,$$

recall that for a reduced complex space with finite many irreducible component, the normalization is a bimeromorphic map. So that $f : X \rightarrow S$ is Moishezon iff the restriction on each component X_i are Moishezon.

For (2), by the Chow lemma (Theorem 3.12), we can find some locally projective morphism such that X^* is smooth and h is a bimeromorphic S -morphism.

$$\begin{array}{ccc} X^* & \xrightarrow{h} & X \\ & \searrow g & \swarrow f \\ & S & \end{array}$$

We then take the inverse image of the analytic subspace X' denote it $Z = h^{-1}(X')$. (we can assume the inverse image has reduced structure). Since the restriction of the projective morphism on $g|_Z : Z \rightarrow S$ is still locally projective. And by construction, clearly the morphism $Z \rightarrow X'$ is surjective. And therefore, by Proposition 3.14, we know that $X' \rightarrow S$ is a Moishezon morphism. \square

Proposition 3.16. When the base is Moishezon then the total space is Moishezon iff the morphism is Moishezon.

Proof. We first prove that morphism between Moishezon space is a Moishezon morphism. Let us define the graph embedding to be

$$\iota : X \rightarrow X \times S, \quad x \mapsto (x, f(x)),$$

since X is Moishezon it's bimeromorphic to a projective variety, as the diagram below shows

$$\begin{array}{ccccc} X & \xrightarrow{\iota} & X \times S & \dashrightarrow & X^p \times S \\ & \searrow f & \downarrow \pi & \swarrow \pi^p & \\ & & S & & \end{array}$$

Clearly, π^p is a projective morphism. And consequently π is a Moishezon morphism. And finally by Proposition 3.15, the morphism $f : X \rightarrow S$ is again Moishezon.

Conversely, if the morphism is Moishezon, and S is Moishezon variety. Then there exist bimeromorphic modifications such that the following diagram commute

$$\begin{array}{ccccc} X^p & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & \swarrow & \\ S^p & \longrightarrow & S & & \end{array}$$

Where $X' \rightarrow S$ is a projective morphism and S^p is a projective variety. Since the base change preserve the projective condition, easy to see that $X^p \rightarrow S^p$ is a projective morphism over S^p . And therefore X^p is a projective variety. By Proposition 3.14, X' is a Moishezon variety. Since $X' \rightarrow X$ is bimeromorphic, this implies that X is also projective. \square

Proposition 3.17 ([Kol22, Lemma 15]). Let $g : X \rightarrow S$ be a proper, generically finite, dominant morphism of normal, complex, analytic spaces. Then $\text{Ex}(g) \rightarrow S$ is Moishezon.

Proof. We will prove the result under the additional assumption that S is Stein. By the geometric Neother normalization theorem, there exist a finite morphism

$$S \rightarrow \mathbb{C}^{\dim S}.$$

After replacing the base by $\mathbb{C}^{\dim S}$, we can assume that smooth locus of S is dense in $g(\text{Ex}(g))$. Note that, by Proposition 3.13, if the restriction on $\mathbb{C}^{\dim S}$ is Moishezon, then so will the restriction on S . We will prove the result by induction on dimension.

We first define the base case $(g_0 : X_0 \rightarrow S_0) := (g : X \rightarrow S)$. Let E_0 be a g_0 exceptional divisor, with the image $Z_0 = g_0(E_0)$. We then inductively define the morphism $g_{i+1} : X_{i+1} \rightarrow S_{i+1}$ as follows. Assume that we already construct $g_i : X_i \rightarrow S_i$, we then blow up S_i along Z_i . We then blow up X_i along Z_i and let S_{i+1} be the normalization of the blow-up $\text{Bl}_{Z_i} S_i$. Since S_i is reduced, this will induce a generic finite map $X_i \dashrightarrow$. So that by the universal property of the normalization, the generic finite morphism $g_i : X_i \rightarrow S_i$ lift to a generic finite morphism $g_{i+1} : X_{i+1} \rightarrow S_{i+1}$, where X_{i+1} is the normalization of the graph of the map $X_{i+1} \dashrightarrow S_{i+1}$.

$$\begin{array}{ccc} X_{i+1} & \xrightarrow{g_{i+1}} & S_{i+1} = \text{Bl}_{Z_i}(S_i)^\nu \\ \downarrow & \nearrow & \downarrow \\ \Gamma_\phi & & \text{Bl}_{Z_i}(S_i) \\ \downarrow & \nearrow \phi & \downarrow \\ X_i & \xrightarrow{g_i} & S_i \end{array}$$

Let $E_{i+1} \subset X_{i+1}$ denote the birational transform of E_i . (Note that $X_{i+1} \rightarrow X_i$ is an isomorphism over an open subset of E_i). We then compute the vanishing order $a(E_i, S_i)$ of Jacobian of g_i along E_i . We claim that

$$a(E_{i+1}, S_{i+1}) \leq a(E_i, S_i) + 1 - \text{codim}(Z_i \subset S_i).$$

Thus eventually we reach the situation when $\text{codim}(Z_i \subset S_i) = 1$, indeed if $\text{codim}(Z_i \subset S_i) \geq 2$ then the Jacobian of g_i along E_i will eventually goes to zero. Contradiction.

Thus by comparing the dimension we know when restrict the morphism $X_i \rightarrow S_i$ to $E_i \rightarrow Z_i$ it will become a generic finite morphism. Since $S_{i+1} \rightarrow S_i$ is projective, the composition $Z_i \rightarrow Z_0$ will be a locally projective morphism.

$$\begin{array}{ccc} E_i & \xrightarrow{\quad} & Z_i \\ & \searrow & \swarrow \\ & Z_0 & \end{array}$$

By Proposition 3.13, we know that $E_i \rightarrow Z_0$ is a Moishezon morphism. Since the strict transform $E_i \rightarrow E_0$ is a dominant morphism, by Proposition 3.14, we know that $E_0 \rightarrow Z_0$ is also Moishezon morphism. Finally, by Proposition 3.15, we know that $\text{Ex}(f) \rightarrow S$ is Moishezon. \square

4.1 The Hironaka's example

Hironaka discovered a bunch of complete non-projective 3-fold which is called the Hironaka's varieties. Note that based on the construction of Hironaka, we can from almost all the projective varieties construct some Moishezon varieties, that is why we said at the beginning that Moishezon varieties are versatile in birational geometry. (However this is not true in dimension 2, since all the smooth Moishezon surface are actually projective, see e.g. [SCV7]). The major reference of this part of note is the paper by Ulrich Thiel (see https://ulthiel.com/math/wp-content/uploads/other/hironakas_example.pdf).

Given a smooth projective threefold, which contains two rational curves transversely intersection at two points. Assume two rational curves are C and D that intersect at point P, Q .

We then take two steps blow up

$$\begin{aligned} X_1 &= \text{Bl}_{(D \setminus P)'} (\text{Bl}_{C \setminus P}(X \setminus P)) \xrightarrow{\pi_2} \text{Bl}_{C \setminus P}(X \setminus P) \xrightarrow{\pi_1} X \setminus P \\ X_2 &= \text{Bl}_{(C \setminus Q)'} (\text{Bl}_{D \setminus Q}(X \setminus Q)) \xrightarrow{\sigma_2} \text{Bl}_{D \setminus Q}(X \setminus Q) \xrightarrow{\sigma_1} X \setminus Q, \end{aligned}$$

Note that if we define $U = X - \{P, Q\}$, then $\pi^{-1}(U) \cong \sigma^{-1}(U)$. In particular, we can glue X_1 and X_2 along $\pi^{-1}(U)$ and $\sigma^{-1}(U)$. In the picture below, we glue the red exceptional surface on the right hand side with the black exceptional surface on the left hand side (denote it S_1) and blue exceptional surface on the left hand side with the black exceptional surface on the right hand side (denote it S_2). (see the picture 1). By gluing lemma, there exists a morphism $f : H \rightarrow X$ and the restriction of the morphism on S_1, S_2 as $f_1 = f|_{S_1} : S_1 \rightarrow C$ and $f_2 = f|_{S_2} : S_2 \rightarrow C$.

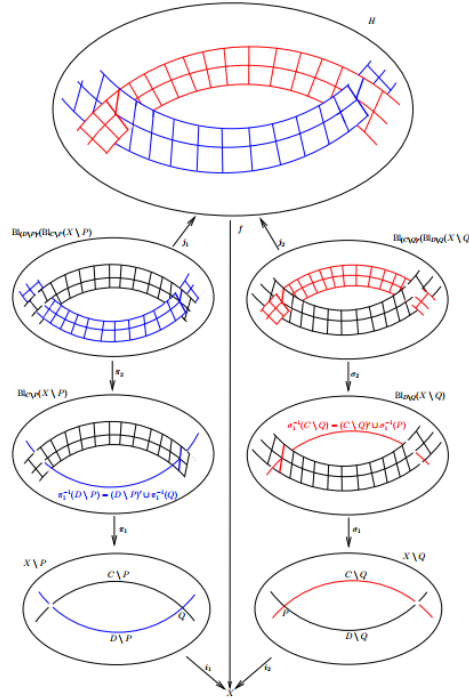


Figure 1: Construction of the Hironaka's variety

We claim that Hironaka's variety is non-projective.

The idea is to find some divisor on the surface $S = S_1 \cup S_2$ on the Hironaka's variety H .

The idea to prove the non-projective is to find some curve on the surface $S = S_1 \cup S_2$. Which has positive degree but add up to 0.

The key observation is that $f^{-1}(P)$ (resp. $f^{-1}(Q)$) decompose into two split projective line L_Q and L'_Q in S_1 (resp. L_P and L'_P in S_2). (see the precise statement below).

Choose two points $A \in C - \{P, Q\}$ and $B \in D - \{P, Q\}$. Since all the point on a rational curve are linear equivalent, therefore

$$\begin{aligned} A \sim_C Q &\implies f_1^{-1}(A) \sim_{S_1} f_1^{-1}(Q) = L_Q + L'_Q \\ B \sim_D P &\implies f_2^{-1}(B) \sim_{S_2} f_2^{-1}(P) = L_P + L'_P \end{aligned}$$

and Push forward of cycle, we get equivalence on S .

$$\begin{aligned} I : f^{-1}(A) &\sim_S f^{-1}(Q) = L_Q + L'_Q \\ II : f^{-1}(B) &\sim_S f^{-1}(P) = L_P + L'_P \end{aligned}$$

On the other hand we also that B, Q lies in the same rational curve, so that

$$III : B \sim_D Q \implies f_2^{-1}(B) \sim_{S_2} f_2^{-1}(Q) \implies f^{-1}(B) \sim_S L'_Q$$

and combined then together, we get

$$\begin{aligned} f^{-1}(A) + f^{-1}(B) &\sim_S f^{-1}(A) + f^{-1}(B) \implies L_Q + L'_Q + L_P + L'_P \sim_S L'_Q + L'_P \\ &\implies L_Q + L_P \sim_S 0 \end{aligned}$$

If there exist some ample divisor on A , then both $L_Q \cdot A > 0$ and $L_P \cdot A > 0$ contradict the linearly trivial relation above. Therefore the only possible case is Hironaka's variety is non-projective.

4.2 Locally Moishezon morphism which is not globally Moishezon: family of K3 surfaces

This example comes from family of K3 surfaces.

4.3 Flip the rational curve on the quintic 3-fold produce Moishezon variety

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