Albanese varieties and Albanese mappings Fall 2025

Note I.2 — 22, 09, 2025 (in progress draft)

Yi Li

The aim of this note is try to give an introduction of Albanese varieties and Albanese mapping with varies applications.

Contents

1 A brief introduction to Albanese Varieties

Let us first construct the Albanese variety. To do this, we need the following proposition.

Proposition 1. Let X be a compact Kähler manifold, with the Kähler form ω.
(1) We have the following well defined map

$$\varphi : H_1(X, \mathbb{Z}) \to (H^0(X, \Omega_X^1))^\vee, \quad [\gamma] \mapsto (\alpha \mapsto \int_\gamma \alpha).$$

(2) The image of $H_1(X, \mathbb{Z})$ forms a lattice in $(H^0(X, \Omega_X^1))^\vee$, hence the quotient is a complex torus of dimension equals to the $\dim H^0(X, \Omega_X^1)$.

Remark 2 (Definition of lattice)**.** Let V denote a complex vector space of dimension g. A lattice in V is by definition a discrete subgroup of maximal rank in V. It is a free abelian group of rank $2g$. That is

$$\dim_{\mathbb{C}} V = g, \quad \operatorname{rk} \Lambda = 2g.$$

Proof of (1). Let (X, ω) be a compact Kähler manifold, by Lemma 3, given a holomorphic p-forms α, it is always closed. On the other hand, if two class $[\gamma] = [\gamma'] \in H_1(X, \mathbb{Z})$, then there exists a singular 2-chain such that

$$\gamma - \gamma' = \partial S.$$

In particular, by the Stoke's formula,

$$\int_{\gamma-\gamma'} \alpha = \int_{\partial S} \alpha = \int_S d\alpha = 0.$$

Thus it's independent of choice of representative. □

Proof of (2). To show that $\mathrm{Alb}(X)$ is a complex torus, we need to prove that $H_1(X, \mathbb{Z})$ forms a lattice in $(H^0(X, \Omega_X^1))^\vee$. The idea is to use the fact that the natural map

$$H^{2n-1}(X, \mathbb{Z}) \to H^{2n-1}(X, \mathbb{C})$$

has image a lattice (since, by the universal coefficient theorem, tensoring with $\mathbb{Q}$ kills torsion). We then apply Serre duality (SD) and Poincaré duality (PD), which gives the following commutative diagram.

$$\begin{tikzcd} H^0(X, \Omega_X^1)^\vee & H^n(X, \Omega_X^{n-1}) \arrow[l, "\overset{\mathrm{SD}}{\simeq}"'] \\ H_1(X, \mathbb{Z}) \arrow[u, "\varphi"] \arrow[r, "\underset{\simeq}{\mathrm{PD}}"'] & H^{2n-1}(X, \mathbb{Z}) \arrow[u, "p"'] \end{tikzcd}$$

Where p is the composition

$$p : H^{2n-1}(X, \mathbb{Z}) \to H^{2n-1}(X, \mathbb{C}) \to H^{n-1,n}(X)$$

whose image is a lattice (since tensoring with $\mathbb{C}$ eliminate torsion).

Note that the image of φ can be identified with image of p, since by definition of PD, we have

$$\int_X \omega \wedge \mathrm{PD}([\gamma]) = \int_\gamma \omega.$$

While by definition SD, we have

$$\mathrm{SD}(\mathrm{PD}([\gamma]))(\omega) := \int \mathrm{PD}([\gamma]) \wedge \omega = \int_\gamma \omega = \varphi([\gamma])(\omega).$$

And hence image of φ is also a lattice. □

Lemma 3. Let X be a compact Kähler manifold with a Kähler metric. Let $\alpha \in H^0(X, \Omega_X^p)$ be a holomorphic p-form, then α is always closed.

Proof. by definition we have $\bar{\partial}\alpha = 0$, on the other hand, by type reason, we know that $\bar{\partial}^*\alpha = 0$ as well, thus

$$\Delta_{\bar{\partial}}\alpha = 0 \implies \Delta_\partial \alpha = 0.\text{(by Kähler identity)}.$$

And consequently,

$$(\partial\partial^*\alpha + \partial^*\partial\alpha, \alpha) = \|\partial\alpha\|^2 + \|\partial^*\alpha\|^2 = 0 \implies \partial\alpha = 0.$$

□

Definition 4 (Albanese variety)**.** Let X be a compact Kähler manifold (or more generally a compact complex manifold). We define the complex torus

$$\mathrm{Alb}(X) = (H^0(X, \Omega_X^1))^\vee / H_1(X, \mathbb{Z})$$

to be the Albanese variety associated to X, which is a complex torus.

Remark 5. For the readers who are interested in the more general construction of Albanese torus for any compact complex manifold, please refer to [Uen75, Theorem 9.7] or [GPR94, Theorem 3.27].

Theorem 6 (Duality Between Albanese variety and Picard variety, [Lan23, Proposition 5.2.6])**.** Let X be a projective manifold, then the Picard variety is dual to the Albanese variety

$$\mathrm{Pic}^0(X) = \widehat{\mathrm{Alb}(X)}$$

Proof. □

When X is projective, we can show that Albanese variety is an Abelian variety.

Theorem 7. Let X be a projective manifold, Then $\mathrm{Pic}^0(X)$ and hence Albanese variety is an Abelian variety.

Proof. Only needs to show the projectivity of the Picard variety $\mathrm{Pic}^0(X)$, then since dual of Abelian variety is Abelian $\mathrm{Alb}(X)$ will be an Abelian variety as well. □

Note that when X is Moishezon manifold, the Albanese variety is still an Abelian variety. In general however it's only a complex torus.

Proposition 8 ([Uen75, Proposition 9.15])**.** Let X be a Moishezon manifold, then the Albanese torus $\mathrm{Alb}(X)$ is a projective manifold.

Proof. □

2 A brief introduction to Albanese mappings

Definition 9 (Albanese mapping)**.** Let $[\omega_1], \cdots, [\omega_k]$ be the basis of $H^0(X, \Omega_X^1)$. Then the representative ω_i are closed $(1,0)$-forms. We then define the Albanese mapping as

$$\mathrm{alb}_X : X \to \mathrm{Alb}(X)., \quad z \mapsto (\int_{z_0}^{z} \omega_1, \cdots, \int_{z_0}^{z} \omega_k).$$

Proposition 10. The Albanese mapping is well defined.

Proof. First, by Lemma 3, the integration does not depend on the real path that we choose. Second, we need to check the linear functional does not depend on the □

When a projective variety admits

Proposition 11 ([BS95, Lemma 2.4.1])**.** Let X be a normal projective variety with rational singularities, then the Albanese map is well defined.

Proof. □

We first introduce the universal property of Albanese mapping.

Proposition 12 ([Lan23, Theorem 5.2.2])**.** Let $\varphi : M \to X$ be a holomorphic map into a complex torus X. There exists a unique homomorphism $\widetilde{\varphi} : \mathrm{Alb}(M) \to X$ of complex tori such that the following diagram is commutative

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, "\alpha_{p_0}"'] & X \arrow[d, "T_{-\varphi(p_0)}"] \\
\mathrm{Alb}(M) \arrow[r, "\widetilde{\varphi}"'] & X
\end{tikzcd}$$

Proof. □

3 Conditions for Albanese mapping to be fibration

Note that in general Albanese mapping is not fibration. However when Kodaira dimension is 0, Kawamata proved the Albanese mapping is actually a fibration.

Theorem 13 ([Kaw81])**.**

4 Applications of Albanese Fibration

4.1 Albanese mapping in Iitaka conjecture

When Albanese mapping is a fibration, we can use it in the proof of Iitaka conjecture.

Theorem 14 ([Nak04, IV.4.9 Corollary])**.** Let X be a normal projective variety and let Δ be a $\mathbb{Q}$-divisor such that (X, Δ) is log-terminal. If $\kappa_\sigma(K_X + \Delta) = 0$, then $\kappa(K_X + \Delta) = 0$.

Theorem 15 ([CPB24,])**.**

4.2 Albanese mapping in the classification problems

References

[BS95] Mauro C. Beltrametti and Andrew J. Sommese, *The adjunction theory of complex projective varieties*, De Gruyter Expositions in Mathematics, vol. 16, Walter de Gruyter & Co., Berlin, 1995. MR 1318687

[CPB24] Junyan Cao, Mihai Paun, and Bo Berndtsson, *On the Ohsawa-Takegoshi extension theorem*, J. Geom. Anal. **34** (2024), no. 1, Paper No. 25, 47. MR 4668049

[GPR94] H. Grauert, Th. Peternell, and R. Remmert (eds.), *Several complex variables. VII*, Encyclopaedia of Mathematical Sciences, vol. 74, Springer-Verlag, Berlin, 1994, Sheaf-theoretical methods in complex analysis, A reprint of *Current problems in mathematics. Fundamental directions. Vol. 74* (Russian), Vseross. Inst. Nauchn. i Tekhn. Inform. (VINITI), Moscow.

[Kaw81] Yujiro Kawamata, *Characterization of abelian varieties*, Compositio mathematica **43** (1981), no. 2, 253–276.

[Lan23] Herbert Lange, *Abelian varieties over the complex numbers—a graduate course*, Grundlehren Text Editions, Springer, Cham, [2023] ©2023.

[Nak04] Noboru Nakayama, *Zariski-decomposition and abundance*, MSJ Memoirs, vol. 14, Mathematical Society of Japan, Tokyo, 2004. MR 2104208

[Uen75] Kenji Ueno, *Classification theory of algebraic varieties and compact complex spaces*, Lecture Notes in Mathematics, vol. Vol. 439, Springer-Verlag, Berlin-New York, 1975, Notes written in collaboration with P. Cherenack. MR 506253