Leverage Cycle over the Life Cycle: A Quantitative Model of Endogenous Leverage

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Abstract

I construct a quantitative model to rationalize two robust features of the U.S. housing market: leverage moves in tandem with housing prices, while mortgage spreads move in the opposite direction. In this model, a large number of overlapping generations accumulate housing assets using leverage. They select mortgage contracts from a menu that specifies interest rates for various levels of loan-to-value ratio (LTV), often called a Credit Surface. Large negative endowment shocks reduce housing prices directly by decreasing households' purchasing power, and reinforce the decline by weakening the ability of houses to serve as collateral for borrowing. The Credit Surface rises and gets steeper as interest rates and spreads increase in downturns. Parameters are calibrated to replicate the 10-percentage-point drop in the leverage of first-time homebuyers that was observed during the Great Recession.

Keywords: endogenous leverage, leverage cycle, household life cycle, Credit Surface, housing prices, default
JEL: E20, E44, G51, C68, D52, D53

1. Introduction

The U.S. housing and mortgage markets exhibit two well-documented trends: leverage tends to move in the same direction as housing prices, whereas mortgage spreads move in the opposite direction. The primary contribution of this paper is the development of a quantitative general equilibrium model with endogenous leverage in an overlapping-generation (OLG) framework to rationalize the observed co-movements of housing prices, leverage, and mortgage spreads.

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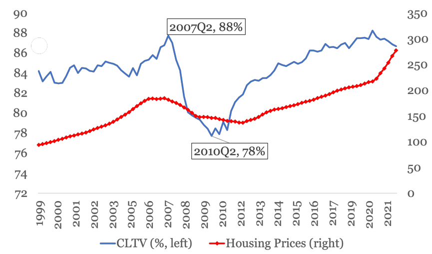
Using quarterly data spanning from 1999Q1 to 2024Q1, table 1 presents the cross correlations of changes in Case-Shiller housing prices with first-time homebuyers' LTV, Combined LTV (CLTV), and mortgage spreads, with the highest absolute correlations underlined. Both LTV and CLTV are positively correlated with housing prices, whereas mortgage spreads are negatively correlated with housing prices. These patterns were especially pronounced in the 2000s, during which the U.S. housing market experienced a leverage cycle. Fostel and Geanakoplos (2014) present the co-movement of leverage and housing prices using data on the average down payment for borrowers below the median in the subprime/Alt-A category. Figure 1 provides further evidence for this pattern by showing trends in housing prices and CLTV for first-time homebuyers in the U.S. Specifically, the CLTV initially increased steadily along with rising housing prices, then experienced a sharp 10-percentage-point drop from its peak in 2007Q2 to its trough in two years. Mortgage spreads measure the relative cost of purchasing housing assets using leverage. Walentin (2014) and Musso, Neri, and Stracca (2011) show that mortgage spreads rise during times of economic stress, especially in the Great Recession. Figure 2 adds to this empirical finding and shows that mortgage spreads fell modestly from 2 in 1999Q1 to 1.5 in 2007Q2, followed by a substantial increase to 2.61 within just one year by 2008Q2.

Table 1: Cross Correlations of Leverage and Mortgage Spreads with Housing prices

	LTV	CLTV	Spread
lead (2)	0.41	0.35	-0.08
lead (1)	0.42	0.39	-0.24
contemporaneous	0.41	0.42	-0.28
lag (1)	0.38	0.42	<u>-0.29</u>
lag (2)	<u>0.53</u>	<u>0.44</u>	-0.08

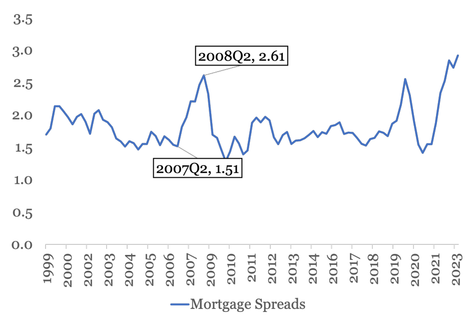
Notes: "lead/lag (i)" indicates that the variable leads/lags housing prices by i quarter(s). LTV is calculated by dividing the balance of the primary mortgage by the appraised value of the property securing the mortgage. CLTV is calculated by summing the balances of all loans secured by a property and then dividing this total by the appraised value of the property. According to Walentin (2014), the duration of a 30-year fixed rate mortgage in the U.S. is 7 to 8 years, Therefore, I define the mortgage spread as the difference between the average 30-year fixed-rate mortgage and the 5-year treasury bill yield. Data source: leverage from Freddie Mac Single-Family Loan-Level Dataset, Case-Shiller housing price index from FRED, index Jan 2000 = 100, mortgage spreads from FRED MORTGAGE30US, DGS5.

To explain these trends, this paper's model considers three key elements that have not been jointly included in the existing literature: large aggregate endowment shocks, the endogenous determination of leverage, and hump-shaped life-cycle endowment profiles of households combined with realistic household lifespans. In the model, the economy is populated by a large number of overlapping generations of households, each deriving utility from both housing and non-housing consumption. Households receive low endowments in early life, which gradually increase and peak in middle age



Data source: CLTV from Freddie Mac Single-Family Loan-Level Dataset, Case-Shiller housing price index from FRED, index Jan 2000=100.

Figure 1: Leverage and Housing Prices



Data source: FRED MORTGAGE30US, DGS10.

Figure 2: Mortgage Spreads

before declining in later stages of life. The consideration of the household life cycle is grounded in micro data. As illustrated in Figure 3, there is an evident age-related pattern in the housing market: housing wealth increases with age then slightly declines, while leverage decreases with age. Households face large aggregate endowment shocks which lead to substantial fluctuations in equilibrium housing prices and mortgage interest rates. Housing assets are perfectly durable and can be used as collateral for issuing debt contracts. A collateral constraint requires that the amount of collateralized contracts issued by an agent must not exceed the quantity of housing stock owned by the agent. Agents can accumulate housing assets using leverage through the simultaneous purchase of housing and issuance of debt contracts. Leverage is measured by LTV, defined as the ratio of the loan amount to the value of the collateral. LTV ranges from 0% to an endogenously determined upper limit, which is always below 100%. Fostel and Geanakopolos (2015) introduce the concept of the "Credit Surface", which represents a menu of leverage levels paired with their corresponding interest rates. Following their methodology, equilibrium prices in each time period define a Credit Surface that specifies the interest rates for different LTV levels. The interest rate increases monotonically with the LTV ratio. For loans with an LTV low enough to rule out default, a uniform risk-free interest rate is applied. However, for loans with an LTV that implies default risk, the interest rate begins to rise as the LTV increases.

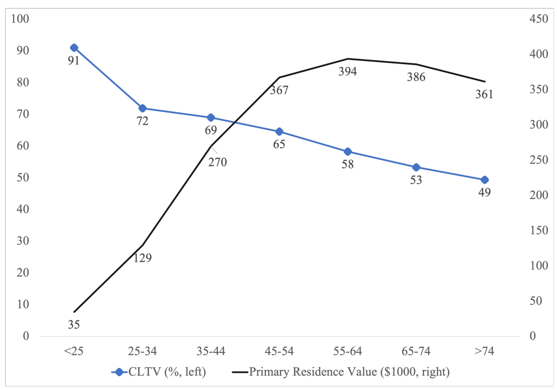


Figure 3: Trends in Age

In this framework, endogenous leverage has a twofold meaning. First, the upper limit of LTV is determined within the model, based on the capacity of housing to serve as collateral for borrowing, and it adjusts in response to changes in economic fundamentals. Second, while agents have infinitely many leverage options, as LTV is a continuous variable, the scarcity of collateral limits equilibrium choices to only a few LTV levels. Borrowers face a trade-off between present liquidity and future obligations: higher leverage provides greater liquidity but comes with higher interest rates. Lenders, driven by consumption-smoothing motives, also chose their optimal LTV levels. Both borrowers and lenders select their optimal positions on the Credit Surface, leaving most LTV levels rationed with zero supply and demand in equilibrium.

My paper resolves two major challenges in numerically solving the proposed model, providing guidance for finding a collateral equilibrium with housing assets and a high degree of agent heterogeneity. The first challenge is conceptual: which LTV levels will be selected in equilibrium? Since LTV is a continuous variable, deriving an infinite number of first-order conditions for each LTV is impractical. Existing literature on endogenous leverage provides theoretical guidelines for finding such equilibria, but these are restricted to cases where agents do not derive utility from collateral assets. Such frameworks are unsuitable for this study, as the demand and supply of leverage are influenced by agents' preferences over housing. To address this challenge, I assume the existence of an equilibrium in which only a finite number of specific LTV levels are selected, then verify whether agents have incentives to deviate by trading contracts with LTV levels outside this finite set. I repeat this procedure until I find an equilibrium. The second challenge is computational in nature. The substantial heterogeneity of agents and the multiplicity of tradable assets result in a high-dimensional state space. In addition, the collateral constraint introduces non-linearity into the policy functions. Azinovic, Gaegauf, and Scheidegger (2022) introduce a methodology that employs neural networks to address the curse of dimensionality and the non-linearity of policy functions. Following their approach, I use a neural network with two hidden layers to directly approximate the policy functions. The resulting maximum normalized Euler equation errors, expressed in consumption units, are less than 1%.

The model has two main implications. First, households start with high leverage and then progressively lower their leverage over the course of their life. In the equilibrium, only two types of contracts are traded, even though infinitely many debt contracts are priced: a risky contract, where agents default during downturns, and a risk-free contract, which ensures no default but offers less liquidity than the risky one. Households transition through four distinct life stages: they start as constrained borrowers, become unconstrained borrowers, transition into a mixed role of borrowers and lenders, and eventually become pure lenders. Second, large aggregate endowment shocks produce leverage cycles. Housing prices and leverage are high when the economy is in a normal state; however, in a crisis state, a negative endowment shock significantly reduces housing prices and leverage while simultaneously increasing mortgage interest rates and spreads.

Life cycle implications. In the first stage, young households receive low endowments. Anticipating an increase in future endowment, they issue risky debt contracts to borrow, pledging all the available housing stock as collateral. As their endowments grow, they transition to using a combination of risky and risk-less contracts for borrowing, without pledging all their owned housing as collateral. In the third stage, as endowments begin to decline, they start investing in risky contracts while also leveraging their housing assets through risk-less contracts. Finally, in the last stage, driven by a strong savings motive, households accumulate more financial wealth. To diversify their investments, they purchase both risky and risk-less debt contracts. On average, younger households are the most leveraged among all age groups, and leverage decreases as households age, a pattern that is consistent with data.

Leverage cycle implications. The second main implication of the model is that leverage and housing prices decline substantially in times of crisis, whereas mortgage spreads surge. These results are consistent with data. The sequence of these events unfolds as follows. When a large negative endowment shock hits the economy, it lowers housing prices directly by reducing the purchasing power of households. In addition, borrowers who had previously issued the risky contract are underwater and default.

This leads to losses for the lenders. As lenders experience losses from defaults and lower endowments, their marginal utility of consumption rises and they start demanding higher returns on lending. Consequently, lenders collectively raise the interest rates charged on all debt contracts, as well as the spreads between risky contracts and riskless contracts. Borrowing becomes more costly for young households in downturns, compromising the ability of housing assets to facilitate borrowing. The upper limit of LTV is reduced significantly as the leveraging capacity of housing assets decreases, thereby amplifying the decline in housing prices.

This paper contributes to two main strands of the literature. First, it builds on the foundational work on endogenous leverage theory by Geanakoplos (1997) and Fostel and Geanakoplos (2008, 2012, 2014). In contrast to the majority of macro-finance models that assume an exogenous cap on leverage, these models take changes in leverage as endogenous responses to shifts in economic fundamentals, and uncover a strong feedback loop between leverage and asset prices. While existing models in this literature are primarily stylized theoretical constructs, this paper bridges the gap between theory and data by incorporating age heterogeneity observed in the housing market.

There are two quantitative works that consider endogenous leverage. Diamond and Landvoigt (2022) develop an OLG model with housing assets as collateral. The key mechanism driving housing booms and busts is a household saving glut, driven by an increased preference for deposits, combined with aggregate negative productivity shocks and idiosyncratic housing wealth shocks that increase housing price dispersion during downturns. The combination of a negative productivity shock and rising housing wealth dispersion leads to substantial defaults and losses for financial intermediaries, prompting them to tighten credit. Brumm, Grill, Kubler, and Schmedders (2015) examine the effects of endogenous leverage on asset price volatility through a model with two infinitely-lived agents with different degrees of risk aversion. In contrast, this paper considers the life-cycle dynamics of households and emphasizes the role of non-financial assets, specifically housing, as collateral, where agents' preferences influence equilibrium prices.

Second, this paper also relates to the macroeconomic literature focusing on the housing market and the households life cycle. Extant models treat the upper limit of LTV as an exogenous credit parameter, and household default is often ruled out. As a result, these models are silent on the factors driving changes in housing finance conditions and overlook the feedback loop between leverage and housing prices. In these settings, variations in the LTV limit have minor effects on housing prices. Kaplan, Mitman, and Violante (2020) analyze the housing boom and bust episode within an OLG model with endogenous housing prices. They conclude that exogenous shifts in the LTV limit do not significantly affect housing prices; instead, fluctuations in housing prices are primarily driven by shifts in households' beliefs about future housing demand. Favilukis, Ludvigson, and Van Nieuwerburgh (2017) also examine the housing market in an OLG model with endogenous housing prices. They find that a higher LTV cap significantly increases housing prices, but this effect is only significant when they introduce heterogeneity in households' bequest preferences, which results in a substantial number of constrained households in equilibrium.

2. Model

2.1. Agents, Commodities and Uncertainty

Time is discrete and indexed by $t = 0, 1, 2, ...$ At each period, one of two possible exogenous aggregate shocks $z_t \in \mathbf{Z} = \{U, D\}$ realizes. The state U stands for "Up", representing normal times. The state D stands for "Down", representing rare crisis states similar to the Great Recession. The aggregate shock z_t affects both the aggregate endowment and the allocation of endowments among different age groups in every period. z_t evolves according to a Markov chain with the transition matrix Γ. Let $\gamma_{z_t,z_{t+1}}$ denote the probability of transitioning from state z_t to state z_{t+1}.

Agents can trade both consumption good (c) and housing (h). The consumption good is perishable, whereas housing assets are perfectly durable and in fixed supply of H. Let the spot price of the consumption good be 1, and the housing price be q_t.

In each period, a continuum of mass 1 identical agents of a new generation is born and lives for A periods. There is no mortality risk; all households die after age A. Age is indexed by $a \in \mathbf{A} = \{1, ..., A\}$. At the beginning of each period, households of age a receive endowments in consumption good $e_t^a = e^a(z_t)$, which depends on the aggregate shock. The aggregate endowment is denoted by $\bar{e}(z_t) = \sum_{a=1}^{A} e^a(z_t)$.

2.2. Preferences

The expected lifetime utility of households born at time t is given by

$$U_t = E_t \sum_{a=1}^{A} \beta^{a-1} u^a(c_t^a, h_t^a), \tag{1}$$

where $\beta > 0$ is the discount factor, c_t^a and h_t^a are the amount of the consumption good and the stock of housing at age a. $u^a(c, h)$ is an age-dependent period utility function. Let $\hat{\mathbf{A}} = \{1, ..., A-1\}$ be the set of agents excluding those who are in the last period of life. All agents have the Cobb-Douglas utility over consumption and housing nested within a constant relative risk aversion utility form when they are at age $a \in \hat{\mathbf{A}}$, and they do not value housing assets in the last period of life, the period utility function is given by

$$u^a(c, h) = \begin{cases} \dfrac{(c^{1-\alpha} h^{\alpha})^{1-\rho}}{1-\rho}, & a \in \hat{\mathbf{A}}, \\ \dfrac{c^{1-\rho}}{1-\rho}, & a = A, \end{cases}$$

where $\alpha > 0$ measures the relative share of housing expenditure, $\rho > 0$ is the coefficient of risk aversion.

2.3. Debt Contracts

Households enter the economy with neither debts or assets. Each period, they meet in anonymous, competitive financial markets to trade collateralized debt contracts. All contracts are one-period. Let J_t denote the set of such contracts available at time t. A financial contract in J_t is defined by an ordered pair representing its promise and collateral requirement, denoted as $(j, 1)$. $j \in \mathbb{R}_+$ is a non-contingent promise to deliver

j units of consumption good in the next period, while the number 1 indicates that the promise j must be backed by one unit of housing as collateral. J_t contains an infinite number of contracts, as j is continuous and unbounded above.

Agents can default on the promise, the consequences of defaulting are limited to losing the collateral they have previously pledged. All contracts are non-recourse, there are no additional penalties in the event of default. An agent who sells one unit of contract j will default on the promise if j exceeds the realized housing price q_{t+1}. Therefore, the delivery of contract j is $\min\{j, q_{t+1}\}$.

The price of contract j is denoted by $\pi_{j,t}$. Let $\theta^a_{j,t} \in \mathbb{R}$ be the number of contract j traded by an agent of age a. $\theta^a_{j,t} < 0 (> 0)$ indicates the agent is shorting (longing) contract j, by doing so the agent borrows (lends) $|\pi_{j,t}\theta^a_{j,t}|$. When agents buy one unit of housing and finance this purchase by selling a debt contract j, they are effectively making a downpayment of $q_t - \pi_{j,t}$ on the house.

Let q^U_{t+1} and q^D_{t+1} denote the realizations of housing prices in states $z_{t+1} = U$ and $z_{t+1} = D$ respectively, assuming that $q^U_{t+1} > q^D_{t+1}$. The following proposition shows that $\pi_{j,t}$ is an increasing function of j.

Proposition 1. *$\pi_{j,t}$ is non-negative, and is strictly increasing in j over $[0, q^U_{t+1}]$, and $\pi_{j,t} = \pi_{q^U_{t+1},t}$ for $j \in (q^U_{t+1}, +\infty)$.*

Proof. Consider in equilibrium, an agent aged a makes a net purchase of contract $(j, 1) \in J_t$, i.e., $\theta^a_{j,t} > 0$, then it must be that

$$\begin{aligned}\pi_{j,t} &= E_t\Big[\beta \frac{Du_c(c^{a+1}_{t+1}, h^{a+1}_{t+1})}{Du_c(c^a_t, h^a_t)} \min\{j, q_{t+1}\}\Big] \\ &= \sum_{z_{t+1} \in \mathbf{Z}} \beta\gamma_{z_t,z_{t+1}} \frac{Du_c(c^{a+1,z_{t+1}}_{t+1}, h^{a+1,z_{t+1}}_{t+1})}{Du_c(c^a_t, h^a_t)} \min\{j, q^{z_{t+1}}_{t+1}\}, \end{aligned} \tag{2}$$

where the right-hand side is the present value of the expected actual delivery of contract $(j, 1)$, discounted by the agent's intertemporal marginal rate of substitution of consumption between time t and $t + 1$. Du_x denotes the derivative of the period utility function with respect to x. y^z_t denotes the variable y at time t in state z. Because this agent is optimizing in equilibrium, increasing or decreasing $\theta^a_{j,t}$ by an infinitesimal amount must yield zero gain in utility. Since $\rho > 0$, $Du_c(c, h) > 0$ for $c > 0$ and $h > 0$. Additionally, $\beta > 0$, therefore the state-dependent intertemporal marginal utility between time t and $t + 1$ $M^{a,z_{t+1}}_{t,t+1} \equiv \beta\gamma_{z_t,z_{t+1}} \frac{Du_c(c^{a+1,z_{t+1}}_{t+1}, h^{a+1,z_{t+1}}_{t+1})}{Du_c(c^a_t, h^a_t)}$, $z_{t+1} \in \mathbf{Z}$ must be strictly positive. At equilibrium prices and quantities, the second line of the equation shows that $\pi_{j,t}$ is a continuous piecewise linear function of j for $j \in \mathbb{R}_+$. The slope is $\sum_{z_{t+1} \in Z} M^{a,z_{t+1}}_{t,t+1} > 0$ for $j \in [0, q^D_{t+1}]$, and $M^{a,U}_{t,t+1} > 0$ for $j \in [q^D_{t+1}, q^U_{t+1}]$. Therefore, $\pi_{j,t}$ is non-negative and is strictly increasing in j for $j \in [0, q^U_{t+1}]$. For contracts with the promise $j \in (q^U_{t+1}, +\infty)$, the actual delivery is q^D_{t+1} in $z_{t+1} = D$, and q^U_{t+1} in $z_{t+1} = U$, which equals to the delivery of contract $j = q^U_{t+1}$. Hence, $\pi_{j,t} = \pi_{q^U_{t+1},t}$ for $j \in (q^U_{t+1}, +\infty)$. □

The gross interest rate for contract j is defined by

$$R_{j,t} = \frac{j}{\pi_{j,t}}.$$

The LTV for contract j is defined by

$$LTV_{j,t} = \frac{\pi_{j,t}}{q_t}.$$

Proposition 1 implies given q_t, $LTV_{j,t}$ is non-negative and strictly increasing in j over $[0, q^U_{t+1}]$, and equals to $LTV_{q^U_{t+1},t}$ for $j \in (q^U_{t+1}, +\infty)$.

Fostel and Geanakoplos (2008) introduce the concept of collateral value and liquidity wedge; Following their discussion, Geanakoplos and Zame (2014) introduce the concept of liquidity value as a way to quantify the benefits of borrowing using different contracts. Following their definition, in this economy, the liquidity value of contract j for an agent of age a at time t, denoted by $LV^a_{j,t}$, is given by contract j's price net of the present value of its delivery, discounted by the agent's stochastic discount factor:

$$LV^a_{j,t} = \pi_{j,t} - E_t\Big[\beta \frac{Du_c(c^{a+1}_{t+1}, h^{a+1}_{t+1})}{Du_c(c^a_t, h^a_t)} \min\{j, q_{t+1}\}\Big]. \tag{3}$$

2.4. Constraints

2.4.1. Budget Constraint

The budget constraint for agents at age a and at time t is given by

$$c^a_t + q_t h^a_t + \int_{R_+} \theta^a_{j,t}\pi_{j,t} dj \leqslant e^a_t + q_t h^{a-1}_{t-1} + \int_{R_+} \theta^{a-1}_{j,t-1}\min\{j, q_t\} dj. \tag{4}$$

On the left-hand side of the budget constraint, there are expenditures on consumption, housing, and the total amount of borrowing (or lending) through trading debt contracts in the financial market. Since j is a continuous variable, the third item is an integral over $j \in \mathbb{R}_+$. On the right-hand side, agents receive their endowments, observe the market value of the housing assets bought in the last period, and clear debts associated with contracts traded in the last period.

The last term on the right-hand side accounts for the total deliveries from contracts traded in the previous period. This term can be written as the sum of two components, as illustrated in equation (5). For all contracts with $j > q_t$, agents default and deliver the value of the collateral, q_t; for contracts with $j \leqslant q_t$, they fulfill their promise and deliver j.

$$\int_{R_+} \theta^{a-1}_{j,t-1}\min\{j, q_t\} = \int_{j>q_t} q_t\theta^{a-1}_{j,t-1} dj + \int_{j\leqslant q_t} j\theta^{a-1}_{j,t-1} dj. \tag{5}$$

2.4.2. Collateral Constraint

Borrowing through the sale of contracts requires collateral. Since each contract sold short requires one unit of housing as collateral, agents cannot sell more units of contracts than their current housing stock. Their choices must satisfy the following collateral constraint:

$$\int_{R_+} \max\{-\theta^a_{j_t}, 0\} dj \leqslant h^a_t. \tag{6}$$

The integrand $\max\{-\theta^a_{j,t}, 0\}$ serves to filter out the contracts on which agents make a net purchase, as these do not require collateral.

2.4.3. No Short-selling Constraint

Agents are prohibited from taking short positions in the housing stock,

$$h^a_t \geqslant 0. \tag{7}$$

2.5. The Credit Surface

Building on the work of Fostel and Geanakopolos (2015), I use the Credit Surface, defined by equilibrium prices, to demonstrate the pricing schedule for leverage. Figure 4 shows a Credit Surface at time t. Since $LTV_{j,t}$ is strictly increasing in j for $j \in [0, q^U_{t+1}]$, each contract in J_t is uniquely characterized by its specific $LTV_{j,t}$ and the corresponding interest rate $R_{j,t}$. Therefore, selecting contracts given contract prices $\pi_{j,t}$is equivalent to choosing leverage (LTV) given the associated interest rates $R_{j,t}$ on the Credit Surface.

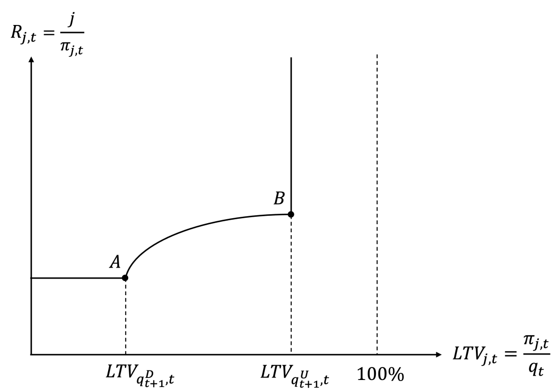


Figure 4: A Credit Surface for a Binomial Economy

Points A and B represent the LTV ratios and interest rates of two contracts promising q^D_{t+1} and q^U_{t+1}, respectively. The Credit Surface is flat until point A, then starts rising until point B, after which the surface becomes a vertical line. Fostel and Geanakopolos (2015) prove that the rising part of the Credit Surface between points A and B is concave. Borrowers will not default on contracts with a promise smaller than q^D_{t+1}; therefore, such contracts will be charged with a uniform risk-free rate of interest. For contracts with a promise between q^D_{t+1} and q^U_{t+1}, borrowers default in state $z_{t+1} = D$ but not in $z_{t+1} = U$. The higher the j, the larger the amount of debt borrowers will default on in the D state; therefore, the interest rate is higher for contracts with a larger j in the interval $[q^D_{t+1}, q^U_{t+1}]$. Lastly, when contracts promise an amount exceeding the housing prices in both states, i.e., $j > q^U_{t+1}$, lenders anticipate that borrowers will default in both states and consequently charge interest rates higher than that of contract $j = q^U_{t+1}$. The transition to a vertical surface after point B can be seen mathematically: for $j > q^U_{t+1}$, as j goes to infinity, $LTV_{j,t}$ remains at $LTV_{q^U_{t+1},t}$, while $R_{j,t} = \frac{j}{\pi_{j,t}} = \frac{j}{\pi_{q^U_{t+1},t}}$ goes to infinity.

Given that housing assets are durable goods, they provide both immediate utility from living in the house and serve as a means of saving. Therefore, even in the case where agents can borrow up to the point where their repayment equals the realized housing price q_{t+1}, they are still obliged to pay for the immediate utility derived from a single period of occupancy. The following proposition shows that agents must make a strictly positive downpayment upfront to acquire housing.

Proposition 2. *The upper limit of $LTV_{j,t}$ is endogenous and is strictly smaller than 100%.*

Proof. Suppose in equilibrium, there exists a contract j such that $j > q^U_{t+1} > q^D_{t+1}$, and $LTV_{j,t} \geqslant 100\%$. By the definition of $LTV_{j,t}$, this is equivalent to $\pi_{j,t} \geqslant q_t$. Since j exceeds housing prices in both states, the actual delivery of this contract, $\min\{j, q_{t+1}\}$, equals to the realized housing price q_{t+1}. In equilibrium, the following must hold for agents who make a net purchase of any contract:

$$\pi_{j,t} = E_t\Big[\beta\frac{Du_c(c^{a+1}_{t+1}, h^{a+1}_{t+1})}{Du_c(c^a_t, h^a_t)}q_{t+1}\Big],$$

$$q_t = \frac{Du_h(c^a_t, h^a_t)}{Du_c(c^a_t, h^a_t)} + E_t\Big[\beta\frac{Du_c(c^{a+1}_{t+1}, h^{a+1}_{t+1})}{Du_c(c^a_t, h^a_t)}q_{t+1}\Big].$$

The second equation says the housing price q_t equals to sum of the immediate utility from housing services and the expected present value of the future housing price q_{t+1}. Take the difference between q_t and $\pi_{j,t}$, we get the following equation, which shows that the downpayment associated with this contract equals to the utility derived from a single period of housing consumption:

$$q_t - \pi_{j,t} = \frac{Du_h(c^a_t, h^a_t)}{Du_c(c^a_t, h^a_t)}.$$

Given that all agents aged $a \in \hat{\mathbf{A}}$ consumes housing, the ratio $\frac{Du_h(c^a_t, h^a_t)}{Du_c(c^a_t, h^a_t)}$ must be strictly greater than zero, which implies $q_t > \pi_{j,t}$. This result contradicts the initial assumption $\pi_{j,t} \geqslant q_t$, therefore, $LTV_{j,t} < 100\%$ for all contracts such that $j > q^U_{t+1} > q^D_{t+1}$. Since the prices of all other contracts in J_t are smaller or equal to $\pi_{j,t}$, it follows that no contract in J_t can have a price $\pi_{j,t}$ that is greater than or equal to q_t. Hence, the upper limit of $LTV_{j,t}$ is strictly less than 100%[1]. □

2.6. Collateral Equilibrium

Definition 1. *A collateral equilibrium of this economy is a collection of agents' allocations of consumption and housing, their portfolio holdings of financial contracts, as well as the prices of housing and financial contracts for all t*

$$\Big((c^a_t, h^a_t, (\theta^a_{j,t})_{j\in J_t})_{a\in\mathbf{A}}; q_t, (\pi_{j,t})_{j\in J_t}\Big)^\infty_{t=0} \tag{8}$$

such that

[1]Nilayamgode (2023) also proves that when non-financial assets serve as collateral, and households who hold housing assets have positive level of consumption, LTV can never go to 100%.

1. *Given* $(q_t, (\pi_{j,t})_{j\in J_t})_{t=0}^{\infty}$, *the choices* $((c_t^a, h_t^a, (\theta_{j,t}^a)_{j\in J_t})_{a\in\mathbf{A}})_{t=0}^{\infty}$ *maximize* (1)*, subject to constraints* (4)*,* (6) *and* (7)*.*
2. *Markets for the consumption good, housing, and all financial contracts in* J_t *clear at each time* t*:*

$$\sum_{a=1}^{A} c_t^a = \bar{e}(z_t),$$

$$\sum_{a=1}^{A} h_t^a = H,$$

$$\sum_{a=1}^{A} \theta_{j,t}^a = 0, \forall j \in J_t.$$

3. Quantitative Analysis

3.1. Parameterization

Table 2: Parameters

Parameter	Value	Interpretation
Uncertainty		
γ_{UD}	0.14	$\gamma_U = 85\%$
γ_{DU}	0.80	$d_U = 7, d_D = 1$
Preferences		
β	0.83	Annual discount rate 0.94
ρ	4.5	Risk-aversion coefficient
α	0.115	Housing share
Endowments		
$\{e^a(U)\}_{a=1}^{20}$		Wage income (2007 SCF)
$\{e^a(D)\}_{a=1}^{20}$		Wage income (2009 SCF)

3.1.1. Demographics

Households live for twenty periods, $A = 20$. In the model, one period is equivalent to three years in real life. Households start their economic life at real age 21 (model age $a = 1$), and live until real age 81 (model age $A = 20$).

3.1.2. Uncertainty

Transition probabilities in Γ are chosen to match two features exhibited by the data: i) the frequency of a Great Recession-like state is around 15%; ii) the average duration of the U state is around 7 times the duration of D state. I collect US GDP per capita data from two sources: FRED (after 1947) and Maddison Project 2020 (before 1947). Then I use Hodrick-Prescott (HP) filter with a smoothing parameter 1600 to get the trend and cycle. In 2009, the US GDP per capita is 2.46% below trend. I then define the US economy as being in the recession if the GDP per capita in that year is below

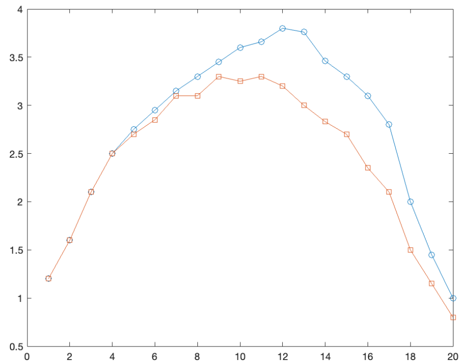


Figure 5: Age Profiles of Endowments

2.4%. Given this threshold, the US economy has historically been in the recession state 14.6% of the time, this frequency serves as one target for the choice of transition probabilities.

3.1.3. Preferences

I set the discount factor β to 0.83, which corresponds to an annual discount factor of 0.94. The coefficient of risk aversion ρ and the weight on housing α are calibrated to 4.5 and 0.115, respectively. This calibration aims to replicate a 10-percentage-point difference in the average LTV for first-time homebuyers between states U and D, thereby capturing the observed change in LTV during the Great Recession.

3.1.4. Endowments

The total supply of housing is set to $H = 20$. Figure 5 presents the life cycle age profiles of endowment, with the line with circle markers representing $\{e^a(U)\}_{a=1}^{20}$ and square representing $\{e^a(D)\}_{a=1}^{20}$. The Survey of Consumer Finances (SCF) provides detailed data on household income and wealth in the U.S. A special panel survey was conducted between 2007 and 2009 to study the aftermath of the Great Recession. In the panel survey, households who had responded to the 2007 survey were invited to participate in a follow-up survey in 2009. The income data collected from the same interviewees in these two years serve as the source for constructing endowment age profiles across different states in the model. To construct the endowment profiles, I divide households from age 21 to 81 into 20 age groups and take the average wage and salary income within each age group, using SCF sample weights. To smooth the resulting profiles, I apply a five-period moving average. The 2007 data represent the endowments in the U state $\{e^a(U)\}_{a=1}^{20}$, while the 2009 data represent the D state $\{e^a(U)\}_{a=1}^{20}$.

3.2. Numerical Solutions

Finding a collateral equilibrium for this economy presents two main challenges: one conceptual and the other computational. Conceptually, it is challenging to derive the optimality conditions for households given that there are an infinite number of contracts in J_t. The presence of the term $\int_{j\in\mathbb{R}_+} \pi_{j,t}\theta^a_{j,t} dj$ in the budget constraint complicates the problem, as it is impractical to derive an infinite number of Euler equations regarding each $\theta^a_{j,t}$.

Geanakoplos (1997) argues that only a limited set of contracts will be traded in equilibrium due to the scarcity of collateral; however, it remains unclear which specific contract(s) will have both non-zero supply and demand in equilibrium. The No-Default Theorem, formalized by Fostel and Geanakoplos (2015), states that in a binomial economy with financial assets as collateral, the contract that makes the maximal promise without defaulting will be the one actively traded in equilibrium. However, the theorem does not apply to models in which non-financial assets, like housing, serve as collateral. Unlike financial assets, non-financial assets provide utility to agents. Consequently, the demand for such assets, and debt contracts tied to them, depends not only on agents' motivations to transfer wealth across time and states, but also on the intrinsic utility derived from the collateral assets.

Geanakoplos (1997) provides an example with two types of agents and housing as collateral, concluding that, in equilibrium, only the contract promising the highest possible realization of housing prices in the next period would be traded. However, in a more complex setting involving 20 types (generations) of agents, such an equilibrium does not exist. Assuming that there exists an equilibrium in which only the contract that promises q^U_{t+1} is traded in each period, I solve for variables that satisfy all the equilibrium conditions under this assumption. In doing so, I find that agents making net purchases of the contract do not agree on the prices of all other contracts in J_t. This disagreement implies that prices have not yet reached an equilibrium, contradicting the initial assumption.

Numerically approximating an equilibrium is challenging in two ways. Firstly, the presence of 20 types of agents who engage in trading multiple assets inevitably leads to a large number of state variables. Secondly, the model features occasionally binding collateral constraints, which induce non-linearity in the policy functions, making them difficult to approximate accurately with linear methods. Following Azinovic, Gaegauf, and Scheidegger (2022), I used a neural network with two hidden layers to directly approximate the the policy functions, addressing the second challenge. They suggest that a neural network with stochastic simulations has the advantage of approximating potentially non-linear policy functions only in the ergodic endogenous state space, even when dealing with a high-dimensional state vector.

To search for a collateral equilibrium, I begin by guessing an equilibrium regime and verifying whether agents would deviate from it. Assuming an equilibrium exists in which only a finite number of contracts are actively traded with non-zero supply and demand in each period, let $\hat{J}_t$ be the set of traded contracts, and N_j be its cardinality. Under this assumption, contracts not included in $\hat{J}_t$ have zero supply and demand, i.e., $(\theta^a_{j,t})_{a\in\hat{\mathbf{A}}, j\notin\hat{J}_t} = 0$ for all t. This simplifies the budget constraint, and reduces the equilibrium conditions to a finite system of Karush-Kuhn-Tucker (KKT) and market clearing conditions.

To numerically approximate the candidate equilibrium, I define the Functional Rational Expectations Equilibrium (FREE), as described below, and use a neural network to approximate the policy and pricing functions such that equilibrium conditions are satisfied within an acceptable tolerance. Upon solving for the functions, I examine whether agents have incentives to deviate and trade contracts not included in $\hat{J}_t$. Specifically, I verify whether the third and fourth conditions in the definition of FREE are satisfied. If they are, I stop the process. If not, I adjust the set of actively traded contracts and repeat the steps until all the conditions described in the definition of a FREE are met. Appendix A outlines the algorithm.

Definition 2. *A FREE, in which only contracts in $\hat{J}_t$ are traded with non-zero demand and supply, consists of $(c_t^a, h_t^a, \boldsymbol{\theta_t^a}, \boldsymbol{\mu_t^a})_{a\in\mathbf{A}}$, and $(q_t, \boldsymbol{\pi_t})$, where:*

- $\boldsymbol{\theta_t^a} = (\theta_{j,t}^a)_{j\in\hat{J}_t}$ *denotes agents' portfolio holdings of contracts in $\hat{J}_t$,*
- $\boldsymbol{\mu_t^a} = (\mu_t^{a,S})_{S\subseteq\hat{J}_t, S\neq\emptyset}$ *denotes the Lagrangian multipliers for the collateral constraint corresponding to all the non-empty subsets S of $\hat{J}_t$,*
- $\boldsymbol{\pi_t} = (\pi_{j,t})_{j\in\hat{J}_t}$ *denotes the prices of contracts in $\hat{J}_t$,*

as time-invariant policy functions and pricing functions of $\mathbf{x_t}$, where

$$\mathbf{x_t} = (z_t, (c_{t-1}^a, h_{t-1}^a, \boldsymbol{\theta_{t-1}^a})_{a\in\boldsymbol{A}}) \in \mathbf{Z} \times \mathbb{R}_+^A \times [0, H]^A \times [-H, H]^{N_j A}$$

is the state vector, such that the following conditions are satisfied.

1. *The following complementary slackness conditions hold for $a \in \hat{\mathbf{A}}$, $j \in \hat{J}_t$, and for any non-empty subset $S \subseteq \hat{J}_t$*[2]*:*

$$\mu_t^{a,S}(\sum_{j\in S} -\theta_{j,t}^a - h_t^a) = 0,$$
$$\sum_{j\in S} -\theta_{j,t}^a - h_t^a \leqslant 0,$$
$$\mu_t^{a,S} \geqslant 0.$$

2. *Define $\mu_{j,t}$ as the sum of Lagrangian multipliers $\mu_t^{a,S}$ for all subsets S that include contract j: $\mu_{j,t}^a \equiv \sum_{S\subseteq\hat{J}_t: j\in S} \mu_t^{a,S}$. Define $\mu_{h,t}^a$ as the sum of all Lagrangian multipliers $\mu_t^{a,S}$: $\mu_{h,t} \equiv \sum_{S\subseteq\hat{J}_t} \mu_t^{a,S}$. The following first-order conditions hold for $a \in \hat{\mathbf{A}}$, $j \in \hat{J}_t$:*

[2]Under the assumption that only contracts $j \in \hat{J}_t$ are traded with non-zero supply and demand, the pertinent collateral constraint simplifies to $\sum_{j\in\hat{J}_t} \max\{-\theta_{j,t}^a, 0\} \leqslant h_t^a$. This condition is equivalent to imposing that the sum of the negative positions for all contracts in any non-empty subset of $\hat{J}_t$ does not exceed the housing stock h_t^a. Mathematically, this yields $\sum_{n=1}^{N_j} \binom{N_j}{n}$ distinct inequalities:

$$\sum_{j\in S} -\theta_{j,t}^a \leqslant h_t^a \quad \text{for all non-empty subsets } S \subseteq \hat{J}_t.$$

$$q_t = \frac{Du_h(c_t^a, h_t^a)}{Du_c(c_t^a, h_t^a)} + E_t\Big[\beta \frac{Du_c(c_{t+1}^{a+1}, h_{t+1}^{a+1})}{Du_c(c_t^a, h_t^a)} q_{t+1}\Big] + \frac{\mu_{h,t}^a}{Du_c(c_t^a, h_t^a)}, \tag{9}$$

$$\pi_{j,t} = E_t\Big[\beta \frac{Du_c(c_{t+1}^{a+1}, h_{t+1}^{a+1})}{Du_c(c_t^a, h_t^a)} min\{j, q_{t+1}\}\Big] + \frac{\mu_{j,t}^a}{Du_c(c_t^a, h_t^a)}. \tag{10}$$

3. *The state-dependent intertemporal marginal utilities between t and t+1 are equal among unconstrained agents with* $\boldsymbol{\mu_t^a} = 0$.[3] *Let* $\bar{\mathbf{A}}$ *be the set of unconstrained agents, then for all* $a, b \in \bar{\mathbf{A}}$, $z_{t+1} \in \mathbf{Z}$*:*

$$M_{t,t+1}^{a,z_{t+1}} = \beta\gamma_{z_t,z_{t+1}} \frac{Du_c(c_{t+1}^{a+1,z_{t+1}}, h_{t+1}^{a+1,z_{t+1}})}{Du_c(c_t^a, h_t^a)} = \beta\gamma_{z_t,z_{t+1}} \frac{Du_c(c_{t+1}^{b+1,z_{t+1}}, h_{t+1}^{b+1,z_{t+1}})}{Du_c(c_t^b, h_t^b)} = M_{t,t+1}^{b,z_{t+1}}.$$

$(\pi_{j,t})_{j \notin \hat{J}_t}$ *are given by the present value of contracts for unconstrained agents:*

$$\pi_{j,t} = E_t\Big[\beta \frac{Du_c(c_{t+1}^{a+1}, h_{t+1}^{a+1})}{Du_c(c_t^a, h_t^a)} min\{j, q_{t+1}\}\Big], where\ a \in \bar{\mathbf{A}}.$$

4. *Constrained agents have one and only one strictly positive Lagrangian multiplier,* $\mu_{j,t}^a$*, among the set* $(\mu_{j,t}^a)_{j \in \hat{J}_t}$*: if* $\mu_{j,t}^a > 0$ *for some contract* j *in* $\hat{J}_t$*, then* $\mu_{i,t}^a = 0$ *for all other contracts* $i \neq j$ *in* $\hat{J}_t$*, and contract* j *has the highest liquidity value among all contracts in* J_t*. i.e.,* $LV_{j,t} \geqslant LV_{i,t}\ \forall i \in J_t$*, where* $i \neq j$*. Liquidity values are given by equation* (3)*.*
5. *The budget constraint is satisfied for all* $a \in \mathbf{A}$*:*

$$c_t^a + q_t h_t^a + \sum_{j \in \hat{J}_t} \theta_{j,t}^a \pi_{j,t} = e_t^a + q_t h_{t-1}^{a-1} + \sum_{j \in \hat{J}_t} \theta_{j,t-1}^{a-1} min\{j, q_t\}.$$

6. *All markets clear:*

$$\sum_{a=1}^{A} c_t^a = \bar{e}(z_t),$$

$$\sum_{a=1}^{A} h_t^a = H,$$

$$\sum_{a=1}^{A} \boldsymbol{\theta_t^a} = \mathbf{0}.$$

[3] In equilibrium, unconstrained agents ($\boldsymbol{\mu_t^a} = 0$) must agree on the prices of all contracts in J_t. Suppose that two unconstrained agents, a and b, have different state-dependent intertemporal marginal utilities, implying different present values for the same contract. Without loss of generality, assume agent a values a contract higher than agent b, agent a would have an incentive to buy more of it, while b would hold less. This incentive to trade indicates that the market is not yet in equilibrium, contradicting the initial assumption. Therefore, unconstrained agents must agree on $(\pi_{j,t})_{j \in \boldsymbol{J_t}}$.

The fourth condition in the definition asserts that constrained agents do not have incentives to trade contracts with lower liquidity values. Liquidity value guides borrowers in selecting among various contracts written against the same collateral, as it quantifies the trade-off between the immediate liquidity provided by a contract against its future obligations. By rearranging the Euler equation (10) and using the definition of $LV^a_{j,t}$, we can see that the liquidity value of contract j is equal to $\mu^a_{j,t}$ divided by agent's current marginal utility of consumption:

$$LV^a_{j,t} = \frac{\mu^a_{j,t}}{Du_c(c^a_t, h^a_t)}.$$

As previously defined, $\mu^a_{j,t}$ is the sum of Lagrangian multipliers corresponding to the collateral constraints for every subset of contracts that contains j, representing the additional gain in utility if agents could marginally increase sales of contract j by relaxing all the relevant collateral constraints. To translate the abstract gain in utility reflected by $\mu^a_{j,t}$ into a more tangible metric, divide $\mu^a_{j,t}$ by $Du_c(c^a_t, h^a_t)$, yielding the liquidity value $LV^a_{j,t}$, which represents the marginal benefit in real consumption good. Holding the marginal utility of consumption fixed at the equilibrium level, comparing liquidity value across j is equivalent to comparing $\mu^a_{j,t}$.

$\mu^a_{j,t} > \mu^a_{k,t}$ indicates that contract j offers a greater marginal benefit per unit of collateral than contract k. If $\mu^a_{j,t} > \mu^a_{k,t}$ for all $k \in J_t$ with $k \neq j$, agents will not waste collateral on any contracts that provide less marginal benefit than contract j. Instead, they increase the issuance of contract j until the collateral constraint binds: $-\theta^a_{j,t} = h^a_t$. If $\mu^a_{j,t} = \mu^a_{k,t} > \mu^a_{l,t}$, for all contracts k in some subset $S_k \subset J_t$, and all contracts l in $J_t \backslash \{j, S_k\}$, then agents are indifferent between issuing contract j and any contract in S_k, and choose $\mu^a_{l,t} = 0$, for all l in $J_t \backslash \{j, S_k\}$. In this case, we can let $-\theta^a_{j,t} = h^a_t$ to simplify the computation. If $\mu^a_{j,t} = 0$ for all $j \in J_t$, agents are indifferent between trading any contracts in J_t without violating the collateral constraint.

A FREE satisfies all the equilibrium conditions of a collateral equilibrium, including agents' optimality conditions and market clearing conditions. Therefore, a FREE effectively induces a collateral equilibrium.

4. Results

The procedure of finding a collateral equilibrium stops when the set of actively traded contracts $\hat{J}_t$ contains only the risk-less contract that promises q^D_{t+1} and the risky contract that promises q^U_{t+1}. Appendix B gives detailed description of the accuracy of the solutions. Let $j_{A,t}$ and $j_{B,t}$ denote the risk-less contract and the risky contract respectively. Their corresponding prices are denoted by $\pi_{A,t}$ and $\pi_{B,t}$. The quantities of these contracts held by an agent of age a at time t are denoted by $\theta^a_{A,t}$ and $\theta^a_{B,t}$. In this equilibrium, $\theta^a_{j,t} = 0$ for all $a \in \mathbf{A}$, $j \in J_t \backslash \hat{J}_t$. I simulate the economy for a total of $T = 500,000$ periods. All subsequent analyses are based on the simulation.

4.1. Life Cycle Implications

Figure 6 presents the mean life cycle profiles of portfolio holdings in risk-less contract j_A (θ_A), risky contract j_B (θ_B), housing assets, and the average amount of housing pledged as collateral. The collateral profile (marked with Diamonds) consistently lies below the housing profile (marked with Crosses), due to the collateral constraints. This figure suggests that on average, agents within the model transition through four distinct life stages.

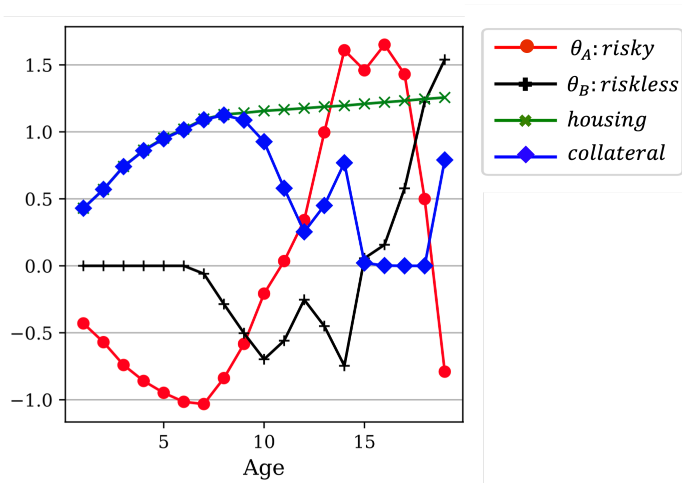


Figure 6: Average Life Cycle Portfolio Holdings

First stage: constrained borrowers. In the initial stage, agents aged between 1 and 6 choose the portfolio holdings where $\theta^a_{A,t} = 0$, and $-\theta^a_{B,t} = h^a_t$, which indicates that they simultaneously accumulate housing h^a_t and pledge all of these housing assets as collateral to borrow through exclusively selling the risky contract $j_{B,t}$. Because their collateral constraints are binding, they are classified as constrained borrowers.

Anticipating a substantial increase in endowments in the $z_{t+1} = U$ state and a lesser rise in the $z_{t+1} = D$ state, these young agents have two consumption-smoothing incentives. First, they would like to transfer wealth from the $z_{t+1} = U$ state, where they expect to be wealthier, to the present by issuing debt contracts secured by their housing. Second, they would like to make small payments in the $z_{t+1} = D$ state in which they feel poor while borrowing as much as possible in the current period. Among all contracts in J_t, the risky contract $j_{B,t}$ allows for the most current borrowing without incurring a high interest rate on the vertical segment of the Credit Surface. Additionally, agents can default on this contract and only deliver the housing collateral with low value q^D_{t+1} in the D state. Consequently, contract $j_{B,t}$ allows for the greatest degree of consumption smoothing across time and states, these agents will find it most advantageous to exclusively issue this particular contract.

Figure 7 examines the liquidity values. In this figure, there are 500,000 curves, each depicting the liquidity value $LV^{a=1}_{j,t}$ for the youngest agent against the promise j across all simulated periods. Each curve corresponds to a distinct time period. All curves are color-coded: red for periods when $z_t = U$ and blue for periods when $z_t = D$. Note that all Up state curves are above the Down state ones, implying that the liquidity value for all contracts are always higher in normal states than crisis states. In each period, the youngest agents find contract $j_{B,t}$ provides the highest liquidity value. Consequently, they use their entire housing collateral to issue this contract, such that $-\theta_{B,t}^{a=1} = h_t^{a=1}$, and $\theta_{j,t}^{a=1} = 0$ for all $j \in J_t \backslash \{j_{B,t}\}$.

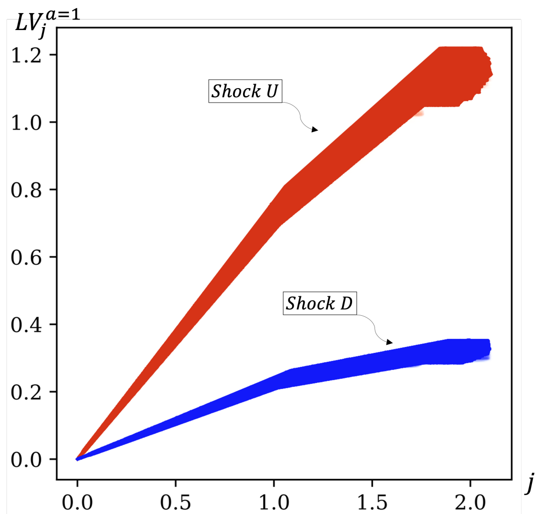


Figure 7: Liquidity Value for $a = 1$ Agents

Second stage: unconstrained borrowers. As agents age into the 7 to 10 bracket, they continue to expect higher future endowments and have the motive to transfer future wealth to the present. However, as the average growth rate of their endowment declines compared to that in the initial life stage, their motivation for consumption smoothing declines correspondingly. They start to shift their portfolio holdings away from exclusively issuing the risky contract $j_{B,t}$ to issuing both the risk-less contract $j_{A,t}$ and the risky contract $j_{B,t}$. Their collateral constraints do not bind throughout this stage.

In equilibrium, these agents find the marginal benefit of a marginal increase in the borrowing through issuing any contracts in J_t to be zero, i.e., $\mu_{j,t}^a = LV_{j,t}^a = 0$ for all $j \in J_t$, $t = 0, ..., T$. They are less liquidity constrained compared to agents in the first life stage, their desired amount of borrowing can be acquired by pledging only a fraction of their housing stock as collateral. As they are indifferent between issuing any contracts in J_t, their portfolio holdings in contract $j_{A,t}$ and $j_{B,t}$ are pinned down in equilibrium by the market clearing conditions.

Third stage: unconstrained borrowers and lenders. As agents progress into the 11 to 14 age bracket, they anticipate their future endowments to be on a downward trajectory. This expected change is reflected in figure 6, where the line of the use of collateral (Diamond marker) exhibits a mild increase at age 12. This uptick suggest that agents, upon experiencing the initial decline in their endowment growth at age 12, use greater leverage in accumulating housing assets than when they were aged 11. During this stage, while utilizing the housing collateral to borrow with contract $j_{A,t}$, agents start to transfer current wealth to the future by investing in the risky contract

$j_{B,t}$.

Last stage: lenders. In the final life stage, agents, driven by a strong saving motive and a motive to smooth consumption across states, stop borrowing and begin to diversify their investments. They acquire housing assets with zero leverage, and diversify their saving by lending to younger agents through buying both the risky contract $j_{B,t}$, and the risk-less contract $j_{A,t}$. Let $\tilde{R}_{A,t}$ and $\tilde{R}_{B,t}$ be the real return on contract $j_{A,t}$ and $j_{B,t}$ respectively, they are given by:

$$\tilde{R}_{A,t} = \frac{\min\{q^D_{t+1}, q_{t+1}\}}{\pi_{A,t}},$$
$$\tilde{R}_{B,t} = \frac{\min\{q^U_{t+1}, q_{t+1}\}}{\pi_{B,t}}.$$

Rearrange agents' Euler equations regarding contract $j_{A,t}$ and $j_{B,t}$ and use the definitions of the real return on both assets, the condition that pins down their portfolio holdings is given by:

$$E_t\Big[M^a_{t,t+1}(\tilde{R}_{B,t} - \tilde{R}_{A,t})\Big] = 0,$$

where $M^a_{t,t+1}$ is the stochastic discount factor and $\tilde{R}_{B,t} - \tilde{R}_{A,t}$ represents the excess return of holding the risky contract over the risk-less contract. This condition says the expected excess return is zero after adjusting for risk by lenders' state-dependent marginal utilities of consumption. $M^a_{t,t+1}$ incorporates lenders' preferences for risk by giving more weight to the excess return in $z_{t+1} = D$ state and less to that in the good state $z_{t+1} = U$.

On average, household leverage decreases with age. To illustrate the life cycle profile of leverage, I follow Fostel and Geanakoplos (2015) and define the leverage of an agent of age a at time t, LTV^a_t, as the ratio of total amount borrowed to the value of the collateral backing this borrowing:

$$LTV^a_t = \frac{\int_{j\in\mathbb{R}_+} \max\{-\theta^a_{j,t}, 0\}\pi_{j,t} dj}{q_t \int_{j\in\mathbb{R}_+} \max\{-\theta^a_{j,t}, 0\} dj}.$$

Recognizing that not all agents fully pledge their housing stock as collateral, LTV^a_t is not well-defined for agents who do not leverage, Fostel and Geanakoplos (2015) propose another measure of leverage, the diluted LTV (DLTV). In this model, it is defined as the total amount of borrowing relative to the value of the agent's entire housing stock:

$$DLTV^a_t = \frac{\int_{j\in\mathbb{R}_+} \max\{-\theta^a_{j,t}, 0\}\pi_{j,t} dj}{q_t h^a_t}.$$

Note that $LTV^a_t \geqslant DLTV^a_t$ as they share the same numerator; however, due to the collateral constraint, LTV^a_t has a smaller denominator than $DLTV^a_t$. Figure 8 presents the mean DLTV for all age groups throughout the simulated periods. DLTV progressively decreases with age and eventually reaches zero, with the exception – an uptick at age 12, where agents begin to face declines in their endowments.

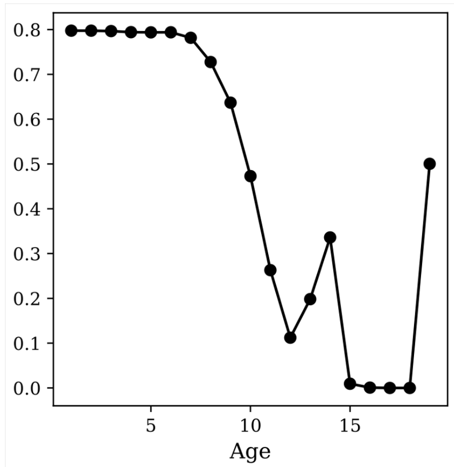


Figure 8: Average DLTV

4.2. Leverage Cycle Implications

This model features infrequent disaster events. Within this framework, lenders collectively provide pricing schedules for leverage through the Credit Surface, which they adjust endogenously in response to fundamental shocks. The model generates leverage cycles characterized by the following dynamics: household leverage is positively correlated with housing prices, and amplifies the impact of fundamental shocks on housing prices; mortgage rates and spreads move in the opposite direction, remaining low during normal times while increasing drastically in downturns.

Figure 9 illustrates the Credit Surfaces for each of the 500,000 simulated periods, table 3 presents the average prices and leverage for state U and D, as well as the differences between these averages. The surfaces in figure 9 are also color-coded to indicate the aggregate state: red represents state U and blue represents state D. The average Credit Surfaces for states U and D are in darker shades of red and blue, respectively. This figure reveals distinct patterns in the Credit Surface based on the aggregate state. Credit Surfaces associated with D states consistently sit above those for U states, with a tendency to start rising at a lower level of LTV and at a faster speed, becoming vertical at comparatively lower LTV levels. When the economy is in a downturn, the Credit Surface rises and becomes steeper as lenders tighten credit. Constrained young households who previously issued the risky contract $j_{B,t}$ find themselves underwater, so they default and deliver the housing collateral, which has a low market price q^D_{t+1}. Lenders experience losses from defaults and receive low endowments. They are reluctant to forgo current consumption for lending and demand higher interest rates on new loans.

The leverage corresponding to the two contracts actively traded in equilibrium differs markedly between states. On average, households can leverage as high as 52.57% at a risk-free interest rate with the risk-less contract $j_{A,t}$ during normal times, compared

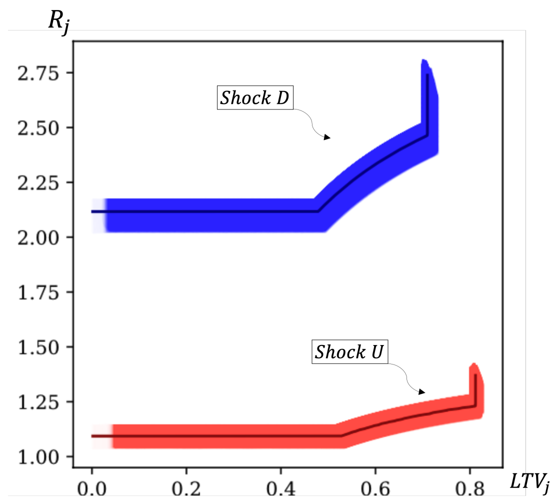


Figure 9: Credit Surfaces

Table 3: Average Prices and Leverage

	D	U	Δ
q: Housing Price	1.03	1.78	-42.13%
π_A: Risk-less Contract Price	0.49	0.94	-47%
π_B: Risky Contract Price	0.73	1.44	-49%
R_A: Risk-less Interest Rate	2.12	1.09	+94%
R_B: Risky Interest Rate	2.46	1.23	+100%
$R_B - R_A$: Mortgage Spread	0.34	0.14	+142%
LTV_A: Risk-less LTV	47.7%	52.57%	-4.87 pp
LTV_B: Risky LTV	71.09%	81.16%	-10.7 pp

Notes: pp stands for percentage points.

to just 47.7% in downturns.[4] Beyond the risk-free leverage threshold, the Credit Surface rises more sharply in downturns than in normal times, reflecting lenders' higher marginal utility of immediate consumption and their demand for greater excess returns to compensate for forgoing current consumption. The LTV associated with the risky contract $j_{B,t}$ endogenously sets the leverage cap. Using contract $j_{B,t}$, households can leverage 81.16% on average in normal times, but only 71.09% in downturns.

In this model, housing plays three critical roles, each influencing the housing prices. First, housing provides immediate utility. As outlined in equation (9), the initial component of housing prices is rent: agents must pay a positive amount of $\frac{Du_h(c_t^a, h_t^a)}{Du_c(c_t^a, h_t^a)}$ to

[4]Note that this feature is attributed to the infrequency of economic rises in the model, a higher frequency of downturns would alter this pattern.

utilize the property. Second, as housing assets are perfectly durable, they serve as a means of savings. Their value today is tied to the present value of future housing prices, as reflected in the second component of housing prices: $E_t\left[\beta\frac{Du_c(c^{a+1}_{t+1},h^{a+1}_{t+1})}{Du_c(c^a_t,h^a_t)}q_{t+1}\right]$. The third component, $\frac{\mu^a_{h,t}}{Du_c(c^a_t,h^a_t)}$, represents the collateral value, which depends on the loan amount the housing collateral can secure. Fostel and Geanakoplos (2008) categorize the sum of the first two components as the fundamental value and the third as the collateral value.

Housing prices significantly decline when the economy faces negative endowment shocks. The average housing price in state D is 42.13% lower compared to state U. This decline is caused by the simultaneous fall in all three components of housing prices. First, negative shocks directly reduce households' purchasing power, resulting in lower rent payments. Second, during downturns, all agents' marginal utility of consumption increases, leading to a heavier discounting of the future and thus lowering the present value of future housing prices. Most importantly, the capacity of housing assets to facilitate loans is substantially reduced. The average prices of risk-less and risky contracts, π_A and π_B, are 47% and 49% lower in state D compared to state U, respectively, exceeding the decline in housing prices (42.13%). Consequently, the role of housing assets as collateral is significantly weakened in downturns, leading to a marked decrease in their collateral value.

In order to demonstrate the amplification effect of leverage on housing prices, I examine the market outcomes in a bond economy, which differs from the baseline model in three ways: there is a single risk-less bond in zero net supply available for trade, agents are assumed to never default, and the LTV ceiling is exogenously fixed at ϕ. Let b^a_t denote the bond holdings for agents of age a at time t, and p_t the price of the bond. Agents, taking prices as given, maximize their life-time utility subject to a budget constraint:

$$c^a_t + q_t h^a_t + p_t b^a_t \leqslant e^a_t + q_t h^{a-1}_{t-1} + b^{a-1}_{t-1},$$

and a borrowing constraint:

$$-p_t b^a_t \leqslant \phi q_t h^a_t.$$

Define the agent's LTV as:

$$LTV^a_t = \frac{\max\{-p_t b_t, 0\}}{q_t h_t},$$

it is straightforward that LTV is capped at ϕ, which does not respond to changes in the fundamentals.

Let $\phi = 0.99$, I solve for the equilibrium and simulate the economy for 510,000 periods, excluding the first 10,000 periods to allow the economy to reach a stationary equilibrium. Table 3 presents the coefficient of variation in housing prices for both the collateral and the bond economies. The standard deviation of housing prices is 16.2% of the mean housing price in the collateral economy, compared to only 11.6% in the bond economy. This result illustrates the amplification effect of leverage on housing price fluctuations.

Collateral Economy	Bond Economy
16.2%	11%

Table 4: Coefficient of Variation of Housing Prices

5. Conclusion

This paper provides a quantitative model to explain co-movements among housing prices, leverage, and mortgage spreads in the context of the U.S. housing market. A critical contribution of this research lies in its endogenous treatment of the LTV ratio in a model featuring a high degree of agent heterogeneity, which challenges the conventional assumption of exogenously given leverage limits. The model shows that both the upper limit of LTV and the entire pricing schedule of leverage, the Credit Surface, are inherently dynamic, reacting to changes in economic fundamentals. This is particularly evident in economic downturns, where the Credit Surface rises and becomes steeper, indicating a tightening of credit conditions. This dynamic response provides a crucial understanding of the decrease in leverage observed during economic crises and highlights a feedback loop between leverage and housing prices.

Looking forward, this research opens several avenues for further inquiries. A key extension would be to incorporate long-term debt contracts. Moreover, comparing the equilibria of economies with and without the endogenous determination of leverage highlights the amplification effect of leverage on housing prices. Using the model as a working horse, we can continue investigating the effects of financial innovations, such as the introduction of credit default swaps (CDS), tranching, and other financial instruments.

Appendix A. Algorithm for Approximating a FREE

1. Set the episode counter $s = 0$. Initialize a neural network $\mathcal{N}^{(s)}$ with two hidden layers. The first layer contains 500 nodes, and the second layer contains 300 nodes.
2. Generate an initial state vector $\mathbf{x_0}^{(s)}$ randomly. This vector includes the aggregate state z_{s-1}, past consumption c^a_{s-1}, housing h^a_{s-1}, and portfolio holdings $\boldsymbol{\theta}^a_{s-1}$ for each age cohort a within the active agent set $\hat{\mathbf{A}}$. The state vector is represented within the space $\mathbf{Z} \times \mathbb{R}^{A-1} \times [0, H]^{A-1} \times [-H, H]^{N_j(A-1)}$.
3. Simulate the economy using $\mathcal{N}^{(s)}$ and state vector $\mathbf{x_0}^{(s)}$ over $12,000$ periods. Discard the first $2,000$ periods to allow the economy to approach a stationary equilibrium (if exists), and construct a training dataset $\mathcal{D}^{(s)}_{\text{train}}$ using the remaining $10,000$ periods. Each simulation counts as one episode within the training sequence.
4. Calculate the loss function, which is the mean squared errors across all equilibrium conditions. These conditions include the Euler equations, market clearing conditions, budget constraints, and complementary slackness conditions, evaluated over the training dataset $\mathcal{D}^{(s)}_{\text{train}}$.
5. Implement the Adam optimization algorithm, a type of mini-batch stochastic gradient descent, to update the neural network parameters from $\mathcal{N}^{(s)}$ to $\mathcal{N}^{(s+1)}$. Update the parameters only once per simulation. Increment the episode counter to $s = s + 1$. Set the new initial state vector $\mathbf{x_0}^{(s+1)}$ as the final state vector from the preceding simulation.
6. Repeat steps 3 to 5 until either the episode counter reaches $100,000$ or the neural network converges. If the loss function does not converge after 100,000 episodes, adjust the learning rate, the size of mini-batches, or the number of nodes in each hidden layers, then return to step 1 with the new parameters.

Appendix B. Accuracy of Numerical Solutions

This appendix examines the accuracy of the numerical solution, based on the simulation spanning 510,000 periods. Throughout the simulation, the budget constraints are enforced, resulting in no errors in this aspect. The average error in the market clearing conditions is 10^{-4}, and the maximum is $10^{-2.7}$. To examine the accuracy of approximating the Euler equations, define the Euler equation error for housing, contract $j_{A,t}$, and contract $j_{B,t}$ as the following:

$$e(h^a_t) = |1 - \frac{(Du_c)^{-1}(\frac{Du_h(c^a_t, h^a_t) + \beta E_t[Du_c(c^{a+1}_{t+1}, h^{a+1}_{t+1}) q_{t+1}] + \mu^a_{h,t}}{q_t})}{c^a_t}|,$$

$$e(\theta^a_{A,t}) = |1 - \frac{(Du_c)^{-1}(\frac{\beta E_t[Du_c(c^{a+1}_{t+1}, h^{a+1}_{t+1}) q^D_{t+1}] + \mu^a_{A,t}}{q_t})}{c^a_t}|,$$

$$e(\theta^a_{B,t}) = |1 - \frac{(Du_c)^{-1}(\frac{\beta E_t[Du_c(c^{a+1}_{t+1}, h^{a+1}_{t+1}) q_{t+1}] + \mu^a_{B,t}}{q_t})}{c^a_t}|.$$

Figure B.10 presents these three kinds of Euler equation errors by age. The errors are displayed in terms of log_{10}, with the solid lines indicating the mean and dash line the maximum. The maximum Euler equation errors is below 10^{-2}, meaning that the maximum percentage loss in consumption due to approximation errors in prices is below 1%.

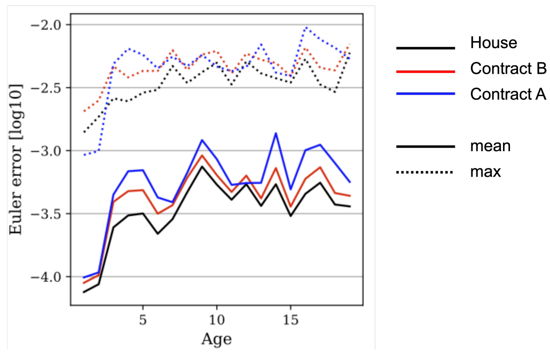


Figure B.10: Euler Equation Errors

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