Leverage Cycle over the Life Cycle: A Quantitative Model of Endogenous Leverage

Yi Ping[1]

[a]ERUNI, Politických Vězňů 1419/11, Nové Město, Praha, 110 00, Czechia

Abstract

I construct a quantitative model to rationalize two robust features of the U.S. housing market: leverage moves in tandem with housing prices, while mortgage spreads move in the opposite direction. The model features overlapping generations who accumulate housing assets using leverage, selecting mortgage contracts from a menu where interest rates vary with loan-to-value ratios (LTV), known as the Credit Surface. Large negative shocks depress housing prices through two connected channels. First, young borrowers default, reducing the wealth of middle-aged lenders and directly lowering housing prices. Second, lenders raise mortgage rates and spreads as they bear losses from both defaults and the falling housing wealth, weakening housing's ability to facilitate borrowing. Reduced leverage feeds back into further declines in housing prices, amplifying volatility in the housing market. The model is calibrated to replicate the 10-percentage-point decline in leverage observed during the Great Recession.

Keywords: endogenous leverage, leverage cycle, household life cycle, Credit Surface, housing prices, default
JEL: E20, E44, G51, C68, D52, D53

1. Introduction

The U.S. housing and mortgage markets exhibit two well-documented trends: leverage tends to move in the same direction as housing prices, whereas mortgage spreads move in the opposite direction. Understanding leverage is crucial for policymakers for two reasons. First, in practice, the majority of U.S. homebuyers finance their housing purchases through mortgages. Using annual data on new house sales by financing type from 1988 to 2023, the average share of mortgages as a financing method is 94% in the U.S.[1] Second, in theory, existing work on endogenous leverage (Geanakoplos (1997); Fostel and Geanakoplos (2008), among others) demonstrates that

⋆I am deeply grateful to my advisor, Ana Fostel, for her intellectual guidance and mentorship, and for dedicating considerable time and effort to my work. I am also sincerely thankful to Eric Young for his insightful and invaluable support throughout this research. I also wish to express my gratitude to Anton Korinek and the University of Virginia faculty and my fellow graduate students for their constructive feedback and comments. I extend my thanks to the UVa Bankard Fund for Political Economy for its financial support.

Email address: `yi.ping@eruni.org` (Yi Ping)

[1]Mortgage types include conventional, FHA-insured, and VA loans. Data source: FRED, Federal Reserve Bank of St. Louis (data labels: HSTFCM, HSTFC, HSTFFHAI, HSTFVAG).

leverage plays a key role in amplifying fluctuations in asset prices. The primary contribution of this paper is to develop a quantitative general equilibrium model with endogenous leverage in an overlapping-generation (OLG) framework. This model rationalizes the observed co-movements of housing prices, leverage, and mortgage spreads.

Using quarterly data spanning from 1999Q1 to 2024Q1, Table 1 presents the cross correlations of changes in Case-Shiller housing prices with the average first-time homebuyers' LTV, Combined LTV (CLTV), and mortgage spreads, with the highest absolute correlations underlined. Both LTV and CLTV are positively correlated with housing prices, whereas mortgage spreads are negatively correlated with housing prices. These patterns were especially pronounced in the 2000s, during which the U.S. housing market experienced a leverage cycle. Fostel and Geanakoplos (2014) present the co-movement of leverage and housing prices using data on the average down payment for borrowers below the median in the subprime/Alt-A category. The left panel of Figure 1 provides further evidence for this pattern by showing trends in housing prices and CLTV for first-time homebuyers in the U.S. Specifically, the CLTV initially increased steadily along with rising housing prices, then experienced a sharp 10-percentage-point drop from its peak in 2007Q2 to its trough in two years. Mortgage spreads measure the relative cost of purchasing housing assets using leverage. Walentin (2014) and Musso, Neri, and Stracca (2011) show that mortgage spreads rise during times of economic stress, especially in the Great Recession. The right panel of Figure 1 adds to this empirical finding and shows that mortgage spreads fell modestly from 2% in 1999Q1 to 1.5% in 2007Q2, followed by a substantial increase to 2.61% within just one year by 2008Q2.

Table 1: Cross Correlations of Leverage and Mortgage Spreads with Housing prices

	LTV	CLTV	Spread
lead (2)	0.41	0.35	-0.08
lead (1)	0.42	0.39	-0.24
contemporaneous	0.41	0.42	-0.28
lag (1)	0.38	0.42	<u>-0.29</u>
lag (2)	<u>0.53</u>	<u>0.44</u>	-0.08

Notes: "lead/lag (i)" indicates that the variable leads/lags housing prices by i quarter(s). LTV is calculated by dividing the balance of the primary mortgage by the appraised value of the property securing the mortgage. CLTV is calculated by summing the balances of all loans secured by a property and then dividing this total by the appraised value of the property. According to Walentin (2014), the duration of a 30-year fixed rate mortgage in the U.S. is 7 to 8 years, Therefore, I define the mortgage spread as the difference between the average 30-year fixed-rate mortgage and the 5-year treasury bill yield. Data source: leverage from Freddie Mac Single-Family Loan-Level Dataset, Case-Shiller housing price index from FRED, index Jan 2000 = 100, mortgage spreads from FRED MORTGAGE30US, DGS5.

To explain these trends, this paper considers three key elements that have not been jointly included in the existing literature: (i) large aggregate endowment shocks, (ii) hump-shaped life-cycle endowment profiles of households combined with realistic household lifespans, and (iii) the endogenous determination of leverage. In the model, the economy is populated by overlapping generations of households, each deriving utility from both housing and non-housing consumption. Households face large aggregate endowment shocks which lead to substantial fluctuations

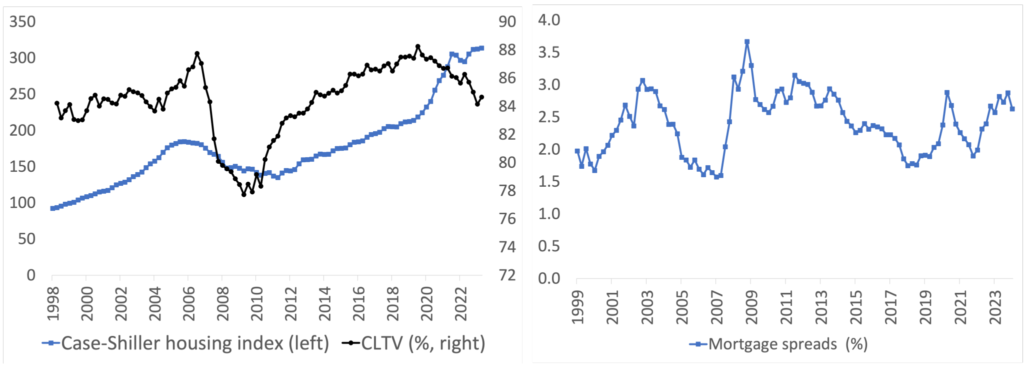

Figure 1: Housing Prices, Leverage, and Mortgage Spreads

in equilibrium housing prices and mortgage interest rates. They receive low endowments in early life, which gradually increase and peak in middle age before declining in later stages of life. The consideration of the household life cycle is grounded in micro data. As illustrated in Figure 2, there is an evident age-related pattern in the housing market: housing wealth increases with age then slightly declines, while leverage decreases with age.

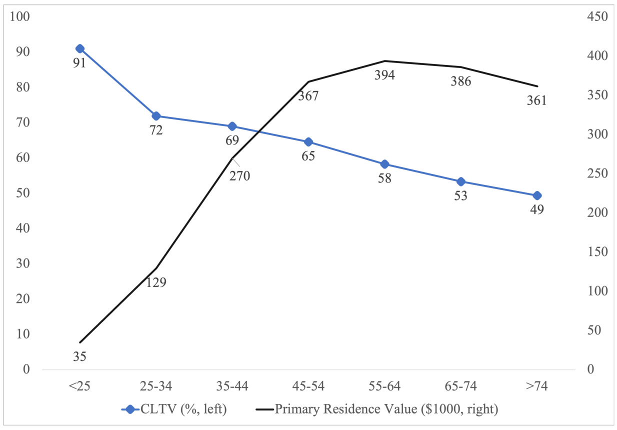

Figure 2: Trends in Age

Notes: The CLTV for the first age group (<25) is averaged based on loans taken for home purchases, and the CLTV for all other age groups is averaged based on loans taken for refinancing purposes. Primary residence values are in 2022 dollars. Data sources: CLTV ratios are from the 2023 HDMA national loan-level dataset, and primary residence values are from the 2022 SCF dataset. These patterns are consistent across years, see supplementary materials for details.

Housing assets are perfectly durable and can be used as collateral for issuing debt contracts. Agents can accumulate housing assets using leverage through the simultaneous purchase of housing and issuance of debt contracts. A collateral constraint prohibits households from issuing a quantity of debt contracts that exceeds their housing stock. Leverage of a contract is measured by LTV, defined as the ratio of the loan amount to the value of the collateral. LTV ranges from 0% to an endogenously determined upper limit, which is always below 100%. Fostel and Geanakoplos (2015) introduce the concept of the "Credit Surface", which represents a menu of leverage levels paired with their corresponding interest rates. Following their methodology, equilibrium prices define a Credit Surface that specifies the interest rates for different LTV levels. The interest rate increases monotonically with the LTV ratio. For loans with an LTV low enough to rule out default, a uniform risk-free interest rate is applied. However, for loans with an LTV that implies default risk, the interest rate begins to rise as the LTV increases.

In this framework, endogenous leverage has a twofold meaning. First, the upper limit of LTV is determined within the model, based on the capacity of housing to serve as collateral for borrowing, and it adjusts in response to changes in economic fundamentals. Second, while agents have infinitely many leverage options, as LTV is a continuous variable, the scarcity of collateral limits equilibrium choices to only a few LTV levels. Borrowers face a trade-off between present liquidity and future obligations: higher leverage provides greater liquidity but comes with higher interest rates. Lenders, driven by consumption-smoothing motives, also chose their optimal LTV levels. Both borrowers and lenders select their optimal positions on the Credit Surface, leaving most LTV levels rationed with zero supply and demand in equilibrium.

The model has two main implications. First, households start with high leverage and then progressively lower their leverage over the course of their life. In the equilibrium, only two types of contracts are traded, even though infinitely many debt contracts are priced: a risky contract, where agents default during downturns, and a risk-free contract, which ensures no default but offers less liquidity than the risky one. Households transition through four distinct life stages: they start as constrained borrowers, become unconstrained borrowers, transition into a mixed role of borrowers and lenders, and eventually become pure lenders. Second, large aggregate endowment shocks produce leverage cycles. In normal states, housing prices and leverage are high, while mortgage rates and spreads remain low. However, during downturns, housing prices and leverage fall sharply, whereas mortgage interest rates and spreads rise.

Life cycle implications. In the first stage, young households receive low endowments. Anticipating an increase in future endowment, they issue risky debt contracts to borrow, pledging all the available housing stock as collateral. As their endowments grow, they transition to using a combination of risky and risk-less contracts for borrowing, without pledging all their owned housing as collateral. In the third stage, as endowments begin to decline, they start investing in risky contracts while also leveraging their housing assets through risk-less contracts. Finally, in the last stage, driven by a strong savings motive, households accumulate more financial wealth. To diversify their investments, they purchase both risky and risk-less debt contracts. On average, younger households are the most leveraged among all age groups, and leverage decreases as households age, a pattern that is consistent with data.

Leverage cycle implications. The second main implication of the model is that leverage and housing prices decline substantially during crises, while mortgage rates and spreads surge. A large negative endowment shock reduces housing prices by impacting the two groups of marginal buyers of housing assets—young borrowers and middle-aged lenders—through distinct but related channels. The sequence of events unfolds as follows: during downturns, young borrowers who previously issued risky contracts find themselves underwater and default, which reduces lenders' wealth. Since lenders are marginal buyers of housing assets, this reduction in their wealth directly depresses housing prices. Lenders, facing significant losses from defaults and declining housing wealth, become reluctant to forego current consumption to lend. Consequently, they demand higher returns and collectively raise interest rates across all debt contracts, as well as spreads between risky and risk-free contracts. Credit Surface rises and steepens, reducing the leveraging capacity of housing assets for young borrowers. This lower leverage then feeds back into further declines in housing prices. Figure 3 illustrates the feedback loop between leverage and housing prices in downturns.

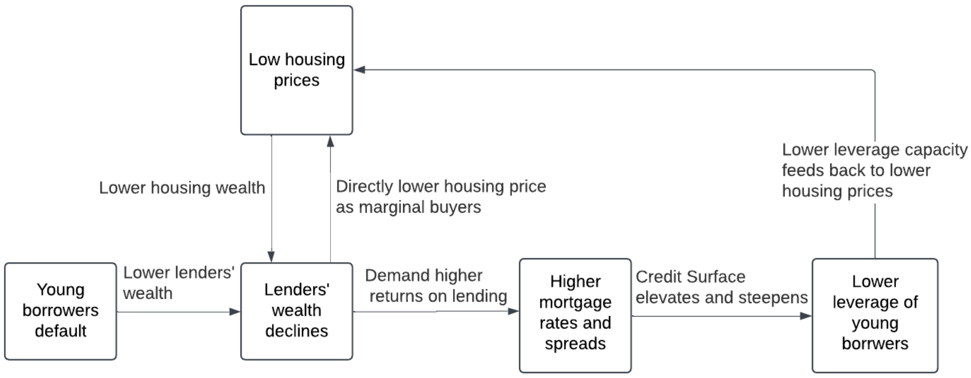

Figure 3: Feedback Loop between Leverage and Housing Prices in Downturns

The majority of debt issuance in the economy is through the risky contract. The key feature of this contract is that its payoff varies significantly across states: borrowers do not default in normal times but default during downturns. This variation in payoffs translates into higher volatility in lenders' wealth across states. As the marginal buyers in the housing market, fluctuations in their wealth greatly amplify housing price volatility.

In contrast, I show that this mechanism is absent in a bond economy, in which the borrowing constraint follows the framework of Kiyotaki and Moore (1997). In this setting, agents borrow by issuing non-contingent bonds, households' LTV is capped exogenously, and default does not occur in equilibrium. Lenders' wealth does not vary with the debt repayment, and aggregate debt is significantly lower in this economy. In addition, the feedback loop between leverage and housing prices is muted, leading to much smaller declines in housing prices during downturns.

Solving for the equilibrium presents two main challenges. This paper addresses these challenges and offers guidance for finding a collateral equilibrium with housing assets and a high degree of agent heterogeneity. The first challenge is conceptual: which LTV levels will be selected in equilibrium? Since LTV is a continuous variable, deriving an infinite number of first-order conditions for each LTV is impractical. Existing literature on endogenous leverage provides theoretical guidelines for finding such equilibria, but these are restricted to cases where agents do not derive utility from collateral assets. Such frameworks are unsuitable for this study, as the demand and supply of leverage are influenced by agents' preferences over housing. To address this challenge, I assume the existence of an equilibrium in which only a finite number of specific LTV levels are selected, then verify whether agents have incentives to deviate by trading contracts with LTV levels outside this finite set. I repeat this procedure until I find an equilibrium.

The second challenge is computational in nature. The substantial heterogeneity of agents and the multiplicity of tradable assets result in a high-dimensional state space. In addition, the collateral constraint introduces non-linearity into the policy functions. Azinovic, Gaegauf, and Scheidegger (2022) introduce a methodology that uses neural networks to address the curse of dimensionality and the non-linearity of policy functions. Following their approach, I use a neural network with two hidden layers to directly approximate the policy functions. The resulting maximum error across all equilibrium conditions is less than 1%.

Related literature. This paper contributes to two main strands of the literature. First, it builds on the foundational work on endogenous leverage theory by Geanakoplos (1997) and Fostel and Geanakoplos (2008, 2012, 2014, 2015). In contrast to the majority of macro-finance models that assume an exogenous cap on leverage, these models take changes in leverage as endogenous responses to shifts in economic fundamentals, and uncover a strong feedback loop between leverage and asset prices. While existing models in this literature are primarily stylized theoretical constructs, this paper bridges the gap between theory and data by incorporating the age heterogeneity observed in the housing market.

There are two quantitative works that consider endogenous leverage. Diamond and Landvoigt (2022) develop an OLG model with housing assets as collateral. The key mechanism driving housing booms and busts is a household saving glut, driven by an increased preference for deposits, combined with aggregate shocks and idiosyncratic housing wealth shocks that increase housing price dispersion during downturns. The combination of a negative productivity shock and rising housing wealth dispersion leads to substantial defaults and losses for financial intermediaries, prompting them to tighten credit. The key distinction between their work and mine lies in the relationship between leverage and housing prices. When driven solely by aggregate endowment shocks, their model predicts a negative correlation, whereas my model produces a positive correlation. Brumm, Grill, Kubler, and Schmedders (2015) examine the effects of endogenous leverage on asset price volatility through a model with two infinitely-lived agents with different degrees of risk aversion. In contrast, this paper considers the life-cycle dynamics of households and emphasizes the role of non-financial assets, specifically housing, as collateral, where agents' preferences of housing influence equilibrium prices.

Second, this paper also relates to the macroeconomic literature focusing on the housing market and the households life cycle. Extant models typically adopt the borrowing constraint framework of Kiyotaki and Moore (1997), where the LTV cap is exogenous, interest rate does not vary with leverage, and household default is often ruled out. As a result, these models are silent on the factors driving changes in housing finance conditions and overlook the feedback loop between leverage and housing prices. In these settings, variations in the LTV limit have minor effects on housing prices. Kaplan, Mitman, and Violante (2020) analyze the housing boom and bust episode within an OLG model with endogenous housing prices. They conclude that exogenous shifts in the LTV limit do not significantly affect housing prices; instead, fluctuations in housing prices are primarily driven by shifts in households' beliefs about future housing demand. Favilukis, Ludvigson, and Van Nieuwerburgh (2017) also examine the housing market in an OLG model with endogenous housing prices. They find that a higher LTV cap significantly increases housing prices, but this effect is only significant when they introduce heterogeneity in households' bequest preferences, which results in a substantial number of constrained households in equilibrium.

2. Model

2.1. Agents, Commodities and Uncertainty

Time is discrete and indexed by $t = 0, 1, 2, ...$ At each period, one of two possible exogenous aggregate shocks $z_t \in \mathbf{Z} = \{U, D\}$ realizes. The state U stands for "Up", representing normal times. The state D stands for "Down", representing rare crisis states similar to the Great

Recession. The aggregate shock z_t affects both the aggregate endowment and the allocation of endowments among different age groups in every period. z_t evolves according to a Markov chain with the transition matrix Γ. Let $\gamma_{z_t,z_{t+1}}$ denote the probability of transitioning from state z_t to state z_{t+1}.

Agents can trade both consumption good (c) and housing (h). The consumption good is perishable, whereas housing assets are perfectly durable and in fixed supply of H. Let the spot price of the consumption good be 1, and the housing price be q_t.

In each period, a continuum of mass 1 identical agents of a new generation is born and lives for A periods. There is no mortality risk; all households die after age A. Age is indexed by $a \in \mathbf{A} = \{1, ..., A\}$. At the beginning of each period, all households receive a strictly positive endowment of the consumption good, which depends on the aggregate shock: $e_t^a = e^a(z_t) > 0, \ \forall a \in \mathbf{A}$. The aggregate endowment is denoted by $\bar{e}(z_t) = \sum_{a=1}^{A} e^a(z_t)$.

2.2. Preferences

The expected lifetime utility of households born at time t is given by

$$U_t = E_t \sum_{a=1}^{A} \beta^{a-1} u^a(c_t^a, h_t^a), \tag{1}$$

where $\beta > 0$ is the discount factor, c_t^a and h_t^a are the amount of the consumption good and the stock of housing at age a. $u^a(c, h)$ is an age-dependent period utility function. Let $\hat{\mathbf{A}} = \{1, ..., A-1\}$ be the set of agents excluding those who are in the last period of life. All agents have the Cobb-Douglas utility over consumption and housing nested within a constant relative risk aversion utility form when they are at age $a \in \hat{\mathbf{A}}$, and they do not value housing assets in the last period of life, the period utility function is given by

$$u^a(c,h) = \begin{cases} \dfrac{(c^{1-\alpha}h^{\alpha})^{1-\rho}}{1-\rho}, & a \in \hat{\mathbf{A}}, \\ \dfrac{c^{1-\rho}}{1-\rho}, & a = A, \end{cases}$$

where $\alpha > 0$ measures the relative share of housing expenditure, $\rho > 0$ is the coefficient of risk aversion.

2.3. Debt Contracts

Households enter the economy with neither debts or assets. Each period, they meet in anonymous, competitive financial markets to trade collateralized debt contracts. All contracts are one-period. Let J_t denote the set of such contracts available at time t. A financial contract in J_t is defined by an ordered pair representing its promise and collateral requirement, denoted as $(j, 1)$. $j \in \mathbb{R}_+$ is a non-contingent promise to deliver j units of consumption good in the next period, while the number 1 indicates that the promise j must be backed by one unit of housing as collateral. For simplicity, I use the terms contract $(j, 1)$ and contract j interchangeably. J_t contains an infinite number of contracts, as j is continuous and unbounded above.

All contracts are non-recourse, meaning that agents can default on their promises without incurring additional penalties beyond losing the collateral they have previously pledged. An agent who sells one unit of contract j will default on the promise if j exceeds the realized housing price q_{t+1}. Therefore, the delivery of contract j is $min\{j, q_{t+1}\}$.

The price of contract j is denoted by $\pi_{j,t}$. Let $\theta^a_{j,t} \in \mathbb{R}$ be the number of contract j traded by an agent of age a. $\theta^a_{j,t} < 0 (> 0)$ indicates the agent is shorting (longing) contract j, by doing so the agent borrows (lends) $|\pi_{j,t}\theta^a_{j,t}|$. When agents buy one unit of housing and finance this purchase by selling a debt contract j, they are effectively making a downpayment of $q_t - \pi_{j,t}$ on the house.

The gross interest rate for contract j is defined as the ratio of its promise to its price:

$$R_{j,t} = \frac{j}{\pi_{j,t}}.$$

The LTV for contract j is given by:

$$LTV_{j,t} = \frac{\pi_{j,t}}{q_t}.$$

While $LTV_{j,t}$ measures the leverage of individual contracts, Fostel and Geanakoplos (2015) propose metrics to assess the leverage of an agent and the economy-wide leverage. Building on their framework, I define these measures as follows:

The leverage of an agent, LTV^a_t, is defined as the ratio of the agent's total debt issuance to the value of the collateral backing this borrowing:

$$LTV^a_t = \frac{\int_{j\in\mathbb{R}_+} max\{-\theta^a_{j,t}, 0\}\pi_{j,t} dj}{q_t \int_{j\in\mathbb{R}_+} max\{-\theta^a_{j,t}, 0\} dj}.$$

LTV^a_t is not well-defined for agents who do not leverage. To measure borrowing relative to the agent's entire housing stock, define DLTV of an agent, $DLTV^a_t$, as:

$$DLTV^a_t = \frac{\int_{j\in\mathbb{R}_+} max\{-\theta^a_{j,t}, 0\}\pi_{j,t} dj}{q_t h^a_t}.$$

The aggregate amount of debt in the economy is the sum of all agents' debt issuance across contracts:

$$\sum_{a\in A} \int \pi_{j,t} max\{-\theta^a_{j,t}, 0\} dj.$$

To measure the economy-wide level of leverage, define the diluted LTV of housing, $DLTV^h_t$, as the ratio of aggregate debt to the aggregate value of housing:

$$DLTV^h_t = \frac{\sum_{a\in A} \int \pi_{j,t} max\{-\theta^a_{j,t}, 0\} dj}{q_t H}.$$

Fostel and Geanakoplos (2008) introduce the concept of collateral value and liquidity wedge; Following their discussion, Geanakoplos and Zame (2014) introduce the concept of liquidity value as a way to quantify the benefits of borrowing using different contracts. Following their definition, in this economy, the liquidity value of contract j for an agent of age a at time t, denoted by $LV^a_{j,t}$, is given by contract j's price net of the present value of its delivery, discounted by the agent's stochastic discount factor:

$$LV^a_{j,t} = \pi_{j,t} - E_t\Big[\beta \frac{Du_c(c^{a+1}_{t+1}, h^{a+1}_{t+1})}{Du_c(c^a_t, h^a_t)} min\{j, q_{t+1}\}\Big]. \quad (2)$$

2.4. Constraints

2.4.1. Budget Constraint

The budget constraint for agents at age a and at time t is given by

$$c_t^a + q_t h_t^a + \int\limits_{R_+} \theta_{j,t}^a \pi_{j,t} dj \leqslant e_t^a + q_t h_{t-1}^{a-1} + \int\limits_{R_+} \theta_{j,t-1}^{a-1} min\{j, q_t\} dj. \tag{3}$$

On the left-hand side of the budget constraint, there are expenditures on consumption, housing, and the total amount of borrowing (or lending) through trading debt contracts in the financial market. Since j is a continuous variable, the third item is an integral over $j \in \mathbb{R}_+$. On the right-hand side, agents receive their endowments, observe the market value of the housing assets bought in the last period, and clear debts associated with contracts traded in the last period.

The last term on the right-hand side accounts for the total deliveries from contracts traded in the previous period. This term can be written as the sum of two components, as illustrated in equation (4). For all contracts with $j > q_t$, agents default and deliver the value of the collateral, q_t; for contracts with $j \leqslant q_t$, they fulfill their promise and deliver j.

$$\int_{R_+} \theta_{j,t-1}^{a-1} min\{j, q_t\} = \int_{j>q_t} q_t \theta_{j,t-1}^{a-1} dj + \int_{j \leqslant q_t} j \theta_{j,t-1}^{a-1} dj. \tag{4}$$

2.4.2. Collateral Constraint

Borrowing through the sale of contracts requires collateral. Since each contract sold short requires one unit of housing as collateral, agents cannot sell more units of contracts than their current housing stock. Their choices must satisfy the following collateral constraint:

$$\int_{R_+} max\{-\theta_{j_t}^a, 0\} dj \leqslant h_t^a. \tag{5}$$

The integrand $max\{-\theta_{j,t}^a, 0\}$ serves to filter out the contracts on which agents make a net purchase, as these do not require collateral.

2.4.3. No Short-selling Constraint

Agents are prohibited from taking short positions in the housing stock,

$$h_t^a \geqslant 0. \tag{6}$$

2.5. The Credit Surface

In each period, the Credit Surface, determined by equilibrium prices, provides a pricing schedule for leverage. Specifically, it maps the LTV of each contract to its associated interest rate for all contracts in J_t. Figure 4 illustrates an example Credit Surface at a given time t. Let q_{t+1}^U and q_{t+1}^D represent the realizations of housing prices in states $z_{t+1} = U$ and $z_{t+1} = D$, respectively, with $q_{t+1}^U > q_{t+1}^D$. The points A and B represent the LTV and interest rates of two contracts promising q_{t+1}^D and q_{t+1}^U, respectively.

A description of the properties of the Credit Surface is in order. Proposition 1 establishes that both $\pi_{j,t}$ and $LTV_{j,t}$ is an increasing function of j, Proposition 2 characterizes the shape of the Credit Surface, and Proposition 3 asserts that the upper limit of $LTV_{j,t}$ is determined within the model and is always strictly below 100%.

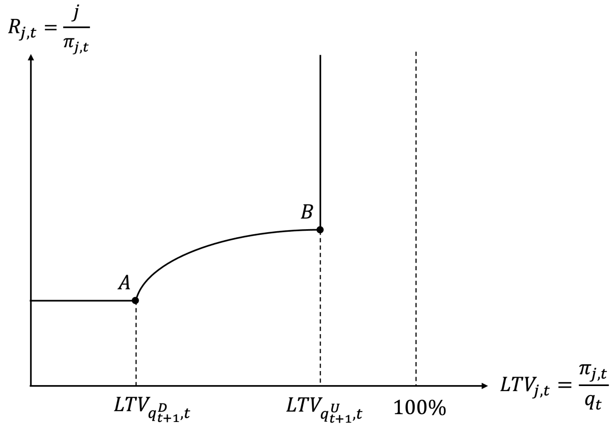

Figure 4: A Credit Surface for a Binomial Economy

Proposition 1. *Both $\pi_{j,t}$ and $LTV_{j,t}$ are non-negative and strictly increase with respect to j over the interval $[0, q^U_{t+1}]$. For $j \in (q^U_{t+1}, +\infty)$, $\pi_{j,t}$ remains constant at $\pi_{q^U_{t+1},t}$ and $LTV_{j,t}$ remains constant at $LTV_{q^U_{t+1},t}$.*

Proof. Let Du_x denote the derivative of the period utility function with respect to x, y^z_t denote the variable y in state z_t. Since households have strictly positive endowments and the period utility function is convex and satisfies $\lim_{c\to 0} Du_c(c^a_t, h^a_t) = \infty$ and $\lim_{h\to 0} Du_h(c^a_t, h^a_t) = \infty \ \forall a \in \hat{\mathbf{A}}$, it follows that $c^a_t > 0, h^a_t > 0$ for all $a \in \hat{\mathbf{A}}$, and that q_t is strictly positive.

Consider in equilibrium, an agent makes a net purchase of contract $j \in J_t$, i.e. $\theta^a_{j,t} > 0$. It must be that

$$\begin{aligned}\pi_{j,t} &= E_t\Big[\beta \frac{Du_c(c^{a+1}_{t+1}, h^{a+1}_{t+1})}{Du_c(c^a_t, h^a_t)} min\{j, q_{t+1}\}\Big] \\ &= \sum_{z_{t+1}\in \mathbf{Z}} \beta\gamma_{z_t,z_{t+1}} \frac{Du_c(c^{a+1,z_{t+1}}_{t+1}, h^{a+1,z_{t+1}}_{t+1})}{Du_c(c^a_t, h^a_t)} min\{j, q^{z_{t+1}}_{t+1}\}, \end{aligned} \tag{7}$$

where the right-hand side is the present value of the expected actual delivery of contract j, discounted by the agent's intertemporal marginal rate of substitution of consumption between time t and $t+1$. Because this agent is optimizing in equilibrium, increasing or decreasing $\theta^a_{j,t}$ by an infinitesimal amount must yield zero gain in utility. Since $Du_c(c, h) > 0$ for $c > 0$ and $h > 0$ and $\beta > 0$, the state-dependent intertemporal marginal utility between time t and $t+1$, $M^{a,z_{t+1}}_{t,t+1} \equiv \beta\gamma_{z_t,z_{t+1}} \frac{Du_c(c^{a+1,z_{t+1}}_{t+1}, h^{a+1,z_{t+1}}_{t+1})}{Du_c(c^a_t, h^a_t)}$ for all $z_{t+1} \in \mathbf{Z}$, must be strictly positive. At equilibrium prices and quantities, the second line of the equation shows that $\pi_{j,t}$ is a continuous piecewise linear function of j for $j \in \mathbb{R}_+$. The slope is $\sum_{z_{t+1}\in Z} M^{a,z_{t+1}}_{t,t+1} > 0$ for $j \in [0, q^D_{t+1}]$, and $M^{a,U}_{t,t+1} > 0$ for $j \in [q^D_{t+1}, q^U_{t+1}]$. Therefore, $\pi_{j,t}$ is non-negative and is strictly increasing in j for $j \in [0, q^U_{t+1}]$. For contracts with the promise $j \in (q^U_{t+1}, +\infty)$, the actual delivery is q^D_{t+1} in $z_{t+1} = D$, and q^U_{t+1} in $z_{t+1} = U$, which equals to the delivery of contract $j = q^U_{t+1}$. Hence, $\pi_{j,t} = \pi_{q^U_{t+1},t}$ for $j \in (q^U_{t+1}, +\infty)$. Since q_t is strictly positive, it follows that $LTV_{j,t}$ is non-negative, strictly increasing in j over the interval $[0, q^U_{t+1}]$, and remains constant at $LTV_{q^U_{t+1},t}$ when $j \in (q^U_{t+1}, +\infty)$.

□

Proposition 1 establishes that there is a strict mapping from j to $LTV_{j,t}$ over $[0, q^U_{t+1}]$ and to $R_{j,t}$ over (q^U_{t+1}, ∞), each contract j corresponds uniquely to a point on the Credit Surface, defined by the pair ($LTV_{j,t}$, $R_{j,t}$). As a result, the market for each contract j effectively functions as a market for leverage at a specific $LTV_{j,t}$. While $\pi_{j,t}$ clears the market for contract j, $R_{j,t}$ serves as the market-clearing price for leverage at $LTV_{j,t}$. Both borrowers and lenders choose their leverage based on the Credit Surface, making leverage an endogenous outcome of the interaction between supply and demand.

Proposition 2. *The relationship between $R_{j,t}$ and $LTV_{j,t}$ is characterized as follows:*

- *For $j \in [0, q^D_{t+1}]$, $R_{j,t}$ is constant over the interval $[0, LTV_{q^D_{t+1},t}]$.*
- *For $j \in (q^D_{t+1}, q^U_{t+1}]$, $R_{j,t}$ is increasing and concave in $LTV_{j,t}$ over $(LTV_{q^D_{t+1},t}, LTV_{q^U_{t+1},t}]$.*
- *For $j \in (q^U_{t+1}, \infty)$, as j goes to infinity, $LTV_{j,t}$ remains constant at $LTV_{q^U_{t+1},t}$ while $R_{j,t}$ goes to infinity.*

The Credit Surface is flat until point A, starts rising until point B, after which the surface becomes a vertical line.

Appendix A outlines the proof. For contracts with a promise smaller than q^D_{t+1}, borrowers will not default, hence, such contracts are charged with a uniform risk-free rate of interest. For contracts with a promise between q^D_{t+1} and q^U_{t+1}, borrowers default in state $z_{t+1} = D$ but not in $z_{t+1} = U$. The higher the j, the larger the amount of debt borrowers will default on in the D state; therefore, the interest rate is higher for contracts with a larger j in the interval $[q^D_{t+1}, q^U_{t+1}]$. Lastly, when contracts promise an amount exceeding the housing prices in both states, i.e., $j > q^U_{t+1}$, lenders anticipate that borrowers will default in both states and will not pay more than the price of contract $j = q^U_{t+1}$. The interest rate goes to infinity as j goes to infinity and $\pi_{j,t}$ remains constant.

Proposition 2 establishes that, at a given point in time, the interest rate increases with LTV. This property aligns with the findings of Geanakoplos and Rappoport (2019), who estimate two Credit Surfaces in 2007Q2 and 2008Q4 and demonstrate that the interest rate increases with both LTV and credit scores. In addition, $R_{j,t}$ is concave in $LTV_{j,t}$ when $j \in (q^D_{t+1}, q^U_{t+1}]$. The concavity implies that as soon as j exceeds q^D_{t+1}, the contract transitions from risk-free to one carrying default risk, leading to a steep increase in the interest rate. Borrowers default in $z_{t+1} = D$, providing small repayment precisely when lenders experience the highest marginal utility. Consequently, lenders become highly sensitive to small increases in leverage and demand a sharp increase in the risk premium to compensate for the perceived default risk.

Proposition 3. *The upper limit of $LTV_{j,t}$ is endogenous and is strictly smaller than* 100%.

Proof. Suppose in equilibrium, there exists a contract j such that $j > q^U_{t+1} > q^D_{t+1}$, and $LTV_{j,t} \geqslant 100\%$. By the definition of $LTV_{j,t}$, this is equivalent to $\pi_{j,t} \geqslant q_t$. Since j exceeds housing prices in both states, the actual delivery of this contract, $min\{j, q_{t+1}\}$, equals to the realized housing price q_{t+1}. In equilibrium, the following must hold for agents whose collateral constraint is not binding:

$$\pi_{j,t} = E_t\Big[\beta \frac{Du_c(c^{a+1}_{t+1}, h^{a+1}_{t+1})}{Du_c(c^a_t, h^a_t)} q_{t+1}\Big],$$

$$q_t = \frac{Du_h(c^a_t, h^a_t)}{Du_c(c^a_t, h^a_t)} + E_t\Big[\beta \frac{Du_c(c^{a+1}_{t+1}, h^{a+1}_{t+1})}{Du_c(c^a_t, h^a_t)} q_{t+1}\Big].$$

The second equation says the housing price q_t equals to sum of the immediate utility from housing services and the expected present value of the future housing price q_{t+1}. Take the difference between q_t and $\pi_{j,t}$, we get the following equation, which shows that the downpayment associated with this contract equals to the utility derived from a single period of housing consumption:

$$q_t - \pi_{j,t} = \frac{Du_h(c_t^a, h_t^a)}{Du_c(c_t^a, h_t^a)}.$$

Given that all agents $a \in \hat{\mathbf{A}}$ consumes housing, the ratio $\frac{Du_h(c_t^a, h_t^a)}{Du_c(c_t^a, h_t^a)}$ must be strictly greater than zero, which implies $q_t > \pi_{j,t}$. This result contradicts the initial assumption $\pi_{j,t} \geqslant q_t$, therefore, $LTV_{j,t} < 100\%$ for all contracts such that $j > q_{t+1}^U > q_{t+1}^D$. Since the prices of all other contracts in J_t are smaller or equal to $\pi_{j,t}$, it follows that no contract in J_t can have a price $\pi_{j,t}$ that is greater than or equal to q_t. Hence, the upper limit of $LTV_{j,t}$ is strictly less than 100%[2]. □

Proposition 3 asserts that agents must make a strictly positive downpayment upfront to acquire housing. Housing assets provide both immediate utility from living in the house and serve as a means of saving. Therefore, even in the case where agents can borrow up to the point where their repayment equals the realized housing price q_{t+1}, they are still obliged to pay for the immediate utility derived from a single period of occupancy. This requirement underscores a key distinction between non-financial assets, like housing, which yield utility to their owners, and financial assets, which do not directly provide utility. If households did not have utility over housing, the downpayment $q_t - \pi_{j,t}$ would have been zero, and $LTV_{j,t}$ could have reached 100%.

2.6. Collateral Equilibrium

Definition 1. *A collateral equilibrium of this economy is a collection of agents' allocations of consumption and housing, their portfolio holdings of financial contracts, as well as the prices of housing and financial contracts for all t*

$$\left((c_t^a, h_t^a, (\theta_{j,t}^a)_{j \in J_t})_{a \in \mathbf{A}}; q_t, (\pi_{j,t})_{j \in J_t}\right)_{t=0}^{\infty} \tag{8}$$

such that

1. *Given* $(q_t, (\pi_{j,t})_{j \in J_t})_{t=0}^{\infty}$, *the choices* $((c_t^a, h_t^a, (\theta_{j,t}^a)_{j \in J_t})_{a \in \mathbf{A}})_{t=0}^{\infty}$ *maximize* (1), *subject to constraints* (3), (5) *and* (6).
2. *Markets for the consumption good, housing, and all financial contracts in* J_t *clear at each time t:*

$$\sum_{a=1}^{A} c_t^a = \bar{e}(z_t),$$
$$\sum_{a=1}^{A} h_t^a = H,$$
$$\sum_{a=1}^{A} \theta_{j,t}^a = 0, \forall j \in J_t.$$

[2] Nilayamgode (2023) also proves that when non-financial assets serve as collateral, and households who hold housing assets have positive level of consumption, LTV can never go to 100%.

3. Quantitative Analysis

3.1. Parameterization

Table 2: Parameters

Parameter	Value	Interpretation
Uncertainty		
γ_{UD}	0.14	$\gamma_U = 85\%$
γ_{DU}	0.80	$d_U = 7, d_D = 1$
Preferences		
β	0.83	Annual discount rate 0.94
ρ	4.5	Risk-aversion coefficient
α	0.115	Housing share
Endowments		
$\{e^a(U)\}_{a=1}^{20}$		Wage income (2007 SCF)
$\{e^a(D)\}_{a=1}^{20}$		Wage income (2009 SCF)

Notes: γ_U denotes the unconditional frequency of the U state. d_U and d_D denote the durations of the U and D states, respectively.

3.1.1. Demographics

Households live for twenty periods, $A = 20$. In the model, one period is equivalent to three years in real life. Households start their economic life at real age 21 (model age $a = 1$), and live until real age 81 (model age $A = 20$).

3.1.2. Uncertainty

Transition probabilities in Γ are chosen to match two features exhibited by the data: (i) the frequency of a Great Recession-like state is around 15%; (ii) the average duration of the U state is around seven times the duration of D state. I collect US GDP per capita data from two sources: FRED (after 1947) and Maddison Project 2020 (before 1947). Then I use Hodrick-Prescott (HP) filter with a smoothing parameter 1600 to get the trend and cycle. In 2009, the US GDP per capita is 2.46% below trend. I then define the US economy as being in the recession if the GDP per capita in that year falls below 2.4% of the trend. Given this threshold, the US economy has historically been in the recession state 14.6% of the time. The transition probabilities are set such that the economy, on average, spends 85% of the time in a normal state.

3.1.3. Preferences

I set the discount factor β to 0.83, which corresponds to an annual discount factor of 0.94. The coefficient of risk aversion ρ and the weight on housing α are calibrated to 4.5 and 0.115, respectively. This calibration aims to replicate a 10-percentage-point difference in the average LTV for first-time homebuyers between states U and D.

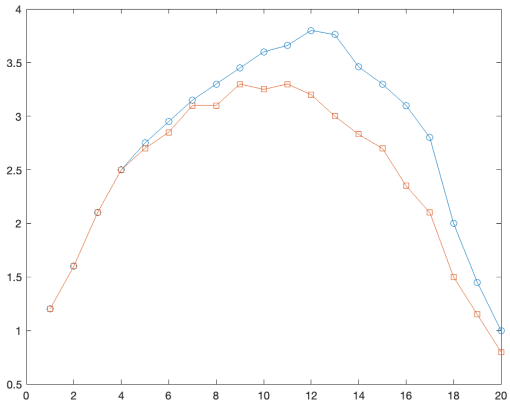

Figure 5: Age Profiles of Endowments

3.1.4. Endowments

The total supply of housing is set to $H = 20$. Figure 5 presents the life cycle age profiles of endowment, with the line with circle markers representing $\{e^a(U)\}_{a=1}^{20}$ and square representing $\{e^a(D)\}_{a=1}^{20}$. The Survey of Consumer Finances (SCF) provides detailed data on household income and wealth in the U.S. A special panel survey was conducted between 2007 and 2009 to study the aftermath of the Great Recession. In the panel survey, households who had responded to the 2007 survey were invited to participate in a follow-up survey in 2009. The income data collected from the same interviewees in these two years serve as the source for constructing endowment age profiles across different states in the model. To construct the endowment profiles, I divide households from age 21 to 81 into 20 age groups and take the average wage and salary income within each age group, using SCF sample weights. The resulting profiles exhibit many spikes. To smooth them, I apply a five-period moving average and make discretionary adjustments. The 2007 data represent the endowments in the U state $\{e^a(U)\}_{a=1}^{20}$, while the 2009 data represent the D state $\{e^a(D)\}_{a=1}^{20}$.

3.2. Numerical Solutions

Finding a collateral equilibrium for this economy presents two main challenges: one conceptual and the other computational. Conceptually, it is challenging to derive the optimality conditions for households given that there are an infinite number of contracts in J_t. The presence of the term $\int_{j\in\mathbb{R}_+} \pi_{j,t}\theta^a_{j,t} dj$ in the budget constraint complicates the problem, as it is impractical to derive an infinite number of Euler equations regarding each $\theta^a_{j,t}$.

Geanakoplos (1997) argues that only a limited set of contracts will be traded in equilibrium due to the scarcity of collateral; however, it remains unclear which specific contract(s) will have both non-zero supply and demand in equilibrium. The No-Default Theorem, formalized by Fostel and Geanakoplos (2015), states that in a binomial economy with financial assets as collateral, the contract that makes the maximal promise without defaulting will be the one actively traded in equilibrium. However, the theorem does not apply to models in which non-financial assets, like housing, serve as collateral. Unlike financial assets, non-financial assets provide utility to agents. Consequently, the demand for such assets, and debt contracts tied to them, depends not only on agents' motivations to transfer wealth across time and states, but also on the intrinsic utility derived from the collateral assets.

Geanakoplos (1997) provides an example with two types of agents and housing as collateral, concluding that, in equilibrium, only the contract promising the highest possible realization of housing prices in the next period would be traded. However, in a more complex setting involving 20 types (generations) of agents, such an equilibrium does not exist. Assuming that there exists an equilibrium in which only the contract that promises q_{t+1}^U is traded in each period, I solve for variables that satisfy all the Karush-Kuhn-Tucker (KKT) and market clearing conditions under this assumption. In doing so, I find that agents making net purchases of the contract do not agree on the prices of all other contracts in J_t. This disagreement implies that prices have not yet reached an equilibrium, contradicting the initial assumption.

Numerically approximating an equilibrium is challenging in two ways. Firstly, the presence of 20 types of agents who engage in trading multiple assets inevitably leads to a large number of state variables. Secondly, the model features occasionally binding collateral constraints, which induces non-linearity in the policy functions, making them difficult to approximate accurately with linear methods. Following Azinovic, Gaegauf, and Scheidegger (2022), I used a neural network with two hidden layers to directly approximate the the policy functions, addressing the second challenge. They suggest that a neural network with stochastic simulations has the advantage of approximating potentially non-linear policy functions only in the ergodic endogenous state space, even when dealing with a high-dimensional state vector.

To search for a collateral equilibrium, I begin by guessing an equilibrium regime and verifying whether agents would deviate from it. Assuming an equilibrium exists in which only a finite number of contracts are actively traded with non-zero supply and demand in each period, let $\hat{J}_t$ be the set of traded contracts, and N_j be its cardinality. Under this assumption, contracts not included in $\hat{J}_t$ have zero supply and demand, i.e., $(\theta_{j,t}^a)_{a\in\hat{\mathbf{A}},j\notin\hat{J}_t} = 0$ for all t. This simplifies the budget constraint, and reduces the equilibrium conditions to a finite system of KKT and market clearing conditions.

To numerically approximate the candidate equilibrium, I follow Spear (1988) and Krueger and Kubler (2004) to define the Functional Rational Expectations Equilibrium (FREE), as described below. I then use a neural network to approximate the policy and pricing functions so that the equilibrium conditions are satisfied within an acceptable tolerance. Upon solving for the functions, I examine whether agents have incentives to deviate and trade contracts not included in $\hat{J}_t$. Specifically, I verify whether the third and fourth conditions in the definition of FREE are satisfied. If they are, I stop the process. If not, I adjust the set of actively traded contracts and repeat the steps until all the conditions described in the definition of a FREE are met. Appendix B outlines the algorithm.

Definition 2. *A FREE, in which only contracts in $\hat{J}_t$ are traded with non-zero demand and supply, consists of $(c_t^a, h_t^a, \boldsymbol{\theta_t^a}, \boldsymbol{\mu_t^a})_{a\in\mathbf{A}}$, and $(q_t, \boldsymbol{\pi_t})$, where:*

- $\boldsymbol{\theta_t^a} = (\theta_{j,t}^a)_{j\in\hat{J}_t}$ *denotes agents' portfolio holdings of contracts in $\hat{J}_t$,*
- $\boldsymbol{\mu_t^a} = (\mu_t^{a,S})_{S\subseteq\hat{J}_t, S\neq\emptyset}$ *denotes the Lagrangian multipliers for the collateral constraints corresponding to all the non-empty subsets S of $\hat{J}_t$,*
- $\boldsymbol{\pi_t} = (\pi_{j,t})_{j\in\hat{J}_t}$ *denotes the prices of contracts in $\hat{J}_t$,*

as time-invariant policy functions and pricing functions of $\boldsymbol{x_t}$, where

$$\boldsymbol{x_t} = (z_t, (h_{t-1}^a, \boldsymbol{\theta_{t-1}^a}, c_{t-1}^a, e_t^a)_{a\in\boldsymbol{A}}, (\gamma_{z_t,U}, \gamma_{z_t,D})) \in \mathbf{Z} \times [0, H]^A \times [-H, H]^{N_j A} \times \mathbb{R}_{++}^A \times \mathbb{R}_{++}^2$$

is the state vector[3]*, such that the following conditions are satisfied.*

1. *The following complementary slackness conditions hold for $a \in \hat{\mathbf{A}}$, $j \in \hat{J}_t$, and for any non-empty subset $S \subseteq \hat{J}_t$*[4]*:*

$$\mu_t^{a,S}(\sum_{j\in S} -\theta_{j,t}^a - h_t^a) = 0,$$
$$\sum_{j\in S} -\theta_{j,t}^a - h_t^a \leqslant 0,$$
$$\mu_t^{a,S} \geqslant 0.$$

2. *Define $\mu_{j,t}$ as the sum of Lagrangian multipliers $\mu_t^{a,S}$ for all subsets S that include contract j: $\mu_{j,t}^a \equiv \sum_{S\subseteq\hat{J}_t:j\in S} \mu_t^{a,S}$. Define $\mu_{h,t}^a$ as the sum of all Lagrangian multipliers $\mu_t^{a,S}$: $\mu_{h,t} \equiv \sum_{S\subseteq\hat{J}_t} \mu_t^{a,S}$. The following first-order conditions hold for $a \in \hat{\mathbf{A}}$, $j \in \hat{J}_t$:*

$$q_t = \frac{Du_h(c_t^a, h_t^a)}{Du_c(c_t^a, h_t^a)} + E_t\Big[\beta \frac{Du_c(c_{t+1}^{a+1}, h_{t+1}^{a+1})}{Du_c(c_t^a, h_t^a)} q_{t+1}\Big] + \frac{\mu_{h,t}^a}{Du_c(c_t^a, h_t^a)}, \tag{9}$$

$$\pi_{j,t} = E_t\Big[\beta \frac{Du_c(c_{t+1}^{a+1}, h_{t+1}^{a+1})}{Du_c(c_t^a, h_t^a)} min\{j, q_{t+1}\}\Big] + \frac{\mu_{j,t}^a}{Du_c(c_t^a, h_t^a)}. \tag{10}$$

3. *The state-dependent intertemporal marginal utilities between t and $t+1$ are equal among unconstrained agents with $\boldsymbol{\mu_t^a} = 0$.*[5] *Let $\bar{\mathbf{A}}$ be the set of unconstrained agents, then for all $a, b \in \bar{\mathbf{A}}$, $z_{t+1} \in \mathbf{Z}$:*

$$M_{t,t+1}^{a,z_{t+1}} = \beta\gamma_{z_t,z_{t+1}} \frac{Du_c(c_{t+1}^{a+1,z_{t+1}}, h_{t+1}^{a+1,z_{t+1}})}{Du_c(c_t^a, h_t^a)} = \beta\gamma_{z_t,z_{t+1}} \frac{Du_c(c_{t+1}^{b+1,z_{t+1}}, h_{t+1}^{b+1,z_{t+1}})}{Du_c(c_t^b, h_t^b)} = M_{t,t+1}^{b,z_{t+1}}.$$

[3] The vector of the minimal state variables is $(z_t, (h_{t-1}^a, \boldsymbol{\theta_{t-1}^a})_{a\in\boldsymbol{A}})$. Azinovic, Gaegauf, and Scheidegger (2022) suggest that augmenting the input vector with additional information can help stabilize the training process. Consistent with their findings, I observe that including the endowment vector and transition probabilities in the input vector reduces the training time. Furthermore, in theory, the intertemporal marginal utility of consumption for unconstrained borrowers and lenders is equal in equilibrium, implying that they share the same consumption growth rate, and that the past consumption could help predict current consumption. Motivated by this result, I add the vector of past consumption to the input and find that it further decreases the training time.

[4] Under the assumption that only contracts $j \in \hat{J}_t$ are traded with non-zero supply and demand, the pertinent collateral constraint simplifies to $\sum_{j\in\hat{J}_t} max\{-\theta_{j,t}^a, 0\} \leqslant h_t^a$. This condition is equivalent to imposing that the sum of the negative positions for all contracts in any non-empty subset of $\hat{J}_t$ does not exceed the housing stock h_t^a. Mathematically, this yields $\sum_{n=1}^{N_j} \binom{N_j}{n}$ distinct inequalities:

$$\sum_{j\in S} -\theta_{j,t}^a \leqslant h_t^a \quad \text{for all non-empty subsets } S \subseteq \hat{J}_t.$$

[5] In equilibrium, unconstrained agents ($\boldsymbol{\mu_t^a} = 0$) must agree on the prices of all contracts in J_t. Suppose that two unconstrained agents, a and b, have different state-dependent intertemporal marginal utilities, implying different present values for the same contract. Without loss of generality, assume agent a values a contract higher than agent b, agent a would have an incentive to buy more of it, while b would hold less. This incentive to trade indicates that the market is not yet in equilibrium, contradicting the initial assumption. Therefore, unconstrained agents must agree on $(\pi_{j,t})_{j\in\boldsymbol{J_t}}$.

$(\pi_{j,t})_{j \notin \hat{J}_t}$ are given by the present value of contracts for unconstrained agents:

$$\pi_{j,t} = E_t \Big[\beta \frac{Du_c(c^{a+1}_{t+1}, h^{a+1}_{t+1})}{Du_c(c^a_t, h^a_t)} min\{j, q_{t+1}\} \Big], where\ a \in \bar{\boldsymbol{A}}.$$

4. *Constrained agents have one and only one strictly positive Lagrangian multiplier, $\mu^a_{j,t}$, among the set $(\mu^a_{j,t})_{j \in \hat{J}_t}$: if $\mu^a_{j,t} > 0$ for some contract j in $\hat{J}_t$, then $\mu^a_{i,t} = 0$ for all other contracts $i \neq j$ in $\hat{J}_t$, and contract j has the highest liquidity value among all contracts in J_t. i.e., $LV_{j,t} \geqslant LV_{i,t}$ $\forall i \in J_t$, where $i \neq j$. Liquidity values are given by equation* (2).
5. *The budget constraint is satisfied for all $a \in \mathbf{A}$:*

$$c^a_t + q_t h^a_t + \sum_{j \in \hat{J}_t} \theta^a_{j,t} \pi_{j,t} = e^a_t + q_t h^{a-1}_{t-1} + \sum_{j \in \hat{J}_t} \theta^{a-1}_{j,t-1} min\{j, q_t\}.$$

6. *All markets clear:*

$$\sum_{a=1}^{A} c^a_t = \bar{e}(z_t),$$

$$\sum_{a=1}^{A} h^a_t = H,$$

$$\sum_{a=1}^{A} \boldsymbol{\theta}^a_t = \mathbf{0}.$$

The fourth condition in the definition asserts that constrained agents do not have incentives to trade contracts with lower liquidity values. Liquidity value guides borrowers in selecting among various contracts written against the same collateral, as it quantifies the trade-off between the immediate liquidity provided by a contract against its future obligations. By rearranging the Euler equation (10) and using the definition of $LV^a_{j,t}$, we can see that the liquidity value of contract j is equal to $\mu^a_{j,t}$ divided by agent's current marginal utility of consumption:

$$LV^a_{j,t} = \frac{\mu^a_{j,t}}{Du_c(c^a_t, h^a_t)}.$$

As previously defined, $\mu^a_{j,t}$ is the sum of Lagrangian multipliers corresponding to the collateral constraints for every subset of contracts that contains j, representing the additional gain in utility if agents could marginally increase sales of contract j by relaxing all the relevant collateral constraints. To translate the abstract gain in utility reflected by $\mu^a_{j,t}$ into a more tangible metric, divide $\mu^a_{j,t}$ by $Du_c(c^a_t, h^a_t)$, yielding the liquidity value $LV^a_{j,t}$, which represents the marginal benefit in real consumption good. Holding the marginal utility of consumption fixed at the equilibrium level, comparing liquidity value across j is equivalent to comparing $\mu^a_{j,t}$.

$\mu^a_{j,t} > \mu^a_{k,t}$ indicates that contract j offers a greater marginal benefit per unit of collateral than contract k. If $\mu^a_{j,t} > \mu^a_{k,t}$ for all $k \in J_t$ with $k \neq j$, agents will not waste collateral on any contracts that provide less marginal benefit than contract j. Instead, they increase the issuance of contract j until the collateral constraint binds: $-\theta^a_{j,t} = h^a_t$. If $\mu^a_{j,t} = \mu^a_{k,t} > \mu^a_{l,t}$, for all contracts k in some subset $S_k \subset J_t$, and all contracts l in $J_t \backslash \{j, S_k\}$, then agents are indifferent between issuing contract j and any contract in S_k, and choose $\mu^a_{l,t} = 0$, for all l in $J_t \backslash \{j, S_k\}$. In this case, we can let $-\theta^a_{j,t} = h^a_t$ to simplify the computation. If $\mu^a_{j,t} = 0$ for all $j \in J_t$, agents are indifferent between trading any contracts in J_t without violating the collateral constraint.

A FREE satisfies all the equilibrium conditions of a collateral equilibrium, including agents' optimality conditions and market clearing conditions. Therefore, a FREE effectively induces a collateral equilibrium.

4. Results

The procedure of finding a collateral equilibrium stops when the set of actively traded contracts $\hat{J}_t$ contains only the risk-less contract that promises q^D_{t+1} and the risky contract that promises q^U_{t+1}. Appendix C gives detailed description of the accuracy of the solutions. Let $j_{A,t}$ and $j_{B,t}$ denote the risk-less contract and the risky contract respectively. Their corresponding prices are denoted by $\pi_{A,t}$ and $\pi_{B,t}$. The quantities of these contracts held by an agent of age a at time t are denoted by $\theta^a_{A,t}$ and $\theta^a_{B,t}$. In this equilibrium, $\theta^a_{j,t} = 0$ for all $a \in \mathbf{A}$, $j \in J_t \backslash \hat{J}_t$. I simulate the economy for a total of $T = 500,000$ periods. All subsequent analyses are based on the 500,000-period simulation.

4.1. Life Cycle Implications

Figure 6 presents the mean life cycle profiles of portfolio holdings in risk-less contract j_A (θ_A), risky contract j_B (θ_B), housing assets, and the average amount of housing pledged as collateral. The collateral profile (marked with Diamonds) consistently lies below the housing profile (marked with Crosses), due to the collateral constraints. This figure suggests that on average, agents within the model transition through four distinct life stages.

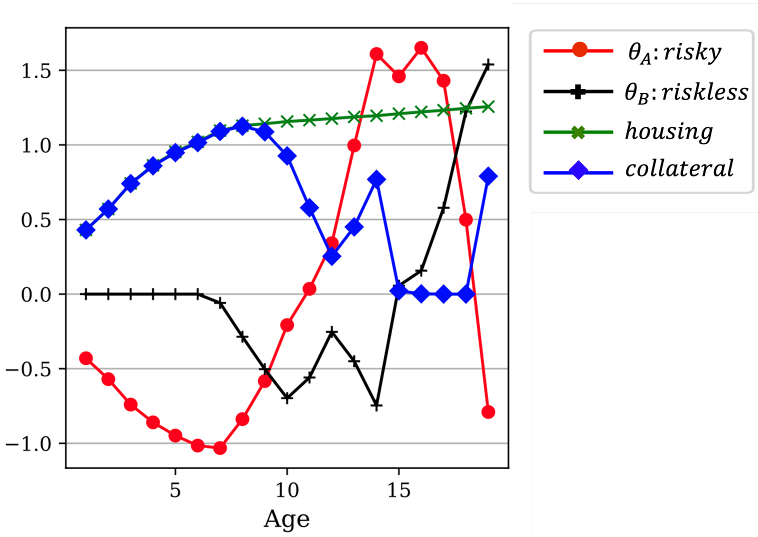

Figure 6: Average Life Cycle Portfolio Holdings

First stage: constrained borrowers. In the initial stage, agents aged between 1 and 6 choose the portfolio holdings where $\theta^a_{A,t} = 0$, and $-\theta^a_{B,t} = h^a_t$, which indicates that they simultaneously accumulate housing h^a_t and pledge all of these housing assets as collateral to borrow through exclusively selling the risky contract $j_{B,t}$. Because their collateral constraints are binding, they are classified as constrained borrowers.

Anticipating a substantial increase in endowments in the $z_{t+1} = U$ state and a lesser rise in the $z_{t+1} = D$ state, these young agents have two consumption-smoothing incentives. First, they would like to transfer wealth from the $z_{t+1} = U$ state, where they expect to be wealthier, to the present by issuing debt contracts secured by their housing. Second, they would like to make small payments in the $z_{t+1} = D$ state in which they feel poor while borrowing as much as possible in the current period. Among all contracts in J_t, the risky contract $j_{B,t}$ allows for the most current borrowing without incurring a high interest rate on the vertical segment of the Credit Surface. Additionally, agents can default on this contract and only deliver the housing collateral with low value q^D_{t+1} in the D state. Consequently, contract $j_{B,t}$ allows for the greatest degree of consumption smoothing across time and states, these agents will find it most advantageous to exclusively issue this particular contract.

Figure 7 examines the liquidity values. In this figure, there are 500,000 curves, each depicting the liquidity value $LV^{a=1}_{j,t}$ for the youngest agent against the promise j across all simulated periods. Each curve corresponds to a distinct time period. All curves are color-coded: red for periods when $z_t = U$ and blue for periods when $z_t = D$. Note that all Up state curves are above the Down state ones, implying that the liquidity value for all contracts are always higher in normal states than crisis states. In each period, the youngest agents find contract $j_{B,t}$ provides the highest liquidity value. Consequently, they use their entire housing collateral to issue this contract, such that $-\theta^{a=1}_{B,t} = h^{a=1}_t$, and $\theta^{a=1}_{j,t} = 0$ for all $j \in J_t \backslash \{j_{B,t}\}$.

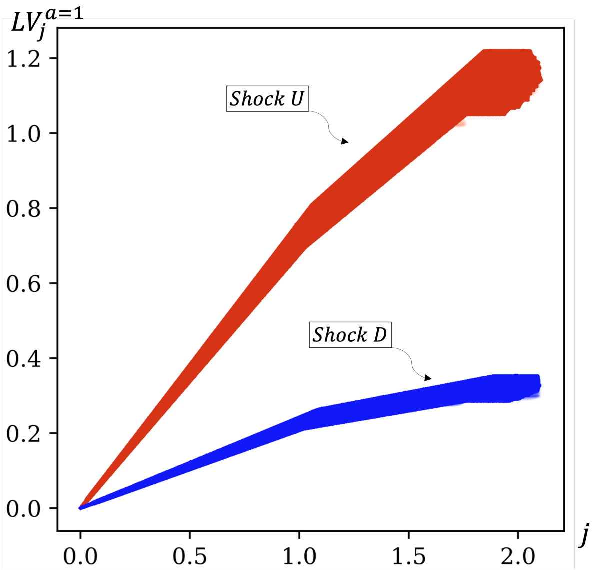

Figure 7: Liquidity Value for $a = 1$ Agents

Second stage: unconstrained borrowers. As agents age into the 7 to 10 bracket, they continue to expect higher future endowments and have the motive to transfer future wealth to the present. However, as the average growth rate of their endowment declines compared to that in the initial life stage, their motivation for consumption smoothing declines correspondingly. They start to shift their portfolio holdings away from exclusively issuing the risky contract $j_{B,t}$ to issuing both the risk-less contract $j_{A,t}$ and the risky contract $j_{B,t}$. Their collateral constraints do not bind throughout this stage.

In equilibrium, these agents find the marginal benefit of a marginal increase in the borrowing through issuing any contracts in J_t to be zero, i.e., $\mu^a_{j,t} = LV^a_{j,t} = 0$ for all $j \in J_t$, $t = 0, ..., T$. They are less liquidity constrained compared to agents in the first life stage, their desired amount of borrowing can be acquired by pledging only a fraction of their housing stock as collateral. As they are indifferent between issuing any contracts in J_t, their portfolio holdings in contract $j_{A,t}$ and $j_{B,t}$ are pinned down in equilibrium by the market clearing conditions.

Third stage: unconstrained borrowers and lenders. As agents progress into the 11 to 14 age bracket, they anticipate their future endowments to be on a downward trajectory. This expected change is reflected in Figure 6, where the line of the use of collateral (Diamond marker) exhibits a mild increase at age 12. This uptick suggest that agents, upon experiencing the initial decline in their endowment growth at age 12, use greater leverage in accumulating housing assets than when they were aged 11. During this stage, while utilizing the housing collateral to borrow with contract $j_{A,t}$, agents start to transfer current wealth to the future by investing in the risky contract $j_{B,t}$.

Last stage: lenders. In the final life stage, agents, driven by a strong saving motive and a motive to smooth consumption across states, stop borrowing and begin to diversify their investments. They acquire housing assets with zero leverage, and diversify their saving by lending to younger agents through buying both the risky contract $j_{B,t}$, and the risk-less contract $j_{A,t}$. Let $\tilde{R}_{A,t}$ and $\tilde{R}_{B,t}$ be the real return on contract $j_{A,t}$ and $j_{B,t}$ respectively, they are given by:

$$\tilde{R}_{A,t} = \frac{min\{q^D_{t+1}, q_{t+1}\}}{\pi_{A,t}},$$
$$\tilde{R}_{B,t} = \frac{min\{q^U_{t+1}, q_{t+1}\}}{\pi_{B,t}}.$$

Rearrange agents' Euler equations regarding contract $j_{A,t}$ and $j_{B,t}$ and use the definitions of the real return on both assets, the condition that pins down their portfolio holdings is given by:

$$E_t\Big[M^a_{t,t+1}(\tilde{R}_{B,t} - \tilde{R}_{A,t})\Big] = 0,$$

where $M^a_{t,t+1}$ is the stochastic discount factor and $\tilde{R}_{B,t} - \tilde{R}_{A,t}$ represents the excess return of holding the risky contract over the risk-less contract. This condition says the expected excess return is zero after adjusting for risk by lenders' state-dependent marginal utilities of consumption. $M^a_{t,t+1}$ incorporates lenders' preferences for risk by giving more weight to the excess return in $z_{t+1} = D$ state and less to that in the good state $z_{t+1} = U$.

To illustrate the life cycle profile of leverage, Figure 8 presents the mean DLTV for all age groups throughout the simulated periods. On average, DLTV progressively decreases with age and eventually reaches zero, with the exception – an uptick at age 12, where agents begin to face declines in their endowments.

4.2. Leverage Cycle Implications

This model features infrequent disaster events. Within this framework, lenders collectively provide pricing schedules for leverage through the Credit Surface, which they adjust endogenously in response to fundamental shocks. The model generates leverage cycles characterized

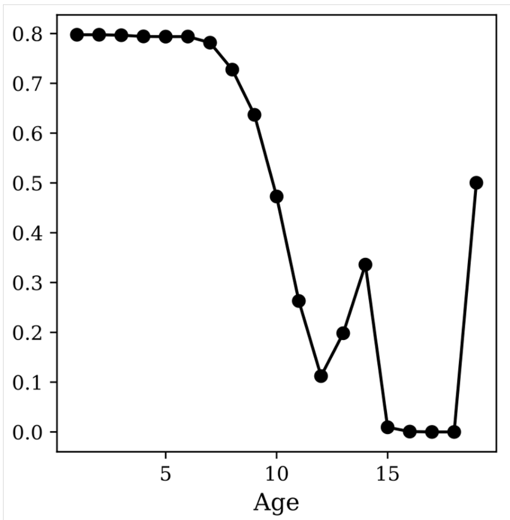

Figure 8: Average DLTV

by the following dynamics: household leverage is positively correlated with housing prices, and amplifies the impact of fundamental shocks on housing prices; mortgage rates and spreads move in the opposite direction, remaining low during normal times while increasing drastically in downturns.

Figure 9 illustrates the Credit Surfaces for each of the 500,000 simulated periods. Table 3 presents the average prices and leverage in states U and D, as well as the differences (Δ), which represent either the absolute change ($D-U$) or the percentage drop ($(D-U)/U$) in the average value of each variable in state D compared to state U. The surfaces in Figure 9 are also color-coded to indicate the aggregate state: red represents state U and blue represents state D. The average Credit Surfaces for states U and D are in darker shades of red and blue, respectively. This figure reveals distinct patterns in the Credit Surface based on the aggregate state. Credit Surfaces associated with D states consistently sit above those for U states, with a tendency to start rising at a lower level of LTV and at a faster speed, becoming vertical at comparatively lower LTV levels. When the economy is in a downturn, the Credit Surface rises and becomes steeper as lenders tighten credit.

The leverage corresponding to the two contracts actively traded in equilibrium differs markedly between states. On average, households can leverage as high as 53% at a risk-free interest rate with the risk-less contract $j_{A,t}$ during normal times, compared to just 48% in downturns. Beyond the risk-free leverage threshold, the Credit Surface rises more sharply in downturns than in normal times, reflecting lenders' higher marginal utility of immediate consumption and their demand for greater excess returns to compensate for forgoing current consumption. The LTV associated with the risky contract $j_{B,t}$ endogenously sets the leverage cap. Using contract $j_{B,t}$, households can leverage 81% on average in normal times, but only 71% in downturns.

In this model, housing plays three critical roles, each influencing the housing prices. First, housing provides immediate utility. As outlined in equation (9), the initial component of housing prices is rent: agents must pay a positive amount of $\frac{Du_h(c_t^a,h_t^a)}{Du_c(c_t^a,h_t^a)}$ to utilize the property. Second, as housing assets are perfectly durable, they serve as a means of savings. Their value today is tied

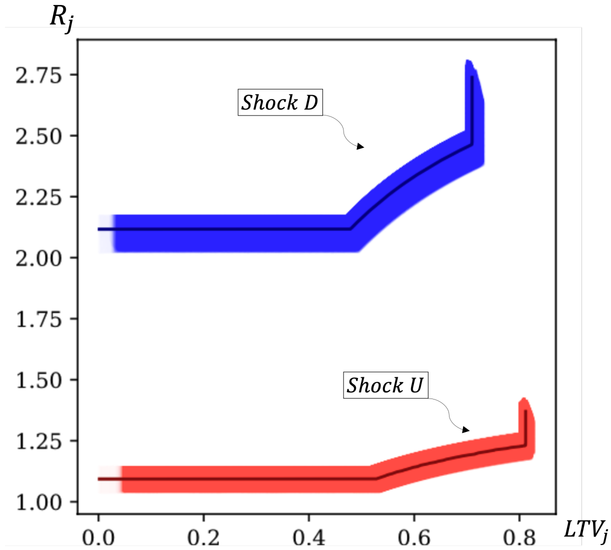

Figure 9: Credit Surfaces

to the present value of future housing prices, as reflected in the second component of housing prices: $E_t\left[\beta\frac{Du_c(c^{a+1}_{t+1},h^{a+1}_{t+1})}{Du_c(c^a_t,h^a_t)}q_{t+1}\right]$. The third component, $\frac{\mu^a_{h,t}}{Du_c(c^a_t,h^a_t)}$ represents the value of housing derived specifically from its role as collateral, which depends on the loan amount the housing can secure. Fostel and Geanakoplos (2008) categorize the sum of the first two components as the fundamental value and the third as the collateral value.

Housing prices significantly decline when the economy experiences negative endowment shocks. The average housing price in state D is 42% lower than in state U. This decline stems from the simultaneous fall in all three components of housing prices, driven by two interconnected channels, each affecting a distinct type of marginal buyer: the reduced wealth of middle-aged marginal buyers and the diminished leverage capacity of constrained young borrowers.

First, households face lower endowments in downturns, and lenders are hit particularly hard

Table 3: Summary Statistics of the Collateral Equilibrium

	D	U	Δ
q: Housing Price	1.03	1.78	-42.13%
π_A: Risk-less Contract Price	0.49	0.94	-47%
π_B: Risky Contract Price	0.73	1.44	-49%
R_A: Risk-less Interest Rate	2.12	1.09	+94%
R_B: Risky Interest Rate	2.46	1.23	+100%
$R_B - R_A$: Mortgage Spread	0.34	0.14	+142%
LTV_A: Risk-less LTV	47.7%	52.57%	-4.87 *pp*
LTV_B: Risky LTV	71.09%	81.16%	-10.1 *pp*

Notes: pp stands for percentage points.

because they save significantly in housing and debt contracts, both of which suffer substantial losses. When young borrowers default, instead of receiving the repayment of $j = q_{t+1}^{U}$, lenders can only liquidate the collateral at a lower housing price, q_{t+1}^{D}. Additionally, their housing wealth declines as housing prices fall during downturns. With diminished wealth, all households, especially lenders, are unable to support high rent payments for housing as they could in normal states. Households experience a sharp increase in the marginal utility of consumption, which leads to a heavier discounting of the future and reduces the present value of future housing prices.

Second, and most importantly, leverage is significantly reduced because the capacity of housing assets to facilitate loans declines sharply. The average prices of risk-less and risky contracts, π_A and π_B, are 47% and 49% lower in state D compared to state U, respectively, exceeding the decline in housing prices (42%). For constrained borrowers, collateral value is a major component of housing prices, while it is zero for unconstrained households and lenders. As a result, this channel impacts constrained borrowers the hardest.

To examine how the collateral value for each constrained borrower responds to a one-time D shock, I simulate each economy 100,000 times over 20 periods. In each simulation, the economy starts in the U state at time 0, experiences a D shock at time 1, and returns to the U state thereafter. For each time period, I calculate the deviation of each variable relative to its value at time 0 and then average these deviations across all simulations. Figure 10 reports the average percentage deviations in collateral value for age groups 1 to 6. On impact, the collateral value for these constrained borrowers declines sharply, ranging from 70% to 100%. As the third component of housing prices, this decline feeds back to further reductions in housing prices.

4.3. Leverage and Housing Price Volatility

To understand the impact of endogenously determined leverage on housing price volatility, I analyze a simple bond economy in which this mechanism is absent. The bond economy differs from the baseline collateral economy in three key ways: the only financial asset available is a risk-free bond in zero net supply; agents are assumed to never default; and the LTV ratio is exogenously capped at ϕ. Let $(b_t^a)_{a\in\mathbf{A}}$ denote the bond holdings of agents, and p_t the price of the bond. Agents, taking prices as given, maximize their life-time utility (1) subject to a budget constraint:

$$c_t^a + q_t h_t^a + p_t b_t^a \leqslant e_t^a + q_t h_{t-1}^{a-1} + b_{t-1}^{a-1},$$

and a borrowing constraint, as is standard in the macroeconomic housing literature:

$$-p_t b_t^a \leqslant \phi q_t h_t^a.$$

LTV of an agent is given by:

$$LTV_t^a = \frac{max\{-p_t b_t, 0\}}{q_t h_t}.$$

In the bond economy, interest rates no longer monotonically increase in leverage. Instead, they are uniform across all levels of leverage and given by:

$$R_t = \frac{1}{q_t}.$$

If the Credit Surface were plotted, it would appear as a flat line, truncated at $LTV = \phi$.

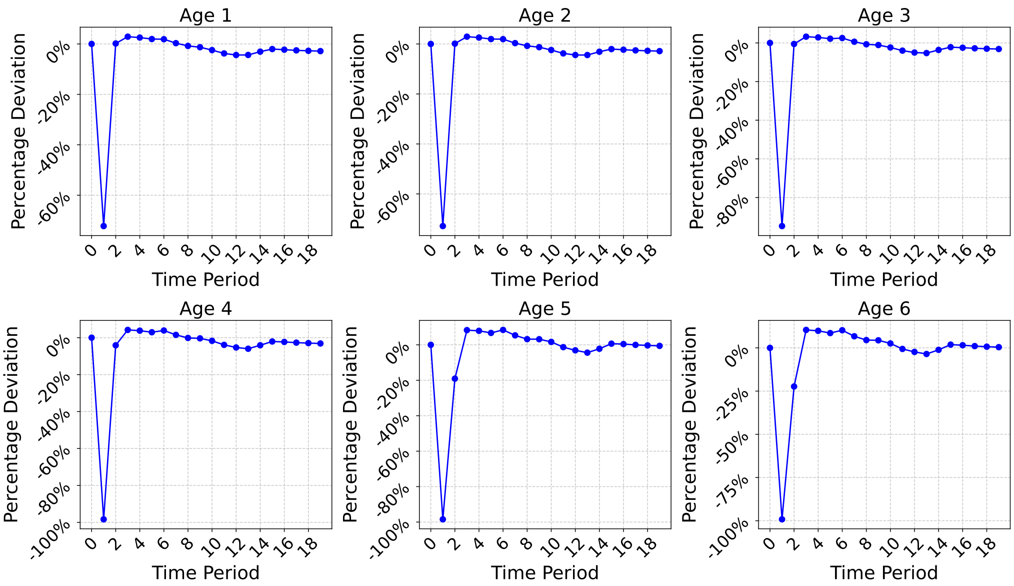

Figure 10: Percentage Deviations in Collateral Value for Constrained Borrowers Under a D shock

Notes: For each simulation, I select the initial state vector from the states generated during the 500,000-period simulation where the exogenous state is U. Therefore, each simulation begins with an initial state within the ergodic set of the state space.

The aggregate amount of debt in the bond economy is given by the product of the bond price and the aggregate quantity of bonds shorted across agents:

$$p_t \sum_{a \in \boldsymbol{A}} max\{-b_t^a, 0\}.$$

The diluted leverage of housing in the bond economy is the ratio of the aggregate debt in the economy to the aggregate value of housing:

$$DLTV_t^h = \frac{p_t \sum_{a \in \boldsymbol{A}} max\{-b_t^a, 0\}}{q_t H}.$$

Using the average upper limit of LTV in the normal state of the collateral economy as a reference point, I set $\phi = 81\%$ and solve for the equilibrium. I then simulate the bond economy for 500,000 periods. Table 4 provides an overview of the average behavior of key variables across the two economies, including average housing prices, aggregate debt, and the diluted LTV of housing in both states, as well as the differences of each variable between the two states.

On average, housing prices are more volatile in the collateral economy compared to the bond economy. In the bond economy, the average housing price in the D state is only 29% lower than in the U state, compared to a 42% decline in the collateral economy. Aggregate debt contracts more sharply during downturns in the collateral economy than in the bond economy. Specifically, the average aggregate debt in the D state is 56% lower than in the U state in the collateral economy, compared to a decline of 26% in the bond economy. In the bond economy,

Table 4: Comparison of Collateral Economy and Bond Economy

	Collateral Economy			Bond Economy		
	U	D	Δ	U	D	Δ
Housing Price	1.78	1.03	-42.1%	1.84	1.3	-29.3%
Aggregate debt	15.29	6.81	-55.5%	11.07	8.23	-25.7%
$DLTV^h$	43%	33.2%	-9.8 *pp*	30.1%	31.6%	1.5 *pp*
Volatility	16.2%			11.1%		

Notes: Volatility refers to the volatility of housing prices, which is measured by the coefficient of variation over the simulation. 16.2% *indicates that housing prices deviate by* 16.2% *from the mean on average.*

aggregate debt does not decline as much as housing prices in downturns, resulting in higher average aggregate leverage of housing during downturns. Aggregate leverage in the D state is nearly 10 percentage points lower than in the U state in the collateral economy, whereas in the bond economy, the average $DLTV^h$ is higher in the D state than in the U state.

Figure 11 compares the unconditional deviations of these key variables caused by the shock across the two economies. The analysis of these results echoes the previous findings from the overview of average behavior across states. On impact, housing prices and aggregate debt decline more sharply in the collateral economy compared to the bond economy. In the collateral economy, housing leverage drops by 10 percentage points, whereas in the bond economy, it increases by 1.5 percentage points. In addition, at time 2, the recovery of housing prices and aggregate debt is slower in the collateral economy relative to the bond economy.

In the collateral economy, the majority of debt is issued through the risky contract $j_{B,t}$ that promises q^U_{t+1}. Over the 500,000-period simulation, risky debt accounts for an average of 77% of aggregate debt issuance[6], while risk-free debt makes up the remaining 23%. While the risky contract facilitates consumption smoothing for the young borrowers, it amplifies the volatility of lenders' wealth. In the collateral economy, borrowers deliver q^U_{t+1} in the U state but default on risky loans and deliver only q^D_{t+1} in the D state, leading to significant wealth volatility for lenders across states. In contrast, this issue does not arise in the bond economy, where lenders receive a fixed payment of 1 unit of the consumption good regardless of the state. Figure 12 illustrates the impact of default. It shows the average percentage deviations of lenders' financial wealth from time 0 under a D shock at time 1. On impact, lenders in the collateral economy experience a significantly sharper decline in wealth and subsequently contract credit more severely compared to those in the bond economy.

5. Conclusion

This paper provides a quantitative model to explain co-movements among housing prices, leverage, and mortgage spreads in the context of the U.S. housing market. A critical contribution of this research lies in its endogenous treatment of the LTV ratio in a model featuring a high degree of agent heterogeneity, which challenges the conventional assumption of exogenously given

[6]The share of risky debt issuance is given by the ratio of the aggregate risky debt to the aggregate debt: $\frac{\pi_{B,t} \sum_{a \in \mathbf{A}} max\{-\theta^a_{B,t}, 0\} dj}{\sum_{a \in \mathbf{A}} \int \pi_{j,t} max\{-\theta^a_{j,t}, 0\} dj}$.

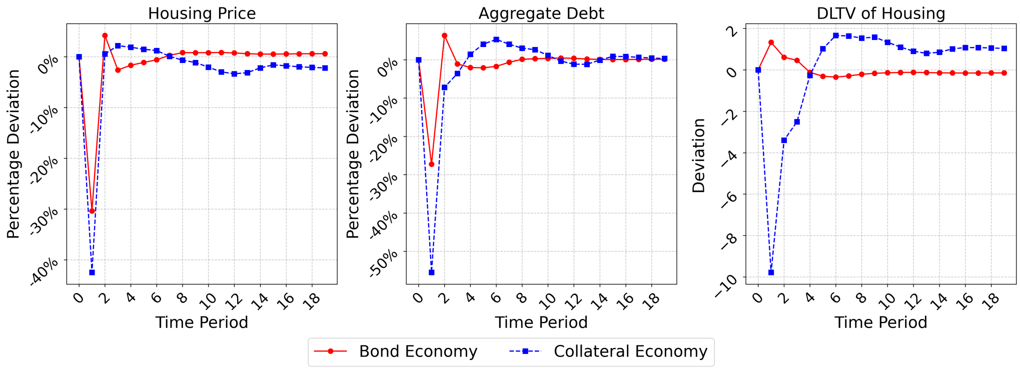

Figure 11: Deviations in Housing Price, Aggregate Debt and Leverage Under a D shock: Collateral vs Bond Economies

Notes: Deviations in housing price and credit are expressed as percentages, while deviations in leverage ($DLTV^h$) are in percentage points.

leverage limits. The model shows that both the upper limit of LTV and the entire pricing schedule of leverage, the Credit Surface, are inherently dynamic, reacting to changes in economic fundamentals. This is particularly evident in economic downturns, where the Credit Surface rises and becomes steeper, indicating a tightening of credit conditions. This dynamic response provides an understanding of the decrease in leverage observed during economic crises and highlights a feedback loop between leverage and housing prices.

Looking forward, this research opens several avenues for further inquiries. A key extension would be to incorporate long-term debt contracts. Moreover, comparing the equilibria of economies with and without the endogenous determination of leverage highlights the amplification effect of leverage on housing prices. Using the model as a working horse, we can continue investigating the effects of financial innovations, such as the introduction of credit default swaps (CDS), tranching, and other financial instruments.

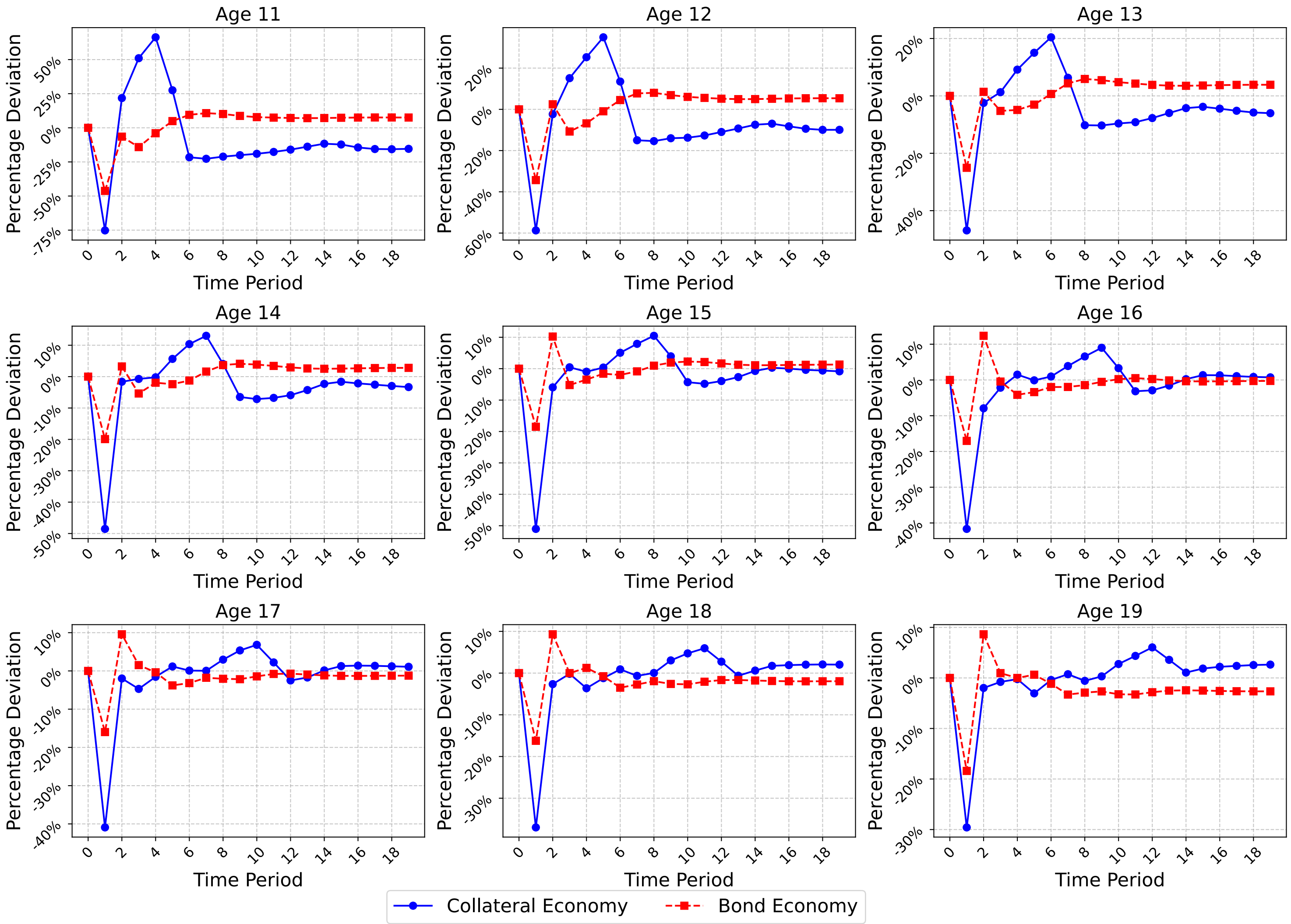

Figure 12: Percentage Deviations in Financial Wealth Under a D shock: Collateral vs Bond Economies

Notes: Financial wealth is the equity after clearing debts: $q_t h_{t-1}^{a-1} + \int_{R_+} \theta_{j,t-1}^{a-1} min\{j, q_t\} dj$ *for the collateral economy, and* $q_t h_{t-1}^{a-1} + b_{t-1}^{a-1}$ *for the bond economy.*

Appendix A. Proof of Proposition 2

Proof. Consider three cases: $j \in [0, q^D_{t+1}]$, $j \in (q^U_{t+1}, +\infty)$, and $j \in (q^D_{t+1}, q^U_{t+1}]$.

1. Case 1: $j \in [0, q^D_{t+1}]$

According to Equation (7), in equilibrium, the interest rate $R_{j,t}$ is constant over this interval, and equals to the inverse of the expected intertemporal marginal utility of consumption:

$$R_{j,t} = \frac{j}{\pi_{j,t}} = \frac{1}{\sum_{z_{t+1}\in Z} M^{a,z_{t+1}}_{t,t+1}}.$$

2. Case 2: $j \in (q^U_{t+1}, +\infty)$

As j goes to infinity, $LTV_{j,t}$ remains constant at $LTV_{q^U_{t+1},t}$, while $R_{j,t}$ approaches infinity. Thus, the Credit Surface becomes a vertical line for $j > q^U_{t+1}$.

3. Case 3: $j \in (q^D_{t+1}, q^U_{t+1}]$

For this interval, $LTV_{j,t}$ is given by:

$$LTV_{j,t} = \frac{M^{a,D}_{t,t+1} q^D_{t+1} + M^{a,U}_{t,t+1} j}{q_t},$$

and the interest rate is:

$$R_{j,t} = \frac{j}{M^{a,D}_{t,t+1} q^D_{t+1} + M^{a,U}_{t,t+1} j}.$$

Using the chain rule, the derivative of $R_{j,t}$ with respect to $\pi_{j,t}$ is:

$$\begin{aligned}\frac{dR_{j,t}}{d\pi_{j,t}} &= \frac{dR_{j,t}}{dj}\frac{dj}{d\pi_{j,t}} \\ &= \frac{\pi_{j,t} - j\frac{d\pi_{j,t}}{dj}}{(\pi_{j,t})^2} \cdot \frac{1}{\frac{d\pi_{j,t}}{dj}} \\ &= \frac{M^{a,D}_{t,t+1} q^D_{t+1}}{(\pi_{j,t})^2 M^{a,U}_{t,t+1}} > 0.\end{aligned}$$

Since $q_t > 0$, it follows that $\frac{dR_{j,t}}{dLTV_{j,t}} > 0$. Furthermore, observe that $\frac{dR_{j,t}}{d\pi_{j,t}}$ is decreasing in $\pi_{j,t}$, which implies that $\frac{dR_{j,t}}{dLTV_{j,t}}$ is also decreasing in $LTV_{j,t}$. Hence, $R_{j,t}$ is increasing and concave in $LTV_{j,t}$ for $j \in (q^D_{t+1}, q^U_{t+1}]$. □

Appendix B. Algorithm for Approximating a FREE

1. Set the episode counter $s = 0$. Initialize a neural network $\mathcal{N}^{(s)}$ with two hidden layers. The first layer contains 500 nodes, and the second layer contains 300 nodes.
2. Solve for the steady state equilibrium using the endowments for $z = U$, and use the resulting equilibrium variables to construct the initial state vector $\boldsymbol{x}^{(s)}_{\mathbf{0}}$. The state vector lies in the space $\mathbf{Z} \times [0, H]^A \times [-H, H]^{N_j A} \times \mathbb{R}^A_{++} \times \mathbb{R}^2_{++}$.

3. Simulate the economy using $\mathcal{N}^{(s)}$ and state vector $\mathbf{x_0}^{(s)}$ over $10,000$ periods. Construct a training dataset $\mathcal{D}_{\text{train}}^{(s)}$ using the 100,000-period simulation. Each simulation counts as one episode within the training sequence.
4. Calculate the loss function, which is the mean squared errors across all equilibrium conditions. These conditions include the Euler equations, market clearing conditions, budget constraints, and complementary slackness conditions, evaluated over the training dataset $\mathcal{D}_{\text{train}}^{(s)}$.
5. Implement the Adam optimization algorithm, a type of mini-batch stochastic gradient descent, to update the neural network parameters from $\mathcal{N}^{(s)}$ to $\mathcal{N}^{(s+1)}$. Update the parameters only once per simulation. Increment the episode counter to $s = s + 1$. Set the new initial state vector $\mathbf{x_0}^{(s+1)}$ as the final state vector from the preceding simulation.
6. Repeat steps 3 to 5 until either the episode counter reaches $200,000$ or the neural network converges. If the loss function does not converge after 200,000 episodes, adjust the learning rate, the size of mini-batches, or the number of nodes in each hidden layers, then return to step 1 with the new parameters.

Appendix C. Accuracy of Numerical Solutions

This appendix examines the accuracy of the numerical solutions for the collateral economy and the bond economy based on simulations spanning 500,000 periods. The equilibrium of each economy is is characterized by four sets of equations: (1) complementary slackness conditions for collateral constraints or borrowing constraints, (2) Euler equations, (3) budget constraints, and (4) market clearing conditions. In both simulations, the budget constraints are enforced, so there's no approximation errors in this aspect.

Tables C.5 and C.6 report the errors for the remaining three types of equilibrium conditions for the collateral and bond economies, respectively. The errors are expressed in terms of log_{10}. Following Judd (1992), I define the relative Euler equation errors for each type of condition in the collateral and bond economies as follows.

For the collateral economy, the relative Euler equation errors for housing(h_t^a), the risk-less contract($\theta_{A,t}^a$), and the risky contract($\theta_{B,t}^a$) are defined as:

$$e(h_t^a) = |1 - \frac{(Du_c)^{-1}(\frac{Du_h(c_t^a,h_t^a)+\beta E_t[Du_c(c_{t+1}^{a+1},h_{t+1}^{a+1})q_{t+1}]+\mu_{h,t}^a}{q_t})}{c_t^a}|,$$

$$e(\theta_{A,t}^a) = |1 - \frac{(Du_c)^{-1}(\frac{\beta E_t[Du_c(c_{t+1}^{a+1},h_{t+1}^{a+1})q_{t+1}^D]+\mu_{A,t}^a}{q_t})}{c_t^a}|,$$

$$e(\theta_{B,t}^a) = |1 - \frac{(Du_c)^{-1}(\frac{\beta E_t[Du_c(c_{t+1}^{a+1},h_{t+1}^{a+1})q_{t+1}]+\mu_{B,t}^a}{q_t})}{c_t^a}|.$$

For the bond economy, the relative Euler equation errors for housing (h_t^a) and bond holdings (b_t^a) are:

$$e(h_t^a) = |1 - \frac{(Du_c)^{-1}(\frac{Du_h(c_t^a,h_t^a)+\phi\mu_t^a q_t+\beta E_t[Du_c(c_{t+1}^{a+1},h_{t+1}^{a+1})q_{t+1}]}{q_t})}{c_t^a}|,$$

$$e(b_t^a) = |1 - \frac{(Du_c)^{-1}(\frac{\mu_t^a p_t + \beta E_t[Du_c(c_{t+1}^{a+1}, h_{t+1}^{a+1})]}{p_t})}{c_t^a}|,$$

where μ_t^a is the Lagrange multiplier associated with the borrowing constraint.

The maximum Euler equation error remains below 10^{-2}, indicating that the largest percentage loss in consumption due to approximation errors in prices is less than 1%. Errors in the market clearing conditions for housing and the two contracts are normalized by the aggregate housing stock H, while errors in consumption are normalized by the aggregate endowment.

Table C.5: Summary Statistics of Errors in Equilibrium Conditions: Collateral Economy

Equilibrium Conditions	Type	25th	Median	99th	Mean	Max
Collateral Constraints		-11.28	-9.80	-3.74	-5.28	-3.46
Euler Equations	Housing	-4.11	-3.62	-2.74	-3.42	-2.22
	Risky	-4.03	-3.54	-2.57	-3.31	-2.15
	Risk-less	-3.95	-3.44	-2.40	-3.18	-2.02
Market Clearing Conditions	Housing	-5.41	-5.12	-4.29	-4.97	-3.80
	Risky	-4.86	-4.65	-3.92	-4.54	-3.42
	Risk-less	-5.15	-4.79	-3.93	-4.60	-3.30
	Consumption	-5.09	-4.85	-4.11	-4.73	-3.59

Notes: The 25th and 99th refer to the 25th and 99th quantiles of each error in the simulation data series.

Table C.6: Summary Statistics of Errors in Equilibrium Conditions: Bond Economy

Equilibrium Conditions	Type	25th	Median	99th	Mean	Max
Borrowing Constraints		-7.88	-5.90	-2.81	-3.88	-2.42
Euler Equations	Housing	-4.20	-3.90	-3.09	-3.76	-2.23
	Bond	-4.12	-3.78	-3.06	-3.69	-2.28
Market Clearing Conditions	Housing	-4.73	-4.63	-4.19	-4.61	-3.69
	Bond	-4.21	-4.18	-3.85	-4.16	-3.52
	Consumption	-4.46	-4.44	-4.27	-4.43	-4.03

References

Marlon Azinovic, Luca Gaegauf, and Simon Scheidegger. Deep equilibrium nets. *International Economic Review*, 63 (4):1471–1525, 2022.

Johannes Brumm, Michael Grill, Felix Kubler, and Karl Schmedders. Collateral requirements and asset prices. *International Economic Review*, 56(1):1–25, 2015.

William Diamond and Tim Landvoigt. Credit cycles with market-based household leverage. *Journal of Financial Economics*, 146(2):726–753, 2022.

Jack Favilukis, Sydney C Ludvigson, and Stijn Van Nieuwerburgh. The macroeconomic effects of housing wealth, housing finance, and limited risk sharing in general equilibrium. *Journal of Political Economy*, 125(1):140–223, 2017.

Ana Fostel and John Geanakoplos. Leverage cycles and the anxious economy. *American Economic Review*, 98(4): 1211–1244, 2008.

Ana Fostel and John Geanakoplos. Why does bad news increase volatility and decrease leverage? *Journal of Economic Theory*, 147(2):501–525, 2012.

Ana Fostel and John Geanakoplos. Endogenous collateral constraints and the leverage cycle. *Annu. Rev. Econ.*, 6(1): 771–799, 2014.

Ana Fostel and John Geanakoplos. Leverage and default in binomial economies: a complete characterization. *Econometrica*, 83(6):2191–2229, 2015.

John Geanakoplos. Promises, promises, the economy as an evolving complex system ii. *Reading*, pages 285–320, 1997.

John Geanakoplos and David Rappoport. Credit surfaces, economic activity, and monetary policy. *Economic Activity, and Monetary Policy (July 29, 2019)*, 2019.

John Geanakoplos and William R Zame. Collateral equilibrium, i: a basic framework. *Economic Theory*, 56:443–492, 2014.

Kenneth L Judd. Projection methods for solving aggregate growth models. *Journal of Economic theory*, 58(2):410–452, 1992.

Greg Kaplan, Kurt Mitman, and Giovanni L Violante. The housing boom and bust: Model meets evidence. *Journal of Political Economy*, 128(9):3285–3345, 2020.

Nobuhiro Kiyotaki and John Moore. Credit cycles. *Journal of political economy*, 105(2):211–248, 1997.

Dirk Krueger and Felix Kubler. Computing equilibrium in olg models with stochastic production. *Journal of Economic Dynamics and Control*, 28(7):1411–1436, 2004.

Alberto Musso, Stefano Neri, and Livio Stracca. Housing, consumption and monetary policy: How different are the us and the euro area? *Journal of Banking & Finance*, 35(11):3019–3041, 2011.

Stephen E Spear. Existence and local uniqueness of functional rational expectations equilibria in dynamic economic models. *Journal of Economic Theory*, 44(1):124–155, 1988.

Karl Walentin. Business cycle implications of mortgage spreads. *Journal of Monetary Economics*, 67:62–77, 2014.