Leverage Cycle over the Life Cycle: A Quantitative Model of Endogenous Leverage

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Abstract

This paper provides a quantitative model that rationalizes two well-established facts on the US housing market: leverage moves in tandem with housing prices, whereas mortgage spreads move in the opposite direction. In this model, a large number of overlapping generations accumulate housing assets using leverage. They select mortgage contracts from a menu that specifies interest rates for various levels of loan-to-value ratio (LTV), often called a Credit Surface. Within this framework, large negative endowment shocks not only reduce housing prices due to households' decreased purchasing power but also reinforce the decline by weakening the ability of houses to serve as collateral for borrowing. The Credit Surface rises and gets steeper as interest rates and spreads increase in downturns. In an application calibrated to the Great Recession, my model matches the 10-percentage-point drop in the leverage of first-time homebuyers that was observed at that time.

JEL Classification: E20, E44, G51, C68, D52, D53

Keywords: endogenous leverage, leverage cycle, life cycle, Credit Surface, housing prices, default

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1 Introduction

The U.S. housing and mortgage markets exhibit two well-established trends: leverage moves in the same direction as housing prices, while mortgage spreads move in the opposite direction. This chapter develops a quantitative model with endogenous leverage in an overlapping-generation economy to explain these trends.

Using quarterly data spanning from 1999Q1 to 2022Q1, table 1 presents the cross correlations of changes in Case-Shiller housing prices with first-time homebuyers' loan-to-value ratio (LTV)[1], Combined LTV (CLTV)[2], and mortgage spreads (mspread)[3], with the highest absolute correlations underlined. The data reveals a positive correlation between both LTV and CLTV with housing prices, with LTV leading and CLTV lagging. In contrast, mortgage spreads exhibit negative correlations and lag one quarter behind housing prices. These patterns were especially pronounced in the 2000s, during which the U.S. housing market experienced a leverage cycle. Fostel and Geanakoplos (2014) present the co-movement of leverage and housing prices using data on the average down payment for borrowers below the median in the subprime/Alt-A category. Figure 1 provides further evidence for this pattern by showing trends in housing prices and CLTV for first-time homebuyers in the U.S. Specifically, the CLTV initially increased steadily along with rising housing prices, then experienced a sharp 10-percentage-point drop from its peak in 2007Q2 to its trough in two years. Regarding the relative cost of leverage, Walentin (2014) and Musso, Neri, and Stracca (2011) show that mortgage spreads tend to rise during times of economic stress, especially in the Great Recession. Figure 2 adds to this empirical finding and shows that mortgage spreads fell modestly from 2 in 1999Q1 to 1.5 in 2007Q2, followed by a substantial increase

[1]LTV is calculated by dividing the balance of the primary mortgage by the appraised value of the property securing the mortgage.

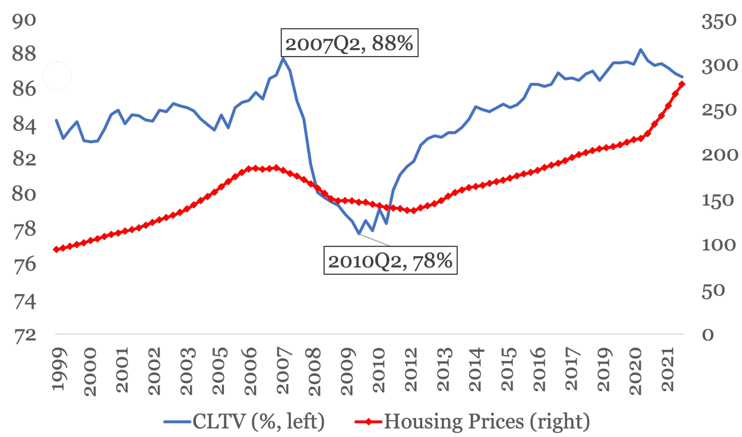
[2]CLTV is calculated by summing the balances of all loans secured by a property and then dividing this total by the appraised value of the property.

[3]According to Walentin (2014), the duration of a 30-year fixed rate mortgage in the U.S. is 7 to 8 years, Therefore, I define the mortgage spread (mspread) as the difference between the average 30-year fixed-rate mortgage and the 10-year treasury bill yield.

to 2.61 within just one year by 2008Q2.

	LTV	CLTV	mspread
lead (2)[4]	0.46	0.40	-0.05
lead (1)	0.44	0.44	-0.23
contemporaneous	0.43	0.49	-0.30
lag (1)	0.40	0.52	-0.31
lag (2)	0.39	0.56	-0.29

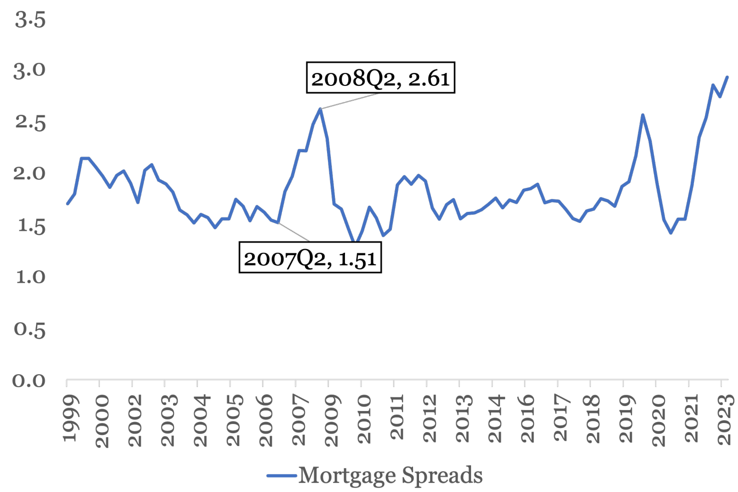
Table 1: Cross Correlations with Housing prices



Data source: CLTV from Freddie Mac Single-Family Loan-Level Dataset, Case-Shiller housing price index from FRED, index Jan 2000=100.

Figure 1: Leverage and Housing Prices

Theoretical models within the endogenous leverage literature, notably the seminal works of Fostel and Geanakoplos (2008, 2012, 2014, etc.), establish that leverage moves in tandem with asset prices and is a critical factor in driving significant fluctuations in asset prices. In contrast to the majority of macro-finance models that assume an exogenous cap on leverage, these models take changes in leverage as endogenous responses to shifts in economic fundamentals. These models uncover a strong feedback loop between leverage and asset prices, which greatly amplifies the effects of exogenous shocks on asset prices, whereas models that rely on exogenous shifts in leverage caps often produce small changes in asset prices. However, existing models within this line of literature are stylized theoretical constructs, which are not suitable for quantitative analysis.

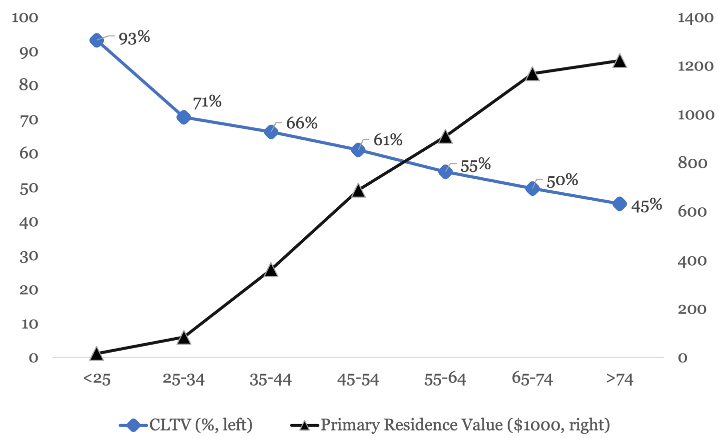


Data source: FRED MORTGAGE30US, DGS10.

Figure 2: Mortgage Spreads

The primary contribution of this paper is to rationalize the co-movements of housing prices, leverage, and mortgage spreads, and reproduce leverage cycles in a quantitative stochastic general equilibrium model. This is the first model that brings together the endogenous determination of leverage and a rich heterogeneity of agents in age using an overlapping generations framework. This model is built on two key elements: the first is large aggregate endowment shocks which lead to substantial fluctuations in equilibrium housing prices, leverage, and mortgage spreads; the second is the life cycle of households, featuring both a realistic lifespan and a hump-shaped endowment profile over that lifespan. The consideration of the household life cycle is grounded in micro data. As illustrated in Figure 3, there is an evident age-related pattern: housing wealth increases with age while leverage monotonically decreases with age.

The economy in the model is populated by a large number of overlapping generations of households, each deriving utility from both housing and non-housing consumption. Household endowments are low in early life, increase and peak at middle age, then decline in older age. Housing assets are perfectly durable assets and can be used as collateral for the issuance of debt contracts. Agents can accumulate housing assets using leverage, which is



Data source: CLTV from Dynamic National Loan-Level Dataset under HMDA in 2022 (Home Mortgage Disclosure Act); Primary Residence Value from Survey of Consumer Finance in 2019.

Figure 3: Trends in Age

measured by LTV – the ratio of the amount of borrowing to the value of the collateral. LTV ranges from 0% to an endogenously determined upper limit, which is always below 100%. Following Fostel and Geanakopolos (2015), in each time period, a menu defined by equilibrium prices, termed the Credit Surface, specifies the interest rates for each LTV level. The interest rate increases monotonically with LTV. For loans with an LTV low enough that rules out future default, a uniform risk-less interest rate is applied. For loans with an LTV that implies a default risk, the interest rate starts to rise. The higher the LTV, the greater the amount agents will default on in the event of a crisis, leading to higher interest rates. In this framework, the meaning of endogenous leverage is twofold: First, the upper limit of LTV is determined within the model, rather than being exogenously imposed. Second, with the same collateral, each LTV level has a separate market and an interest rate that clears the market. Both borrowers and lenders choose where they want to be on the Credit Surface; therefore leverage is determined by both supply and demand forces.

My paper resolves two major challenges in numerically solving the proposed model, providing guidance for finding a collateral equilibrium with non-financial assets and a high degree of agent heterogeneity. The first is a conceptual challenge: LTV is a continuous variable, and it is unclear ex ante which LTV level agents will select given the same housing collateral. While the existing literature on endogenous leverage provides theoretical guidelines when the collateral is a financial asset, these frameworks are not applicable to this study, as the housing collateral in this paper is a non-financial asset (because it provides utility to agents). To proceed, I assume that there exists an equilibrium where only a finite number of contracts are traded, then I verify whether agents have incentives to deviate and trade contracts not included in the finite set. I repeat this procedure until I find an equilibrium. The second challenge is computational in nature. Due to the substantial heterogeneity of agents and the multiplicity of tradable assets, the state space is highly dimensional. To tackle the curse of dimensionality, I employ a neural network with two hidden layers to directly approximate the policy and pricing functions, following the methodology introduced by Azinovic, Gaegauf, and Scheidegger (2022).

The model has two main implications. First, households start with high leverage and then progressively lower their leverage over the course of their life. In the model, households transition through four distinct life stages: they start as constrained borrowers, become unconstrained borrowers, transition into a mixed role of borrowers and lenders, and eventually become pure lenders. Second, large aggregate endowment shocks produce a leverage cycle. Housing prices and leverage are high when the economy is in a normal state; however, in a crisis state, a negative endowment shock significantly reduces housing prices and leverage while simultaneously increasing mortgage spreads.

Life cycle implications. In the first stage, households are young and receive low endowments. Anticipating an increase in future endowment, they issue risky debt contracts to borrow, pledging all the available housing stock as collateral. As their endowments grow, they transition to using a combination of risky and riskless contracts for borrowing, without pledging all their owned housing as collateral. In the third stage, as endowments begin to decline, they start investing in risky contracts while also leveraging their housing assets through riskless contracts. Finally, in the last stage, driven by a strong savings motive, households accumulate more financial wealth. To diversify their investments, they purchase both risky and riskless debt contracts. On average, younger households are the most leveraged among all age groups, and leverage tends to decrease as households age, a pattern that is consistent with empirical data.

Leverage cycle implications. The second main implication of the model is that leverage and housing prices decline substantially in times of crisis, whereas mortgage spreads surge. These results are consistent with data. The sequence of these events unfolds as follows. When a large negative endowment shock hits the economy, it lowers housing prices directly by reducing the purchasing power of households. In addition, borrowers who had previously issued the risky contract are underwater and default. This leads to losses for the lenders. As lenders experience losses from defaults and lower endowments, their marginal utility of consumption rises and they start demanding higher returns. Consequently, lenders collectively raise the interest rates charged on all debt contracts, as well as the spreads between risky contracts and riskless contracts. The elevated interest rates and spreads make borrowing more costly for young households, compromising the ability of housing assets to facilitate borrowing. The upper limit of LTV is reduced significantly as the leveraging capacity of housing assets is undermined, thereby amplifying the decline in housing prices.

This paper is closely related to two strands of literature. First, the paper extends the foundational work on endogenous leverage by Geanakoplos (1997) and Fostel and Geanakoplos (2008, 2014) etc., by adapting their theoretical models to an infinite-horizon overlapping generations framework. This adaptation allows for the age heterogeneity of homebuyers and sellers and is well-suited for quantitative analysis, thereby bridging the gap between theory and data. Brumm, Grill, Kubler and Schmedders (2015) examine the impact of endogenous leverage on asset price volatility using a model with two infinitely-lived agents with different degrees of risk aversion, and they use financial assets as collateral. This paper is different from their work in two key ways: 1) it has a much richer heterogeneity of agents with the consideration of households' life cycle; 2) it focuses on the use of non-financial assets, specifically housing, as collateral, which leads to different implications. Diamond and Landvoigt (2022) also consider an overlapping generations model with housing assets serving as collateral. In their model, agents can choose leverage from a menu provided by a financial intermediary, and they derive utility from holding deposits. However, their model, when driven solely by productivity shocks, produces a negative correlation between output and leverage, which is at odds with the trends observed during the Great Recession.

This paper also relates to the macroeconomic literature focusing on the housing market. Many extant models in this line of research treat the upper limit of LTV as an exogenous credit parameter, and household default is often ruled out. As a result, these models are silent on the factors driving changes in housing finance conditions and overlook the feedback loop between leverage and housing prices. In these settings, variations in the LTV limit have minor effects on housing prices. Kaplan, Mitman and Violante (2020) analyze the housing boom and bust episode within an overlapping-generation model with endogenous housing prices. They conclude that exogenous shifts in the LTV limit do not significantly affect housing prices; instead, fluctuations in housing prices are primarily driven by shifts in households' beliefs about future housing demand. Favilukis, Ludvigson and Van Nieuwerburgh (2017) also examine the housing market in an overlapping-generation model with endogenous housing prices. They find a higher LTV cap significantly increases housing prices, but this effect is only significant when they introduce heterogeneity in households' bequest preferences, which results in a substantial number of constrained households in equilibrium.

2 Model

2.1 Agents, Commodities and Uncertainty

Time is discrete and indexed by $t = 0, 1, 2, ...$ At each period, one of two possible exogenous aggregate shocks $z_t \in \mathbf{Z} = \{U, D\}$ realizes. The state U stands for "Up", representing normal times. The state D stands for "Down", representing rare crisis states similar to the Great Recession. The aggregate shock z_t affects both the aggregate endowment and the allocation of endowments among different age groups in every period. z_t evolves according to a Markov chain with the transition matrix Γ. Let $\gamma_{z_t, z_{t+1}}$ denote the probability of transitioning from state z_t to state z_{t+1}.

Agents can trade both consumption good (c) and housing (h). The consumption good is perishable, whereas housing assets are perfectly durable and in fixed supply of H. Let the spot price of the consumption good be 1, and the housing price be q_t.

In each period, a continuum of mass 1 identical agents of a new generation is born and lives for A periods. There is no mortality risk; all households die after age A. Age is indexed by $a \in \mathbf{A} = \{1, ..., A\}$. At the beginning of each period, households of age a receive endowments in consumption good $e^a_t = e^a(z_t)$, which depends on the aggregate shock. The aggregate endowment is denoted by $\bar{e}(z_t) = \sum_{a=1}^{A} e^a(z_t)$.

2.2 Preferences

The expected lifetime utility of households born at time t is given by

$$U_t = E_t \sum_{a=1}^{A} \beta^{a-1} u^a(c^a_t, h^a_t), \tag{1}$$

where $\beta > 0$ is the discount factor, c_t^a and h_t^a are the amount of the consumption good and the stock of housing at age a. $u^a(c,h)$ is an age-dependent period utility function.

Let $\hat{\mathbf{A}} = \{1, ..., A-1\}$ be the set of agents excluding those who are in the last period of life. All agents have the Cobb-Douglas utility over consumption and housing nested within a constant relative risk aversion utility form when they are at age $a \in \hat{\mathbf{A}}$, and they do not value housing assets in the last period of life, the period utility function is given by

$$u^a(c,h) = \begin{cases} \dfrac{(c^{1-\alpha}h^{\alpha})^{1-\rho}}{1-\rho}, & a \in \hat{\mathbf{A}}, \\ \dfrac{c^{1-\rho}}{1-\rho}, & a = A, \end{cases}$$

where $\alpha > 0$ measures the relative share of housing expenditure, $\rho > 0$ is the coefficient of risk aversion. It follows that $c_t^a > 0$ for all agents, $h_t^a > 0$ for $a \in \hat{\mathbf{A}}$, and $h_t^A = 0$.

2.3 Debt Contracts

Households enter the economy with neither debts or assets. Each period, they meet in anonymous, competitive financial markets to trade collateralized debt contracts. All contracts are one-period. Let J_t denote the set of such contracts available at time t. A financial contract in J_t is defined by an ordered pair representing its promise and collateral requirement, denoted as $(j, 1)$. $j \in \mathbb{R}_+$ is a non-contingent promise to deliver j units of consumption good in the next period, while the number 1 indicates that the promise j must be backed by one unit of housing as collateral. J_t contains an infinite number of contracts, as j is continuous and unbounded above.

Agents can default on the promise, the consequences of defaulting are limited to losing the collateral they have previously pledged. All contracts are non-recourse, there are no additional penalties in the event of default. An agent who sells one unit of contract j will default on the promise if j exceeds the realized housing price q_{t+1}. Therefore, the delivery of contract j is $\min\{j, q_{t+1}\}$.

The price of contract j is denoted by $\pi_{j,t}$. Let $\theta^a_{j,t} \in \mathbb{R}$ be the number of contract j traded by agent of age a. $\theta^a_{j,t} < 0 (> 0)$ indicates the agent is shorting (longing) contract j, by doing so the agent borrows (lends) $|\pi_{j,t}\theta^a_{j,t}|$. When agents buy one unit of housing and finance this purchase by selling a debt contract j, they are effectively making a downpayment of $q_t - \pi_{j,t}$.

Let q^U_{t+1} and q^D_{t+1} denote the realizations of housing prices in states $z_{t+1} = U$ and $z_{t+1} = D$ respectively, assuming that $q^U_{t+1} > q^D_{t+1}$. The following proposition shows that $\pi_{j,t}$ must be strictly increasing in j for $0 \leqslant j \leqslant q^U_{t+1}$, and for all $j > q^U_{t+1}$, $\pi_{j,t}$ remains constant at $\pi_{q^U_{t+1},t}$.

Proposition 1. *$\pi_{j,t}$ is strictly increasing in j over $[0, q^U_{t+1}]$, and $\pi_{j,t} = \pi_{q^U_{t+1},t}$ for $j \in (q^U_{t+1}, +\infty)$.*

Proof. Consider an agent aged a buys a contract promising $j \in \mathbb{R}_+$, i.e., $\theta^a_{j,t} > 0$, then it must be that

$$\begin{aligned}\pi_{j,t} &= E_t\left[\beta \frac{Du_c(c^{a+1}_{t+1}, h^{a+1}_{t+1})}{Du_c(c^a_t, h^a_t)} \min\{j, q_{t+1}\}\right] \\ &= \sum_{z_{t+1}\in\mathbf{Z}} \beta\gamma_{z_t,z_{t+1}} \frac{Du_c(c^{a+1,z_{t+1}}_{t+1}, h^{a+1,z_{t+1}}_{t+1})}{Du_c(c^a_t, h^a_t)} \min\{j, q^{z_{t+1}}_{t+1}\},\end{aligned} \tag{2}$$

where the right-hand side is the present value of the expected actual delivery of contract j, discounted by agent a's intertemporal marginal rate of substitution of consumption between time t and $t+1$. Du_x denotes the derivative of the period utility function with respect to x. y^z_t denotes the variable y at time t in state z. Because this agent is optimizing in equilibrium, increasing or decreasing $\theta^a_{j,t}$ by an infinitesimal amount must yield zero gain in utility. Since $\rho > 0$, $Du_c(c, h) > 0$ for $c > 0$ and $h > 0$. Additionally, $\beta > 0$, therefore the state-dependent intertemporal marginal utility between time t and $t+1$ $M^{a,z_{t+1}}_{t,t+1} \equiv \beta\gamma_{z_t,z_{t+1}} \frac{Du_c(c^{a+1,z_{t+1}}_{t+1}, h^{a+1,z_{t+1}}_{t+1})}{Du_c(c^a_t, h^a_t)}$, $z_{t+1} \in \mathbf{Z}$ must be strictly positive. At equilibrium prices and quantities, the second line of the equation shows that $\pi_{j,t}$ is a continuous piecewise linear function of j for $j \in \mathbb{R}_+$. The slope is $\sum_{z_{t+1}\in Z} M^{a,z_{t+1}}_{t,t+1} > 0$ for $j \in [0, q^D_{t+1}]$, and $M^{a,U}_{t,t+1} > 0$ for $j \in [q^D_{t+1}, q^U_{t+1}]$. Therefore, $\pi_{j,t}$ and is strictly increasing in j for $j \in [0, q^U_{t+1}]$. For contracts with the promise $j \in (q^U_{t+1}, +\infty)$, the actual delivery is q^D_{t+1} in $z_{t+1} = D$, and q^U_{t+1} in $z_{t+1} = U$, which equals to the delivery of contract $j = q^U_{t+1}$. Hence, $\pi_{j,t} = \pi_{q^U_{t+1},t}$. □

The gross interest rate for contract j is defined by

$$R_{j,t} = \frac{j}{\pi_{j,t}}.$$

The LTV for contract j is defined by

$$LTV_{j,t} = \frac{\pi_{j,t}}{q_t}.$$

Note that because $\pi_{j,t}$ is non-negative and strictly increasing in j over $[0, q^U_{t+1}]$, and $\pi_{j,t} = \pi_{q^U_{t+1},t}$ for $j \in (q^U_{t+1}, +\infty)$, given q_t, $LTV_{j,t}$ is non-negative and strictly increasing in j over $[0, q^U_{t+1}]$, and equals to $LTV_{q^U_{t+1},t}$ for $j \in (q^U_{t+1}, +\infty)$.

Fostel and Geanakoplos (2008) introduce the concept of collateral value and liquidity wedge; Following their discussion, Geanakoplos and Zame (2014) introduce the concept of liquidity value as a way to quantify the benefits of borrowing using different contracts. Following their definition, in this economy, the liquidity value of contract j for an agent of age a at time t, denoted by $LV^a_{j,t}$, is given by contract j's price net of the present value of its delivery, discounted by the agent's stochastic discount factor:

$$LV^a_{j,t} = \pi_{j,t} - E_t\left[\beta\frac{Du_c(c^{a+1}_{t+1}, h^{a+1}_{t+1})}{Du_c(c^a_t, h^a_t)}\min\{j, q_{t+1}\}\right].$$

2.4 Constraints

2.4.1 Budget Constraint

The budget constraint for agents at age a and at time t is given by

$$c^a_t + q_t h^a_t + \int\limits_{R_+} \theta^a_{j,t}\pi_{j,t}dj \leqslant e^a_t + q_t h^{a-1}_{t-1} + \int\limits_{R_+} \theta^{a-1}_{j,t-1}\min\{j, q_t\}dj. \tag{3}$$

On the left-hand side of the budget constraint, there are expenditures on consumption, housing, and the total amount of borrowing (or lending) through trading debt contracts in the financial market. Since j is a continuous variable, the third item is an integral over $j \in \mathbb{R}_+$. On the right-hand side, agents receive their endowments, observe the market value of the housing assets bought in the last period, and clear debts associated with contracts traded in the last period.

The last term on the right-hand side accounts for the total deliveries from contracts traded in the previous period. This term can be written as the sum of two components, as illustrated in the equation below: the first component ensures that agents deliver the value of the collateral for contracts on which they default (i.e., when $j > q_t$); the second component makes sure that they fulfill the promised delivery for contracts on which they do not default (i.e., when $j \leqslant q_t$).

$$\int_{R_+} \theta^{a-1}_{j,t-1}\min\{j, q_t\} = \int_{j>q_t} q_t\theta^{a-1}_{j,t-1}dj + \int_{j\leqslant q_t} j\theta^{a-1}_{j,t-1}dj.$$

2.4.2 Collateral Constraint

Borrowing through the sale of contracts requires collateral. Since each contract sold short requires one unit of housing as collateral, agents cannot sell more units of contracts than their current housing stock. Their choices must satisfy the following collateral constraint:

$$\int_{R_+} \max\{-\theta^a_{jt}, 0\} dj \leqslant h^a_t. \tag{4}$$

The integrand $\max\{-\theta^a_{j,t}, 0\}$ serves to filter out the contracts on which agents assume a long position, as these do not require collateral.

2.4.3 No Short-selling Constraint

Agents are prohibited from taking short positions in the housing stock,

$$h^a_t \geqslant 0. \tag{5}$$

2.5 The Credit Surface

Building on the work of Fostel and Geanakopolos (2015), where they define the Credit Surface as a menu that specifies the relationship between interest rates and various levels of LTV, this paper adopts their modeling approach to use the Credit Surface as the pricing schedule for leverage in equilibrium. Figure 4 shows a Credit Surface at time t. Since $LTV_{j,t}$ is strictly increasing in j for $j \in [0, q^U_{t+1}]$, each contract in J_t is uniquely characterized by its specific $LTV_{j,t}$ and the corresponding interest rate $R_{j,t}$. Consequently, selecting a contract is tantamount to choosing a point (a particular LTV) on the prevailing Credit Surface.

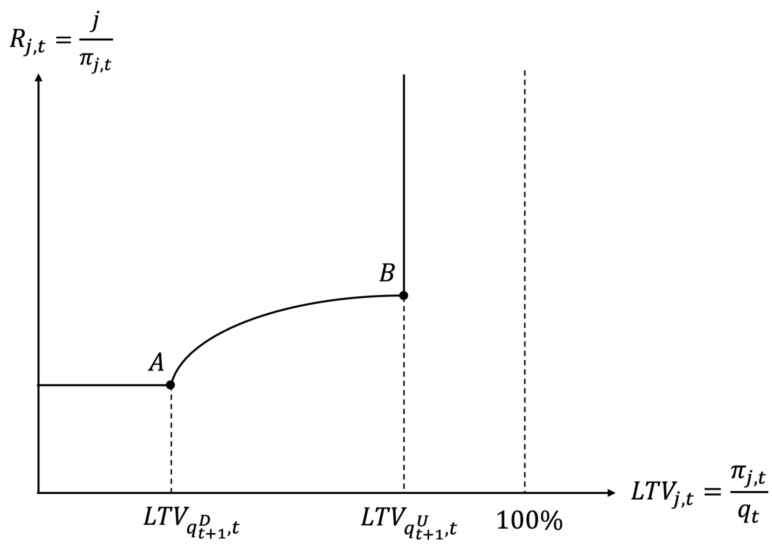


Figure 4: Credit Surface for a Binomial Economy

Points A and B represent the LTVs and interest rates of two contracts promising q^D_{t+1} and q^U_{t+1}, respectively. The Credit Surface is flat until point A, then starts rising until point B, after which the Surface becomes a vertical line. Fostel and Geanakopolos (2015) prove that the rising part of the Credit Surface between points A and B is concave. Borrowers will not default on contracts with a promise smaller than q^D_{t+1}; therefore, such contracts will be charged with a uniform risk-free rate of interest. For contracts with a promise between q^D_{t+1} and q^U_{t+1}, borrowers default in the D state but not in U. The higher the j, the larger the amount of debt borrowers will default on in the D state; therefore, the interest rate is higher for contracts with a larger j in the interval $[q^D_{t+1},\ q^U_{t+1}]$. Lastly, when contracts promise an amount exceeding the housing prices in both states, i.e., $j > q^U_{t+1}$, lenders anticipate that borrowers will default in both states and consequently charge an infinitely high interest rate on such contracts. The transition to a vertical surface after point B can be seen mathematically: for $j > q^U_{t+1}$, as j goes to infinity, $LTV_{j,t}$ remains at $LTV_{q^U_{t+1},t}$ (because $\pi_{j,t}$ remains at $\pi_{q^U_{t+1},t}$), while $R_{j,t} = \frac{j}{\pi_{j,t}} = \frac{j}{\pi_{q^U_{t+1},t}}$ goes to infinity.

Given that housing assets are durable goods, they provide both immediate utility from living in the house and serve as a means of saving. Therefore, even in the case where agents can borrow up to the point where their repayment equals the realized housing price q_{t+1}, they are still obliged to pay for the immediate utility derived from a single period of occupancy. The following proposition asserts that agents must make a strictly positive downpayment upfront to acquire housing.

Proposition 2. *In this economy, the upper limit of $LTV_{j,t}$ is an endogenous variable and is strictly smaller than* 100%*.*

Proof. Suppose in equilibrium, there exists a contract j such that $LTV_{j,t} \geqslant 100\%$, where $j > q^U_{t+1}$. This is equivalent to $\pi_{j,t} \geqslant q_t$. The actual delivery of this contract, $\min\{j, q_{t+1}\}$, equals to the realized housing price q_{t+1}. For lenders at age $a < A$, the following must hold:

$$\pi_{j,t} = E_t\left[\beta \frac{Du_c(c^{a+1}_{t+1}, h^{a+1}_{t+1})}{Du_c(c^a_t, h^a_t)} q_{t+1}\right] \geqslant \frac{Du_h(c^a_t, h^a_t)}{Du_c(c^a_t, h^a_t)} + E_t\left[\beta \frac{Du_c(c^{a+1}_{t+1}, h^{a+1}_{t+1})}{Du_c(c^a_t, h^a_t)} q_{t+1}\right] = q_t.$$

The right-hand side of the equation says the housing price q_t equals to sum of the immediate utility from housing services and the expected present value of the future housing price q_{t+1}. Rearranging this equation shows that the downpayment associated with this contract equals to the utility derived from a single period of housing consumption:

$$q_t - \pi_{j,t} = \frac{Du_h(c^a_t, h^a_t)}{Du_c(c^a_t, h^a_t)}.$$

Given that all agents aged $a < A$ consumes housing, the ratio $\frac{Du_h(c^a_t, h^a_t)}{Du_c(c^a_t, h^a_t)}$ must be strictly greater than zero, which implies $q_t > \pi_{j,t}$. This result contradicts the initial assumption $\pi_{j,t} \geqslant q_t$. Furthermore, since the prices of all other contracts in J_t are smaller or equal to $\pi_{j,t}$, it follows that no contract in J_t can have a price $\pi_{j,t}$ that is greater than or equal to q_t. Hence, the upper limit of $LTV_{j,t}$ is strictly less than 100%[5]. $\square$

[5] Nilayamgode (2023) also proves that when non-financial assets serve as collateral, and households who hold housing assets have positive level of consumption, LTV can never go to 100%.

2.6 Collateral Equilibrium

Definition 1. *A collateral equilibrium of this economy is a collection of agents' allocations of consumption and housing, their portfolio holdings of financial contracts, as well as the prices of housing and financial contracts for all* t

$$\left((c_t^a, h_t^a, (\theta_{j,t}^a)_{j\in J_t})_{a\in\mathbf{A}}; q_t, (\pi_{j,t})_{j\in J_t}\right)_{t=0}^{\infty} \tag{6}$$

such that

1. *Given* $(q_t, (\pi_{j,t})_{j\in J_t})_{t=0}^{\infty}$*, the choices* $((c_t^a, h_t^a, (\theta_{j,t}^a)_{j\in J_t})_{a\in\mathbf{A}})_{t=0}^{\infty}$ *maximize* (1)*, subject to constraints* (3)*,* (4) *and* (5)*.*

2. *Markets for the consumption good, housing, and all financial contracts in* J_t *clear at each time* t*:*

$$\sum_{a=1}^{A} c_t^a = \sum_{a=1}^{A} e_t^a,$$
$$\sum_{a=1}^{A} h_t^a = H,$$
$$\sum_{a=1}^{A} \theta_{j,t}^a = 0, \forall j \in J_t.$$

Parameter	Value	Interpretation
Uncertainty		
γ_{UD}	0.14	$\gamma_U = 85\%$
γ_{DU}	0.80	$d_U = 7, d_D = 1$
Preferences		
β	0.83	Annual discount rate 0.94
ρ	4.5	Risk-aversion coefficient
α	0.115	Housing share
Endowments		
$\{e^a(U)\}_{a=1}^{20}$		Wage income (2007 SCF)
$\{e^a(D)\}_{a=1}^{20}$		Wage income (2009 SCF)

Table 2: Parameters

3 Quantitative Analysis

3.1 Parameterization

3.1.1 Demographics

Households live for twenty periods, $A = 20$. In the model, one period is equivalent to three years in real life. Households start their economic life at real age 21 (model age $a = 1$), and live until real age 81 (model age $A = 20$).

3.1.2 Uncertainty

Transition probabilities in Γ are chosen to match two features exhibited by the data: i) the frequency of a Great Recession-like state is around 15%; ii) the average duration of the U state is around 7 times the duration of D state. I collect US GDP per capita data from two sources: FRED (after 1947) and Maddison Project 2020 (before 1947). Then I use Hodrick-Prescott (HP) filter with a smoothing parameter 1600 to get the trend and cycle. In 2009, the US GDP per capita is 2.46% below trend. I then define the US economy as being in the recession if the GDP per capita in that year is below 2.4%. Given this threshold, the US economy has historically been in the recession state 14.6% of the time, this frequency serves as one target for the choice of transition probabilities.

3.1.3 Preferences

I set the discount factor β to 0.83, which corresponds to an annual discount factor of 0.94. The coefficient of risk aversion ρ and the weight on housing α are calibrated to 4.5 and 0.115, respectively. This calibration aims to replicate a 10-percentage-point difference in the average LTV for first-time homebuyers between states U and D, thereby capturing the observed change in LTV during the Great Recession.

3.1.4 Endowments

The total supply of housing is normalized to $H = 20$. Figure 5 presents the life cycle age profiles of endowment, with the line with circle markers representing $\{e^a(U)\}_{a=1}^{20}$ and square representing $\{e^a(D)\}_{a=1}^{20}$. The Survey of Consumer Finances (SCF) provides detailed data on household income and wealth in the U.S. A special panel survey was conducted between 2007 and 2009 to study the aftermath of the Great Recession. In the panel survey, households who had responded to the 2007 survey were invited to participate in a follow-up survey in 2009. The income data collected from the same interviewees in these two years serve as the source for constructing endowment age profiles across different states in the model. To create the endowment profiles, I divide households from age 21 to 81 into 20 age groups and take the average wage and salary income within each age group, using SCF sample weights. The data from 2007 represent the endowments in the U state $\{e^a(U)\}_{a=1}^{20}$, while the 2009 data represent the D state $\{e^a(U)\}_{a=1}^{20}$.

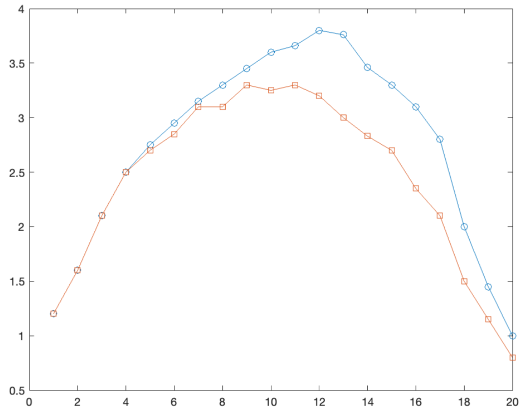


Figure 5: Age Profiles of Endowments

3.2 Numerical Solution

Finding a collateral equilibrium for this economy presents two main challenges: one conceptual and the other computational. Conceptually, it is challenging to derive the optimality conditions for households given that there are an infinite number of contracts in J_t. The presence of the term $\int_{j\in\mathbb{R}_+} \pi_{j,t}\theta^a_{j,t}dj$ in the budget constraint complicates the problem, as it is impractical to derive an infinite number of Euler equations regarding each contract $\theta^a_{j,t}$.

Geanakoplos (1997) argues that only a limited set of contracts will be traded in equilibrium due to the scarcity of collateral; however, it remains unclear which specific contract(s) will have both non-zero supply and demand in equilibrium. The No-Default Theorem, formalized by Fostel and Geanakoplos (2015), states that in a binomial economy with financial assets as collateral, the contract that makes the maximal promise without defaulting will be the one actively traded in equilibrium. However, the theorem does not apply to models in which non-financial assets, like housing, serve as collateral. Unlike financial assets, non-financial assets provide utility to agents. Consequently, the demand for such assets, and debt contracts tied to them, depends not only on agents' motivations to transfer wealth across time and states, but also on the intrinsic utility derived from the collateral assets.

Geanakoplos (1997) presents an example with two types of agents and houses as collateral, suggesting that only the contract promising the highest housing price in the next period would be traded. However, in a more complex setup involving 20 types (generations) of agents, such an equilibrium does not exist. In my analysis, I assume that there exists an equilibrium in which only the contract that promises q^U_{t+1} is traded in each period. Upon solving for variables that satisfy all the equilibrium conditions under this assumption, I find that the break-even condition (2) for lenders does not hold for all the other contracts in J_t. This implies that lenders do not agree on contract prices and consequently the equilibrium does not exist under the assumption that only one contract $j = q^U_{t+1}$ is traded.

In addition, numerically approximating an equilibrium is challenging in two ways. Firstly, the presence of 20 types of agents who engage in trading multiple assets inevitably leads to a large number of state variables. Secondly, the model features occasionally binding collateral constraints, which induce non-linearity in the policy functions, making them difficult to approximate accurately with linear methods. Following Azinovic, Gaegauf, and Scheidegger (2022), I used a neural network with two hidden layers to directly approximate the the policy functions, addressing the second challenge. They suggest that a neural network with stochastic simulations has the advantage of approximating potentially non-linear policy functions only in the ergodic endogenous state space, even when dealing with a high-dimensional state vector.

To search for a collateral equilibrium, I iterate on the following steps:

1. Guess an equilibrium regime. Assume that there exists an equilibrium in which only N_j contracts in $\hat{J}_t$ will have non-zero supply and demand in each time period. This assumption implies that $(\theta^a_{j,t})_{a \in \hat{\mathbf{A}}, j \notin \hat{J}_t} = 0$, thereby reducing the equilibrium conditions to a finite number of Karush-Kuhn-Tucker (KKT) and market clearing conditions.

2. Under the assumption in step 1, use a neural network to numerically approximate

the candidate equilibrium, i.e., find the policy and pricing functions as defined in a Functional Rational Expectations Markov Equilibrium (FREE) such that all the KKT and market clearing conditions are satisfied within an acceptable tolerance.

3. Examine whether agents have incentives to deviate and trade contracts not included in $\hat{J}_t$. Specifically, verify whether the third and fourth conditions are satisfied in the definition of FREE. If yes, stop the process. If not, adjust the set of actively traded contracts and repeat the steps until all the conditions described in the definition of a FREE are met.

Definition 2. *A FREE, in which only contracts $j \in \hat{J}_t$ are traded with non-zero demand and supply, consists of $(c_t^a, h_t^a, \boldsymbol{\theta_t^a}, \boldsymbol{\tilde{\mu}_t^a})_{a\in\mathbf{A}}$, and $(q_t, \boldsymbol{\pi_t})$, where $\boldsymbol{\theta_t^a} = (\theta_{j,t}^a)_{j\in\hat{J}_t}$ denotes the portfolio holdings of contracts $j \in \hat{J}_t$, $\boldsymbol{\tilde{\mu}_t^a} = (\mu_{h,t}^a, (\mu_{j,t}^a)_{j\in\hat{J}_t})$ denotes the lagrangian multipliers, $\boldsymbol{\pi_t} = (\pi_{j,t})_{j\in\hat{J}_t}$ denotes the prices of contracts $j \in \hat{J}_t$, as time-invariant policy functions and pricing functions of $\mathbf{x_t}$, where $\mathbf{x_t} = (z_t, (c_{t-1}^a, h_{t-1}^a, \boldsymbol{\theta_{t-1}^a})_{a\in\hat{\mathbf{A}}}) \in \mathbf{Z} \times \mathbb{R}^{A-1} \times [0,1]^{A-1} \times [-1,1]^{N_j(A-1)}$ is the state vector, such that:*

1. *The following complementary slackness conditions hold for $a \in \hat{\mathbf{A}}$, $j \in \hat{J}_t$, and for any non-empty subset $S \subseteq \hat{J}_t$*[6]*,*

[6] Under the assumption that only contracts $j \in \hat{J}_t$ are traded with non-zero supply and demand, the pertinent collateral constraint simplifies to $\sum_{j\in\hat{J}_t} \max\{-\theta_{j,t}^a, 0\} \leqslant h_t^a$. This condition is equivalent to imposing that the sum of the negative positions for all contracts in any non-empty subset of $\hat{J}_t$ does not exceed the housing stock h_t^a. Mathematically, this yields $\sum_{n=1}^{N_j} \binom{N_j}{n}$ distinct inequalities:

$$\sum_{j\in S} -\theta_{j,t}^a \leq h_t^a \quad \text{for all non-empty subsets } S \subseteq \hat{J}_t.$$

$$\mu_t^{a,S}(\sum_{j\in S} -\theta_{j,t}^a - h_t^a) = 0, \tag{7}$$

$$\sum_{j\in S} -\theta_{j,t}^a - h_t^a \leqslant 0, \tag{8}$$

$$\mu_t^{a,S} \geqslant 0. \tag{9}$$

2. *The following Euler equations hold for $a \in \hat{\mathbf{A}}$, $j \in \hat{J}_t$* [7]*:*

$$q_t = \frac{Du_h(c_t^a, h_t^a)}{Du_c(c_t^a, h_t^a)} + E_t\left[\beta \frac{Du_c(c_{t+1}^{a+1}, h_{t+1}^{a+1})}{Du_c(c_t^a, h_t^a)} q_{t+1}\right] + \frac{\mu_{h,t}^a}{Du_c(c_t^a, h_t^a)}, \tag{10}$$

$$\pi_{j,t} = E_t\left[\beta \frac{Du_c(c_{t+1}^{a+1}, h_{t+1}^{a+1})}{Du_c(c_t^a, h_t^a)} min\{j, q_{t+1}\}\right] + \frac{\mu_{j,t}^a}{Du_c(c_t^a, h_t^a)}. \tag{11}$$

3. *All agents with $\tilde{\boldsymbol{\mu}}_t^a = 0$ agree on the prices of all other contracts. For all $j \in \mathbb{R}_+$ such that $j \notin \hat{J}_t$:*

$$\pi_{j,t} = E_t\left[\beta \frac{Du_c(c_{t+1}^{a+1}, h_{t+1}^{a+1})}{Du_c(c_t^a, h_t^a)} min\{j, q_{t+1}\}\right], \text{ provided } \tilde{\boldsymbol{\mu}}_t^a = 0. \tag{12}$$

4. *If $\mu_{j,t}^a > 0$ for some contract j in $\hat{J}_t$, then $\mu_{i,t}^a = 0$ for all other contracts $i \neq j$ in $\hat{J}_t$, and contract j has the highest liquidity value among all contracts in J_t. i.e., $LV_{j,t} \geqslant LV_{i,t}$ $\forall i \in J_t$, s.t. $i \neq j$.*

By definition, a FREE satisfies all the equilibrium conditions of a collateral equilibrium, including agents' optimality conditions and market clearing conditions. The fourth condition

[7] Let $\mu_t^{a,S}$ denote the Lagrangian multiplier for the collateral constraint corresponding to each non-empty subset S of $\hat{J}_t$. Then, define $\mu_{j,t}$ as the sum of Lagrangian multipliers $\mu_t^{a,S}$ for all subsets S that include contract j. Formally, $\mu_{j,t}^a \equiv \sum_{S\subseteq \hat{J}_t: j\in S} \mu_t^{a,S}$. Furthermore, define $\mu_{h,t}^a$ as the sum of all Lagrangian multipliers $\mu_t^{a,S}$: $\mu_{h,t} \equiv \sum_{S\subseteq \hat{J}_t} \mu_t^{a,S}$.

in the definition ensures the break-even condition for lenders are satisfied for all contracts in J_t. Therefore, a FREE effectively induces a collateral equilibrium.

4 Results

The procedure of finding a collateral equilibrium stops when the set of actively traded contracts $\hat{J}_t$ contains only the riskless contract that promises q^D_{t+1} and the risky contract that promises q^U_{t+1}. Detailed description of the accuracy of the solution can be found in appendix B. Let $j_{A,t}$ and $j_{B,t}$ denote the riskless contract and the risky contract respectively. Their corresponding prices are denoted by $\pi_{A,t}$ and $\pi_{B,t}$. The quantities of these contracts held by an agent of age a at time t are denoted by $\theta^a_{A,t}$ and $\theta^a_{B,t}$. In this equilibrium, $\theta^a_{j,t} = 0$ for all $a \in \mathbf{A}$, $j \in J_t \backslash \hat{J}_t$. I simulate the economy for a total of $T + 10,000 = 510,000$ periods. The first $10,000$ periods are discarded to allow the economy to reach a stationary equilibrium. All subsequent analyses are based on the simulation over the remaining $500,000$ periods.

4.1 Life Cycle Implications

Figure 6 depicts the mean life cycle profiles of portfolio holdings in riskless contract j_A (denoted by θ_A) and risky contract j_B (denoted by θ_B), along with housing assets, and the average amount of housing pledged as collateral. The collateral profile (marked with Diamonds) consistently lies below the housing profile (marked with Crosses), due to the collateral constraints. This figure suggests that on average, agents within the model transition through four distinct life stages.

First stage: constrained borrowers. In the initial stage, agents aged between 1 and 6 choose the portfolio holdings where $\theta^a_{A,t} = 0$, and $-\theta^a_{B,t} = h^a_t$, which indicates that they simultaneously accumulate housing h^a_t and pledge all of these housing assets as collateral to

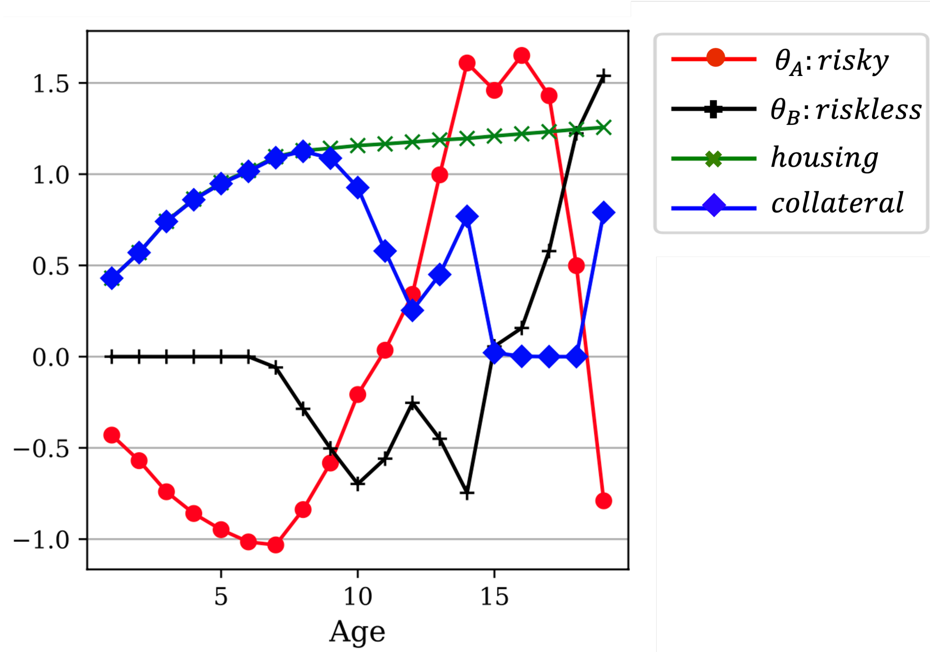


Figure 6: Average Life Cycle Portfolio Holdings and Collateral

borrow through exclusively selling the risky contract $j_{B,t}$. Because their collateral constraints are binding, they are classified as constrained borrowers.

Anticipating a substantial increase in endowments in the $z_{t+1} = U$ state and a lesser rise in the $z_{t+1} = D$ state, these young agents have two consumption-smoothing incentives. First, they would like to transfer wealth from the $z_{t+1} = U$ state – where they expect to be wealthier – to the present by issuing debt contracts secured by their housing. Second, they would like to make small payments in the $z_{t+1} = D$ state in which they feel poor while borrowing as much as possible in the current period. Among all contracts in J_t, the risky contract $j_{B,t}$ allows for the most current borrowing without triggering an infinitely high interest rate, for $\pi_{j,t}$ strictly increases with j over the interval $[0, q^U_{t+1}]$. Additionally, agents can default on this contract and only deliver the housing collateral with low value q^D_{t+1} in the D state. Consequently, contract $j_{B,t}$ allows for the greatest degree of consumption smoothing across time and states, these agents will find it most advantageous to exclusively issue this particular contract.

As previously mentioned, liquidity value quantifies the trade-off between the immediate liquidity provided by a contract against its future obligations, guiding borrowers in selecting among various contracts written against the same collateral. By rearranging the Euler equation for contract j and using the definition of $LV^a_{j,t}$, we can see that the liquidity value of contract j is equal to $\mu^a_{j,t}$ divided by agent's current marginal utility of consumption: $LV^a_{j,t} = \frac{\mu^a_{j,t}}{Du_c(c^a_t, h^a_t)}$. As previously defined, $\mu^a_{j,t}$ is the sum of Lagrangian multipliers corresponding to the collateral constraints for every subset of contracts that contains j, representing the additional gain in utility if agents could marginally increase sales of contract j by relaxing all the relevant collateral constraints. To translate the abstract gain in utility reflected by $\mu^a_{j,t}$ into a more tangible metric, divide $\mu^a_{j,t}$ by $Du_c(c^a_t, h^a_t)$, yielding the liquidity value $LV^a_{j,t}$, which represents the marginal benefit in real consumption good. Holding the marginal utility of consumption fixed at the equilibrium level, comparing liquidity value across j is equivalent to comparing $\mu^a_{j,t}$.

$\mu^a_{j,t} > \mu^a_{k,t}$ indicates that contract j offers a greater marginal benefit per unit of collateral than contract k. If $\mu^a_{j,t} > \mu^a_{k,t}$ for all $k \in J_t$ with $k \neq j$, agents will not waste collateral on any contracts that provide less marginal benefit than contract j. Instead, they increase the issuance of contract j until the collateral constraint binds: $-\theta^a_{j,t} = h^a_t$. If $\mu^a_{j,t} = \mu^a_{k,t} > \mu^a_{l,t}$, for all contracts k in some subset S_k, and all contract l in $J_t \backslash \{j, S_K\}$, then agents are indifferent between issuing contract j and any contracts in S_k, and choose $\mu^a_{l,t} = 0$, for all l in $J_t \backslash \{j, S_K\}$. If $\mu^a_{j,t} = 0$ for all $j \in J_t$, agents are indifferent between trading any contracts in J_t without violating the collateral constraint.

Figure 7 presents a series of curves, each depicting the liquidity value $LV^{a=1}_{j,t}$ for the youngest agent against the promise $j \in [0, q^U_{t+1}]$ across all simulated periods from $t = 1$ to T. Each curve corresponds to a distinct time period. All curves are color-coded: red for periods when $z_t = U$ and blue for periods when $z_t = D$. Note that all red curves are positioned above the blue ones, implying that the liquidity value for all contracts are always higher in normal states than crisis states. In each period, the youngest agents find contract $j_{B,t}$ provides the highest liquidity value. Consequently, they use their entire housing collateral to issue this contract, such that $-\theta_{B,t}^{a=1} = h_t^{a=1}$, and do not issue any other contracts, with $\theta_{j,t}^{a=1} = 0$ for all $j \in J_t \backslash \{j_{B,t}\}$.

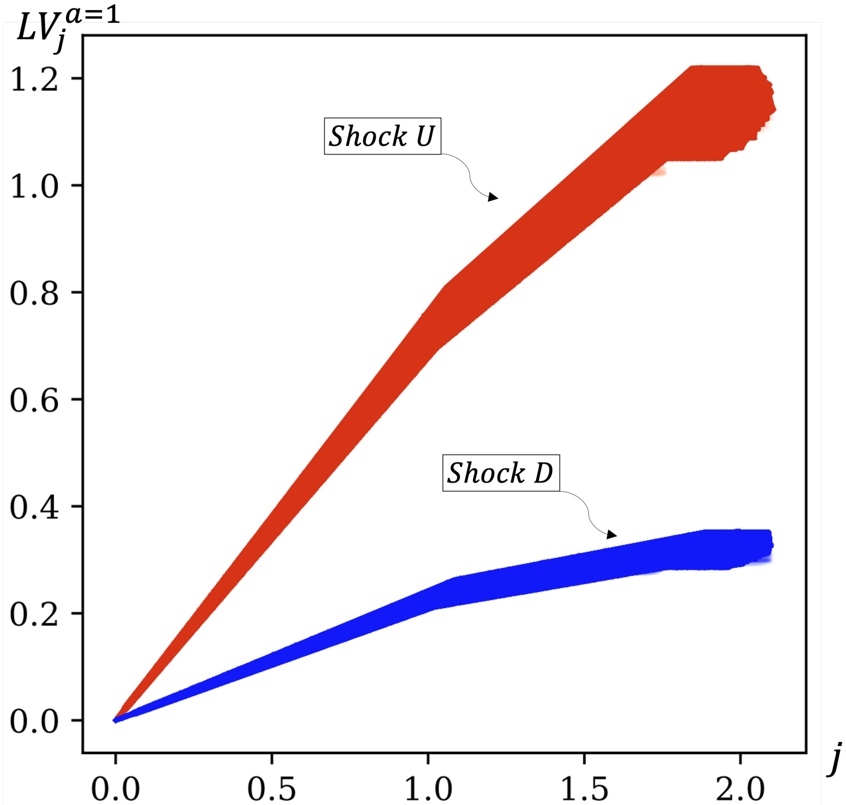


Figure 7: Liquidity Value for $a = 1$ Agents

Second stage: unconstrained borrowers. As agents age into the 7 to 10 bracket, they continue to expect higher future endowments and have the motive to transfer future wealth to the present. However, as the average growth rate of their endowment declines compared to that in the initial life stage, their motivation for consumption smoothing declines correspondingly. They start to shift their portfolio holdings away from exclusively issuing the risky contract $j_{B,t}$ to issuing both the riskless contract $j_{A,t}$ and the risky contract $j_{B,t}$. Their collateral constraints do not bind throughout this stage.

In equilibrium, these agents find the marginal benefit of a marginal increase in the borrowing through issuing any contracts in J_t to be zero, i.e., $\mu^a_{j,t} = LV^a_{j,t} = 0$ for all $t = 0, ..., T$. They are less liquidity constrained compared to agents in the first life stage, their desired amount of borrowing can be acquired by pledging only a fraction of their housing stock as collateral. As they are indifferent between issuing any contracts in J_t, their portfolio holdings in contract $j_{A,t}$ and $j_{B,t}$ are pinned down in equilibrium by the market clearing conditions.

Third stage: unconstrained borrowers and lenders. As agents progress into the 11 to 14 age bracket, they anticipate their future endowments to be on a downward trajectory. This expected change is reflected in figure 6, where the line of the use of collateral (Diamond marker) exhibits a mild increase at age 12. This uptick suggest that agents, upon experiencing the initial decline in their endowment growth at age 12, use greater leverage in accumulating housing assets than when they were aged 11.

During this stage, while utilizing the housing collateral to borrow with contract $j_{A,t}$, agents start to transfer current wealth to the future by investing in the risky contract $j_{B,t}$.

Last stage: lenders. In the final life stage, agents, driven by a strong saving motive and a motive to smooth consumption across states, stop borrowing and begin to diversify their investments. They accumulate housing assets with zero leverage, and diversify their saving by lending to younger agents through buying both the risky contract $j_{B,t}$, and the riskless contract $j_{A,t}$.

Let $\tilde{R}_{A,t}$ and $\tilde{R}_{B,t}$ be the real return on contract $j_{A,t}$ and $j_{B,t}$ respectively, they are given by:

$$\tilde{R}_{A,t} = \frac{\min\{q^D_{t+1}, q_{t+1}\}}{\pi_{A,t}},$$
$$\tilde{R}_{B,t} = \frac{\min\{q^U_{t+1}, q_{t+1}\}}{\pi_{B,t}}.$$

Rearrange agents' Euler equations regarding contract $j_{A,t}$ and $j_{B,t}$ and use the definitions of the real return on both assets, the condition that pins down their portfolio holdings is given by:

$$E_t\left[M^a_{t,t+1}(\tilde{R}_{B,t} - \tilde{R}_{A,t})\right] = 0,$$

where $M^a_{t,t+1}$ is the stochastic discount factor and $\tilde{R}_{B,t} - \tilde{R}_{A,t}$ represents the excess return of holding the risky contract over the riskless contract. This condition says the expected excess return is zero after adjusting for risk by lenders' state-dependent ratio of marginal utility of consumption. $M^a_{t,t+1}$ incorporates lenders' preferences for risk by giving more weight to the excess return in $z_{t+1} = D$ state and less to that in the good state $z_{t+1} = U$.

On average, household leverage decreases with age. To illustrate the life cycle profile of leverage, I follow Fostel and Geanakoplos (2015) and define the leverage of an agent of age a at time t, LTV^a_t, as the ratio of total amount borrowed to the value of the collateral backing this borrowing:

$$LTV^a_t = \frac{\int_{j\in\mathbb{R}_+} \max\{-\theta^a_{j,t}, 0\}\pi_{j,t}dj}{q_t \int_{j\in\mathbb{R}_+} \max\{-\theta^a_{j,t}, 0\}dj}.$$

Recognizing that not all agents fully pledge their housing stock as collateral, the denominator is not well-defined for agents who do not leverage, Fostel and Geanakoplos (2015) propose another measure of leverage, the diluted LTV (DLTV). In this model, it is defined as the total amount of borrowing relative to the value of the agent's entire housing stock:

$$DLTV^a_t = \frac{\int_{j\in\mathbb{R}_+} \max\{-\theta^a_{j,t}, 0\}\pi_{j,t}dj}{q_t h^a_t}.$$

Note that $LTV^a_t \geqslant DLTV^a_t$ as they share the same numerator; however, due to the collateral constraint, LTV^a_t has a smaller denominator than $DLTV^a_t$. Figure 8 presents the mean DLTV for all age groups throughout the simulated periods. DLTV progressively decreases with age and eventually reaches zero, with the exception – an uptick at age 12, where agents begin to face declines in their endowments.

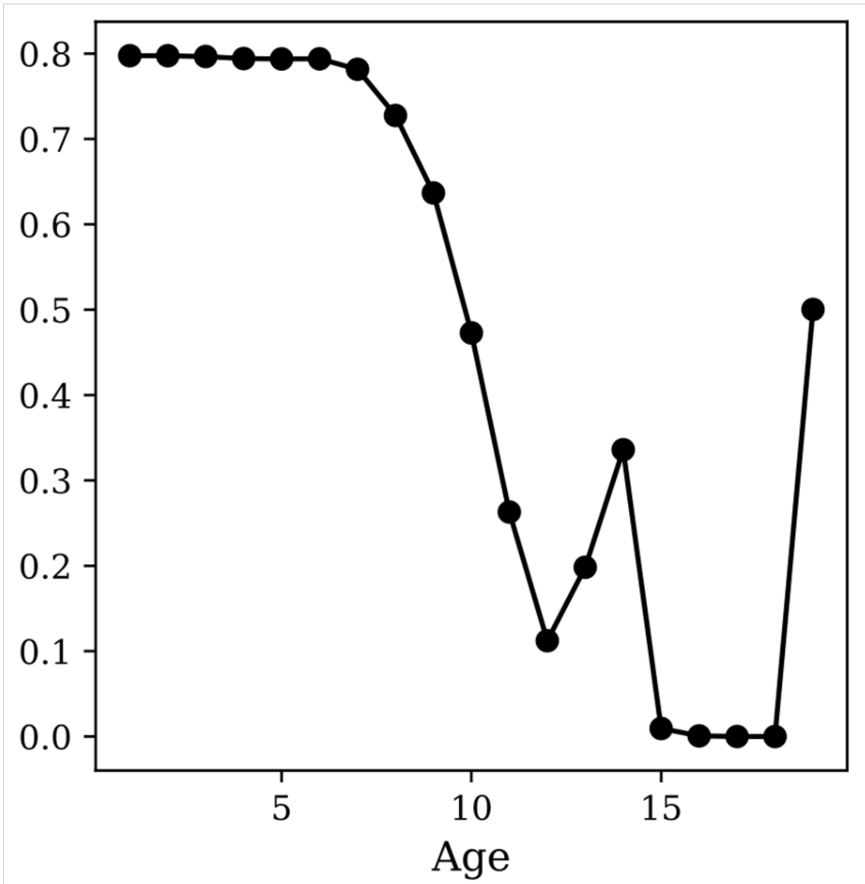


Figure 8: Average DLTV

4.2 Leverage Cycle Implications

This model features large yet infrequent crisis events. Within this framework, lenders collectively provide pricing schedules for leverage through the Credit Surface, which they adjust endogenously in response to fundamental shocks. The model generates leverage cycles characterized by the following dynamics: household leverage is positively correlated with housing prices, and amplifies the impact of fundamental shocks on housing prices; mortgage rates and spreads move in the opposite direction, remaining low during normal times while escalating during downturns.

Figure 9 illustrates the Credit Surfaces for each of the 500,000 simulated periods, table 3 presents the average prices and leverage for state U and D, as well as the differences between these averages. The surfaces in figure 9 are also color-coded to indicate the aggregate state: red represents state U and blue represents state D. The average Credit Surfaces for states U and D are in darker shades of red and blue, respectively. This figure reveals distinct patterns in the Credit Surface based on the aggregate state. Specifically, Credit Surfaces associated with D states consistently sit above those for U states, with a tendency to start rising at a lower level of LTV and at a faster speed, becoming vertical at comparatively lower LTV levels.

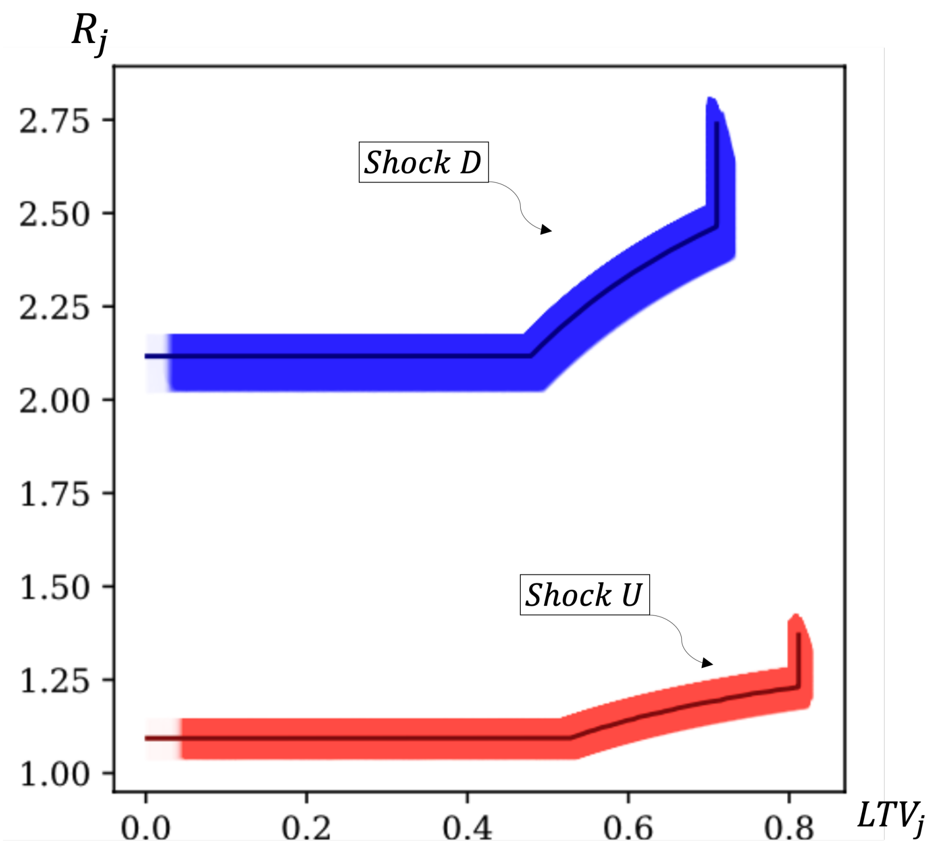


Figure 9: Credit Surfaces

The elevated position of Credit Surfaces in downturns (D states) stems from lenders'

[8]Percentage points.

	D	U	Δ
q: Housing Price	1.03	1.78	-42.13%
π_A: Riskless Contract Price	0.49	0.94	-47%
π_B: Risky Contract Price	0.73	1.44	-49%
R_A: Riskless Interest Rate	2.12	1.09	+94%
R_B: Risky Interest Rate	2.46	1.23	+100%
$R_B - R_A$: Mortgage Spread	0.34	0.14	+142%
LTV_A: Riskless LTV	47.7%	52.57%	-4.87 *pp*[8]
LTV_B: Risky LTV	71.09%	81.16%	-10.7 *pp*

Table 3: Average Prices and Leverage

optimality conditions. To illustrate, consider the the real return of contract j, defined as:

$$\tilde{R}_{j,t} = \frac{\min\{j, q_{t+1}\}}{\pi_{j,t}},$$

which, according to lenders' Euler equation, must satisfy the condition:

$$E_t\left[M^a_{t,t+1}\tilde{R}_{j,t}\right] = 1. \tag{13}$$

Note that $\tilde{R}_{j,t} \leqslant R_{j,t}$. Condition (13) requires that the risk-adjusted expected real return of any contract j in J_t must be one. Given the same history, the marginal utility of consumption $Du_c(c^a_t, h^a_t)$ is lower when $z_t = U$ compared to when $z_t = D$. As a result, the stochastic discount factor $M^a_{t,t+1}$ is higher in $z_t = U$ than in $z_t = D$. During downturns, lenders are reluctant to forgo current consumption to lend to borrowers; consequently they demand higher real returns on all contracts. This requirement leads to an increase in $\tilde{R}_{j,t}$, and, consequently, causes the Credit Surface in $z_t = D$ to be positioned higher than that in $z_t = U$.

The leverage corresponding to the two contracts actively traded in equilibrium differs markedly between states. On average, households can leverage as high as 52.57% at a risk-free interest rate with the riskless contract $j_{A,t}$ during normal times, compared to just 47.7%

in downturns.[9] Beyond the risk-free leverage threshold, the Credit Surface rises more sharply in downturns than in normal times, reflecting lenders' higher marginal utility of immediate consumption and their demand for greater excess returns to compensate for forgoing current consumption. The LTV associated with the risky contract $j_{B,t}$ endogenously sets the leverage cap, as promises beyond this lead to an infinitely high interest rate. Using contract $j_{B,t}$, households can leverage 81.06% on average in normal times, but only 71.09% in downturns.

In this model, housing plays three critical roles, each influencing the housing prices. First, housing provides immediate utility. As outlined in equation (10), the initial component of housing prices is rent: agents must pay a positive amount of $\frac{Du_h(c_t^a,h_t^a)}{Du_c(c_t^a,h_t^a)}$ to utilize the property. Second, as housing assets are perfectly durable, they serve as a means of savings. Their value today is tied to the present value of future housing prices, as reflected in the second component of housing prices: $E_t\left[\beta\frac{Du_c(c_{t+1}^{a+1},h_{t+1}^{a+1})}{Du_c(c_t^a,h_t^a)}q_{t+1}\right]$. The third component, $\frac{\mu_{h,t}^a}{Du_c(c_t^a,h_t^a)}$, represents the collateral value, which depends on the loan amount the housing collateral can secure. Fostel and Geanakoplos (2008) categorize the sum of the first two components as the fundamental value and the third as the collateral value.

Housing prices significantly decline when the economy faces negative endowment shocks. The average housing price in state D is 42.13% lower compared to state U. This decline is attributable to the simultaneous fall in all three components of housing prices. First, negative shocks directly reduce households' purchasing power, resulting in lower rent payments. Second, during downturns, all agents' marginal utility of consumption increases, leading to a heavier discounting of the future and thus lowering the present value of future housing prices. Most importantly, the capacity of housing assets to facilitate loans is substantially reduced. The average prices of riskless and risky contracts, π_A and π_B, are 47% and 49% lower in state D compared to state U, respectively, exceeding the decline in housing prices (42.13%). Consequently, the role of housing assets as collateral is significantly weakened in

[9] Note that this feature is attributed to the infrequency of economic rises in the model, a higher frequency of downturns would alter this pattern.

downturns, leading to a marked decrease in their collateral value.

In order to demonstrate the amplification effect of leverage on housing prices, I examine the market outcomes in a bond economy, which differs from the baseline model in three ways: there is a single riskless bond in zero net supply available for trade, agents are assumed to never default, and the LTV ceiling is exogenously fixed at ϕ. Let b_t^a denote the bond holdings for agents of age a at time t, and p_t the price of the bond. Agents, taking prices as given, maximize their life-time utility subject to a budget constraint:

$$c_t^a + q_t h_t^a + p_t b_t^a \leqslant e_t^a + q_t h_{t-1}^{a-1} + b_{t-1}^{a-1},$$

and a borrowing constraint:

$$-p_t b_t^a \leqslant \phi q_t h_t^a.$$

Define the agent's LTV as:

$$LTV_t^a = \frac{\max\{-p_t b_t, 0\}}{q_t h_t},$$

it is straightforward that LTV is capped at ϕ, which does not respond to changes in the fundamentals.

Let $\phi = 0.99$, I solve for the equilibrium and simulate the economy for 510,000 periods, excluding the first 10,000 periods to allow the economy to reach a stationary equilibrium. Table 3 presents the coefficient of variation in housing prices for both the collateral and the bond economies. The standard deviation of housing prices is 16.2% of the mean housing price in the collateral economy, compared to only 11.6% in the bond economy. This result illustrates the amplification effect of leverage on housing price fluctuations.

Collateral Economy	Bond Economy
16.2%	11.6%

Table 4: Coefficient of Variation of Housing Prices

5 Conclusion

This paper has explored the dynamic interplay between leverage, housing prices, and mortgage spreads in the context of the U.S. housing market, using a quantitative general equilibrium model within an overlapping-generation framework. The proposed model produces leverage cycles, characterized by the co-movements among housing prices, leverage, and and mortgage spreads.

A critical contribution of this research lies in its endogenous treatment of the LTV ratio in a model featuring a high degree of agent heterogeneity, which challenges the conventional assumption of exogenously given leverage limits. The model shows that both the upper limit of LTV and the entire pricing schedule of leverage, the Credit Surface, are inherently dynamic, reacting to changes in economic conditions. This is particularly evident in economic downturns, where the Credit Surface rises and becomes steeper, indicating a tightening of credit conditions. This dynamic response provides a crucial understanding of the decrease in leverage observed during economic crises and highlights a feedback loop between leverage and housing prices.

Looking forward, this research opens several avenues for further inquiries. A key extension would be to incorporate a more realistic lifespan for debt contracts, akin to actual mortgages. Moreover, comparing the equilibria of economies with and without the endogenous determination of leverage highlights the amplification effect of leverage on housing prices. Using the model as a working horse, we can continue investigating the effects of financial innovations, such as the introduction of credit default swaps (CDS), tranching, and other financial instruments.

Appendix A: Algorithm for Approximating a FREE

1. Set the episode counter $s = 0$. Initialize a neural network $\mathcal{N}^{(s)}$ with two hidden layers. The first layer contains 500 nodes, and the second layer contains 300 nodes.

2. Generate an initial state vector $\mathbf{x_0}^{(s)}$ randomly. This vector includes the aggregate state z_{s-1}, past consumption c^a_{s-1}, housing h^a_{s-1}, and portfolio holdings $\boldsymbol{\theta}^a_{s-1}$ for each age cohort a within the active agent set $\hat{\mathbf{A}}$. The state vector is represented within the space $\mathbf{Z} \times \mathbb{R}^{A-1} \times [0,1]^{A-1} \times [-1,1]^{N_j(A-1)}$.

3. Simulate the economy using $\mathcal{N}^{(s)}$ and state vector $\mathbf{x_0}^{(s)}$ over $12,000$ periods. Discard the first $2,000$ periods to allow the economy to approach a stationary equilibrium (if exists), and construct a training dataset $\mathcal{D}^{(s)}_{\text{train}}$ using the remaining $10,000$ periods. Each simulation counts as one episode within the training sequence.

4. Calculate the loss function, which is the mean squared errors across all equilibrium conditions. These conditions include the Euler equations, market clearing conditions, budget constraints, and complementary slackness conditions, evaluated over the training dataset $\mathcal{D}^{(s)}_{\text{train}}$.

5. Implement the Adam optimization algorithm, a type of mini-batch stochastic gradient descent, to update the neural network parameters from $\mathcal{N}^{(s)}$ to $\mathcal{N}^{(s+1)}$. Update the parameters only once per simulation. Increment the episode counter to $s = s + 1$. Set the new initial state vector $\mathbf{x_0}^{(s+1)}$ as the final state vector from the preceding simulation.

6. Repeat steps 3 to 5 until either the episode counter reaches $100,000$ or the neural network converges. If the loss function does not converge after 100,000 episodes, adjust the learning rate, the size of mini-batches, or the number of nodes in each hidden layers, then return to step 1 with the new parameters.

Appendix B: Accuracy of Numerical Solutions

This appendix examines the accuracy of the numerical solution, based on the simulation spanning 510,000 periods. Throughout the simulation, the budget constraints are enforced, resulting in no errors in this aspect. The average error in the market clearing conditions is 10^{-4}, and the maximum is $10^{-2.7}$. To examine the accuracy of approximating the Euler equations, define the Euler equation error for housing, contract $j_{A,t}$, and contract $j_{B,t}$ as the following:

$$e(h_t^a) = |1 - \frac{(Du_c)^{-1}(\frac{Du_h(c_t^a,h_t^a)+\beta E_t[Du_c(c_{t+1}^{a+1},h_{t+1}^{a+1})q_{t+1}]+\mu_{h,t}^a}{q_t})}{c_t^a}|,$$

$$e(\theta_{A,t}^a) = |1 - \frac{(Du_c)^{-1}(\frac{\beta E_t[Du_c(c_{t+1}^{a+1},h_{t+1}^{a+1})q_{t+1}^D]+\mu_{A,t}^a}{q_t})}{c_t^a}|,$$

$$e(\theta_{B,t}^a) = |1 - \frac{(Du_c)^{-1}(\frac{\beta E_t[Du_c(c_{t+1}^{a+1},h_{t+1}^{a+1})q_{t+1}]+\mu_{B,t}^a}{q_t})}{c_t^a}|.$$

Figure 10 presents these three kinds of Euler equation errors by age. The errors are displayed in terms of log_{10}, with the solid lines indicating the mean and dash line the maximum. The maximum Euler equation errors is below 10^{-2}, meaning that the maximum percentage loss in consumption due to approximation errors in prices is below 1%.

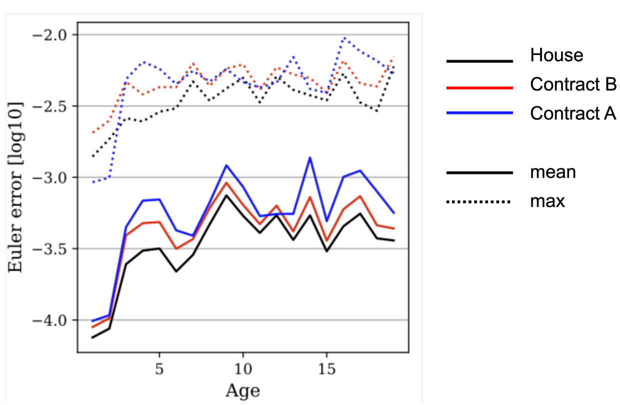


Figure 10: Euler Equation Errors