Leverage Cycle over the Life Cycle: A Quantitative Model of Endogenous Leverage

By Yi Ping*

Draft: January 1, 2025

I construct a quantitative model to rationalize two key features of the U.S. housing market: leverage positively correlates with housing prices, while mortgage spreads move inversely. In the model, overlapping generations use leverage to accumulate housing, facing a schedule, the Credit Surface, where interest rates increase with leverage. Negative aggregate shocks depress housing prices as young borrowers default, reducing lenders' wealth and driving up interest rates and spreads. The higher cost of borrowing reduces leverage, feeding back into further declines in housing prices. The model is calibrated to replicate the observed 10-percentage-point drop in leverage during the Great Recession.

JEL: E20,E44, G51, C68, D52, D53

Keywords: endogenous leverage, housing prices, leverage cycle, household life cycle, Credit Surface, default

The U.S. housing and mortgage markets exhibit two well-documented trends: leverage tends to move in the same direction as housing prices, whereas mortgage spreads move in the opposite direction. Understanding leverage is crucial for policymakers for two reasons. First, in practice, the majority of U.S. homebuyers finance their housing purchases through mortgages. Using annual data on new house sales by financing type from 1988 to 2023, the average share of mortgages as a financing method is 94% in the U.S.[1] Second, in theory, existing work on endogenous leverage (Geanakoplos (1997); Fostel and Geanakoplos (2008), among others) demonstrates that leverage plays a key role in amplifying fluctuations in asset prices. The primary contribution of this paper is to develop a quantitative general equilibrium model with endogenous leverage in an overlapping-generation (OLG) framework. This model rationalizes the observed co-movements of housing prices, leverage, and mortgage spreads.

Using quarterly data spanning from 1999Q1 to 2024Q1, Table 1 presents the cross correlations of changes in Case-Shiller housing prices with the average first-time home-

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[1]Mortgage types include conventional, FHA-insured, and VA loans. Data source: FRED, Federal Reserve Bank of St. Louis (data labels: HSTFCM, HSTFC, HSTFFHAI, HSTFVAG).

buyers' LTV, Combined LTV (CLTV), and mortgage spreads, with the highest absolute correlations underlined. Both LTV and CLTV are positively correlated with housing prices, whereas mortgage spreads are negatively correlated with housing prices. These patterns were especially pronounced in the 2000s, during which the U.S. housing market experienced a leverage cycle. Fostel and Geanakoplos (2014) present the co-movement of leverage and housing prices using data on the average down payment for borrowers below the median in the subprime/Alt-A category. The left panel of Figure 1 provides further evidence for this pattern by showing trends in housing prices and CLTV for first-time homebuyers in the U.S. Specifically, the CLTV initially increased steadily along with rising housing prices, then experienced a sharp 10-percentage-point drop from its peak in 2007Q2 to its trough in two years. Mortgage spreads measure the relative cost of purchasing housing assets using leverage. Walentin (2014) and Musso, Neri and Stracca (2011) show that mortgage spreads rise during times of economic stress, especially in the Great Recession. The right panel of Figure 1 adds to this empirical finding and shows that mortgage spreads fell modestly from 2% in 1999Q1 to 1.5% in 2007Q2, followed by a substantial increase to 2.61% within just one year by 2008Q2.

Table 1—Cross Correlations of Leverage and Mortgage Spreads with Housing prices

	LTV	CLTV	Spread
lead (2)	0.41	0.35	-0.08
lead (1)	0.42	0.39	-0.24
contemporaneous	0.41	0.42	-0.28
lag (1)	0.38	0.42	-0.29
lag (2)	0.53	0.44	-0.08

Note: "lead/lag (i)" indicates that the variable leads/lags housing prices by i quarter(s). LTV is calculated by dividing the balance of the primary mortgage by the appraised value of the property securing the mortgage. CLTV is calculated by summing the balances of all loans secured by a property and then dividing this total by the appraised value of the property. According to Walentin (2014), the duration of a 30-year fixed rate mortgage in the U.S. is 7 to 8 years, Therefore, I define the mortgage spread as the difference between the average 30-year fixed-rate mortgage and the 5-year treasury bill yield.
Source: LTV and CLTV: Freddie Mac Single-Family Loan-Level Dataset. Case-Shiller housing price index: FRED. Mortgage spreads: FRED, data label: MORTGAGE30US, DGS5.

To explain these trends, this paper considers three key elements that have not been jointly included in the existing literature: (i) large aggregate endowment shocks, (ii) hump-shaped life cycle endowment profiles of households combined with realistic household lifespans, and (iii) the endogenous determination of leverage. In the model, the economy is populated by overlapping generations of households, each deriving utility from both housing and non-housing consumption. Households face large aggregate endowment shocks which lead to substantial fluctuations in equilibrium housing prices and mortgage interest rates. They receive low endowments in early life, which gradually increase and peak in middle age before declining in later stages of life. The consideration of the household life cycle is grounded in micro data. As illustrated in Figure 2, there is an evident age-related pattern in the housing market: housing wealth increases with age then slightly declines, while leverage decreases with age.

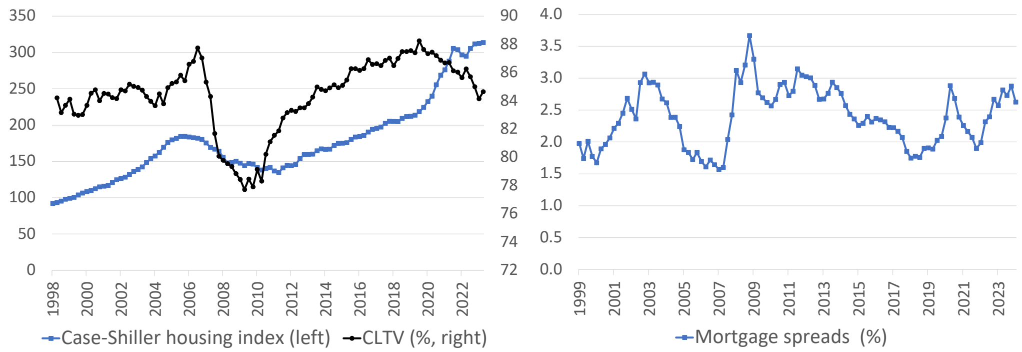

FIGURE 1. HOUSING PRICES, LEVERAGE, AND MORTGAGE SPREADS

Note: Case-Shiller housing index: Jan 2000 = 100.

Housing assets are perfectly durable and can be used as collateral for issuing debt contracts. Agents can accumulate housing assets using leverage through the simultaneous purchase of housing and issuance of debt contracts. A collateral constraint prohibits households from issuing a quantity of debt contracts that exceeds their housing stock. Leverage of a contract is measured by LTV, defined as the ratio of the loan amount to the value of the collateral. LTV ranges from 0% to an endogenously determined upper limit, which is always below 100%. Fostel and Geanakoplos (2015) introduce the concept of the "Credit Surface", which represents a menu of leverage levels paired with their corresponding interest rates. Following their methodology, equilibrium prices define a Credit Surface that specifies the interest rates for different LTV levels. The interest rate increases with the LTV ratio. For loans with an LTV low enough to rule out default, a uniform risk-free interest rate is applied. However, for loans with an LTV that implies default risk, the interest rate begins to rise as the LTV increases.

In this framework, endogenous leverage has a twofold meaning. First, the upper limit of LTV is determined within the model, based on the capacity of housing to serve as collateral for borrowing, and it adjusts in response to changes in economic fundamentals. Second, while agents have infinitely many leverage options, as LTV is a continuous variable, the scarcity of collateral limits equilibrium choices to only a few LTV levels. Borrowers face a trade-off between present liquidity and future obligations: higher leverage provides greater liquidity but comes with higher interest rates. Lenders, driven by consumption-smoothing motives, also chose their optimal LTV levels. Both borrowers and lenders select their optimal positions on the Credit Surface, leaving most LTV levels rationed with zero supply and demand in equilibrium.

The model has two main implications. First, households start with high leverage and then progressively lower their leverage over the course of their life. Households transition through four distinct life stages: they start as constrained borrowers, become unconstrained borrowers, transition into a mixed role of borrowers and lenders, and eventu-

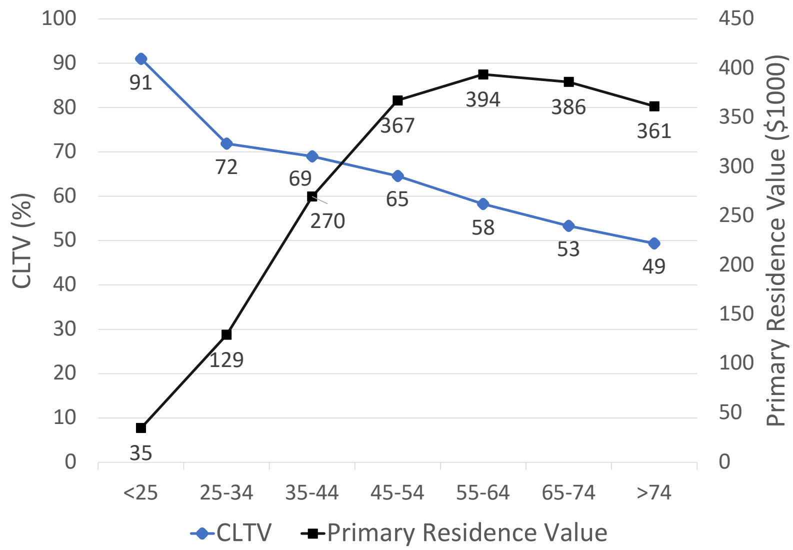

FIGURE 2. TRENDS IN AGE

Note: CLTV for the first age group (<25) is averaged based on loans taken for home purchases, and the CLTV for all other age groups is averaged based on loans taken for refinancing purposes. Primary residence values are in 2022 dollars. These patterns are consistent across years, see supplementary materials for details.
Source: CLTV: 2023 HDMA national loan-level dataset. Primary residence values: 2022 Survey of Consumer Finances dataset.

ally become pure lenders. Second, large aggregate endowment shocks produce leverage cycles. In normal states, housing prices and leverage are high, while mortgage rates and spreads remain low. However, during downturns, housing prices and leverage fall sharply, whereas mortgage interest rates and spreads rise.

Life cycle implications. In the equilibrium considered in the paper, only two types of debt contracts are traded, even though infinitely many debt contracts are priced: a risky contract, where agents default during downturns, and a risk-free contract, which ensures no default but offers less liquidity than the risky one. In the first stage, young households receive low endowments. Anticipating an increase in future endowment, they issue risky debt contracts to borrow, pledging all the available housing stock as collateral. As their endowments grow, they transition to using a combination of risky and risk-less contracts for borrowing, without pledging all their owned housing as collateral. In the third stage, as endowments begin to decline, they start investing in risky contracts while also leveraging their housing assets through risk-less contracts. Finally, in the last stage, driven by a strong savings motive, households accumulate more financial wealth. To diversify their investments, they purchase both risky and risk-less debt contracts. On average, younger households are the most leveraged among all age groups, and leverage decreases as households age, a pattern that is consistent with data.

Leverage cycle implications. The second main implication of the model is that leverage and housing prices decline substantially during crises, while mortgage rates and spreads surge. A large negative endowment shock reduces housing prices by impacting the two groups of marginal buyers of housing assets—young borrowers and middle-aged lenders—through distinct but related channels. The sequence of events unfolds as follows: during downturns, young borrowers who previously issued risky contracts find themselves underwater and default, which reduces lenders' wealth. Since lenders are marginal buyers of housing assets, this reduction in their wealth directly depresses housing prices. Lenders, facing significant losses from defaults and declining housing wealth, become reluctant to forego current consumption to lend. Consequently, they demand higher returns and collectively raise interest rates across all debt contracts, as well as spreads between risky and risk-free contracts. Credit Surface rises and steepens, reducing the leveraging capacity of housing assets for young borrowers. This lower leverage then feeds back into further declines in housing prices. Figure 3 illustrates the feedback loop between leverage and housing prices in downturns.

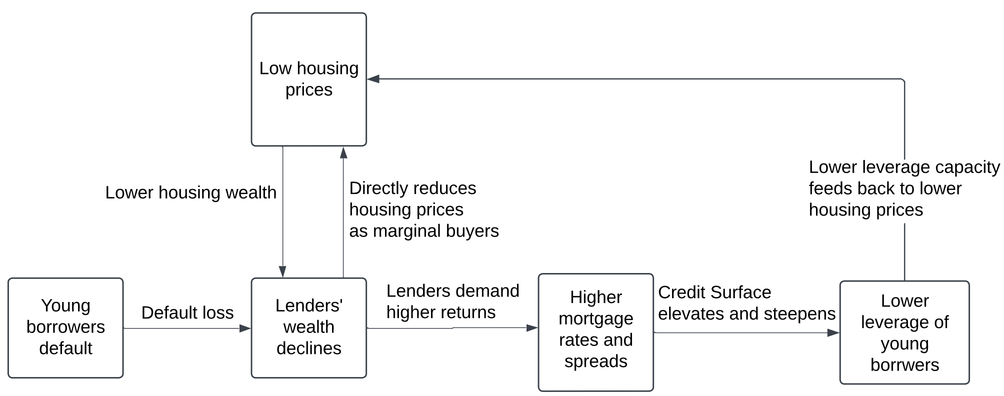

FIGURE 3. FEEDBACK LOOP BETWEEN LEVERAGE AND HOUSING PRICES IN DOWNTURNS

The majority of debt issuance in the economy is through the risky contract. The key feature of this contract is that its payoff varies significantly across states: borrowers do not default in normal times but default during downturns. This variation in payoffs translates into higher volatility in lenders' wealth across states. As the marginal buyers in the housing market, fluctuations in their wealth greatly amplify housing price volatility.

In contrast, I show that this mechanism is absent in a bond economy, in which the borrowing constraint follows the framework of Kiyotaki and Moore (1997). In this setting, agents borrow by issuing non-contingent bonds, households' LTV is capped exogenously, and default does not occur in equilibrium. Lenders' wealth does not vary with the debt repayment, and aggregate debt is significantly lower in this economy. In addition, the feedback loop between leverage and housing prices is muted, leading to much smaller declines in housing prices during downturns.

Solving for the equilibrium presents two main challenges. This paper addresses these challenges and offers guidance for finding a collateral equilibrium with housing assets and a high degree of agent heterogeneity. The first challenge is conceptual: which LTV levels will be selected in equilibrium? Since LTV is a continuous variable, deriving an infinite number of first-order conditions for each LTV is impractical. Existing literature on endogenous leverage provides theoretical guidelines for finding such equilibria, but these are restricted to cases where agents do not derive utility from collateral assets. Such frameworks are unsuitable for this study, as the demand and supply of leverage are influenced by agents' preferences over housing. To address this challenge, I assume the existence of an equilibrium in which only a finite number of specific LTV levels are selected, then verify whether agents have incentives to deviate by trading contracts with LTV levels outside this finite set. I repeat this procedure until I find an equilibrium.

The second challenge is computational in nature. The substantial heterogeneity of agents and the multiplicity of tradable assets result in a high-dimensional state space. In addition, the collateral constraint introduces non-linearity into the policy functions. Azinovic, Gaegauf and Scheidegger (2022) introduce a methodology that uses neural networks to address the curse of dimensionality and the non-linearity of policy functions. Following their approach, I use a neural network with two hidden layers to directly approximate the policy functions. The resulting maximum error across all equilibrium conditions is less than 1%.

Related literature. This paper contributes to two main strands of the literature. First, it builds on the foundational work on endogenous leverage theory by Geanakoplos (1997) and Fostel and Geanakoplos (2008, 2012, 2014, 2015). In contrast to the majority of macro-finance models that assume an exogenous cap on leverage, these models take changes in leverage as endogenous responses to shifts in economic fundamentals, and uncover a strong feedback loop between leverage and asset prices. While existing models in this literature are primarily stylized theoretical constructs, this paper bridges the gap between theory and data by incorporating the age heterogeneity observed in the housing market.

There are two quantitative works that consider endogenous leverage. Diamond and Landvoigt (2022) develop an OLG model with housing assets as collateral. The key mechanism driving housing booms and busts is a household saving glut, driven by an increased preference for deposits, combined with aggregate shocks and idiosyncratic housing wealth shocks that increase housing price dispersion during downturns. The combination of a negative productivity shock and rising housing wealth dispersion leads to substantial defaults and losses for financial intermediaries, prompting them to tighten credit. The key distinction between their work and mine lies in the relationship between leverage and housing prices. When driven solely by aggregate endowment shocks, their model predicts a negative correlation, whereas my model produces a positive correlation. Brumm et al. (2015) examine the effects of endogenous leverage on asset price volatility through a model with two infinitely-lived agents with different degrees of risk aversion. In contrast, this paper considers the life cycle dynamics of households and emphasizes the role of non-financial assets, specifically housing, as collateral, where agents' preferences of housing influence equilibrium prices.

Second, this paper also relates to the macroeconomic literature focusing on the housing market and the households life cycle. Extant models typically adopt the borrowing constraint framework of Kiyotaki and Moore (1997), where the LTV cap is exogenous, interest rate does not vary with leverage, and household default is often ruled out. As a result, these models are silent on the factors driving changes in housing finance conditions and overlook the feedback loop between leverage and housing prices. In these settings, variations in the LTV limit have minor effects on housing prices. Kaplan, Mitman and Violante (2020) analyze the housing boom and bust episode within an OLG model with endogenous housing prices. They conclude that exogenous shifts in the LTV limit do not significantly affect housing prices; instead, fluctuations in housing prices are primarily driven by shifts in households' beliefs about future housing demand. Favilukis, Ludvigson and Van Nieuwerburgh (2017) also examine the housing market in an OLG model with endogenous housing prices. They find that a higher LTV cap significantly increases housing prices, but this effect is only significant when they introduce heterogeneity in households' bequest preferences, which results in a substantial number of constrained households in equilibrium.

I. Model

I will hereafter refer to the baseline economy described in this section as the collateral economy, to differentiate it from a benchmark—a bond economy—introduced later in Section III.C.

A. Agents, Commodities and Uncertainty

Time is discrete and indexed by $t = 0, 1, 2, \ldots$. At each period, one of two possible exogenous aggregate shocks $z_t \in \boldsymbol{Z} = \{U, D\}$ realizes. The state U stands for "Up", representing normal times. The state D stands for "Down", representing rare crisis states similar to the Great Recession. The aggregate shock z_t affects both the aggregate endowment and the allocation of endowments among different age groups in every period. z_t evolves according to a Markov chain with the transition matrix Γ. Let $\gamma_{z_t, z_{t+1}}$ denote the probability of transitioning from state z_t to state z_{t+1}.

Agents can trade both consumption good (c) and housing (h). The consumption good is perishable, whereas housing assets are perfectly durable and in fixed supply of H. Let the spot price of the consumption good be 1, and the housing price be q_t.

In each period, a continuum of mass 1 identical agents of a new generation is born and lives for A periods. There is no mortality risk; all households die after age A. Age is indexed by $a \in \boldsymbol{A} = \{1, \ldots, A\}$. At the beginning of each period, all households receive a strictly positive endowment of the consumption good, which depends on their age and the aggregate shock: $e_t^a = e^a(z_t) > 0$, $\forall a \in \boldsymbol{A}$. The aggregate endowment is denoted by $\bar{e}(z_t) = \sum_{a=1}^{A} e^a(z_t)$.

B. Preferences

The expected lifetime utility of households born at time t is given by

$$U_t = E_t \sum_{a=1}^{A} \beta^{a-1} u^a(c_t^a, h_t^a), \tag{1}$$

where $\beta > 0$ is the discount factor, c_t^a and h_t^a are the amount of the consumption good and the stock of housing at age a. $u^a(c,h)$ is an age-dependent period utility function. Let $\hat{\boldsymbol{A}} = \{1,...,A-1\}$ be the set of agents excluding those who are in the last period of life. All agents have the Cobb-Douglas utility over consumption and housing nested within a constant relative risk aversion utility form when they are at age $a \in \hat{\boldsymbol{A}}$, and they do not value housing assets in the last period of life, the period utility function is given by

$$u^a(c,h) = \begin{cases} \dfrac{(c^{1-\alpha}h^{\alpha})^{1-\rho}}{1-\rho}, & a \in \hat{\boldsymbol{A}}, \\ \dfrac{c^{1-\rho}}{1-\rho}, & a = A, \end{cases} \tag{2}$$

where $\alpha > 0$ measures the relative share of housing expenditure, $\rho > 0$ is the coefficient of risk aversion.

C. Debt Contracts

Households enter the economy with neither debts or assets. Each period, they meet in anonymous, competitive financial markets to trade collateralized debt contracts. All contracts are one-period: if traded at time t, they are settled at time $t+1$. Let J_t denote the set of such contracts available at time t. A contract in J_t is defined by an ordered pair representing its promise and collateral requirement, denoted as $(j,1)$. $j \in \mathbb{R}_+$ is a non-contingent promise to deliver j units of consumption good in the next period, while the number 1 indicates that the promise j must be backed by one unit of housing as collateral. For simplicity in notation, I will refer to contract $(j,1)$ as simply "contract j" hereafter. J_t contains an infinite number of contracts, as j is continuous and unbounded above.

All contracts are non-recourse, meaning that agents can default on their promises without incurring additional penalties beyond losing the collateral they have previously pledged. An agent who sells one unit of contract j will default on the promise if j exceeds the realized housing price q_{t+1}. Therefore, the delivery of contract j is $min\{j, q_{t+1}\}$.

The price of contract j is denoted by $\pi_{j,t}$. Let $\theta_{j,t}^a \in \mathbb{R}$ be the number of contract j traded by an agent of age a. $\theta_{j,t}^a < 0 (> 0)$ indicates the agent is shorting (longing) contract j, by doing so the agent borrows (lends) $|\pi_{j,t}\theta_{j,t}^a|$. When agents buy one unit of housing and finance this purchase by selling a debt contract j, they are effectively making a downpayment of $q_t - \pi_{j,t}$ on the house.

The gross interest rate for contract j is defined as the ratio of its promise to its price:

$$R_{j,t} = \frac{j}{\pi_{j,t}}. \tag{3}$$

The LTV for contract j is the ratio of its price to the value of the collateral:

$$LTV_{j,t} = \frac{\pi_{j,t}}{q_t}. \tag{4}$$

While $LTV_{j,t}$ measures the leverage of individual contracts, Fostel and Geanakoplos (2015) propose metrics to assess the leverage of an agent and the economy-wide leverage. Building on their framework, I define these measures as follows:

The leverage of an agent, LTV_t^a, is defined as the ratio of the agent's total debt issuance to the value of the collateral backing this borrowing:

$$LTV_t^a = \frac{\int_{j\in\mathbb{R}_+} \pi_{j,t} max\{-\theta_{j,t}^a, 0\} dj}{q_t \int_{j\in\mathbb{R}_+} max\{-\theta_{j,t}^a, 0\} dj}. \tag{5}$$

LTV_t^a is not well-defined for agents who do not leverage. To measure borrowing relative to the agent's entire housing stock, define the diluted LTV of an agent, $DLTV_t^a$, as:

$$DLTV_t^a = \frac{\int_{j\in\mathbb{R}_+} \pi_{j,t} max\{-\theta_{j,t}^a, 0\} dj}{q_t h_t^a}. \tag{6}$$

The aggregate amount of debt in the economy is the sum of all agents' debt issuance across contracts:

$$\sum_{a\in\boldsymbol{A}} \int_{j\in\mathbb{R}_+} \pi_{j,t} max\{-\theta_{j,t}^a, 0\} dj. \tag{7}$$

To measure the economy-wide level of leverage, define the LTV of housing, LTV_t^h, and the diluted LTV of housing, $DLTV_t^h$, as the ratio of aggregate debt to the value of housing used as collateral, and aggregate debt to the total value of housing, respectively, as follows:

$$LTV_t^h = \frac{\sum_{a\in\boldsymbol{A}} \int_{j\in\mathbb{R}_+} \pi_{j,t} max\{-\theta_{j,t}^a, 0\} dj}{q_t \sum_{a\in\boldsymbol{A}} \int_{j\in\mathbb{R}_+} max\{-\theta_{j,t}^a, 0\} dj}. \tag{8}$$

$$DLTV_t^h = \frac{\sum_{a\in\boldsymbol{A}} \int_{j\in\mathbb{R}_+} \pi_{j,t} max\{-\theta_{j,t}^a, 0\} dj}{q_t H}. \tag{9}$$

Fostel and Geanakoplos (2008) introduce the concept of collateral value and liquidity wedge; Following their discussion, Geanakoplos and Zame (2014) introduce the concept of liquidity value as a way to quantify the benefits of borrowing using different contracts. Following their definition, in this economy, the liquidity value of contract j for an agent of age a at time t, denoted by $LV^a_{j,t}$, is given by contract j's price net of the present value of its delivery, discounted by the agent's stochastic discount factor[2]. Let Du^a_x denote the derivative of the period utility function u^a with respect to x, liquidity value of age-a agents is given by

$$LV^a_{j,t} = \pi_{j,t} - E_t\left[\beta \frac{Du_c^{a+1}(c^{a+1}_{t+1}, h^{a+1}_{t+1})}{Du^a_c(c^a_t, h^a_t)} min\{j, q_{t+1}\}\right]. \tag{10}$$

D. Constraints

BUDGET CONSTRAINT

The budget constraint for age-a agents and at time t is given by

$$c^a_t + q_t h^a_t + \int\limits_{j\in\mathbb{R}_+} \theta^a_{j,t}\pi_{j,t}dj \leqslant e^a_t + q_t h^{a-1}_{t-1} + \int\limits_{j\in\mathbb{R}_+} \theta^{a-1}_{j,t-1} min\{j, q_t\}dj. \tag{11}$$

On the left-hand side of the budget constraint, there are expenditures on consumption, housing, and the total amount of borrowing (or lending) through trading debt contracts in the financial market. Since j is a continuous variable, the third item is an integral over $j \in \mathbb{R}_+$. On the right-hand side, agents receive their endowments, observe the market value of the housing assets bought in the last period, and clear debts associated with contracts traded in the last period.

The last term on the right-hand side accounts for the total deliveries from contracts traded in the previous period. This term can be written as the sum of two components, as illustrated in equation (12). For all contracts with $j > q_t$, agents default and deliver the value of the collateral, q_t; for contracts with $j \leqslant q_t$, they fulfill their promise and deliver j.

$$\int_{j\in\mathbb{R}_+} \theta^{a-1}_{j,t-1} min\{j, q_t\} = \int_{j>q_t} q_t \theta^{a-1}_{j,t-1} dj + \int_{j\leqslant q_t} j\theta^{a-1}_{j,t-1} dj. \tag{12}$$

COLLATERAL CONSTRAINT

Borrowing through the sale of contracts requires collateral. Since each contract sold short requires one unit of housing as collateral, agents cannot sell more units of contracts than their current housing stock. Their choices must satisfy the following collateral constraint:

$$\int_{j\in\mathbb{R}_+} max\{-\theta^a_{j_t}, 0\} dj \leqslant h^a_t. \tag{13}$$

[2]I discuss liquidity value in detail in Section II.B, within the proof of Proposition 4.

The term $max\{-\theta^a_{j,t}, 0\}$ in the integrand filters out the contracts on which agents make a net purchase, as these do not require collateral.

NO SHORT-SELLING CONSTRAINT

Agents are prohibited from taking short positions in the housing stock,

$$h^a_t \geqslant 0. \tag{14}$$

E. The Credit Surface

In each period, the Credit Surface, determined by equilibrium prices, provides a pricing schedule for leverage. Specifically, it maps each contract to its LTV and interest rate. Figure 4 illustrates an example Credit Surface at a given time t. Let q^U_{t+1} and q^D_{t+1} represent the realizations of housing prices in states $z_{t+1} = U$ and $z_{t+1} = D$, respectively, with $q^U_{t+1} > q^D_{t+1}$. The points A and B represent the LTV and interest rates of two contracts promising q^D_{t+1} and q^U_{t+1}, respectively.

A description of the properties of the Credit Surface is in order. Proposition 1 establishes that both $\pi_{j,t}$ and $LTV_{j,t}$ are increasing in j, Proposition 2 characterizes the shape of the Credit Surface, and Proposition 3 asserts that the upper limit of $LTV_{j,t}$ is determined within the model and is always strictly below 100%.

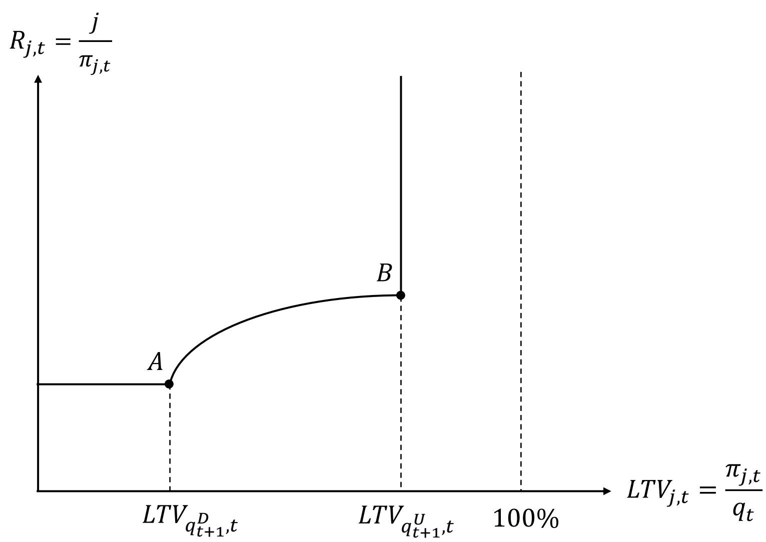

FIGURE 4. A CREDIT SURFACE FOR A BINOMIAL ECONOMY

PROPOSITION 1: *Both $\pi_{j,t}$ and $LTV_{j,t}$ are non-negative and piecewise strictly increasing functions of j over the interval $j \in [0, q^U_{t+1}]$. For $j \in (q^U_{t+1}, \infty)$, $\pi_{j,t}$ remains constant at $\pi_{q^U_{t+1},t}$ and $LTV_{j,t}$ remains constant at $LTV_{q^U_{t+1},t}$.*

PROOF:

Let y_t^z denote the variable y in state z_t. Since households have strictly positive endowments and the period utility functions are convex and satisfy $\lim_{c\to 0} Du_c^a = \infty$ and $\lim_{h\to 0} Du_h^a = \infty$ for $a \in \hat{\boldsymbol{A}}$, and $\lim_{c\to 0} Du_c^A = \infty$, it follows that $c_t^a > 0$ and $Du_c^a > 0$ for all $a \in \boldsymbol{A}$, $h_t^a > 0$ and $Du_h^a > 0$ for all $a \in \hat{\boldsymbol{A}}$, and that q_t is strictly positive.

Consider in equilibrium, an age-a agent makes a net purchase of contract j, i.e., $\theta_{j,t}^a > 0$. It must be that the price of the contract equals to its present value of the actual delivery. The delivery is discounted by the agent's intertemporal marginal rate of substitution of consumption between time t and $t+1$:

$$\begin{aligned}\pi_{j,t} &= E_t\left[\beta \frac{Du_c^{a+1}(c_{t+1}^{a+1}, h_{t+1}^{a+1})}{Du_c^a(c_t^a, h_t^a)} min\{j, q_{t+1}\}\right] \\ &= \sum_{z_{t+1}\in \boldsymbol{Z}} \beta \gamma_{z_t, z_{t+1}} \frac{Du_c^{a+1}(c_{t+1}^{a+1, z_{t+1}}, h_{t+1}^{a+1, z_{t+1}})}{Du_c^a(c_t^a, h_t^a)} min\{j, q_{t+1}^{z_{t+1}}\}. \end{aligned} \tag{15}$$

Define the state-dependent intertemporal marginal utility between z_t and z_{t+1} as follows:

$$M_{t,t+1}^{a, z_{t+1}} \equiv \beta \gamma_{z_t, z_{t+1}} \frac{Du_c^{a+1}(c_{t+1}^{a+1, z_{t+1}}, h_{t+1}^{a+1, z_{t+1}})}{Du_c^a(c_t^a, h_t^a)}.$$

Because this agent is optimizing in equilibrium, increasing or decreasing $\theta_{j,t}^a$ by an infinitesimal amount must yield zero gain in utility, the price must therefore reflect the marginal value of the contract to this agent. Note that $M_{t,t+1}^{a,z_{t+1}}$ is strictly positive for all $z_{t+1} \in \boldsymbol{Z}$.

At equilibrium prices and quantities, the second line of (15) shows that $\pi_{j,t}$ is a continuous piecewise linear function of j for $j \in \mathbb{R}_+$. The slope is $M_{t,t+1}^{a,U} + M_{t,t+1}^{a,D}$ for $j \in [0, q_{t+1}^D]$, and $M_{t,t+1}^{a,U}$ for $j \in [q_{t+1}^D, q_{t+1}^U]$. Therefore, $\pi_{j,t}$ is non-negative and strictly increasing in j over $[0, q_{t+1}^U]$.

For contracts with the promise $j \in (q_{t+1}^U, \infty)$, the actual delivery is q_{t+1}^D in $z_{t+1} = D$, and q_{t+1}^U in $z_{t+1} = U$, which equals to the delivery of contract $j = q_{t+1}^U$. Hence, $\pi_{j,t} = \pi_{q_{t+1}^U, t}$ for $j \in (q_{t+1}^U, \infty)$.

Since q_t is strictly positive, it follows that $LTV_{j,t}$ is non-negative and piecewise strictly increasing in j over the interval $[0, q_{t+1}^U]$, with its slope proportional to the slope of $\pi_{j,t}$ with respect to j, scaled by the inverse of q_t. For $j \in (q_{t+1}^U, \infty)$, $LTV_{j,t}$ remains constant at $LTV_{q_{t+1}^U, t}$.

Proposition 1 establishes that there is a *strict* mapping from j to $LTV_{j,t}$ over $[0, q_{t+1}^U]$ and to $R_{j,t}$ over (q_{t+1}^U, ∞), each contract j corresponds uniquely to a point on the Credit Surface, defined by the pair ($LTV_{j,t}$, $R_{j,t}$). As a result, the market for each contract j effectively functions as a market for leverage at a specific $LTV_{j,t}$. While $\pi_{j,t}$ clears the market for contract j, $R_{j,t}$ serves as the market clearing price for leverage at $LTV_{j,t}$. Both borrowers and lenders choose their leverage based on the Credit Surface, making leverage an endogenous outcome of the interaction between supply and demand.

PROPOSITION 2: *The relationship between $R_{j,t}$ and $LTV_{j,t}$ is characterized as follows:*

- *For $j \in [0, q^D_{t+1}]$, $R_{j,t}$ is constant over the interval $[0, LTV_{q^D_{t+1},t}]$.*
- *For $j \in (q^D_{t+1}, q^U_{t+1}]$, $R_{j,t}$ is increasing and concave in $LTV_{j,t}$ over $(LTV_{q^D_{t+1},t}, LTV_{q^U_{t+1},t}]$.*
- *For $j \in (q^U_{t+1}, \infty)$, as j goes to infinity, $LTV_{j,t}$ remains constant at $LTV_{q^U_{t+1},t}$ while $R_{j,t}$ goes to infinity.*

The Credit Surface is flat until point A, starts rising until point B, after which the surface becomes a vertical line.

Appendix A outlines the proof.

Proposition 2 establishes that, at a given point in time, the interest rate increases with LTV. For contracts with a promise smaller than q^D_{t+1}, borrowers will not default, hence, such contracts are charged with a uniform risk-free rate of interest. For contracts with a promise between q^D_{t+1} and q^U_{t+1}, borrowers default in state $z_{t+1} = D$ but not in $z_{t+1} = U$. The higher the j, the larger the amount of debt borrowers will default on in the D state; therefore, the interest rate is higher for contracts with a larger j. Lastly, when contracts promise an amount exceeding the housing prices in both states, i.e., $j > q^U_{t+1}$, lenders anticipate that borrowers will default in both states and will not pay more than the price of contract $j = q^U_{t+1}$. The interest rate goes to infinity as j goes to infinity and $\pi_{j,t}$ remains constant. This property aligns with the empirical findings of Geanakoplos and Rappoport (2019), who estimate two Credit Surfaces in 2007Q2 and 2008Q4 and demonstrate that the interest rate increases with both LTV and credit scores.

In addition, $R_{j,t}$ is concave in $LTV_{j,t}$ when $j \in (q^D_{t+1}, q^U_{t+1}]$. The concavity implies that as soon as j exceeds q^D_{t+1}, the contract transitions from risk-free to one carrying default risk, leading to a steep increase in the interest rate. Borrowers default in $z_{t+1} = D$, providing small repayment precisely when lenders experience the highest marginal utility. Consequently, lenders become highly sensitive to small increases in leverage and demand a sharp increase in the risk premium to compensate for the perceived default risk.

PROPOSITION 3: *The upper limit of $LTV_{j,t}$ is endogenous and is strictly smaller than 100%.*

PROOF:

Suppose in equilibrium, there exists a contract $k \in J_t$ such that $k > q^U_{t+1} > q^D_{t+1}$, and $LTV_{k,t} \geqslant 100\%$. By the definition of $LTV_{k,t}$, this is equivalent to $\pi_{k,t} \geqslant q_t$. Since k exceeds housing prices in both states, the actual delivery of this contract, $min\{k, q_{t+1}\}$, equals to the realized housing price q_{t+1}. In equilibrium, the following must hold for agents whose collateral constraint is not binding:

$$\pi_{k,t} = E_t\left[\beta \frac{Du_c^{a+1}(c_{t+1}^{a+1}, h_{t+1}^{a+1})}{Du_c^a(c_t^a, h_t^a)} q_{t+1}\right],$$
$$q_t = \frac{Du_h^a(c_t^a, h_t^a)}{Du_c^a(c_t^a, h_t^a)} + E_t\left[\beta \frac{Du_c^{a+1}(c_{t+1}^{a+1}, h_{t+1}^{a+1})}{Du_c^a(c_t^a, h_t^a)} q_{t+1}\right].$$

The second equation says the housing price q_t equals to sum of the immediate utility from housing services and the expected present value of the future housing price. Take the difference between q_t and $\pi_{k,t}$, we get the following equation, which shows that the downpayment associated with this contract equals to the utility derived from a single period of housing consumption:

$$q_t - \pi_{k,t} = \frac{Du_h^a(c_t^a, h_t^a)}{Du_c^a(c_t^a, h_t^a)}.$$

Since $\frac{Du_h^a(c_t^a, h_t^a)}{Du_c^a(c_t^a, h_t^a)}$ must be strictly greater than zero, it must be $q_t > \pi_{k,t}$. This result contradicts the initial assumption $\pi_{k,t} \geqslant q_t$, therefore, $LTV_{k,t} < 100\%$ for all contracts such that $k > q_{t+1}^U > q_{t+1}^D$. Since the prices of all other contracts in J_t are smaller or equal to $\pi_{k,t}$ by Proposition 1, it follows that no contract in J_t can have a price that is greater than or equal to q_t. Hence, the upper limit of $LTV_{j,t}$ is strictly less than 100%[3].

Proposition 3 asserts that agents must make a strictly positive downpayment upfront to acquire housing. Housing assets provide both immediate utility from living in the house and serve as a means of saving. Therefore, even in the case where agents can borrow up to the point where their repayment equals the realized housing price q_{t+1}, they are still obliged to pay for the immediate utility derived from a single period of occupancy. This requirement underscores a key distinction between non-financial assets, like housing, which yield utility to their owners, and financial assets, which do not directly provide utility. If households did not have utility over housing, the downpayment $q_t - \pi_{k,t}$ would have been zero, and $LTV_{j,t}$ could have reached 100%.

F. Collateral Equilibrium

DEFINITION 1: *A collateral equilibrium of the collateral economy is a collection of agents' allocations of consumption and housing, their portfolio holdings of debt contracts, as well as the prices of housing and financial contracts for all t*

$$\Big((c_t^a, h_t^a, (\theta_{j,t}^a)_{j\in J_t})_{a\in \mathbf{A}}; q_t, (\pi_{j,t})_{j\in J_t}\Big)_{t=0}^{\infty} \tag{16}$$

such that

[3]Nilayamgode (2023) also demonstrates that in a two-period model, where non-financial assets serve as collateral and households holding housing assets maintain positive levels of consumption, the LTV ratio can never reach 100%.

1) *Given* $(q_t, (\pi_{j,t})_{j\in J_t})_{t=0}^{\infty}$*, the choices* $((c_t^a, h_t^a, (\theta_{j,t}^a)_{j\in J_t})_{a\in \mathbf{A}})_{t=0}^{\infty}$ *maximize households' lifetime utility* (1)*, subject to the budget constraint* (11)*, the collateral constraint* (13)*, and the no short-selling constraint* (14)*.*

2) *Markets for the consumption good, housing, and all debt contracts in* J_t *clear at each time* t*:*

$$\sum_{a=1}^{A} c_t^a = \bar{e}(z_t), \tag{17}$$

$$\sum_{a=1}^{A} h_t^a = H, \tag{18}$$

$$\sum_{a=1}^{A} \theta_{j,t}^a = 0, \forall j \in J_t. \tag{19}$$

II. Quantitative Analysis

A. Parameterization

TABLE 2—PARAMETERS

Parameter	Value	Interpretation
Uncertainty		
γ_{UD}	0.14	$\gamma_U = 85\%$
γ_{DU}	0.80	$d_U = 7, d_D = 1$
Preferences		
β	0.83	Annual discount rate 0.94
ρ	4.5	Risk-aversion coefficient
α	0.115	Housing share
Endowments		
$\{e^a(U)\}_{a=1}^{20}$		Income (2007)
$\{e^a(D)\}_{a=1}^{20}$		Income (2009)

Note: γ_U denotes the unconditional frequency of the U state. d_U and d_D denote the durations of the U and D states, respectively.

DEMOGRAPHICS

Households live for twenty periods, $A = 20$. In the model, one period is equivalent to three years in real life. Households start their economic life at real age 22 (model age $a = 1$), and live until real age 81 (model age $A = 20$).

Uncertainty

Transition probabilities in Γ are chosen to match two features exhibited by the data: (i) the frequency of a Great Recession-like state is around 15%; (ii) the average duration of the U state is around seven times the duration of D state. I collect US GDP per capita data from two sources: FRED (after 1947) and Maddison Project 2020 (before 1947). Then I use Hodrick-Prescott (HP) filter with a smoothing parameter 1600 to get the trend and cycle. In 2009, the US GDP per capita is 2.46% below trend. I then define the US economy as being in the recession if the GDP per capita in that year falls below 2.4% of the trend. Given this threshold, the US economy has historically been in the recession state 14.6% of the time. The transition probabilities are set such that the economy, on average, spends 85% of the time in a normal state.

Preferences

I set the discount factor β to 0.83, which corresponds to an annual discount factor of 0.94. The coefficient of risk aversion ρ and the weight on housing α are calibrated to 4.5 and 0.115, respectively. This calibration aims to replicate a 10-percentage-point difference in the average leverage of first-time homebuyers (age-1 agents) between states U and D.

Endowments

The total supply of housing is set to $H = 20$. Figure 5 shows the life cycle age profiles of endowments for both states. The Survey of Consumer Finances (SCF) provides detailed data on household income and wealth in the United States. To study the aftermath of the Great Recession, the SCF conducted a special panel survey between 2007 and 2009. Households that participated in the 2007 survey were invited to a follow-up survey in 2009. The income data collected from these households in both years forms the basis for constructing the endowment age profiles across states in the model. Specifically, I divide households aged 22 to 81 into 20 age groups and calculate the average income net of interest and dividend income[4] for each group, using SCF sample weights. Only households who were homeowners in 2007 are included. The dotted lines show the original profiles, which display significant spikes. To ease the computation of equilibria and reflect the general hump-shaped life cycle income profile, I smooth the data, as shown by the solid lines. The smoothed 2007 data represent the endowments in the U state, $\{e^a(U)\}_{a=1}^{20}$, while the smoothed 2009 data represent the D state, $\{e^a(D)\}_{a=1}^{20}$.

[4]The SCF defines total income as the sum of income from the following sources: wages and salaries; business, sole proprietorship, and farm income; interest and dividend income; Social Security and pension benefits; various transfer payments; and capital gains or losses. For the purpose of parametrizing endowments, I exclude interest and dividend income, assuming that these are automatically reinvested in households' stock and bond holdings.

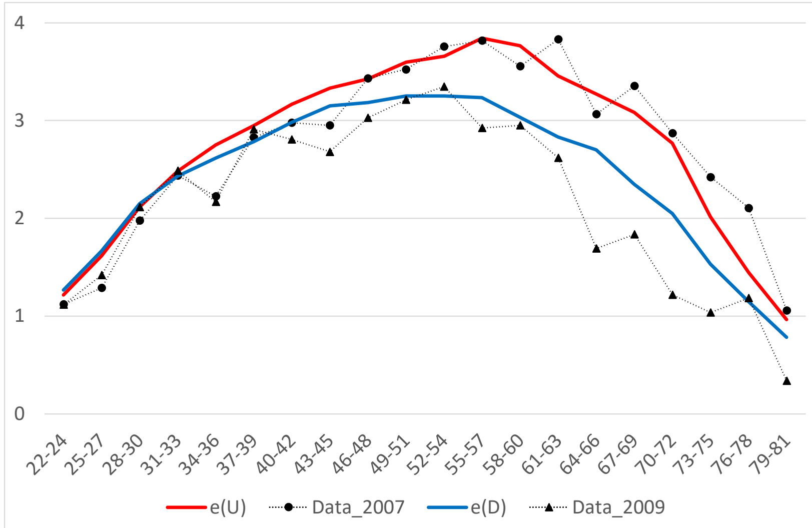

FIGURE 5. LIFE CYCLE AGE PROFILES OF ENDOWMENTS

B. *Numerical Solutions*

CHALLENGES OF LOCATING EQUILIBRIA

Finding a collateral equilibrium for this economy presents two main challenges: one conceptual and the other computational. Conceptually, it is challenging to derive the optimality conditions for households given that there are an infinite number of contracts in J_t. The presence of the term $\int_{j\in\mathbb{R}_+} \pi_{j,t}\theta^a_{j,t}dj$ in the budget constraint complicates the problem, as it is impractical to derive an infinite number of Euler equations regarding each $\theta^a_{j,t}$.

According to Geanakoplos (1997), only a limited set of contracts will be traded in equilibrium due to the scarcity of collateral. However, it remains unclear which specific contract(s) will have both non-zero supply and demand in equilibrium. The No-Default Theorem, formalized by Fostel and Geanakoplos (2015), states that in a binomial economy where financial assets serve as collateral, the contract that maximizes the promised payment without defaulting will be actively traded in equilibrium. However, this theorem does not extend to models in which non-financial assets, such as housing, serve as collateral. A key prerequisite of the theorem is that collateral provides no intrinsic utility to agents, allowing them to freely reshuffle their portfolio holdings to achieve the desired wealth transfer. In this paper, agents cannot freely reshuffle housing holdings because the demand for housing depends not only on its role in transferring wealth across time and states, but also on the intrinsic utility it provides.

Geanakoplos (1997) presents an example with two types of agents and housing as collateral, concluding that, in equilibrium, only the contract promising the highest possible realization of housing prices in the next period would be traded. However, in a more complex setting with 20 types (generations) of agents, such an equilibrium does not exist. Assuming an equilibrium where only the contract that promises q^U_{t+1} is traded in each period, I solve for variables that satisfy all the Karush-Kuhn-Tucker (KKT) conditions and market clearing constraints under this assumption. The results reveal that agents making net purchases of the contract do not agree on the prices of all other contracts in J_t. This disagreement implies the presence of arbitrage opportunities, indicating that prices have not reached equilibrium and thus contradicting the initial assumption.

To address this challenge, I start by guessing an equilibrium regime where only a finite number of contracts are traded and verify whether agents would deviate from it.

Numerically approximating an equilibrium is challenging for two reasons. Firstly, the presence of 20 types of agents trading multiple assets inevitably leads to a large number of state variables. Secondly, the model features occasionally binding collateral constraints, which introduces non-linearity in the policy functions, making them difficult to approximate accurately with linear methods. Azinovic, Gaegauf and Scheidegger (2022) establishes that neural networks, combined with approximating the equilibrium only within the ergodic state space, have the potential to address the challenges of high dimensionality and non-linearity while being computationally efficient. Following their approach, I use a neural network with two hidden layers to approximate the equilibrium.

Functional Rational Expectations Equilibrium

Assuming that an equilibrium exists in which only a finite number of contracts are actively traded with non-zero supply and demand in each period, let $\hat{J}_t$ denote the set of traded contracts, and N_j its cardinality. Under this assumption, contracts not included in $\hat{J}_t$ have zero supply and demand, i.e., $(\theta^a_{j,t})_{\forall a \in \boldsymbol{A}, j \in J_t \setminus \hat{J}_t} = 0$ for all t. This simplifies the budget constraint and reduces the equilibrium conditions to a finite system of Karush-Kuhn-Tucker (KKT) and market clearing conditions.

To numerically approximate the candidate equilibrium, I follow Spear (1988) and Krueger and Kubler (2004) to define the Functional Rational Expectations Equilibrium (FREE), as described below. I then use a neural network to approximate the policy and pricing functions, ensuring that the equilibrium conditions are satisfied within an acceptable tolerance. Upon solving for these functions, I examine whether agents have incentives to deviate and trade contracts not included in $\hat{J}_t$. Specifically, I verify whether the third and fourth conditions in the definition of FREE are satisfied. If these conditions are met, the process concludes. If not, I adjust the set of actively traded contracts and repeat the steps until all conditions in the definition of a FREE are fulfilled. The algorithm of the approximation process is detailed in Appendix B.

DEFINITION 2: *A FREE of the collateral economy, in which only contracts in $\hat{J}_t$ are traded with non-zero demand and supply, consists of $(c^a_t, h^a_t, \theta^a_t, \mu^a_t)_{a \in \boldsymbol{A}}$, and (q_t, π_t), where:*

- $\theta^a_t = (\theta^a_{j,t})_{j \in \hat{J}_t}$ *denotes agents' portfolio holdings of contracts in* $\hat{J}_t$*,*

- $\mu_t^a = (\mu_t^{a,S})_{S \subseteq \hat{J}_t, S \neq \emptyset}$ *denotes the Lagrangian multipliers for the collateral constraints corresponding to all the non-empty subsets S of* $\hat{J}_t$*,*

- $\pi_t = (\pi_{j,t})_{j \in \hat{J}_t}$ *denotes the prices of contracts in* $\hat{J}_t$*,*

as time-invariant policy functions and pricing functions of x_t*, where*

$$x_t = (z_t, (h_{t-1}^a, \theta_{t-1}^a, c_{t-1}^a, e_t^a)_{a \in \mathbf{A}}, (\gamma_{z_t,U}, \gamma_{z_t,D})) \in \mathbf{Z} \times [0,H]^A \times ([-H,H]^{N_j})^A \times ([0, \sum_a e^a(U)]^A)^2 \times [0,1]^2$$

is the state vector[5]*, such that the following conditions are satisfied.*

1) *The following complementary slackness conditions hold for* $a \in \hat{\mathbf{A}}$*,* $j \in \hat{J}_t$*, and for any non-empty subset* $S \subseteq \hat{J}_t$*:*

$$\mu_t^{a,S}(\sum_{j \in S} -\theta_{j,t}^a - h_t^a) = 0, \tag{20}$$

$$\sum_{j \in S} -\theta_{j,t}^a - h_t^a \leqslant 0, \tag{21}$$

$$\mu_t^{a,S} \geqslant 0. \tag{22}$$

2) *Define* $\mu_{j,t}$ *as the sum of Lagrangian multipliers* $\mu_t^{a,S}$ *for all subsets S that include contract j:* $\mu_{j,t}^a \equiv \sum_{S \subseteq \hat{J}_t : j \in S} \mu_t^{a,S}$*. Define* $\mu_{h,t}^a$ *as the sum of Lagrangian multipliers for all non-empty subset S:* $\mu_{h,t} \equiv \sum_{S \subseteq \hat{J}_t} \mu_t^{a,S}$*. The following first-order conditions hold for all* $a \in \hat{\mathbf{A}}$*,* $j \in \hat{J}_t$*:*

$$q_t = \frac{Du_h^a(c_t^a, h_t^a)}{Du_c^a(c_t^a, h_t^a)} + E_t\left[\beta \frac{Du_c^{a+1}(c_{t+1}^{a+1}, h_{t+1}^{a+1})}{Du_c^a(c_t^a, h_t^a)} q_{t+1}\right] + \frac{\mu_{h,t}^a}{Du_c^a(c_t^a, h_t^a)}, \tag{23}$$

$$\pi_{j,t} = E_t\left[\beta \frac{Du_c^{a+1}(c_{t+1}^{a+1}, h_{t+1}^{a+1})}{Du_c^a(c_t^a, h_t^a)} min\{j, q_{t+1}\}\right] + \frac{\mu_{j,t}^a}{Du_c^a(c_t^a, h_t^a)}. \tag{24}$$

Following Fostel and Geanakoplos (2008), I define the fundamental value of housing as the sum of the first two components of q_t *in* (23)*, and the collateral value as the third component.*

[5](i) The vector of the minimal state variables is $(z_t, (h_{t-1}^a, \theta_{t-1}^a)_{a \in \mathbf{A}})$. Azinovic, Gaegauf and Scheidegger (2022) suggest that augmenting the input vector with additional information can help stabilize the training process. Consistent with their findings, I observe that including the endowment vector and transition probabilities in the input vector reduces the training time. Furthermore, in theory, the intertemporal marginal utility of consumption for unconstrained borrowers and lenders is equal in equilibrium, implying that they share the same consumption growth rate, and that the past consumption could help predict current consumption. Motivated by this result, I add the vector of past consumption to the input and find that it further decreases the training time.
(ii) The collateral constraint and the market clearing conditions for contracts restrict θ_{t-1}^a to lie within $[-H,H]^{N_j}$.

3) *The state-dependent intertemporal marginal utilities between t and $t+1$ are equal among unconstrained agents. Unconstrained agents are defined as those for whom the Lagrangian multipliers associated with the collateral constraint are zero, i.e., $\mu_t^a = 0$.*[6] *Let $\bar{\mathbf{A}}$ be the set of unconstrained agents, then for all $a, b \in \bar{\mathbf{A}}$, $z_{t+1} \in \mathbf{Z}$:*

$$\beta \gamma_{z_t, z_{t+1}} \frac{Du_c^{a+1}(c_{t+1}^{a+1, z_{t+1}}, h_{t+1}^{a+1, z_{t+1}})}{Du_c^a(c_t^a, h_t^a)} = \beta \gamma_{z_t, z_{t+1}} \frac{Du_c^{b+1}(c_{t+1}^{b+1, z_{t+1}}, h_{t+1}^{b+1, z_{t+1}})}{Du_c^b(c_t^b, h_t^b)}. \tag{25}$$

$(\pi_{j,t})_{j \in J_t \setminus \hat{J}_t}$ are given by the present value of contracts for unconstrained agents:

$$\pi_{j,t} = E_t \left[\beta \frac{Du_c^{a+1}(c_{t+1}^{a+1}, h_{t+1}^{a+1})}{Du_c^a(c_t^a, h_t^a)} min\{j, q_{t+1}\} \right], where\ a \in \bar{\mathbf{A}}. \tag{26}$$

4) *Define constrained agents as those for whom the collateral constraint is binding, with at least one element of μ_t^a strictly positive, i.e., $\mu_t^a \neq 0$ and $\mu_t^a \geqslant 0$. Such agents exclusively issue one optimal contract, which has the highest liquidity value among all contracts in J_t, and its liquidity value equals to the collateral value*[7]*. Formally, let $j_t^{a,*}$ denote the optimal contract for age-a agent, and $\tilde{\mathbf{A}}$ the set of constrained agents, then for all $a \in \tilde{\mathbf{A}}$:*

$$-\theta_{j_t^{a,*}, t}^a - h_t^a = 0,\ (\theta_{j,t}^a)_{\forall j \in J_t \setminus \{j_t^{a,*}\}} = 0, \tag{27}$$

$$LV_{j_t^{a,*}, t}^a = \frac{\mu_{h,t}^a}{Du_c^a(c_t^a, h_t^a)} \geqslant LV_{j,t}^a,\ \forall j \in J_t \setminus \{j_t^{a,*}\}. \tag{28}$$

5) *The budget constraint is satisfied for all $a \in \mathbf{A}$:*

$$c_t^a + q_t h_t^a + \sum_{j \in \hat{J}_t} \theta_{j,t}^a \pi_{j,t} = e_t^a + q_t h_{t-1}^{a-1} + \sum_{j \in \hat{J}_t} \theta_{j,t-1}^{a-1} min\{j, q_t\}. \tag{29}$$

[6] In equilibrium, unconstrained agents (with $\mu_t^a = 0$) must agree on the prices of all contracts in J_t. Suppose that two unconstrained agents, a and b, have different state-dependent intertemporal marginal utilities, implying different present values for the same contract. Without loss of generality, assume agent a values a contract higher than agent b, agent a would have an incentive to buy more of it, while b would hold less. This incentive to trade indicates that the market is not yet in equilibrium, contradicting the initial assumption. Therefore, unconstrained agents must agree on $(\pi_{j,t})_{j \in J_t}$.

[7] As shown by Geanakoplos and Zame (2014), the liquidity value of the optimal contract equals the collateral value. See Appendix C for a proof of this result in the context of the analyses of a FREE.

6) All markets clear:

$$\sum_{a=1}^{A} c_t^a = \bar{e}(z_t), \tag{30}$$

$$\sum_{a=1}^{A} h_t^a = H, \tag{31}$$

$$\sum_{a=1}^{A} \theta_t^a = 0. \tag{32}$$

PROPOSITION 4: *A FREE for the collateral economy induces a collateral equilibrium.*

PROOF:

To prove that a FREE sufficiently induces a collateral equilibrium, we must establish that, in a FREE, all agents are maximizing their utility subject to the budget constraint, collateral constraint, and the no short-selling constraint, and that all markets clear. The proof proceeds by addressing each condition of a FREE, verifying agent optimization with the absence of incentives to deviate and market clearing.

First, I examine, in a FREE, agents optimize subject to the collateral constraint specified in (13).

In a FREE, under the condition that only a finite number of contracts in the set $\hat{J}_t$ are traded with non-zero supply and demand, the pertinent collateral constraint simplifies to

$$\sum_{j \in \hat{J}_t} max\{-\theta_{j,t}^a, 0\} \leqslant h_t^a.$$

This condition is equivalent to imposing that the sum of the negative positions for all contracts in any non-empty subset of $\hat{J}_t$ does not exceed the housing stock h_t^a. Mathematically, this yields $\sum_{n=1}^{N_j} \binom{N_j}{n}$ distinct inequalities:

$$\sum_{j \in S} -\theta_{j,t}^a \leqslant h_t^a \quad \text{for all non-empty subsets } S \subseteq \hat{J}_t. \tag{33}$$

By reducing the original constraint (13), which involves an integral and a maximization operator, to a set of standard inequality constraints, we make it possible to formulate the Lagrangian problem for the households. The first condition of a FREE specifies the complementary slackness conditions associated with the inequalities in (33).

The second condition of a FREE corresponds to the first-order conditions of the Lagrangian with respect to households' choices of consumption, housing, and contracts in $\hat{J}_t$. The fifth and sixth conditions ensure that the budget constraint and market clearing conditions are satisfied. The no short-selling constraint is satisfied as households derive utility from housing, and the convexity of the utility function ensures non-negative consumption and housing holdings.

The final step is to show that, in a FREE, households have no incentive to deviate by trading contracts not included in the finite set $\hat{J}_t$. We discuss two groups of households based on whether the collateral constraint is binding.

Unconstrained households. For unconstrained agents, the marginal benefit of borrowing using any contract is zero. Consequently, these agents are indifferent to trading any contracts in J_t as long as the collateral constraint is not violated, and thus, they have no incentive to deviate. The third condition of a FREE asserts the absence of arbitrage opportunities: no agent can make a profit in any state given the prevailing prices, and contracts not included in $\hat{J}_t$ are priced based on the intertemporal marginal utility of consumption for unconstrained agents.

Constrained households. The fourth condition in the definition of a FREE asserts that $\hat{J}_t$ contains an optimal contract $j_t^{a,*}$ for constrained agents, which has the highest liquidity value among all contracts in J_t. Consequently, constrained agents have no incentive to issue contracts outside $\hat{J}_t$.

Liquidity value guides borrowers in selecting among various contracts written against the same collateral, as it quantifies the trade-off between the immediate liquidity provided by a contract against its future obligations. By rearranging the Euler equation (24) and using the definition of $LV_{j,t}^a$, we can see that the liquidity value of contract j is equal to $\mu_{j,t}^a$ divided by agent's current marginal utility of consumption[8]:

$$LV_{j,t}^a = \frac{\mu_{j,t}^a}{Du_c^a(c_t^a, h_t^a)}.$$

As previously defined, $\mu_{j,t}^a$ is the sum of Lagrangian multipliers associated with the collateral constraints for every subset of $\hat{J}_t$ that includes contract j. It represents the additional utility agents would gain if they could marginally increase the sales of contract j by relaxing all relevant collateral constraints. Dividing $\mu_{j,t}^a$ by $Du_c^a(c_t^a, h_t^a)$ translates this abstract utility gain into a tangible metric: the liquidity value $LV_{j,t}^a$, which quantifies the marginal benefit in terms of real consumption goods.

If $\mu_{j,t}^a > \mu_{k,t}^a$ for all $k \in J_t$ with $k \neq j$, contract j offers the highest marginal benefit per unit of collateral. In this case, agents will allocate collateral exclusively to contract j until the collateral constraint binds, i.e., $-\theta_{j,t}^a = h_t^a$. If $\mu_{j,t}^a = \mu_{k,t}^a > \mu_{l,t}^a$ for $k \in S_k \subset J_t$ and $l \in J_t \backslash \{j, S_k\}$, agents are indifferent between issuing contract j and any contract in S_k and will set $\theta_{l,t}^a = 0$ for all $l \in J_t \backslash \{j, S_k\}$. For simplicity, we assume that agents exclusively issue contract j in equilibrium. Ensuring that $\hat{J}_t$ contains the optimal contract guarantees that constrained agents do not deviate.

In summary, we have verified that in a FREE, where a finite set of contracts is traded in equilibrium, agents' optimization and market clearing conditions in a collateral equilibrium are satisfied.

[8] For contracts not included in $\hat{J}_t$, their liquidity value can be computed using the definition provided in (10).

III. Results

The procedure of finding a collateral equilibrium stops when the set of actively traded contracts $\hat{J}_t$ contains only the risk-less contract that promises q^D_{t+1} and the risky contract that promises q^U_{t+1}. Appendix D gives detailed description of the accuracy of the solutions. Let $j_{A,t}$ and $j_{B,t}$ denote the risk-less contract and the risky contract respectively. Their corresponding prices are denoted by $\pi_{A,t}$ and $\pi_{B,t}$. The quantities of these contracts held by an age-a agent are denoted by $\theta^a_{A,t}$ and $\theta^a_{B,t}$. In this equilibrium, $\theta^a_{j,t} = 0$ for all $a \in \boldsymbol{A}$, $j \in J_t \backslash \hat{J}_t$. I simulate the economy for a total of $T = 500,000$ periods. All subsequent analyses are based on the 500,000-period simulation[9].

A. Life Cycle Implications

Figure 6 presents the mean life cycle profiles of portfolio holdings in risk-less contract j_A (θ_A), risky contract j_B (θ_B), housing assets, and the average amount of housing pledged as collateral. The collateral profile (marked with Diamonds) consistently lies below the housing profile (marked with Crosses), due to the collateral constraints. This figure suggests that on average, agents within the model transition through four distinct life stages.

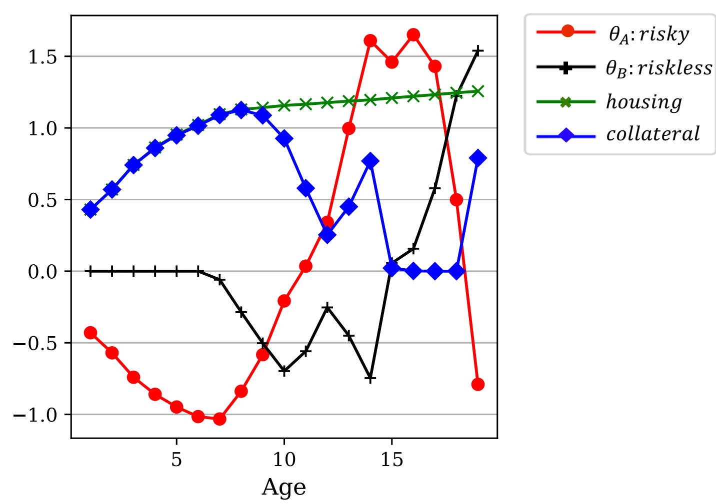

Figure 6. Average Life Cycle Portfolio Holdings

[9]The starting state is selected within the ergodic set of the state space, eliminating the need to discard the initial portion of the simulation to allow the economy to reach its stationary equilibrium.

FIRST STAGE: CONSTRAINED BORROWERS

In the initial stage, agents aged between 1 and 6 choose the portfolio holdings where $\theta^a_{A,t} = 0$, and $-\theta^a_{B,t} = h^a_t$, which indicates that they simultaneously accumulate housing h^a_t and pledge all of these housing assets as collateral to borrow through exclusively selling the risky contract $j_{B,t}$. Because their collateral constraints are binding, they are classified as constrained borrowers.

Anticipating a substantial increase in endowments in the $z_{t+1} = U$ state and a lesser rise in the $z_{t+1} = D$ state, these young agents have two consumption-smoothing motives. First, they would like to transfer wealth from the $z_{t+1} = U$ state, where they expect to be wealthier, to the present by issuing debt contracts secured by their housing. Second, they would like to make small payments in the $z_{t+1} = D$ state in which they feel poor while borrowing as much as possible in the current period. Among all contracts in J_t, the risky contract $j_{B,t}$ allows for the most current borrowing without incurring a high interest rate on the vertical segment of the Credit Surface. Additionally, agents can default on this contract and only deliver the housing collateral with low value q^D_{t+1} in the D state. Consequently, contract $j_{B,t}$ allows for the greatest degree of consumption smoothing across time and states, these agents will find it most advantageous to exclusively issue this particular contract.

Figure 7 examines the liquidity values for age-1 agents. In this figure, there are 500,000 curves, each depicting the liquidity value $LV^{a=1}_{j,t}$ against the promise j across all simulated periods. Each curve corresponds to a distinct time period. All curves are color-coded: red for periods when $z_t = U$ and blue for periods when $z_t = D$. Note that all Up state curves are above the Down state ones, implying that the liquidity value for all contracts are always higher in normal states than crisis states. In each period, the youngest agents find contract $j_{B,t}$ provides the highest liquidity value. Consequently, they use their entire housing collateral to issue this contract.

SECOND STAGE: UNCONSTRAINED BORROWERS

As agents age into the 7 to 10 bracket, they continue to expect higher future endowments and have the motive to transfer future wealth to the present. However, as the average growth rate of their endowment declines compared to that in the initial life stage, their motivation for consumption smoothing declines correspondingly. They start to shift their portfolio holdings away from exclusively issuing the risky contract $j_{B,t}$ to issuing both the risk-less contract $j_{A,t}$ and the risky contract $j_{B,t}$. Their collateral constraints do not bind throughout this stage.

In equilibrium, these agents find the marginal benefit of a marginal increase in the borrowing through issuing any contracts in J_t to be zero. Their desired amount of borrowing can be acquired by pledging only a fraction of their housing stock as collateral. As they are indifferent between issuing any contracts in J_t, their portfolio holdings in contract $j_{A,t}$ and $j_{B,t}$ are pinned down in equilibrium by the market clearing conditions.

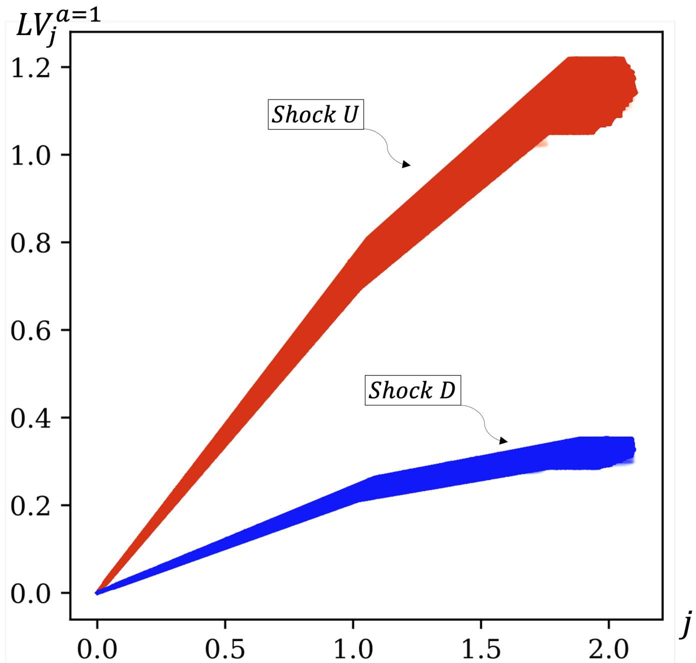

FIGURE 7. LIQUIDITY VALUE FOR $a = 1$ AGENTS

THIRD STAGE: UNCONSTRAINED BORROWERS AND LENDERS

As agents progress into the 11 to 14 age bracket, they anticipate their future endowments to be on a downward trajectory. This expected change is reflected in Figure 6, where the line of the use of collateral (Diamond marker) exhibits a mild increase at age 11. This uptick suggest that agents, upon experiencing the initial decline in their endowment growth at age 11, use greater leverage in accumulating housing assets than when they were age-10. During this stage, while utilizing the housing collateral to borrow with contract $j_{A,t}$, agents start to transfer current wealth to the future by investing in the risky contract $j_{B,t}$.

LAST STAGE: LENDERS

In the final life stage, agents, driven by a strong saving motive and a motive to smooth consumption across states, stop borrowing and begin to diversify their investments. They acquire housing assets with zero leverage, and diversify their saving by lending to younger agents through buying both the risky contract $j_{B,t}$, and the risk-less contract $j_{A,t}$. Let $\tilde{R}_{A,t}$

and $\tilde{R}_{B,t}$ be the real return on contract $j_{A,t}$ and $j_{B,t}$ respectively, they are given by:

$$\tilde{R}_{A,t} = \frac{q^D_{t+1}}{\pi_{A,t}},$$
$$\tilde{R}_{B,t} = \frac{min\{q^U_{t+1}, q_{t+1}\}}{\pi_{B,t}}.$$

Rearrange agents' Euler equations regarding contract $j_{A,t}$ and $j_{B,t}$ and use the definitions of the real return on both assets, the conditions that pin down their portfolio holdings is given by

$$E_t\left[M^a_{t,t+1}(\tilde{R}_{B,t} - \tilde{R}_{A,t})\right] = 0, \tag{34}$$

and the market clearing conditions. In (34), $\tilde{R}_{B,t} - \tilde{R}_{A,t}$ represents the excess real return of holding the risky contract over the risk-less contract. This condition ensures that portfolios are allocated optimally by requiring that the expected excess return of the risky contract, adjusted for lenders' risk preferences, equals zero. Due to their risk aversion, lenders are indifferent between the two assets only after accounting for the trade-off between risk and return, as reflected in their state-dependent marginal utilities of consumption.

LIFE CYCLE PROFILE OF LEVERAGE

Figure 8 presents the mean DLTV for all age groups throughout the simulated periods. On average, DLTV progressively decreases with age and eventually reaches zero, with the exception – an uptick at age 11-12, where agents begin to face declines in their endowments.

B. Leverage Cycle Implications

This model features infrequent disaster events. Within this framework, lenders collectively provide pricing schedules for leverage through the Credit Surface, which they adjust endogenously in response to fundamental shocks. The model generates leverage cycles characterized by the following dynamics: household leverage is positively correlated with housing prices, and amplifies the impact of fundamental shocks on housing prices; mortgage rates and spreads move in the opposite direction, remaining low during normal times while increasing drastically in downturns.

Figure 9 illustrates the Credit Surfaces for each of the 500,000 simulated periods. Table 3 presents the average prices and leverage in states U and D, as well as the differences (Δ), which represent either the absolute change ($D-U$) or the percentage drop ($(D-U)/U$) in the average value of each variable in state D compared to state U. The surfaces in Figure 9 are also color-coded to indicate the aggregate state: red represents state U and blue represents state D. The average Credit Surfaces for states U and D are

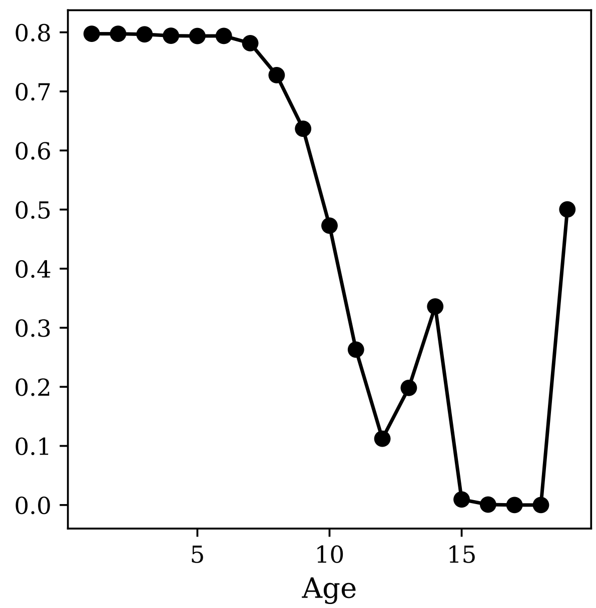

FIGURE 8. AVERAGE DLTV

in darker shades of red and blue, respectively. This figure reveals distinct patterns in the Credit Surface based on the aggregate state. Credit Surfaces associated with D states consistently sit above those for U states, with a tendency to start rising at a lower level of LTV and at a faster speed, becoming vertical at comparatively lower LTV levels. When the economy is in a downturn, the Credit Surface rises and becomes steeper as lenders tighten credit.

The leverage corresponding to the two contracts actively traded in equilibrium differs markedly between states. On average, households can leverage as high as 53% at a risk-free interest rate with the risk-less contract $j_{A,t}$ during normal times, compared to just 48% in downturns. Beyond the risk-free leverage threshold, the Credit Surface rises more sharply in downturns than in normal times, reflecting lenders' higher marginal utility of immediate consumption and their demand for greater excess returns to compensate for forgoing current consumption. The LTV associated with the risky contract $j_{B,t}$ endogenously sets the leverage cap. Using contract $j_{B,t}$, households can leverage 81% on average in normal times, but only 71% in downturns.

In this model, housing plays three critical roles, each influencing the housing prices. First, housing provides immediate utility. As outlined in equation (23), the initial component of housing prices is rent: agents must pay a positive amount of $\frac{Du_h^a(c_t^a,h_t^a)}{Du_c^a(c_t^a,h_t^a)}$ to utilize the property. Second, as housing assets are perfectly durable, they serve as a means of savings. Their value today is tied to the present value of future housing prices, as re-

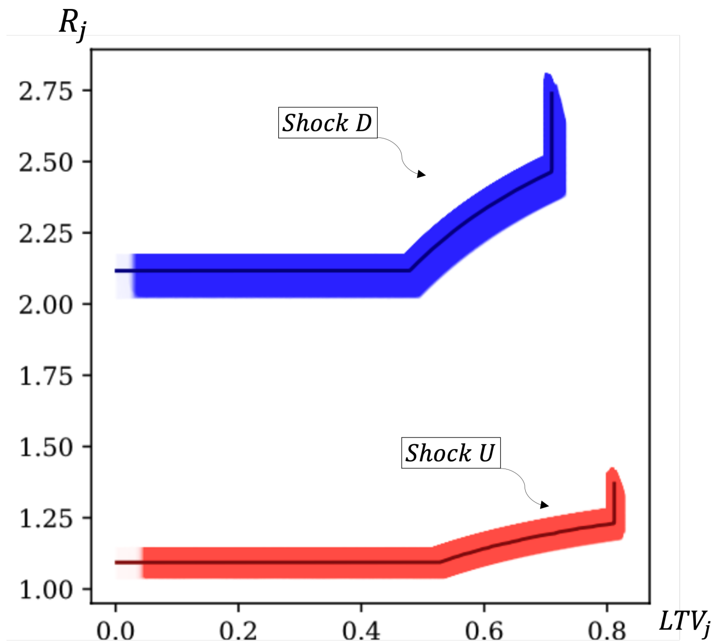

FIGURE 9. CREDIT SURFACES

flected in the second component of housing prices: $E_t\left[\beta\frac{Du_c^{a+1}(c_{t+1}^{a+1},h_{t+1}^{a+1})}{Du_c^a(c_t^a,h_t^a)}q_{t+1}\right]$. The third component, $\frac{\mu_{h,t}^a}{Du_c^a(c_t^a,h_t^a)}$ represents the collateral value of housing, which is derived specifically from its role as collateral.

Housing prices significantly decline when the economy experiences negative endowment shocks. The average housing price in state D is 42% lower than in state U. This decline stems from the simultaneous fall in all three components of housing prices, driven by two interconnected channels, each affecting a distinct type of marginal buyers: the reduced wealth of middle-aged marginal buyers and the diminished leverage capacity of constrained young borrowers.

First, households face lower endowments in downturns, and lenders are hit particularly hard because they save significantly in housing and debt contracts, both of which suffer substantial losses. When young borrowers default, instead of receiving the repayment of $j = q_{t+1}^U$, lenders can only liquidate the collateral at a lower housing price, q_{t+1}^D. Additionally, their housing wealth declines as housing prices fall during downturns. With diminished wealth, all households, especially lenders, are unable to support high rent payments for housing as they could in normal states. Households experience a sharp increase in the marginal utility of consumption, which leads to a heavier discounting of the future and reduces the present value of future housing prices.

Second, and most importantly, leverage is significantly reduced because the capacity

TABLE 3—SUMMARY STATISTICS OF THE COLLATERAL EQUILIBRIUM

	D	U	Δ
q: Housing Price	1.03	1.78	-42.13%
π_A: Risk-less Contract Price	0.49	0.94	-47%
π_B: Risky Contract Price	0.73	1.44	-49%
R_A: Risk-less Interest Rate	2.12	1.09	+94%
R_B: Risky Interest Rate	2.46	1.23	+100%
$R_B - R_A$: Mortgage Spread	0.34	0.14	+142%
LTV_A: Risk-less LTV	47.7%	52.57%	-4.87 pp
LTV_B: Risky LTV	71.09%	81.16%	-10.1 pp

Note: pp stands for percentage points. Column U and column D show the mean of each variable conditional on the exogenous state, while column δ presents their differences across states.

of housing assets to facilitate loans declines sharply. The average prices of risk-less and risky contracts, π_A and π_B, are 47% and 49% lower in state D compared to state U, respectively, exceeding the the magnitude of the decline in housing prices (42%). For constrained borrowers, collateral value is a major component of housing prices, while it is zero for unconstrained borrowers and lenders. As a result, this channel impacts constrained borrowers the hardest.

To examine how the collateral value for each constrained borrower responds to a one-time D shock, I simulate each economy 100,000 times over 20 periods. In each simulation, the economy starts in the U state at time 0, experiences a D shock at time 1, and returns to the U state thereafter. For each time period, I calculate the deviation of each variable relative to its value at time 0 and then average these deviations across all simulations. Figure 10 reports the average percentage deviations in collateral value for age groups 1 to 6. On impact, the collateral value for these constrained borrowers declines sharply, ranging from 70% to 100%. As the third component of housing prices, this decline feeds back to further reductions in housing prices.

C. A Benchmark: The Bond Economy

To understand the impact of endogenously determined leverage on housing price volatility, I analyze a simple bond economy in which this mechanism is absent. The bond economy differs from the baseline collateral economy in three key ways: the only financial asset available is a risk-free bond in zero net supply; agents are assumed to never default; and the LTV ratio is exogenously capped at ϕ.

ENVIRONMENT

Let $(b_t^a)_{a\in\boldsymbol{A}}$ denote the bond holdings of agents, and p_t the price of the bond. Agents, taking prices as given, maximize their life-time utility (1) subject to a budget constraint:

$$c_t^a + q_t h_t^a + p_t b_t^a \leqslant e_t^a + q_t h_{t-1}^{a-1} + b_{t-1}^{a-1},$$

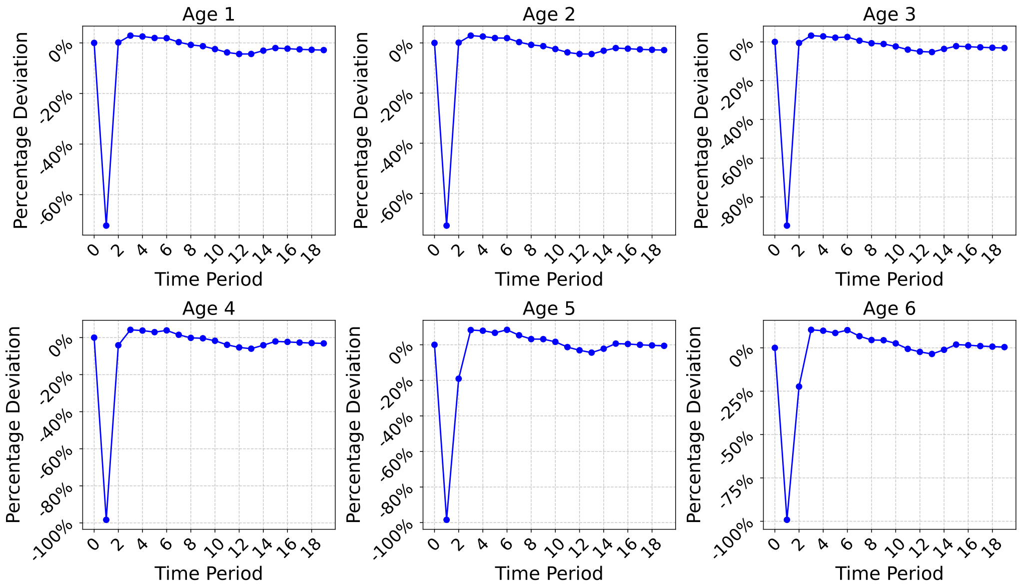

FIGURE 10. PERCENTAGE DEVIATIONS IN COLLATERAL VALUE FOR CONSTRAINED BORROWERS UNDER A *D* SHOCK

Note: For each simulation, I select the initial state vector from the states generated during the 500,000-period simulation where the exogenous state is U. Therefore, each simulation begins with an initial state within the ergodic set of the state space.

and a borrowing constraint, as is standard in the macroeconomic housing literature:

$$-p_t b_t^a \leqslant \phi q_t h_t^a.$$

LTV of an agent is given by:

$$LTV_t^a = \frac{max\{-p_t b_t, 0\}}{q_t h_t}.$$

In the bond economy, interest rates no longer increase in leverage. Instead, they are uniform across all levels of leverage and given by:

$$R_t = \frac{1}{q_t}.$$

If the Credit Surface were plotted, it would appear as a flat line, truncated at $LTV = \phi$.

The aggregate amount of debt in the bond economy is given by the product of the bond price and the aggregate quantity of bonds shorted across agents:

$$p_t \sum_{a \in \boldsymbol{A}} max\{-b_t^a, 0\}.$$

Since bond issuance is not directly tied to the quantity of housing assets, I do not discuss the LTV of housing in this economy. Instead, I define the diluted LTV of housing as the ratio of aggregate debt in the economy to the aggregate value of housing:

$$DLTV_t^h = \frac{p_t \sum_{a \in \mathbf{A}} max\{-b_t^a, 0\}}{q_t H}.$$

NUMERICAL RESULTS AND COMPARISONS

Using the average upper limit of LTV in the normal state of the collateral economy as a reference point, I set $\phi = 81\%$ and solve for the equilibrium. I then simulate the bond economy for 500,000 periods. Table 4 provides an overview of the average behavior of key variables across the two economies, including average housing prices, aggregate debt, and the diluted LTV of housing in both states, as well as the differences of each variable between the two states.

TABLE 4—COMPARISON OF COLLATERAL ECONOMY AND BOND ECONOMY

	Collateral Economy			Bond Economy		
	U	D	Δ	U	D	Δ
Housing Price	1.78	1.03	-42.1%	1.84	1.3	-29.3%
Aggregate debt	15.29	6.81	-55.5%	11.07	8.23	-25.7%
$DLTV^h$	43%	33.2%	-9.8 pp	30.1%	31.6%	1.5 pp
Volatility	16.2%			11.1%		

Note: Volatility refers to the volatility of housing prices, which is measured by the coefficient of variation over the simulation. 16.2% indicates that housing prices deviate by 16.2% from the mean on average.

On average, housing prices are more volatile in the collateral economy compared to the bond economy. In the bond economy, the average housing price in the D state is only 29% lower than in the U state, compared to a 42% decline in the collateral economy. Aggregate debt contracts more sharply during downturns in the collateral economy than in the bond economy. Specifically, the average aggregate debt in the D state is 56% lower than in the U state in the collateral economy, compared to a decline of 26% in the bond economy. In the bond economy, aggregate debt does not decline as much as housing prices in downturns, resulting in higher average aggregate leverage of housing during downturns. Aggregate leverage in the D state is nearly 10 percentage points lower than in the U state in the collateral economy, whereas in the bond economy, the average $DLTV^h$ is higher in the D state than in the U state.

Figure 11 compares the unconditional deviations of these key variables caused by a one-time D shock at time 1 across the two economies. The results echoes the previous findings from the overview of average behavior across states. On impact, housing prices and aggregate debt decline more sharply in the collateral economy compared to the bond economy. In the collateral economy, housing leverage drops by 10 percentage points, whereas in the bond economy, it increases by 1.5 percentage points. In addition, at time

2, the recovery of housing prices and aggregate debt is slower in the collateral economy relative to the bond economy.

In the collateral economy, the majority of debt is issued through the risky contract $j_{B,t}$ that promises q^U_{t+1}. Over the 500,000-period simulation, risky debt accounts for an average of 77% of aggregate debt issuance[10], while risk-free debt makes up the remaining 23%. While the risky contract facilitates consumption smoothing for the young borrowers, it amplifies the volatility of lenders' wealth. In the collateral economy, borrowers deliver q^U_{t+1} in the U state but default on risky loans and deliver only q^D_{t+1} in the D state, leading to significant wealth volatility for lenders across states.

In contrast, this issue does not arise in the bond economy, where lenders receive a fixed payment of 1 unit of the consumption good regardless of the state. Figure 12 illustrates the impact of default. It shows the average percentage deviations of lenders' financial wealth from time 0 under a D shock at time 1. On impact, lenders in the collateral economy experience a significantly sharper decline in wealth and subsequently contract credit more severely compared to those in the bond economy.

The sharp contrast between the behavior of the collateral economy and the bond economy underscores the importance of endogenizing leverage by modeling the Credit Surface and incorporating the possibility of default. Endogenizing leverage provides a clear rationale for the fluctuations in housing finance conditions and explains how exogenous shocks can lead to excess volatility in housing prices.

D. Discussion on Multiple Equilibria in the Collateral Economy

In the collateral economy, agents have infinitely many types of debt contracts as financial instruments to achieve their desired levels of current housing and consumption, as well as state-contingent payoffs in the next period, by adjusting their portfolio holdings. The high degree of agent heterogeneity and the availability of infinitely many assets introduce the possibility of multiple equilibria.

In the previous section, I present a FREE of the collateral economy where the finite set of traded contracts is $\hat{J}_t = \{j_{A,t}, j_{B,t}\}$. Starting from this original equilibrium, it is possible to construct a new equilibrium as follows: constrained agents continue to exclusively issue the optimal contract $j_{B,t}$. Unconstrained agents, however, are free to trade other contracts not included in $\hat{J}_t$, while not changing their housing holdings, as these are determined by their preferences. As long as these adjustments result in the same desired amount of current borrowing or lending, state-contingent next-period wealth, do not violate the collateral constraint, and ensure market clearing for $j_{B,t}$ and the newly traded contracts, the candidate equilibrium satisfies all the conditions specified in the definition of a FREE and thus qualifies as a collateral equilibrium.

However, this new equilibrium is not fundamentally different from the original equilibrium. In this new equilibrium, the allocations of consumption, housing, housing prices, and contract prices remain identical to those in the original equilibrium. The portfolio

[10] The share of risky debt issuance is given by the ratio of the aggregate risky debt to the aggregate debt: $\frac{\pi_{B,t} \sum_{a \in \mathbf{A}} max\{-\theta^a_{B,t}, 0\} dj}{\sum_{a \in \mathbf{A}} \int \pi_{j,t} max\{-\theta^a_{j,t}, 0\} dj}$.

of constrained agents also remains unchanged, while only the portfolio holdings of unconstrained agents may differ. Since households derive utility solely from consumption and housing, portfolio holdings are merely instruments to achieve desired levels of these variables. Therefore, the new equilibrium is economically equivalent to the original one in terms of welfare and macroeconomic implications.

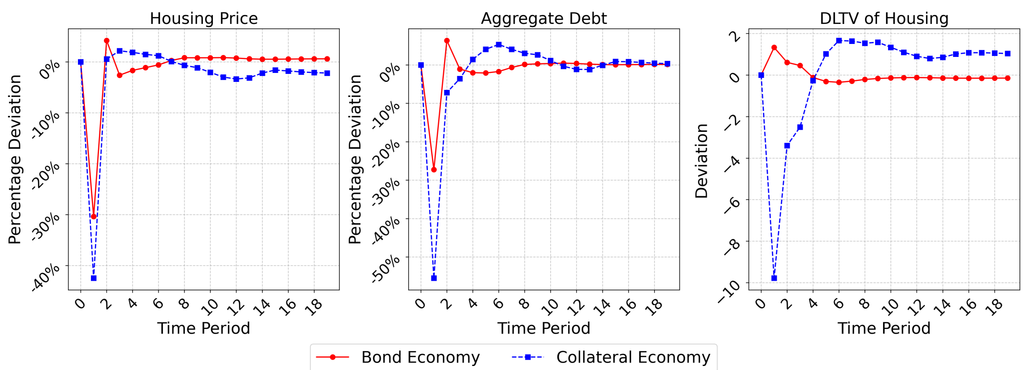

FIGURE 11. DEVIATIONS IN HOUSING PRICE, AGGREGATE DEBT AND LEVERAGE UNDER A D SHOCK: COLLATERAL VS BOND ECONOMIES

Note: Deviations in housing price and credit are expressed as percentages, while deviations in leverage ($DLTV^h$) are in percentage points.

IV. Conclusion

This paper provides a quantitative model to explain co-movements among housing prices, leverage, and mortgage spreads in the context of the U.S. housing market. A critical contribution of this research lies in its endogenous treatment of the LTV ratio in a model featuring a high degree of agent heterogeneity, which challenges the conventional assumption of exogenously given leverage limits. The model shows that both the upper limit of LTV and the entire pricing schedule of leverage, the Credit Surface, are inherently dynamic, reacting to changes in economic fundamentals. This is particularly evident in economic downturns, where the Credit Surface rises and becomes steeper, indicating a tightening of credit conditions. This dynamic response provides an understanding of the decrease in leverage observed during economic crises and highlights a feedback loop between leverage and housing prices.

Looking forward, this research opens several avenues for further inquiries. A key extension would be to incorporate long-term debt contracts. Moreover, comparing the equilibria of economies with and without the endogenous determination of leverage highlights the amplification effect of leverage on housing prices. Using the model as a working horse, we can continue investigating the effects of financial innovations, such as the introduction of credit default swaps (CDS), tranching, and other financial instruments.

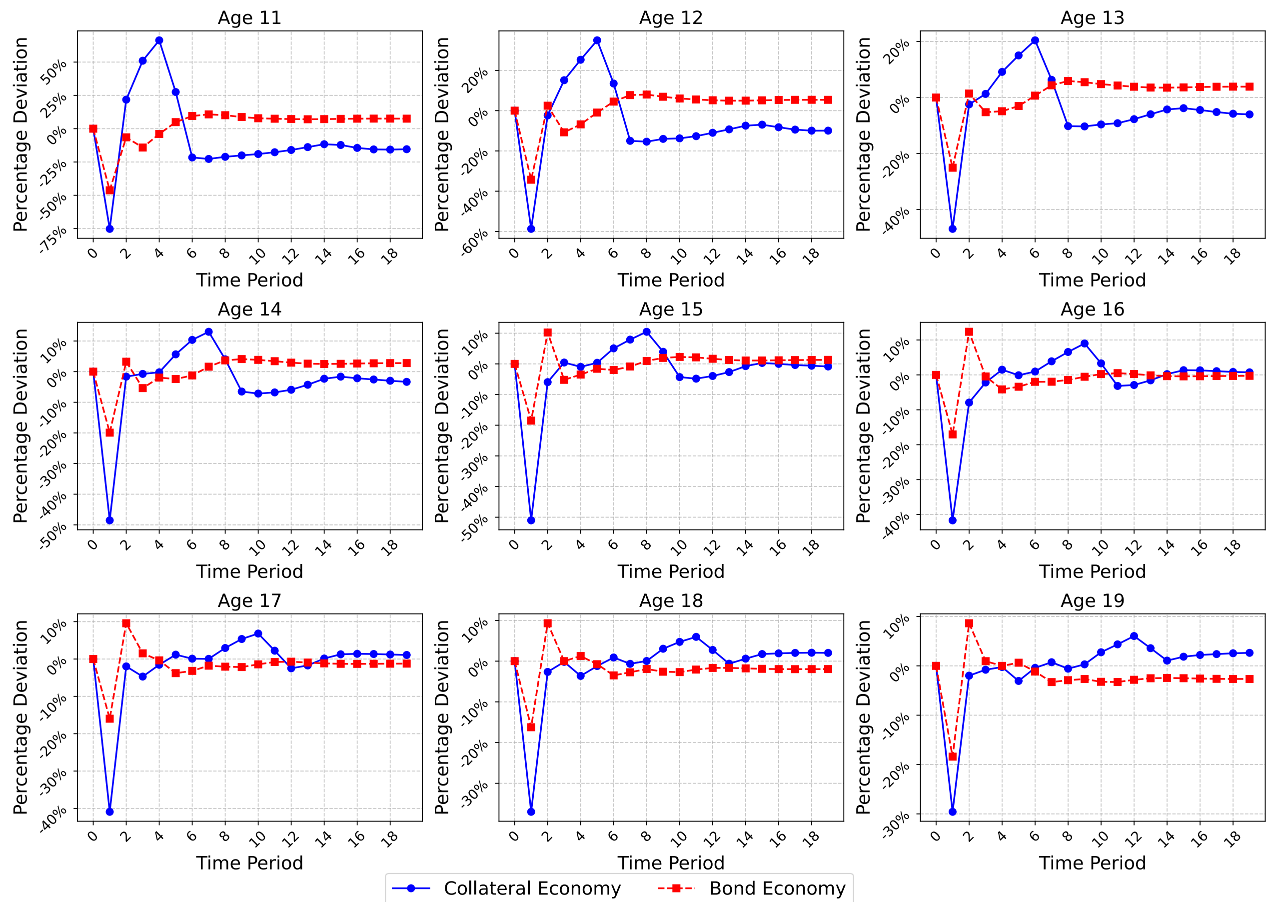

FIGURE 12. PERCENTAGE DEVIATIONS IN FINANCIAL WEALTH UNDER A D SHOCK: COLLATERAL VS BOND ECONOMIES

Note: Financial wealth is the equity after clearing debts: $q_t h_{t-1}^{a-1} + \int_{R_+} \theta_{j,t-1}^{a-1} min\{j, q_t\} dj$ for the collateral economy, and $q_t h_{t-1}^{a-1} + b_{t-1}^{a-1}$ for the bond economy.

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Appendix A: Proof of Proposition 2

PROOF:

Consider three cases: $j \in [0, q^D_{t+1}]$, $j \in (q^U_{t+1}, \infty)$, and $j \in (q^D_{t+1}, q^U_{t+1}]$.

1. Case 1: $j \in [0, q^D_{t+1}]$

According to Equation (15), in equilibrium, the interest rate $R_{j,t}$ is constant over this interval, and equals to the inverse of the expected intertemporal marginal utility of consumption:

$$R_{j,t} = \frac{j}{\pi_{j,t}} = \frac{1}{\sum\limits_{z_{t+1} \in Z} M^{a,z_{t+1}}_{t,t+1}}.$$

2. Case 2: $j \in (q^U_{t+1}, \infty)$

As j goes to infinity, $LTV_{j,t}$ remains constant at $LTV_{q^U_{t+1},t}$, while $R_{j,t}$ approaches infinity. Thus, the Credit Surface becomes a vertical line for $j > q^U_{t+1}$.

3. Case 3: $j \in (q^D_{t+1}, q^U_{t+1}]$

For this interval, $LTV_{j,t}$ is given by:

$$LTV_{j,t} = \frac{M^{a,D}_{t,t+1} q^D_{t+1} + M^{a,U}_{t,t+1} j}{q_t},$$

and the interest rate is:

$$R_{j,t} = \frac{j}{M^{a,D}_{t,t+1} q^D_{t+1} + M^{a,U}_{t,t+1} j}.$$

Using the chain rule, the derivative of $R_{j,t}$ with respect to $\pi_{j,t}$ is:

$$\begin{aligned}
\frac{dR_{j,t}}{d\pi_{j,t}} &= \frac{dR_{j,t}}{dj}\frac{dj}{d\pi_{j,t}} \\
&= \frac{\pi_{j,t} - j\frac{d\pi_{j,t}}{dj}}{(\pi_{j,t})^2} \cdot \frac{1}{\frac{d\pi_{j,t}}{dj}} \\
&= \frac{M^{a,D}_{t,t+1} q^D_{t+1}}{(\pi_{j,t})^2 M^{a,U}_{t,t+1}} > 0.
\end{aligned}$$

Since $q_t > 0$, it follows that $\frac{dR_{j,t}}{dLTV_{j,t}} > 0$. Furthermore, observe that $\frac{dR_{j,t}}{d\pi_{j,t}}$ is decreasing in $\pi_{j,t}$, which implies that $\frac{dR_{j,t}}{dLTV_{j,t}}$ is also decreasing in $LTV_{j,t}$. Hence, $R_{j,t}$ is increasing and concave in $LTV_{j,t}$ for $j \in (q^D_{t+1}, q^U_{t+1}]$.

Appendix B: Algorithm for Approximating a FREE

1) Set the episode counter $s = 0$. Initialize a neural network $\mathcal{N}^{(s)}$ with two hidden layers. The first layer contains 500 nodes, and the second layer contains 300 nodes.

2) Solve for the steady state equilibrium using the endowments for $z = U$, and use the resulting equilibrium variables to construct the initial state vector $x_0{}^{(s)}$. The state vector lies in the space $\boldsymbol{Z} \times [0,H]^A \times ([-H,H]^{N_j})^A \times ([0, \sum_a e^a(U)]^A)^2 \times [0,1]^2$.

3) Simulate the economy using $\mathcal{N}^{(s)}$ and state vector $x_0{}^{(s)}$ over $10,000$ periods. Construct a training dataset $\mathscr{D}^{(s)}_{\text{train}}$ using the 100,000-period simulation. Each simulation counts as one episode within the training sequence.

4) Calculate the loss function, which is the mean squared errors across all equilibrium conditions. These conditions include the Euler equations, market clearing conditions, budget constraints, and complementary slackness conditions, evaluated over the training dataset $\mathscr{D}^{(s)}_{\text{train}}$.

5) Implement the Adam optimization algorithm, a type of mini-batch stochastic gradient descent, to update the neural network parameters from $\mathcal{N}^{(s)}$ to $\mathcal{N}^{(s+1)}$. Update the parameters only once per simulation. Increment the episode counter to $s = s+1$. Set the new initial state vector $x_0{}^{(s+1)}$ as the final state vector from the preceding simulation.

6) Repeat steps 3 to 5 until either the episode counter reaches $200,000$ or the neural network converges. If the loss function does not converge after 200,000 episodes, adjust the learning rate, the size of mini-batches, or the number of nodes in each hidden layers, then return to step 1 with the new parameters.

APPENDIX C: PROOF THAT THE LIQUIDITY VALUE OF THE OPTIMAL CONTRACT EQUALS THE COLLATERAL VALUE OF HOUSING

In this appendix, using proof by contradiction, I show that the liquidity value of the optimal contract, $j_t^{a,*}$, equals the collateral value of housing for constrained agents. For this proof, agents with both a binding collateral constraint and zero Lagrangian multipliers are categorized as unconstrained agents, which are not within the scope of this analysis.
PROOF:

For constrained agents with optimal contract $j_t^{a,*}$, the Lagrangian multiplier associated with the issuance of contract $j_t^{a,*}$ must be strictly positive, and the collateral constraint binds:

$$\mu^a_{j_t^{a,*},t} > 0, \quad -\theta^a_{j_t^{a,*},t} - h^a_t = 0.$$

The multiplier $\mu^a_{j_t^{a,*},t}$ is the sum of the Lagrangian multipliers associated with all non-empty subsets S of $\hat{J}_t$ that include $j_t^{a,*}$:

$$\mu^a_{j_t^{a,*},t} = \sum_S \mu_t^{a,S}, \quad \forall S \text{ such that } S \neq \emptyset,\ j_t^{a,*} \in S,\ S \subseteq \hat{J}_t.$$

Suppose $\mu^a_{j_t^{a,*},t} \neq \mu^a_{h,t}$. Since $\mu^a_{h,t}$ is the sum of the Lagrangian multipliers of all non-empty subsets of $\hat{J}_t$, it must be that:

$$\mu^a_{j_t^{a,*},t} < \mu^a_{h,t}.$$

It follows that the difference between $\mu^a_{h,t}$ and $\mu^a_{j_t^{a,*},t}$ is strictly positive:

$$\mu^a_{h,t} - \mu^a_{j_t^{a,*},t} = \sum_K \mu_t^{a,K} > 0,$$

where K represents all non-empty subsets of $\hat{J}_t$ that do not contain $j_t^{a,*}$.

The condition that $\sum_{K \subseteq \hat{J}_t : j_t^{a,*} \notin K} \mu_t^{a,K} > 0$ indicates that there exists at least one non-empty subset $K \subseteq \hat{J}_t$, with $j_t^{a,*} \notin K$, such that the collateral constraint binds for the set K. This implies that the agent could issue contracts in K to fully utilize their collateral. However, this contradicts the assumption that $j_t^{a,*}$ is the optimal contract that the constrained agent issues exclusively.

Therefore, it must be the case that:

$$\mu^a_{j_t^{a,*},t} = \mu^a_{h,t}.$$

APPENDIX D: ACCURACY OF NUMERICAL SOLUTIONS

This appendix examines the accuracy of the numerical solutions for the collateral economy and the bond economy based on simulations spanning 500,000 periods. The equilibrium of each economy is is characterized by four sets of equations: (1) complementary slackness conditions for collateral constraints or borrowing constraints, (2) Euler equations, (3) budget constraints, and (4) market clearing conditions. In both simulations, the budget constraints are enforced, so there's no approximation errors in this aspect.

Tables D1 and D2 report the errors for the remaining three types of equilibrium conditions for the collateral and bond economies, respectively. The errors are expressed in terms of log_{10}. Following Judd (1992), I define the relative Euler equation errors for each type of condition in the collateral and bond economies as follows.

For the collateral economy, the relative Euler equation errors for housing(h_t^a), the riskless contract($\theta_{A,t}^a$), and the risky contract($\theta_{B,t}^a$) are defined as:

$$e(h_t^a) = |1 - \frac{(Du_c^a)^{-1}\left(\frac{Du_h^a(c_t^a,h_t^a)+\beta E_t[Du_c^{a+1}(c_{t+1}^{a+1},h_{t+1}^{a+1})q_{t+1}]+\mu_{h,t}^a}{q_t}\right)}{c_t^a}|,$$

$$e(\theta_{A,t}^a) = |1 - \frac{(Du_c^a)^{-1}\left(\frac{\beta E_t[Du_c^{a+1}(c_{t+1}^{a+1},h_{t+1}^{a+1})q_{t+1}^D]+\mu_{A,t}^a}{q_t}\right)}{c_t^a}|,$$

$$e(\theta_{B,t}^a) = |1 - \frac{(Du_c^a)^{-1}\left(\frac{\beta E_t[Du_c^{a+1}(c_{t+1}^{a+1},h_{t+1}^{a+1})q_{t+1}]+\mu_{B,t}^a}{q_t}\right)}{c_t^a}|.$$

For the bond economy, the relative Euler equation errors for housing (h_t^a) and bond holdings (b_t^a) are:

$$e(h_t^a) = |1 - \frac{(Du_c^a)^{-1}\left(\frac{Du_h^a(c_t^a,h_t^a)+\phi\mu_t^a q_t+\beta E_t[Du_c^{a+1}(c_{t+1}^{a+1},h_{t+1}^{a+1})q_{t+1}]}{q_t}\right)}{c_t^a}|,$$

$$e(b_t^a) = |1 - \frac{(Du_c^a)^{-1}\left(\frac{\mu_t^a p_t+\beta E_t[Du_c^{a+1}(c_{t+1}^{a+1},h_{t+1}^{a+1})]}{p_t}\right)}{c_t^a}|,$$

where μ_t^a is the Lagrange multiplier associated with the borrowing constraint.

The maximum Euler equation error remains below 10^{-2}, indicating that the largest percentage loss in consumption due to approximation errors in prices is less than 1%. Errors in the market clearing conditions for housing and the two contracts are normalized by the aggregate housing stock H, while errors in consumption are normalized by the aggregate endowment.

TABLE D1—SUMMARY STATISTICS OF ERRORS IN EQUILIBRIUM CONDITIONS: COLLATERAL ECONOMY

Equilibrium Conditions	Type	25th	Median	99th	Mean	Max
Collateral Constraints		-11.28	-9.80	-3.74	-5.28	-3.46
Euler Equations	Housing	-4.11	-3.62	-2.74	-3.42	-2.22
	Risky	-4.03	-3.54	-2.57	-3.31	-2.15
	Risk-less	-3.95	-3.44	-2.40	-3.18	-2.02
Market Clearing Conditions	Housing	-5.41	-5.12	-4.29	-4.97	-3.80
	Risky	-4.86	-4.65	-3.92	-4.54	-3.42
	Risk-less	-5.15	-4.79	-3.93	-4.60	-3.30
	Consumption	-5.09	-4.85	-4.11	-4.73	-3.59

Note: The 25th and 99th refer to the 25th and 99th quantiles of each error in the simulation data series.

TABLE D2—SUMMARY STATISTICS OF ERRORS IN EQUILIBRIUM CONDITIONS: BOND ECONOMY

Equilibrium Conditions	Type	25th	Median	99th	Mean	Max
Borrowing Constraints		-7.88	-5.90	-2.81	-3.88	-2.42
Euler Equations	Housing	-4.20	-3.90	-3.09	-3.76	-2.23
	Bond	-4.12	-3.78	-3.06	-3.69	-2.28
Market Clearing Conditions	Housing	-4.73	-4.63	-4.19	-4.61	-3.69
	Bond	-4.21	-4.18	-3.85	-4.16	-3.52
	Consumption	-4.46	-4.44	-4.27	-4.43	-4.03