Moishezon space and Moizhezon morphism Summer 2025

Note 1 — 2025-07-06 (draft version)

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1 Overview

The aim of this note is to give a brief introduction to Moishezon vairety and Moishezon morphism. The major references of this note are [Kol22], [Fuj83], and [Uen75].

Why we study the Moishezon variety/morphism? First, Moishezon variety has more functorial behavior (compared with projective variety), as we will see in Section 2. Secondly, from almost any projective variety we can construct some Moishezon variety via bimeromorphic modification so that Moishezon varieties are versatile in birational geometry. And finally let us cite a sentence by Shigefumi Mori: Projective variety like the classical paintings, after modern paining are introduced we can enjoy the classical paintings more. To me non-projective complete variety (maybe he mean Moishezon variety) seems like the modern paintings.

Contents

2 Moishezon varieties

Definition 2.1 (Meromorphic map)**.** Let X, Y be reduced complex space. We call the S-map a meromorphic S-map (not necessary morphism) if

$$\begin{tikzcd}
 & \Gamma \arrow[dl, "p"'] \arrow[dr, "q"] & \\
X \arrow[rr, dashed, "\alpha"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
 & S &
\end{tikzcd}$$

with $\Gamma \subset X \times_S Y$, and $p : \Gamma \to X$ is a proper bimeromosphic morphism. Moreover if q is also proper bimeromorphic morphism, then we all α proper bimeromorphic S-map.

Definition 2.2 (Moishezon variety, 1st definition)**.** A proper, irreducible, reduced analytic space X is Moishezon if it is bimeromorphic to a projective variety X^{p}.

Definition 2.3 (Moishezon variety, 2nd definition)**.** A Moishezon variety is a complex variety such that

$$a(X) := \operatorname{tr}\deg_{\mathbb{C}} M(X) = \dim(X)$$

that is it has $\dim X$ number of algebraic dependent meromorphic function.

Definition 2.4 (Moishezon variety, 3rd version)**.** A compact connected complex space X is Moishezon if and only if it carries a big rank 1 reflexive sheaf $\mathscr{F}$.

Proposition 2.5. Three different definition for Moishezon varieties are equivalent.

Proof. $\square$

The first important property for Moishezon variety is that it locally looks like quasi-projective scheme up to étale cover.

Proposition 2.6 ([Kol22, Proposition 8.2])**.** For every $x \in X$ there is a pointed quasi-projective scheme (x', X') and an etale morphism $(x', X') \to (x, X)$.

Proof. $\square$

Proposition 2.7 ([Kol22, Proposition 8.3])**.** If X is normal then there is a proper variety Y and a finite group G acting on Y such that $X \cong Y/G$. (Note that usually Y can not be chosen projective.)

Proof. $\square$

Artin [Art70] shows the following equivalence of category, which shows the importance of the category of Moishezon variety in the moduli theory.

Proposition 2.8 ([Art70, Theorem 7.3])**.** There is a "natural" functor

$$\text{an} : (\text{algebraic space of finite type over } \mathbb{C}) \to (\text{complex spaces})$$

extending the functor an on the category (schemes of finite type $/\mathbb{C}$). This functor induces an equivalence of categories

$$(\text{complex algebraic schemes of finite type}/\mathbb{C}) \to (\text{Moishezon spaces})$$

In other words, every Moishezon spaces "is" in an unique way an algebraic space.

Proposition 2.9 ([Nam02])**.** Let X be a Moishezon variety with 1-rational singularities. If X is Kähler, then X is projective.

Before proving the theorem, let us first state two results that will be used in the proof.

Lemma 2.10. Let X be a compact Moishezon variety with 1-rational singularities, that is, X is normal and has a resolution $\pi : Y \to X$ such that $R^1\pi_*\mathcal{O}_Y = 0$. Then an analytic homology class $b \in A_2(X, \mathbb{Q})$ is zero if it is numerically equivalent to 0 . In particular,

$$A_2(X, \mathbb{Q}) = N_1(X)_{\mathbb{Q}}$$

Lemma 2.11 (Nakai-Moishezon criterion for $\mathbb{Q}$-line bundles over Kahler morphism variety)**.** Let X be a Kähler Moishezon variety with a Kähler form ω. Assume that an element $L \in \operatorname{Pic}(X)_{\mathbf{Q}}$ satisfies the equality for any curve $C \subset X$:

$$(C.L) = \int_C \omega.$$

Then L is ample.

Proof of the Proposition 2.9. Since the numerical equivalence and the homological equivalence coincide for (analytic) 1-cycle by Lemma 2.10, we have a natural map

$$\alpha : N^1(X)_{\mathbf{Q}} \to (A_2(X, \mathbf{Q}))^*, \quad d \mapsto (- \cdot d),$$

and α is an isomorphism.

On the other hand, we have have the morphism

$$\alpha : N^1(X)_{\mathbb{Q}} \to (A_2(X, \mathbb{Q}))^*$$

and the theorem shows this is actually an isomorphism.

Note that $\omega \in H^2(X, \mathbb{R})$ Kähler form as an element of $(A_2(X, \mathbf{R}))^*$. By simply define

$$\alpha_\omega : A_2(X, \mathbb{R}) \to \mathbb{R}, \quad C \mapsto \omega \cdot C = \int_C \omega$$

Since $\alpha_{\mathbf{R}}$ is surjective, there is an element $d \in N^1(X)_{\mathbf{R}}$ such that

$$(C \cdot d) = \int_C \omega,$$

for every curve $C \subset X$.

Apporximate $d \in N^1(X)_{\mathbb{R}}$ by $\mathbb{Q}$-coefficient Approximate $d \in N^1(X)_{\mathbf{R}}$ by a convergent sequence $\{d_m\}$ of rational elements $d_m \in N^1(X)_{\mathbf{Q}}$.

Let us fix the basis $b_1, \dots, b_l$ of the vector space $N^1(X)_{\mathbf{Q}}$. Each b_i is represented by an element $B_i \in \operatorname{Pic}(X)$. Here

$$\operatorname{Pic}(X)_{\mathbb{Q}} \to N^1(X)_{\mathbb{Q}}, \quad B_i \mapsto b_i,$$

Now d (respectively d_m) is represented by an element in $\mathrm{Pic}(X)_{\mathbf{R}}$. Respectively $\mathrm{Pic}(X)_{\mathbf{Q}}$,

$$D := \Sigma x_i B_i,$$

(respectively $D_m := \Sigma x_i^{(m)} B_i$) such that $\lim x_i^{(m)} = x_i$. Put $E_m := D_m - D$. Then there are d closed $(1,1)$-forms α_m corresponding to E_m such that $\{\alpha_m\}$ uniformly converge to 0 .

If m is chosen sufficiently large, then $\omega_m := \omega + \alpha_m$ is a Kähler form. Since

$$(C.D_m) = \int_C \omega_m,$$

for every curve $C \subset X$. We see that D_m is ample by Lemma 2.11 (Note that we have D_m being a $\mathbb{Q}$-divisor, so that it's possible to apply the Nakai-Moishezon criterion). In particular, X is projective. □

Remark 2.12. There exist Kähler Moishezon variety with bad singularity which is not projective.

Proposition 2.13. [[Kol22]]
(1) Let X be a proper Moishezon space, if $Z \to X$ be finite then Z is Moishezon.

(2) If M be a Moishezon variety, and $f : M \to N$ be a surjective morphism of complex varieties. Then N is also Moishezon.

(3) Subvarieties of a Moishezon variety are Moishezon varieties.

This is sometimes called functorial property of Moishezon variety, note that (2) needs not always be true for projective variety.

Proof of (1). By definition

$$\mathrm{trdeg}_{\mathbb{C}} K(X) = \dim X,$$

and if Z is finite map then

$$K(X) \hookrightarrow K(Z),$$

is a finite field extension and therefore by additive property for a tower of field extension, we have

$$\mathrm{trdeg}_{\mathbb{C}}(K(Z)) = \mathrm{trdeg}_{\mathbb{C}}(K(X)) + \mathrm{trdeg}_{K(X)} K(Z) = \mathrm{trdeg}_{\mathbb{C}}(K(X)).$$

□

Proof of (2). We will give a proof of the result in the relative version in Proposition 3.12. □

Proof of (3). Consider the following pull back diagram.

$$\begin{tikzcd} Z^p = f^{-1}(Z) \arrow[r] \arrow[d, "f_Z"'] & X^p \arrow[d, "f"] \\ Z \arrow[r, hook] & X \end{tikzcd}$$

Clearly Z^p is projective (as subvariety of X^p), and f_Z is surjective. And therefore apply (2), we know that Z is again Moishezon. □

The following proposition shows that Moishezon manifold admits strong Hodge decomposition.

Proposition 2.14. If X is a Moishezon manifold, then the Hodge decomposition holds indeed it admits strong Hodge decomposition.

Before proving the theorem, let us first define what is strong Hodge decomposition. We say that a compact manifold admits a strong Hodge decomposition if the natural maps

$$H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^{p,q}(X,\mathbb{C}),\ [\alpha^{p,q}]_{\mathrm{BC}} \mapsto [\alpha^{p,q}]_{\bar\partial} \quad \bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) \longrightarrow H^k(X,\mathbb{C}), \quad \sum[\alpha^{p,q}] \mapsto \sum \alpha^{p,q},$$

are isomorphisms.

Proof. The idea of the proof comes from [Dem97, Proposition (12.3)]. We first take the projective modification

$$\mu : \widetilde{X} \to X,$$

such that X' is a projective manifold. And therefore X' admits a strong Hodge decomposition. On the other hand

We first observe that $\mu_\star\mu^\star\beta = \beta$ for every smooth form β on Y.

In fact, this property is equivalent to the equality

$$\int_Y (\mu_\star\mu^\star\beta) \wedge \alpha = \int_X \mu^\star(\beta \wedge \alpha) = \int_Y \beta \wedge \alpha.$$

for every smooth form α on Y, and this equality is clear because μ is a biholomorphism outside sets of Lebesgue measure 0.

Consequently, the induced cohomology morphism $\mu_\star$ is surjective and $\mu^\star$ is injective (but these maps need not be isomorphisms).

$$\begin{array}{ccccccc}
H^{p,q}_{\mathrm{BC}}(\widetilde{X},\mathbb{C}) & \longrightarrow & H^{p,q}(\widetilde{X},\mathbb{C}), & \quad & \bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(\widetilde{X},\mathbb{C}) & \longrightarrow & H^k(\widetilde{X},\mathbb{C}) \\
\mu_\star\downarrow\uparrow\mu^\star & & \mu_\star\downarrow\uparrow\mu^\star & & \mu_\star\downarrow\uparrow\mu^\star & & \mu_\star\downarrow\uparrow\mu^\star \\
H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) & \longrightarrow & H^{p,q}(X,\mathbb{C}), & & \bigoplus_{p+q=k} H^{p,q}_{\mathrm{BC}}(X,\mathbb{C}) & \longrightarrow & H^k(X,\mathbb{C})
\end{array}$$

Now, we have commutative diagrams with either upward or downward vertical arrows. Hence the surjectivity or injectivity of the top horizontal arrows implies that of the bottom horizontal arrows.

□

We next introduce Campana's Moishezon criterion.

Proposition 2.15. Let X be a compact complex variety in the Fujiki class $\mathscr{C}$. Then X is Moishezon if and only if X is algebraically connected.

Proof. □

Proposition 2.16. Let $f : X \to B$ be a fibration over an algebraically connected variety (e.g. a projective curve). Assume that X is in the Fujiki class $\mathscr{C}$ and the general fiber of f is algebraically connected, then X is Moishezon if and only if f has a multi-section.

Proof. □

3 Basic Properties of Moishezon morphisms

Let us first recall the definition of projective morphism.

Definition 3.1 (Projective morphism, first definition)**.** Let $X \to S$ be a proper morphism between complex spaces. f is projective if there exist an f-ample line bundle $\mathcal{L}$ and a positive integer $m \in \mathbb{Z}_{>0}$ such that there exist a closed S-immersion $X \hookrightarrow \mathbb{P}_S(f_*\mathcal{L}^{\otimes m})$, in short we have the following diagram:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \mathbb{P}_S(f_*\mathcal{L}^{\otimes m}) \arrow[dl] \\ & S & \end{tikzcd}$$

Definition 3.2 (Projective morphism, second definition)**.** Let $X \to S$ be a proper morphism between complex spaces. f is projective if X can be embedded in $\mathbb{P}^N \times S$ for some N, with the following the diagram commute.

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & \mathbb{P}^N \times S \arrow[dl] \\ & S & \end{tikzcd}$$

Definition 3.3 (Locally projective morphism)**.** Let $f : X \to S$ be a proper morphism of complex spaces. We call f locally projective if for every relatively compact open subset Q of S the restriction $f_Q : X_Q \to Q$ is a projective morphism.

Remark 3.4. Easy to see the second definition will immediate implies the first definition. Converse direction also holds when the base is Stein.

Proof. Let $\mathcal{E} = f_*\mathcal{L}^{\otimes m}$, $f : X \to S$ and $g : Y = \mathbb{P}_S(f_*\mathcal{L}^{\otimes m}) \to S$. Let A be an g-ample line bundle. And, therefore by Serre vanishing theorem over some Stein compact subset $W \subset S$, for some sufficient large $n \gg 0$, we have

$$g^*g_*(\mathcal{E} \otimes A^{\otimes n}) \to \mathcal{E} \otimes A^{\otimes n}.$$

Since the base S is Stein, by Cartan A theorem, $g_*(\mathcal{E} \otimes A^{\otimes n})$ is global generated. And therefore so it's the pull back $g^*g_*(\mathcal{E} \otimes A^{\otimes n})$. Since the surjective map sends global generated coherent sheaf to global generated coherent sheaf. This means that $\mathcal{E} \otimes A^{\otimes n}$ is global generated.

By coherence of $\mathcal{E} \otimes A^{\otimes n}$, the cohomology group $V = H^0(Y, \mathcal{E} \otimes A^{\otimes n})$ is finite dimensional. And there is a surjection

$$V \otimes \mathcal{O}_Y \to \mathcal{E} \otimes A^{\otimes n}.$$

And therefore it will induce an embedding

$$X \hookrightarrow \mathbb{P}_B(\mathcal{E}) = \mathbb{P}_B\left(\mathcal{E} \otimes A^{\otimes m}\right) \hookrightarrow \mathbb{P}(V) \times B$$

□

Remark 3.5. When the total space has only finite number of irreducible components then locally projective morphism is bimeromorphic to a projective morphism. (see [Fuj83, Lemma 1.3.1]).

In what follows, we may assume that the base S is reduced.

Definition 3.6 (Moishezon morphism, 1st definition)**.** A proper morphism of analytic spaces $g : X \to S$ is Moishezon if $g : X \to S$ is bimeromorphic to a projective morphism $g^{\mathrm{p}} : X^{\mathrm{p}} \to S$.

That is, there is a closed subspace $Y \subset X \times_S X^{\mathrm{p}}$ such that the coordinate projections $Y \to X$ and $Y \to X^{\mathrm{p}}$ are bimeromorphic.

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed] \arrow[dr] & & X^p \arrow[dl] \\ & S & \end{tikzcd}$$

Definition 3.7 (Moishezon morphism, 2nd definition)**.** A proper morphism of analytic spaces $g : X \to S$ is Moishezon if There is a projective morphism of algebraic varieties $G : \mathbf{X} \to \mathbf{S}$ and a meromorphic $\phi_S : S \dashrightarrow \mathbf{S}$ such that X is bimeromorphic to $\mathbf{X} \times_{\mathbf{S}} S$, the fiber product of rational maps is defined where the maps are defined, so on a dense open set.

Remark 3.8. Let us say few words about the fiber product for a rational map $\phi_S : S \dashrightarrow \mathbf{S}$, the fiber product is defined on the place that ϕ_S is holomorphic map.

Definition 3.9 (Moishezon morphism, 3rd definition)**.** A proper morphism of analytic spaces $g : X \to S$ is Moishezon if there is a rank 1, reflexive sheaf L on X such that the natural map $X \dashrightarrow \mathrm{Proj}_S(g_*L)$ is bimeromorphic onto the closure of its image.

Proposition 3.10. Three definitions of Moishezon morphism are equivalent.

Proof. **Definition 3.7 implies Definition 3.6** is clear.

□

Proposition 3.11. Suppose that there exist a locally projective morphism $g : Y \to S$ and a generically finite meromorphic S-map $h : X \dashrightarrow Y$. Then f is Moishezon.

$$\begin{tikzcd} X \arrow[rr, dashed, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

Proof. □

Proposition 3.12. Let $f : X \to S$ be a Moishezon morphism, and $g : Y \to S$ a proper morphism, of reduced complex spaces. Suppose that there is a generically surjective meromorphic S-map $h : X \dashrightarrow Y$. Then g also is Moishezon.

Proof. This Proposition can be viewed as a generalization of the Proposition 2.13. □

Proposition 3.13.
(1) The morphism $f : X \to S$ is Moishezon if and only if for each irreducible component X_i of X the restriction $f = f|_{X_i} : X_i \to S$ is Moishezon.

(2) Let $f : X \to S$ be a Moishezon morphism. Then: For every reduced analytic subspace $X' \subseteq X$ the induced morphism $f' = f|_{X'} : X' \to S$ is Moishezon.

Proof. □

Proposition 3.14. When the base is Moishezon then the total space is Moishezon iff the morphism is Moishezon.

Proof. We first prove that morphism between Moishezon space is a Moishezon morpihsm. Let us define the graph embedding to be

$$\iota : X \to X \times S, \quad x \mapsto (x, f(x)),$$

since X is Moishezon it's bimeromorphic to a projective variety, as the diagram below shows

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & X \times S \arrow[r, dashed] \arrow[d, "\pi"] & X^p \times S \arrow[dl, "\pi^p"] \\ & S & \end{tikzcd}$$

Clearly, π^p is a projective morphism. And consequently π is a Moishezon morphism. And finally by Proposition 3.13, the morphism $f : X \to S$ is again Moishezon.

Conversely, if the morphism is Moishezon, and S is Moishezon variety. Then there exist bimeromorphic modifications such that the following diagram commute

$$\begin{tikzcd} X^p \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[dl] \\ S^p \arrow[r] & S & \end{tikzcd}$$

Where $X' \to S$ is a projective morphism and S^p is a projective variety. Since the base change preserve the projective condition, easy to see that $X^p \to S^p$ is a projective morphism over S^p. And therefore X^p is a projective variety. By Proposition 2.13, X' is a Moishezon variety. Since $X' \to X$ is bimeromorphic, this implies that X is also projective. □

Proposition 3.15 ([Kol22, Lemma 15])**.** Let $g : X \to S$ be a proper, generically finite, dominant morphism of normal, complex, analytic spaces. Then $\mathrm{Ex}(g) \to S$ is Moishezon.

Proof. We will prove the result under the additional assumption that S is Stein. By the geometric Neother normalization theorem, there exist a finite morphism

$$S \to \mathbb{C}^{\dim S}.$$

After replacing the base by $\mathbb{C}^{\dim S}$, we can assume that smooth locus of S is dense in $g(\mathrm{Ex}(g))$. Note that, by Proposition 3.11, if the restriction on $\mathbb{C}^{\dim S}$ is Moishezon, then so will the restriction on S. We will prove the result by induction on dimension.

We first define the base case $(g_0 : X_0 \to S_0) := (g : X \to S)$. Let E_0 be a g_0 exceptional divisor, with the image $Z_0 = g_0(E_0)$. We then inductivly define the morphism $g_{i+1} : X_{i+1} \to S_{i+1}$ as follows. Assume that we already construct $g_i : X_i \to S_i$, we then blow up S_i along Z_i. We then blow up S_i along Z_i and let S_{i+1} be the normalization of the blow-up $\mathrm{Bl}_{Z_i} S_i$. Since S_i is reduced, this will induce a generic finite map $X_i \dashrightarrow$. So that by the universal property of the normalization, the generic finite morphism $g_i : X_i \to S_i$ lift to a generic finite morphism $g_{i+1} : X_{i+1} \to S_{i+1}$, where X_{i+1} is the normalization of the graph of the map $X_{i+1} \dashrightarrow S_{i+1}$.

$$\begin{tikzcd}
X_{i+1} \arrow[r, "g_{i+1}"] \arrow[d] & S_{i+1} = \mathrm{Bl}_{Z_i}(S_i)^{\nu} \arrow[d] \\
\Gamma_{\phi} \arrow[d] \arrow[ur] & \mathrm{Bl}_{Z_i}(S_i) \arrow[d] \\
X_i \arrow[r, "g_i"'] \arrow[uur, dashed, "\phi"] & S_i
\end{tikzcd}$$

Let $E_{i+1} \subset X_{i+1}$ denote the bimeromorphic transform of E_i. (Note that $X_{i+1} \to X_i$ is an isomorphism over an open subset of E_i). We then compute the vanishing order $a(E_i, S_i)$ of Jacobian of g_i along E_i. We claim that

$$a\left(E_{i+1}, S_{i+1}\right) \leq a\left(E_i, S_i\right) + 1 - \operatorname{codim}\left(Z_i \subset S_i\right).$$

Thus eventually we reach the situation when $\operatorname{codim}\left(Z_i \subset S_i\right) = 1$, indeed if $\operatorname{codim}\left(Z_i \subset S_i\right) \geq 2$ then the Jacobian of g_i along E_i will eventually goes to zero. Contradiction.

Thus by comparing the dimension we know when restrict the morphism $X_i \to S_i$ to $E_i \to Z_i$ it will become a generic finite morphism. Since $S_{i+1} \to S_i$ is projective, the restriction $Z_i \to Z_0$ will be a locally projective morphism.

$$\begin{tikzcd}
E_i \arrow[rr] \arrow[dr] & & Z_i \arrow[dl] \\
& Z_0 &
\end{tikzcd}$$

By Proposition 3.11, we know that $E_i \to Z_0$ is a Moishezon morphism. Since the strict transform $E_i \to E_0$ is a dominant morphism, by Proposition 2.13, we know that $E_0 \to Z_0$ is also Moishezon morphism. Finally, by Proposition 3.13, we know that $\mathrm{Ex}(f) \to S$ is Moishezon. □

Theorem 3.16 (Fibers of the Moishezon morphism are Moishezon varieties, [Kol22, Corollary 16])**.** The fibers of a proper, Moishezon morphism are Moishezon.

Proof. Let $g : X \to S$ be a proper, Moishezon morphism. It is bimeromorphic to a projective morphism $X^{\mathrm{p}} \to S$. We may assume X^{p} to be normal. Let Y be the normalization of the closure of the graph of $X \dashrightarrow X^{\mathrm{p}}$.

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed] \arrow[dr] & & X^p \arrow[dl] \\ & S & \end{tikzcd}$$

Fix now $s \in S$. Let $Z_s \subset X_s$ be an irreducible component, since given a proper dominant morphism, there exist at least one irreducible component dominant the base, there exist $W_s \subset Y_s$ an irreducible component that dominates Z_s. And by Proposition 2.13 and Proposition 3.13, it's enough to show that W_s is Moishezon. We divide the problem into two cases:

If $\pi : Y \to X^{\mathrm{p}}$ is generically an isomorphism along W_s, then W_s is bimeromorphic to an irreducible component of X^{p}_s, hence Moishezon.

Otherwise $W_s \subset \mathrm{Ex}(\pi)$. Now $\mathrm{Ex}(\pi) \to X^{\mathrm{p}}$ is Moishezon by Proposition 3.15. And by induction on dimension, since $\dim \mathrm{Ex}(\pi) < \dim X = \dim Y$, the fiber W_s is Moishezon. □

The deterministic property that a Moishezon morphism satisfies is the following Chow type lemma. Before proving the result, let us first recall the relative Iitaka fibration.

Theorem 3.17 (Relative Iitaka fibration)**.**

Theorem 3.18 ([DH20, Lemma 2.18])**.** Let $f : X \to S$ be a proper surjective morphism of analytic varieties, and let L be a f-big line bundle on X and D a $\mathbb{Q}$-divisor.

Then over any relatively compact open subset $V \subset S$, there exists a proper bimeromorphic morphism $\alpha : W \to f^{-1}V$ from a smooth analytic variety W such that $\beta = f|_{f^{-1}V} \circ \alpha : W \to V$ is a projective morphism and $\left(W, \alpha_*^{-1}\left(D|_{f^{-1}V}\right) + \mathrm{Ex}(\alpha)\right)$ is a log smooth pair.

Proof. Proof. Let $\phi : X \to Y$ be the relative Iitaka fibration of L over S and $g : Y \to S$ the induced projective morphism. Since L is f-big, $\phi : X \to Y$ is bimeromorphic. Let $p : \Gamma \to X$ and $q : \Gamma \to Y$ be the resolution of indeterminacy of ϕ so that p is proper.

Now fix a relatively compact open subset $V \subset S$. Choose another relatively compact open set $U \subset S$ containing V such that $\bar{V} \subset U$. Note that U is σ-compact, since it is relatively compact. Since f and g are both proper morphisms, it follows that $X_U := f^{-1}U$ and $Y_U := g^{-1}U$ are both σ-compact. Let $\Gamma_U := q^{-1}\left(g^{-1}U\right) = p^{-1}\left(f^{-1}U\right)$. Then from the commutative diagram above it follows that $q|_{\Gamma_U} : \Gamma_U \to g^{-1}U$ is a proper morphism. In particular, Γ_U is σ-compact. Note that $q|_{\Gamma_U}$ is bimeromorphic. Therefore by Theorem 2.17 there is a projective bimeromorphic morphism $h : Z \to \Gamma_U$ from an analytic variety Z such that $q|_{\Gamma_U} \circ h : Z \to Y_U$ is a projective bimeromorphic morphism. Since g is projective, so is $Z \to U$. Now we replace U by our previously fixed open set V. Then $Z_V := (g \circ q \circ h)^{-1}V$ is a relatively compact open subset of Z. Let $r : W \to Z_V$ be the log resolution of $\left(Z_V, (p \circ h)_*^{-1}\left(D|_{f^{-1}V}\right)\right)$ as in Theorem 2.16. Let $\alpha := p|_{\Gamma_V} \circ h\Big|_{h^{-1}\Gamma_V} \circ r$

and $\beta := g|_{g^{-1}V} \circ q\Big|_{\Gamma_V} \circ h\Big|_{h^{-1}\Gamma_V} \circ r$, where $\Gamma_V := p^{-1}\left(f^{-1}V\right) = q^{-1}\left(g^{-1}V\right)$. Note that β is a projective morphism, since it is a componsition of projective morphisms over relatively compact bases.

□

Proposition 3.19 ([Kol22, Example 13])**.** Let Z be a normal, projective variety with discrete automorphism group. Let $g : X \to S$ be a fiber bundle with fiber Z over a connected base S. Then g is Moishezon $\Leftrightarrow$ g is projective $\Leftrightarrow$ the monodromy is finite.

Proof. Only needs to show that (1) implies (3) and (3) implies (2). For (3) implies (2), we try to take the etale base change so that the fiber bundle becomes trivial bundle, since the monodromy is finite, we can take the symmetric product of all the ample bundle on the fiber Z,

$$L' = \bigotimes_{g \in G} g^* L$$

since it's monodromy invariant. The ample line bundle will descend to the original family $g : X \to S$ and thus g is a projective morphism.

For (1) implies (3). Since $g : X \to S$ is Moishezon, by Definition 3.9, there exists a g-big H on X. We claim that there are only finite many ample line bundle M such that the intersection number $H \cdot M^{n-1}$ is fixed. □

4 Examples

In this section, we will present varies examples related to the Moishezon space and Moishezon morphism.

4.1 The Hironaka's example

Hironaka discovered a bunch of complete non-projective 3 dimension variety which is now called the Hironaka's variety. Note that based on the construction of Hironaka, we can from almost all the projective variety construct some Moishezon variety, that is why we said at the beginning that Moishezon varieties are versatile in birational geometry. (However this is not true in dimension 2, since all the smooth Moishezon surface are actually projective, see [**SCV7**]).

The major reference of this part of note is the well written note by Ulrich Thiel (see `https://ulthiel.com/math/wp-content/uploads/other/hironakas_example.pdf`).

Given a smooth projective theefold, which contains two rational curves transversely intersection at two points. Assume two rational curves are C and D that intersect at point P, Q.

We then take two steps blow up

$$X_1 = \mathrm{Bl}_{(D\setminus P)'}\left(\mathrm{Bl}_{C\setminus P}(X\setminus P)\right) \xrightarrow{\pi_2} \mathrm{Bl}_{C\setminus P}(X\setminus P) \xrightarrow{\pi_1} X\setminus P$$
$$X_2 = \mathrm{Bl}_{(C\setminus Q)'}\left(\mathrm{Bl}_{D\setminus Q}(X\setminus Q)\right) \xrightarrow{\sigma_2} \mathrm{Bl}_{D\setminus Q}(X\setminus Q) \xrightarrow{\sigma_1} X\setminus Q,$$

Note that if we define $U = X - \{P, Q\}$, then $\pi^{-1}(U) \cong \sigma^{-1}(U)$. In particular, we can glue X_1 and X_2 along $\pi^{-1}(U)$ and $\sigma^{-1}(U)$. In the picture below, we glue the red exceptional surface on the right hand side with the black exceptional surface on the left hand side (denote it S_1) and blue exceptional surface on the left hand side with the black exceptional surface on the right hand side (denote it S_2). (see the pictrue 1). By gluing lemma, there exists a morphism $f : H \to X$ and the restriction of the morphism on S_1, S_2 as $f_1 = f|_{S_1} : S_1 \to C$ and $f_2 = f|_{S_2} : S_2 \to C$.

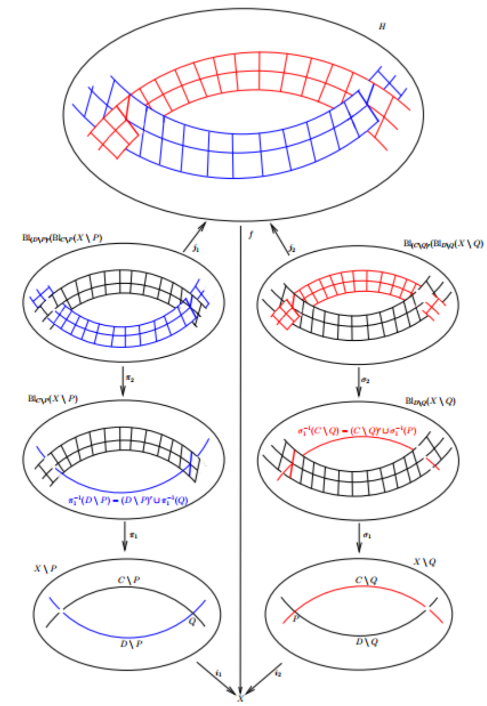

Figure 1: Construction of the Hironaka's variety

We claim that Hironaka's variety is non-projective.

The idea is to find some divisor on the surface $S = S_1 \cup S_2$ on the Hironaka's variety H.

The idea to prove the non-projective is to find some curve on the surface $S = S_1 \cup S_2$. Which has positive degree but add up to 0.

The key observation is that $f^{-1}(P)$ (resp. $f^{-1}(Q)$) decompose into two split projective line L_Q and L'_Q in S_1 (resp. L_P and L'_P in S_2). (see the precise statement below).

Choose two points $A \in C - \{P, Q\}$ and $B \in D - \{P, Q\}$. Since all the point on a rational curve are linear equivalent, therefore

$$A \sim_C Q \Longrightarrow f_1^{-1}(A) \sim_{S_1} f_1^{-1}(Q) = L_Q + L'_Q$$
$$B \sim_D P \Longrightarrow f_2^{-1}(B) \sim_{S_2} f_2^{-1}(P) = L_P + L'_P$$

and Push forward of cycle, we get equivalence on S.

$$I : f^{-1}(A) \sim_S f^{-1}(Q) = L_Q + L'_Q$$
$$II : f^{-1}(B) \sim_S f^{-1}(P) = L_P + L'_P$$

On the other hand we also that B, Q lies in the same rational curve, so that

$$III : B \sim_D Q \Rightarrow f_2^{-1}(B) \sim_{S_2} f_2^{-1}(Q) \Rightarrow f^{-1}(B) \sim_S L'_Q$$

and combined then together, we get

$$\begin{aligned} f^{-1}(A) + f^{-1}(B) \sim_S f^{-1}(A) + f^{-1}(B) &\Rightarrow L_Q + L'_Q + L_P + L'_P \sim_S L'_Q + L'_P \\ &\Rightarrow L_Q + L_P \sim_S 0 \end{aligned}$$

If there exist some ample divisor on A, then both $L_Q \cdot A > 0$ and $L_P \cdot A > 0$ contradict the linearly trivial relation above. Therefore the only possible case is Hironaka's variety is non-projective.

4.2 Locally Moishezon morphism which is not globally Moishezon: family of K3 surfaces

This example comes from family of K3 surfaces.

4.3 Flip the rational curve on the quintic 3-fold produce Moishezon variety

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