Decentralized supply chain coordination under disruption risks

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1 Model framework

Consider a supply chain involving a supplier and a buyer, which is facing disruption risks and operating under the wholesale contract with penalty, see fig.1.

We assume that supplier can source raw materials from a perfectly reliable external source at price c. The supplier has unlimited regular production capacity that transforms the input to products. Any unsold RMI can be salvaged by an external entity after one period at price s.

A buyer, who faces a demand D during a single period for a product, can source this product from a supplier at a wholesale price w. The buyer sells the product to its customers at an exogenous market price $r \geq w$ to gain profits, and backlog is not considered.

The regular production capacity of the supplier and the sourcing ability of the buyer are subject to disruption risk. When a disruption occurs at the supplier, the primary production capacity is no longer available (reduced to 0), and the regular goods are also destroyed. Risk Mitigation Inventory (RMI) at the supplier, I_S, is assumed to be held in the form of products, which can be used to supply the buyer in the event of the disruption. Analogously, when a disruption occurs at the buyer, it is impossible to source the product from the supplier so that the buyer has to sell its RMI, I_B, to meet the demand.

Should the buyer fail to fulfill the market demand, a penalty of p per unit of lost sales will be incurred. Similarly, if the supplier fails to meet the demand, buyer will impose a penalty of γ per unit on the supplier, based on the quantity of demand unmet.

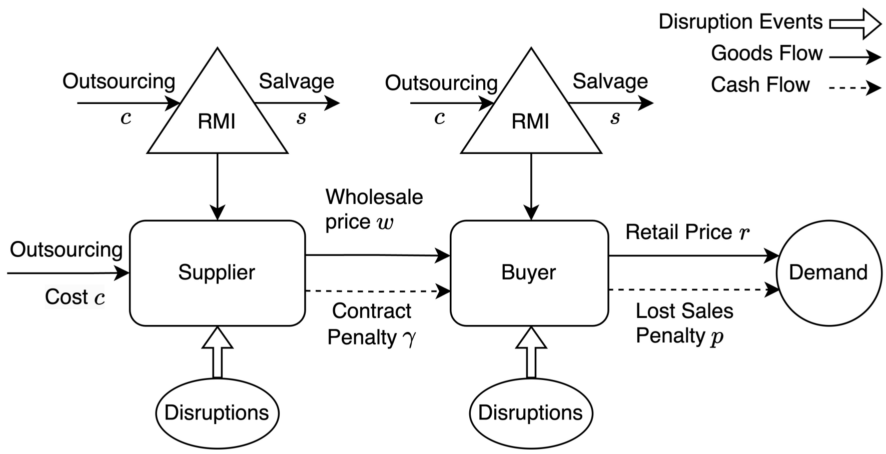

Figure 1: a wholesale contract with penalty between a supplier and a buyer

2 Notations

Decision variables:

I_S : Risk Mitigation Inventory level at the supplier.

I_B : Risk Mitigation Inventory level at the buyer.

w : Whole sale price per unit set by the buyer

γ : Contract penalty per unit for non-delivered goods, charged to the supplier.

Parameters:

$z_S, z_B > 0$: Disruption probability at the supplier and the buyer.

$Y_S, Y_B > 0$: The distribution of the supplier's and buyer's output

$h_S, h_B > 0$: RMI holding cost per unit at the supplier and the buyer.

$p > 0$: the lost sales penalty per unit for unmet market demand incurred to the buyer.

$r > 0$: The fixed retail price per unit

$s > 0$: The salvage value per unit

$c > 0$: The production cost of the supplier

$D > 0$: Demand during a single period

$F(\cdot)$: Continuous distribution function of demand

$f(\cdot)$: Probability density function of demand

$\overline{F}(\cdot)$: Inverse distribution function of demand

Objective function :

Π : Expected profits of the integrated firm during one period

π_S : Expected profit of the supplier in the decentralized problem

π_B : Expected profit of the supplier in the decentralized problem

(may consider availability objective)

3 Assumptions and constraints

- The supplier obtains raw materials from a perfectly reliable external source.
- The supplier initially procures RMI for both the supplier and the buyer. Subsequently, the RMI can be utilized to address disruptions and demand uncertainty.
- The disruption follows a two-point distribution.
- Regular production capacity at supplier is infinite.
- Disruptions will completely destroy the regular production capacity of the supplier and the buyer, as well as the ordered goods, but do not impact the Risk Mitigation Inventory (RMI).
- All RMI incurs charges for holding costs.
- The buyer and supplier enter into a wholesale contract that incorporates a penalty mechanism, wherein the buyer determines a penalty cost p to the supplier for each unit of non-delivered goods.
- Unmet market demand incurs a unit lost sales penalty p_{LS} to the buyer.
- The buyer is facing the retail price r greater than the wholesale price w to pursue profits, and the supplier also sells the products at a price higher than its sourcing cost c. Any unsold inventory can be salvaged at a value s per unit, i.e. assuming that $s < c < w < r$.

4 Deterministic demand model

4.1 Supplier disruption

First of all, consider a supply chain consisting of a supplier and a buyer with a deterministic demand, which operates under a wholesale contract with penalty, see Fig.2. In a deterministic demand model, it is assumed that the buyer faces a constant demand D so that the optimal ordering quantity is always D. The supply lead time is 0, and the supplier is subject to a random

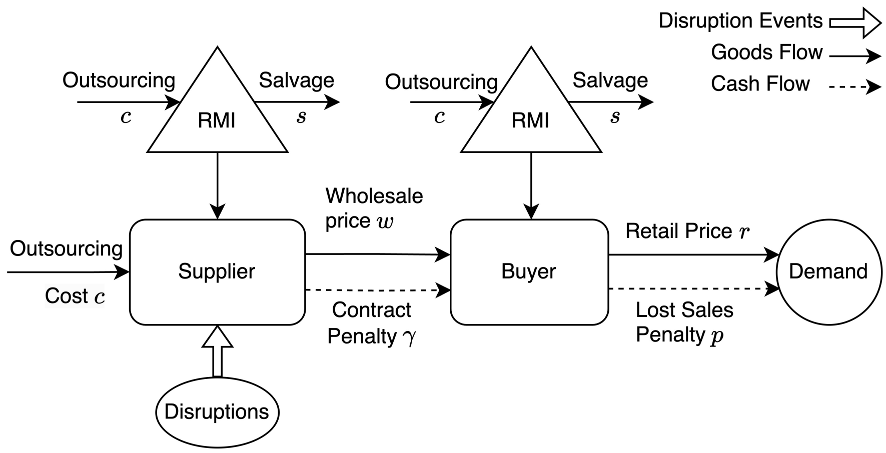

Figure 2: disruptions occur at the supplier

two-point disruption: while facing a deterministic demand D, the output is Y, where Y has the following distribution:

$$Y = \begin{cases} D, & \text{with probability } 1 - z_S, \\ 0, & \text{with probability } z_S. \end{cases} \tag{1}$$

4.1.1 Centralized Optimization problem

Let us compute the expected profits during a single period for supplier's disruption for an integrated firm, i.e., one that controls both manufacturing and sales to the market. When disruptions arise, the optimal strategy for the firm involves initially utilizing the buyer's RMI, followed by engaging with the supplier's RMI. It can be easily verified by the fact that the supplier's holding cost is assumed to be cheaper than the buyer's. Therefore, the objective is to maximize the expected profit for the integrated system, which is given by:

$$\begin{aligned}
\max_{I_S, I_B \geq 0} \Pi(I_S, I_B) = & \underbrace{(1 - z_S) \cdot rD + z_S \cdot [r \min\{I_S + I_B, D\}}_{\text{revenue}} - \underbrace{p \max\{D - I_S - I_B, 0\}]}_{\text{lost sales penalty}} \\
& + \underbrace{(1 - z_S) \cdot s(I_B + I_S) + z_S \cdot s \max\{I_S + I_B - D, 0\}}_{\text{salvage}} \\
& - \underbrace{cD - c(I_S + I_B)}_{\text{procurement cost}} \\
& - \underbrace{h_S I_S - h_B I_B}_{\text{disrupted holding cost}}
\end{aligned} \tag{2}$$

Before the regular production of D starts, the firm will procure all RMI $I_B + I_S$ at cost c to mitigate the disruption risk. Afterwards when the demand arrives, the firm initiates regular production faced with the potential disruptions. When non-disrupted, the total revenue of the integrated firm is rD, and the entirety RMI $I_B + I_S$ can be salvaged at a value s.

In disruption scenarios, when the demand is less than the combined RMI $I_B + I_S$, the firm earn a profit of rD and salvages the unused portion $I_B + I_S - D$. Conversely, all inventory is utilized, resulting in no salvage. Should the demand D exceed the combined RMIs, the firm additionally incurs a penalty $p(D - I_B - I_S)$ for the lost sales.

Solve the optimization problem 2 gives us the optimal RMI, and proposition 1 characterizes the optimal strategy for the supplier and the buyer.

The optimal strategy can be concluded as the following insights: 1)When the disruptions only occurs at the supplier, it is often optimal to hold all RMI at the supplier's end because it is cheaper. 2) In the event that the holding of the supplier cost becomes prohibitively high, surpassing the combined cost of penalties for lost sales and the potential revenue generated by the inventory, both the supplier and the buyer would opt for a zero inventory strategy.

Proposition 1. *When facing a supplier's disruption in a centralized setting, the optimal RMI level of the supplier and buyer $(I_S, I_B)^*$ is:*

$$(I_S, I_B)^* = (0, 0) \cdot \mathbf{I}(h_S \geq (1 - z_S)s + z_S(r + p) - c) + (D, 0) \cdot \mathbf{I}(h_S < (1 - z_S)s + z_S(r + p) - c) \tag{3}$$

where $\mathbf{1}$ *is the indicator function.*

4.1.2 Decentralized Optimization Problem

The supplier's problem In a decentralized problem, the buyer, acting as the Stackelberg game leader, set a penalty contract (w, γ) to the supplier. Here, w represents the wholesale price per unit, and γ denotes the contract penalty per unit for goods that are not delivered. Consequently, the expected profit for the supplier can be given as follows:

$$
\begin{aligned}
\max_{I_S \geq 0} \pi_S(I_S; \gamma, w) = & \underbrace{wI_B + (1 - z_S) \cdot wD + z_S \cdot [w \min\{\max\{D - I_B, 0\}, I_S\}]}_{\text{revenue}} \\
& + \underbrace{(1 - z_S) \cdot sI_S + z_S \cdot s \max\{I_S + I_B - D, 0\})}_{\text{salvage}} \\
& - \underbrace{z_S \cdot \gamma \max\{D - I_B - I_S, 0\}}_{\text{contract penalty}} \\
& - \underbrace{cD - c(I_S + I_B)}_{\text{procurement cost}} \\
& - \underbrace{h_S I_S}_{\text{holding cost}}
\end{aligned}
$$

The supplier initially produce all RMI $I_B + I_S$ at cost c, and then sells I_B to the buyer at a wholesale price w to help them build up inventory, regardless of the presence of disruptions. When non-disrupted, the supplier sells D at price w to the buyer to meet the demand, and all supplier's RMI I_S will be salvaged at price s. When a disruption occurs with a probability z_S, the supplier's raw material inventory (RMI) is utilized to fulfill demand and generate profits only if the demand exceeds the buyer's inventory $(D - I_B > 0)$. In such instances, the buyer can order the excess amount $D - I_B$ or up to the maximum I_S, if the demand surpasses the total combined inventory. If this situation occurs, the supplier is then required to compensate for the undelivered portion with a penalty of $\gamma(D - I_B - I_S)$, as indicated in the contract.

proposition 2 characterizes the optimal RMI for the supplier. The optimal solution can be construed as follows: if the holding cost, h_S, is lower than the sum of the revenue generated from RMI and the contract penalty when disruptions occurs, the supplier is inclined to hold an RMI level equivalent to the demand. Conversely, if the holding cost exceeds this sum, the supplier would prefer to hold no RMI, opting instead to incur the penalty for non-delivery, as this is more economically viable than bearing the holding costs. In addition, it is important to recognize that one can modify the supplier's optimal response by adjusting the wholesale price w, and the penalty γ.

Proposition 2. *When facing a supplier's disruption in a decentralized setting, the optimal RMI of the supplier is:*

$$
(I_S)^* = D \cdot \mathbf{I}(h_S \leq z_S(w + \gamma) + (1 - z_S)s - c) + 0 \cdot \mathbf{I}(h_S > z_S(w + \gamma) + (1 - z_S)s - c) \quad (4)
$$

where $\mathbf{1}$ *is the indicator function.*

The buyer's problem The buyer, taking into account the supplier's best response I_S^* to the contract terms (w, γ), solves the following profit maximization problem with respect to the I_B, wholesale price w and contract penalty γ:

$$
\begin{aligned}
\max_{I_B,w,\gamma\geq 0} \pi_B(I_B,w,\gamma;I_S) =& \underbrace{(1-z_S)\cdot rD + z_S\cdot[r\min\{D,I_B\}+r\min\{\max\{D-I_B,0\},I_S\}]}_{\text{revenue}} \\
&+\underbrace{z_S\cdot[\gamma\max\{D-I_S-I_B,0\}]}_{\text{compensation from supplier}} \\
&+\underbrace{(1-z_S)\cdot sI_B + z_S\cdot s\max\{I_B-D,0\}}_{\text{salvage}} \\
&\underbrace{-z_S\cdot p\max\{D-I_S-I_B,0\}}_{\text{lost sales penalty}} \\
&\underbrace{-wI_B - wD - z_S\cdot w\min\{\max\{D-I_B,0\},I_S\}}_{\text{procurement cost}} \\
&- \underbrace{(1-z_S)(h_B I_B)}_{\text{non-disrupted holding cost}} \underbrace{-z_S\cdot[h_B\max\{D-I_B,0\}]}_{\text{disrupted holding cost}}
\end{aligned} \tag{5}
$$

During a supplier disruption, the buyer has the opportunity to generate profits by utilizing its own RMI, I_B, and additionally by consuming the supplier's RMI, I_S, if the demand surpasses the buyer's inventory. Furthermore, the buyer is entitled to receive compensation from the supplier for any shortfall in quantity, denoted as n (calculated as $D - I_S$).At the beginning, the buyer will always place an order of I_B and D for RMI and regular production, and possibly $D - I_B$ or up to I_S if there is a disruption. However, the buyer is also subject to a penalty for any portion of market demand that remains unsatisfied by the combined RMI, quantified as $p \cdot \max\{D - I_S - I_B, 0\}$.

proposition 3 characterizes the optimal RMI for the supplier.

Proposition 3. *When facing a supplier's disruption in a decentralized setting, the optimal policy of the supplier and the buyer is:*

$$
\begin{aligned}
(I_S,I_B)^* =\ & (D,0)\cdot\mathbf{I}\,(h_S\leq z_S(w+\gamma)+(1-z_S)s-c) \\
&+(0,D)\cdot\mathbf{I}\,(h_S\geq z_S(w+\gamma)+(1-z_S)s-c)\cap I\,(h_B\leq z_S(r-s-\gamma+p)+s-w) \\
&+(0,0)\cdot\mathbf{I}\,(h_S> z_S(w+\gamma)+(1-z_S)s-c)\cap I\,(h_B> z_S(r-s-\gamma+p)+s-w)
\end{aligned}
$$

4.2 Buyer disruption

Consider the buyer is subject to a random two-point disruption, as shown in Fig.3. while facing a deterministic demand D, the output is Y, where Y has the following distribution:

$$
Y = \begin{cases} D, & \text{with probability } 1-z_B, \\ 0, & \text{with probability } z_B. \end{cases} \tag{6}
$$

4.2.1 Centralized Optimization problem

It is imperative to compute the expected profits for a single period in the event of a buyer's disruption within an integrated firm. This scenario differs from a supplier's disruption, as it necessitates the exclusive use of the buyer's RMI, because disruption will destroy the buyer's capacity to outsource from the supplier. The aim is to maximize the expected profit for the integrated system, which is expressed as follows:

$$
\begin{aligned}
\max_{I_S,I_B\geq 0} \Pi(I_S,I_B) =& \underbrace{(1-z_B)\cdot rD + z_B\cdot[r\min\{D,I_B\}}_{\text{revenue}} - \underbrace{p\max\{D-I_B,0\}]}_{\text{lost sales penalty}} \\
&+\underbrace{(1-z_B)\cdot s(I_B+I_S) + z_B[\cdot s\max\{I_B-D,0\}+I_S]}_{\text{salvage}} \\
&-\underbrace{cD - c(I_S+I_B)}_{\text{procurement cost}} \\
&-\underbrace{h_S I_S - h_B I_B}_{\text{holding cost}}
\end{aligned} \tag{7}
$$

When non-disrupted, the total revenue is rD as in the previous section. In the buyer's disruption scenarios, only the buyer's RMI, I_B, can be utilized and yielding a revenue, while the supplier's

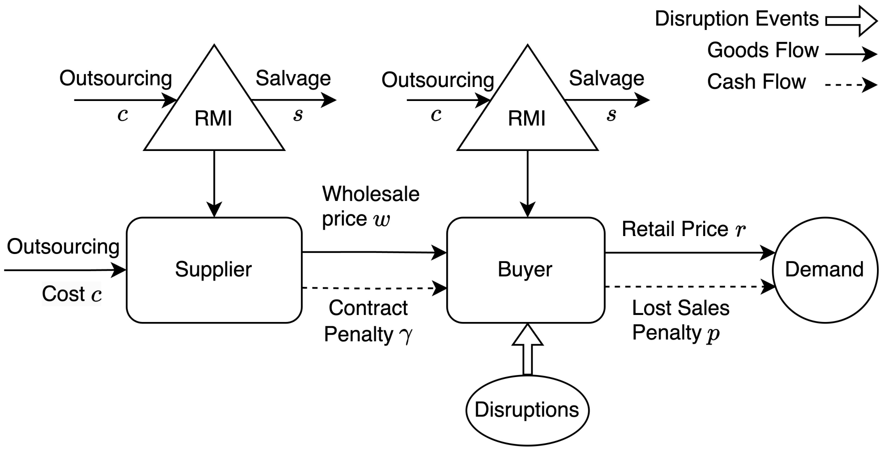

Figure 3: disruptions occur at the buyer

RMI, can not be delivered and used. Should the demand D exceed the combined RMIs, the firm incurs a penalty $p(D - I_B)$ for the lost sales. All supplier's RMI are salvaged in this case because it is impossible to use them.

Solve the optimization problem 7 gives us the optimal RMI, and proposition 4 characterizes the optimal strategy for the supplier and the buyer.

The insights behind are intuitively comprehensible: the supplier will anyhow hold zero RMI because it does not contribute the disruption. And if the holding cost of the buyer becomes prohibitively high, surpassing the sum of the lost sales penalties and the potential revenue generated by the RMI, the buyer would opt for a zero inventory strategy.

Proposition 4. *When facing a buyer's disruption in a centralized setting, the optimal RMI level of the supplier and buyer $(I_S, I_B)^*$ is:*

$$(I_S, I_B)^* = (0, D) \cdot I\left(h_B < z_B(r+p) + s - c\right) + (0, 0) \cdot I\left(h_B \geq z_B(r+p) + s - c\right)$$

Proof. □

4.2.2 Decentralized Optimization Problem

The supplier's problem In a decentralized problem, the buyer, acting as the Stackelberg game leader, set a penalty contract (w, γ) to the supplier. Consequently, the expected profit for the supplier can be given as follows:

$$\begin{aligned}\max_{I_S \geq 0} \pi_S(I_S; \gamma, w) = &\underbrace{wI_B + (1 - z_B) \cdot wD}_{\text{revenue}} \\ &+ \underbrace{(1 - z_B) \cdot sI_S + z_B \cdot sI_S}_{\text{salvage}} \\ &- \underbrace{cD - c(I_S + I_B)}_{\text{procurement cost}} \\ &- \underbrace{h_S I_S}_{\text{holding cost}}\end{aligned}$$

when a disruption occurs at the buyer with respect to a probability z_B, as the supplier can not deliver its RMI and thus gain zero revenue. And the supplier is then not obligated to pay a penalty for the non-delivered part $(D - I_S)$.

Proposition 5 indicates the optimal RMI strategy for the supplier. It is easy to see that the supplier will maintain a zero RMI in instances of disruption at the buyer's end. This is attributed to the fact that augmenting I_S incurs additional holding costs without yielding any corresponding increase in profits. Under this circumstance, penalty contract lose its incentivizing effect on the supplier's RMI.

Proposition 5. *When facing a supplier's disruption in a decentralized setting, the optimal RMI of the supplier is:*

$$(I_S)^* = 0 \qquad (\pi_S = (1 - z_B)(w - c)D)$$

Proof. To solve problem , take the derivative of I_S, we immediately obtain

$$\frac{\partial \pi_S}{\partial I_S} = -h_S < 0$$

the derivative of I_S is $-h_S$, which is negative. This implies that the optimal solution is $I_S^* = 0$. □

The buyer's problem The buyer, taking into account the supplier's best response $I_S^* = 0$ to the contract terms (w, p), solves the following profit maximization problem with respect to the I_B, :

$$\begin{aligned}\max_{I_B,w,\gamma\geq 0} \pi_B(I_B, w, \gamma; I_S) = &\underbrace{(1-z_B)\cdot rD + z_B\cdot[r\min\{I_S, D\}]}_{\text{revenue}} \\ &+ \underbrace{(1-z_S)\cdot sI_B + z_S\cdot s\max\{I_B - D, 0\}}_{\text{salvage}} \\ &\underbrace{-z_S\cdot p\max\{D-I_B, 0\}}_{\text{lost sales penalty}} \\ &\underbrace{-wI_B - wD}_{\text{procurement cost}} \\ &\underbrace{-h_B I_B}_{\text{holding cost}}\end{aligned} \tag{8}$$

In the event of a buyer's disruption, the buyer is able to obtain profits only through the utilization of its own RMI, denoted as I_B. RMI will be salvaged totally if there is no disruption, or the unused part $I_B - D$ is salvaged if disrupted. Additionally, the buyer is precluded from receiving compensation from the supplier's contract penalty, as even in scenarios where the supplier has delivered its RMI, it remains unusable by the buyer. Moreover, the buyer faces penalties for lost sales, which are quantified as $p_M \cdot \max\{D - I_S - I_B, 0\}$.

proposition 6 characterizes the optimal RMI for the buyer.

Proposition 6. *When facing a buyer's disruption in a decentralized setting, the optimal policy of the supplier and the buyer is:*

$$(I_S, I_B)^* = (0, D)\cdot \mathbf{I}\left(h_B < z_B(r+p) + s - w\right) + (0,0)\cdot \mathbf{I}\left(h_B \geq z_B(r+p) + s - w\right)$$

Proof. □

4.3 Buyer and Supplier Disruption

Consider the buyer and the supplier are both subject to a random two-point disruption, as shown in Fig.4. While facing a deterministic demand D, the output of the supplier and the buyer are Y_S and Y_B, which follow the distribution:

$$Y_S = \begin{cases} D, & \text{with probability } 1 - z_S, \\ 0, & \text{with probability } z_S. \end{cases} Y_B = \begin{cases} D, & \text{with probability } 1 - z_B, \\ 0, & \text{with probability } z_B. \end{cases} \tag{9}$$

4.3.1 Centralized Optimization Problem

When disruptions can occur at both the buyer's end and the supplier's end within an integrated firm, and the objective is to maximize the expected profit for the integrated system, which is expressed as follows:

$$\begin{aligned}\max_{I_S,I_B\geq 0} \Pi(I_S, I_B) = &\underbrace{(1-z_S)(1-z_B)\cdot rD}_{\text{non-disrupted revenue}} + \underbrace{z_S(1-z_B)\cdot[r\min\{I_S + I_B, D\}]}_{\text{only disrupted supplier's revenue}} + \underbrace{z_B\cdot r\min\{D, I_B\}}_{\text{disrupted buyer's revenue}} \\ &- \underbrace{z_S(1-z_B)\cdot[p\max\{D - I_S - I_B, 0\}] - z_B\cdot[p\max\{D - I_B, 0\}]}_{\text{lost sales penalty}} \\ &+ \underbrace{(1-z_S)(1-z_B)\cdot s(I_B + I_S) + z_S(1-z_B)\cdot s\max\{I_B + I_S - D, 0\} + z_B\cdot s[\max\{I_B - D, 0\} + I_S]}_{\text{salvage}} \\ &- \underbrace{cD - c(I_S + I_B)}_{\text{procurement cost}} \\ &- \underbrace{h_S I_S - h_B I_B}_{\text{holding cost}}\end{aligned}$$

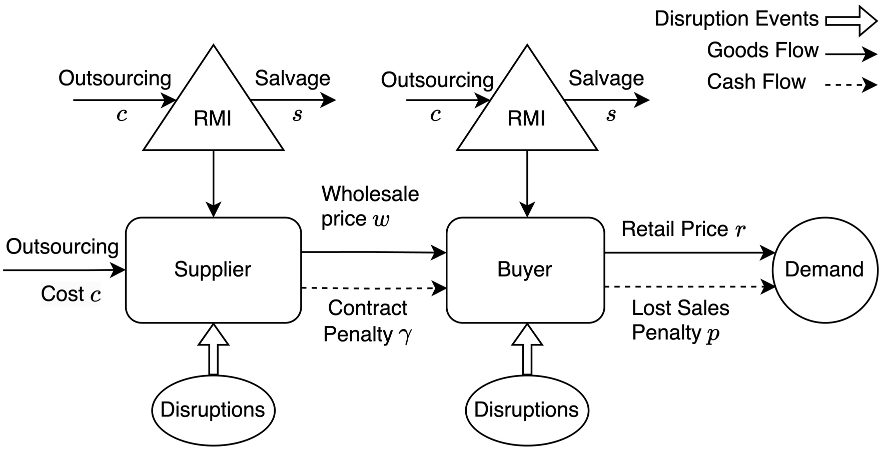

Figure 4: disruptions occur at the supplier and the buyer

The probability of no disruption is computed as $(1-z_S)(1-z_B)$, reflecting the assumption that disruptions are independent and identically distributed (i.i.d.). In cases where the disruption occurs at the supplier with probability z_S, both the supplier's and buyer's RMIs, denoted as I_S and I_B respectively, can be utilized. And the supplier's inventory can be used only if $D > I_S$. Conversely, during a disruption at the buyer with probability z_B, the supplier's RMI cannot be delivered. Only the buyer's RMI, I_B, is operational to yield the revenue. In scenarios where the demand D surpasses the total available RMIs, the firm incurs a penalty, represented as $p(D - I_B - I_S)$ or $p(D - I_B)$, depending on where do the disruptions occur. If there is a buyer's disruption, all of supplier's RMI will be salvaged and the buyer's RMI can be partially salvaged, i.e. an amount of $\max\{I_B - D, 0\} + I_S$. Note that this problem is a combined problem of previous section, if we set $z_B = 0$ and this problem is degenerated into the only supplier disruption problem, and set $z_S = 0$ it becomes the only buyer disruption problem.

Proposition 7. *When facing a buyer's disruption in a centralized setting, the optimal RMI level of the supplier and buyer $(I_S, I_B)^*$ is:*

$$\begin{aligned}(I_S, I_B)^* = {} & (D,0)\cdot I\left(h_S \le z_S(1-z_B)(r+p) + (1-z_S+z_Bz_S)s - c \cap h_B > z_B(r+p-s) + h_S\right) \\ & + (0,D)\cdot I\left(h_S > h_B - z_B(r+p-s)\right) \cap I\left(h_B \le (z_B + z_S(1-z_B))(r+p) + (1-z_S)(1-z_B)s - c\right) \\ & + (0,0)\cdot I([h_S > z_S(1-z_B)(r+p) + (1-z_S+z_Bz_S)s - c \cap h_B > z_B(r+p-s) + h_S] \\ & \cup [h_S > h_B - z_B(r+p-s)] \cap [h_B > (z_B + z_S(1-z_B))(r+p) + (1-z_S)(1-z_B)s - c])\end{aligned}$$

Proof. □

4.3.2 Decentralized Optimization Problem

The supplier's problem In a decentralized problem, the buyer, acting as the Stackelberg game leader, set a penalty contract (w, γ) to the supplier. Consequently, the expected profit for the supplier can be given as follows:

$$\begin{aligned}\max_{I_S, I_B \ge 0} \Pi(I_S, I_B) = {} & wI_B + \underbrace{(1-z_S)(1-z_B)\cdot wD}_{\text{non-disrupted revenue}} + \underbrace{z_S \cdot w\min\{\max\{D - I_B, 0\}, I_S\}}_{\text{disrupted supplier's revenue}}] \\ & - \underbrace{z_S \cdot \gamma\max\{D - I_S - I_B, 0\}}_{\text{contract penalty}} \\ & + \underbrace{(1-z_S)(1-z_B)\cdot s(I_B + I_S) + z_S \cdot s[I_S + \min\{I_B - D, 0\}] + z_B \cdot sI_S}_{\text{salvage}} \\ & - \underbrace{cD - c(I_S + I_B)}_{\text{procurement cost}} \\ & - \underbrace{h_SI_S}_{\text{holding cost}}\end{aligned} \tag{10}$$

when a disruption occurs at the supplier with respect to a probability z_S, and the supplier can utilize its RMI and thus gain revenue when $D > I_B$. However, in a buyer's disruption the revenue is zero. And the supplier needs to pay a penalty for the non-delivered part $(D - I_B - I_S)$ when there is a supplier's disruption, when it comes to the buyer's disruption, the penalty becomes zero. When there is a supplier's disruption, the supplier will salvage RMI of $I_S + I_B - D$ when $I_B < D$, or I_S when $I_B \geq D$. In a buyer's disruption, the supplier will always salvage all of the RMI I_S.

Proposition 8. *When facing both the supplier's and the buyer's disruption in a decentralized setting, the optimal RMI of the supplier is:*

$$(I_S)^* =$$

Proof. $\square$

The buyer's problem The buyer, taking into account the supplier's best response I_S^* to the contract terms (w, p), solves the following profit maximization problem with respect to the I_B, :

$$\begin{aligned}
\max_{I_B, w, \gamma \geq 0} \pi_B(I_B, w, \gamma; I_S) =& (1 - z_S)(1 - z_B) \cdot rD + z_S \cdot \underbrace{[r \min\{D, I_B\} + r \min\{\max\{D - I_B, 0\}, I_S\}]}_{\text{supplier disruption revenue}} \\
& \underbrace{+ z_B \cdot r \min\{I_B, D\}}_{\text{buyer disruption revenue}} + \underbrace{(1 - z_S) \cdot sI_B + z_S \cdot s \max\{I_B - D, 0\}}_{\text{salvage}} \\
& + \underbrace{z_S \cdot \gamma \max\{D - I_S - I_B, 0\}}_{\text{contract compensation from supplier}} \\
& \underbrace{- z_S \cdot p \max\{D - I_S - I_B, 0\} - z_B \cdot p \max\{D - I_B, 0\}}_{\text{lost sales penalty of supplier and buyer disruption}} \\
& \underbrace{- wI_B - wD - z_S \cdot w \min\{\max\{D - I_B, 0\}, I_S\}}_{\text{procurement cost}} \\
& \underbrace{- h_B I_B}_{\text{holding cost}}
\end{aligned} \tag{11}$$

when a supplier's disruption occurs, the buyer can sell their own RMI to gain profit and can also purchase more RMI from the supplier to profit if $D > I_B$. In the case of a buyer's disruption, the buyer is limited to selling their own RMI for profit. Additionally, they will receive compensation from the supplier if the supplier fails to deliver.

proposition 9 characterizes the optimal RMI for the buyer.

Proposition 9. *When facing both the supplier's and the buyer's disruption in a decentralized setting, the optimal RMI of the supplier i, the optimal policy of the buyer is:*

Proof. $\square$

5 Stochastic Demand

5.1 Supplier disruption with stochastic demand

Now let us consider the centralized problem with a stochastic demand D, RMI can be also utilized to mitigate the demand uncertainty. Consider the supplier are subject to a random two-point disruption, as shown in Fig.4.

While facing a stochastic demand D, the single demand realization is drawn from a continuous distribution F with density f. F has a finite mean and an inverse F^{-1}. Let $\overline{F}(\xi) = 1 - F(\xi)$. In the event of a stock out, unmet demand has a penalty.

The output of the supplier are Y_S, where Y_S has the following distribution:

$$Y_S = \begin{cases} D, & \text{with probability } 1 - z_S, \\ 0, & \text{with probability } z_S. \end{cases} \tag{12}$$

5.1.1 Centralized Optimization problem

When disruptions can occur at the supplier's end within an integrated firm, the objective is to maximize the expected profit for the integrated system, which is expressed as follows:

$$
\begin{aligned}
\max_{Q,I_S,I_B\geq 0}\Pi(Q,I_S,I_B) = \; & \underbrace{(1-z_S)}_{\text{non-disrupted}} \cdot \left[\underbrace{r\int_0^{Q+I_S+I_B} \xi f(\xi)d\xi}_{\text{available expected in-stock}} + r\underbrace{(Q+I_S+I_B)\overline{F}(Q+I_S+I_B)}_{\text{stock out}} - p\underbrace{\int_{Q+I_S+I_B}^{\infty} \xi f(\xi)d\xi}_{\text{lost sales penalty}} \right] \\
& + \underbrace{z_S}_{\text{disrupted}} \left[\underbrace{r\int_0^{I_S+I_B} \xi f(\xi)d\xi}_{\text{RMI in stock}} + r\underbrace{(I_S+I_B)\overline{F}(I_S+I_B)}_{\text{stock out}} - \underbrace{p\int_{I_S+I_B}^{\infty} \xi f(\xi)d\xi}_{\text{lost sales penalty}} \right] \\
& + \underbrace{(1-z_S)\cdot s\int_0^{Q+I_S+I_B} (Q+I_S+I_B-\xi)f(\xi)d\xi}_{\text{non-disruption salvage}} + \underbrace{z_S\cdot s\int_0^{I_S+I_B} (I_S+I_B-\xi)f(\xi)d\xi}_{\text{disruption salvage}} \\
& - \underbrace{cQ - c(I_S+I_B)}_{\text{procurement cost}} \\
& - \underbrace{h_S I_S - h_B I_B}_{\text{holding cost}}
\end{aligned}
\tag{13}
$$

Proposition 10. *When facing a supplier's disruption in a centralized setting with stochastic demand, the optimal conditions of RMI level of the supplier and buyer are:*

$$
\begin{aligned}
&(1-z_S)\left[r+(s-r)F(Q+I_S+I_B)\right]-c=0\\
&z_S\left[r+(s-r)F(I_S+I_B)\right]-h_S=0\\
&z_S\left[r+(s-r)F(I_S+I_B)\right]-h_B=0
\end{aligned}
$$

Proof. For the profit function $\Pi(Q,I_S,I_B)$, apply the first-order condition and take the derivative with respect to Q and let it be 0:

$$
\begin{aligned}
\frac{\partial\Pi}{\partial Q} &= (1-z_S)\,(r(1-F(Q+I_S+I_B)) + r(Q+I_S+I_B)f(Q+I_S+I_B) \\
&\quad -r(Q+I_S+I_B)\cdot\left.\frac{dF}{d\xi}\right|_{\xi=Q+I_S+I_B} + s\int_0^{Q+I_S+I_B} f(\xi)d\xi\Bigg) - c - h_S \\
&= (1-z_S)\left(r(1-F(Q+I_S+I_B)) + s\int_0^{Q+I_S+I_B} f(\xi)d\xi\right) - c = 0 \\
&= (1-z_S)\left[r+(s-r)F(Q+I_S+I_B)\right]-c=0
\end{aligned}
$$

where we get the condition with respect to Q, and take the derivative with respect to I_S gives:

$$
\begin{aligned}
\frac{\partial\Pi}{\partial I_S} =& -c - h_S + z_S\,(r(1-F(I_S+I_B)) + r(I_S+I_B)f(I_S+I_B) \\
&-r(I_S+I_B)\cdot\left.\frac{dF}{d\xi}\right|_{\xi=I_S+I_B} + s\int_0^{I_S+I_B} f(\xi)d\xi\Bigg) \\
&+(1-z_S)\,(r(1-F(Q+I_S+I_B)) + r(Q+I_S+I_B)f(Q+I_S+I_B) \\
&-r(Q+I_S+I_B)\cdot\left.\frac{dF}{d\xi}\right|_{\xi=Q+I_S+I_B} + s\int_0^{Q+I_S+I_B} f(\xi)d\xi\Bigg)
\end{aligned}
$$

simplify it further, and and let it be 0 we obtain:

$$
\begin{aligned}
\frac{\partial \Pi}{\partial I_S} =& - c - h_B + z_S \left(r(1 - F(I_S + I_B)) + r(I_S + I_B) f(I_S + I_B) \right. \\
& \left. - r(I_S + I_B) \cdot \left. \frac{dF}{d\xi} \right|_{\xi = I_S + I_B} + s \int_0^{I_S + I_B} f(\xi) d\xi \right) \\
& + (1 - z_S) \left(r(1 - F(Q + I_S + I_B)) + r(Q + I_S + I_B) f(Q + I_S + I_B) \right) \\
=& (1 - z_S) \Big(r(1 - F(Q + I_S + I_B)) + s \int_0^{Q + I_S + I_B} f(\xi) d\xi \Big) \\
& + z_S \Big(r(1 - F(I_S + I_B)) + s \int_0^{I_S + I_B} f(\xi) d\xi \Big) - c - h_S \\
=& (1 - z_S) \left[r + (s - r) F(Q + I_S + I_B) \right] + z_S \left[r + (s - r) F(I_S + I_B) \right] - c - h_S = 0
\end{aligned}
$$

which we get the condition with respect to I_S. Again, take the derivative with respect to I_B, since I_S and I_B always come in pair, thus the only difference here is the holding cost term:

$$
\frac{\partial \Pi}{\partial I_B} = (1 - z_S) \left[r + (s - r) F(Q + I_S + I_B) \right] + z_S \left[r + (s - r) F(I_S + I_B) \right] - c - h_B = 0
$$

combine the three conditions with respect to Q, I_S and I_B, we can recover the two conditions in proposition 10:

$$
\begin{aligned}
z_S \left[r + (s - r) F(I_S + I_B) \right] - h_S = 0 \\
z_S \left[r + (s - r) F(I_S + I_B) \right] - h_B = 0
\end{aligned}
$$

□

Proof. In order to determine whether this problem is concave or not, we need to compute the Hessian matrix, and the definition is:

$$
H = \begin{bmatrix}
\frac{\partial^2 \Pi}{\partial Q^2} & \frac{\partial^2 \Pi}{\partial Q \partial I_S} & \frac{\partial^2 \Pi}{\partial Q \partial I_B} \\
\frac{\partial^2 \Pi}{\partial I_S \partial Q} & \frac{\partial^2 \Pi}{\partial I_S^2} & \frac{\partial^2 \Pi}{\partial I_S \partial I_B} \\
\frac{\partial^2 \Pi}{\partial I_B \partial Q} & \frac{\partial^2 \Pi}{\partial I_B \partial I_S} & \frac{\partial^2 \Pi}{\partial I_B^2}
\end{bmatrix}
$$

The Hessian matrix calculated is as follows:

$$
H = \begin{bmatrix}
(1 - z_S)(-r + s) \frac{d^2 F}{d\xi^2} & (1 - z_S)(-r + s) \frac{d^2 F}{d\xi^2} & (1 - z_S)(-r + s) \frac{d^2 F}{d\xi^2} \\
(1 - z_S)(-r + s) \frac{d^2 F}{d\xi^2} & z_S(-r + s) \frac{d^2 F}{d\xi^2} + (1 - z_S)(-r + s) \frac{d^2 F}{d\xi^2} & z_S(-r + s) \frac{d^2 F}{d\xi^2} + (1 - z_S)(-r + s) \frac{d^2 F}{d\xi^2} \\
(1 - z_S)(-r + s) \frac{d^2 F}{d\xi^2} & z_S(-r + s) \frac{d^2 F}{d\xi^2} + (1 - z_S)(-r + s) \frac{d^2 F}{d\xi^2} & z_S(-r + s) \frac{d^2 F}{d\xi^2} + (1 - z_S)(-r + s) \frac{d^2 F}{d\xi^2}
\end{bmatrix}
$$

Here, $\frac{d^2 F}{d\xi^2}$ represents the second derivative of the function F, which depends on the specific form of the variables $Q + I_S + I_B$ or $I_S + I_B$. □

5.1.2 Decentralized Optimization problem

Supplier's problem In a decentralized problem, the buyer, acting as the Stackelberg game leader, set a penalty contract (w, γ) to the supplier. Here, w represents the wholesale price per unit, and γ denotes the contract penalty per unit for goods that are not delivered. Consequently, the expected profit for the supplier can be given as follows:

$$
\begin{aligned}
\max_{Q,I_S,I_B\geq 0} \Pi(Q,I_S,I_B) = wI_B + (1-z_S)\cdot & \left[\underbrace{w\int_{Q+I_B}^{Q+I_S+I_B} \xi f(\xi)d\xi}_{\text{available expected in-stock}} + w\underbrace{I_S\overline{F}(Q+I_S+I_B)}_{\text{stock out}} \right] \\
& + z_S\left[\underbrace{w\int_{I_B}^{I_S+I_B} \xi f(\xi)d\xi}_{\text{RMI in stock}} + w\underbrace{I_S\overline{F}(I_S+I_B)}_{\text{stock out}} - \underbrace{\gamma\int_{I_S+I_B}^{\infty} \xi f(\xi)d\xi}_{\text{contract penalty}} \right] \\
& + \underbrace{(1-z_S)\cdot s\int_0^{Q+I_S+I_B} (Q+I_S+I_B-\xi)f(\xi)d\xi}_{\text{non-disruption salvage}} + \underbrace{z_S\cdot s\int_0^{I_S+I_B} (I_S+I_B-\xi)f(\xi)d\xi}_{\text{disruption salvage}} \\
& - \underbrace{cQ - c(I_S+I_B)}_{\text{procurement cost}} \\
& - \underbrace{h_S I_S}_{\text{holding cost}}
\end{aligned}
\tag{14}
$$

Proposition 11. *when facing a supplier's disruption in a decentralized setting with stochastic demand, the optimal RMI level of the supplier I_S^* is:*

$$I_S^* = \{$$

Buyer's problem The buyer, taking into account the supplier's best response I_S^* to the contract terms (w,p), solves the following profit maximization problem with respect to the I_B, :

$$
\begin{aligned}
\max_{I_B,w,\gamma\geq 0} \pi_B(I_B,w,\gamma;I_S) = \underbrace{(1-z_S)}_{\text{non-disrupted}} \cdot & \left[\underbrace{r\int_0^{Q+I_B} \xi f(\xi)d\xi}_{\text{available expected in-stock}} + r\underbrace{(Q+I_B)\overline{F}(Q+I_B)}_{\text{stock out}} - \underbrace{p\int_{Q+I_B}^{\infty} \xi f(\xi)d\xi}_{\text{lost sales penalty}} \right] \\
+ \underbrace{z_S}_{\text{disrupted}} & \left[\underbrace{r\int_0^{I_S+I_B} \xi f(\xi)d\xi}_{\text{RMI in stock}} + r\underbrace{(I_S+I_B)\overline{F}(I_S+I_B)}_{\text{stock out}} + \underbrace{\gamma\int_{I_S+I_B}^{\infty} \xi f(\xi)d\xi}_{\text{contract conpensation}} - \underbrace{p\int_{I_S+I_B}^{\infty} \xi f(\xi)d\xi}_{\text{lost sales penalty}} \right] \\
& + \underbrace{(1-z_S)\cdot s\int_0^{Q+I_B} (Q+I_B-\xi)f(\xi)d\xi}_{\text{non-disruption salvage}} + \underbrace{z_S\cdot s\int_0^{I_B} (I_B-\xi)f(\xi)d\xi}_{\text{disruption salvage}} \\
& - \underbrace{wQ - wI_B - w\int_{I_B}^{I_B+I_S} \xi f(\xi)d\xi}_{\text{procurement cost}} \\
& - \underbrace{h_B I_B}_{\text{holding cost}}
\end{aligned}
\tag{15}
$$

when facing a supplier's disruption in a decentralized setting with stochastic demand, the optimal RMI level of the supplier and the buyer is $(I_S, I_B)^*$ is:

6 Extension

- availability in profits, service level
- stochastic demand
- consider more contracts

A Appendix

A.1 Proof of Proposition 1 for the centralized system with disrupted supplier

Proof. Since the problem is linear to I_S and I_B, one can solve the the centralized problem 2 by computing the partial derivative of I_S and I_S.

After rearrangement, the objective function of 2 can be rewritten as:

$$\Pi(I_S, I_B) = (1 - z_S)(r - c)D + z_S((r - c - c_T)I_S + (r - c)I_B) - p_{LS}(D - I_S - I_B) - h_B I_B - h_S I_S$$

Take the partial derivative of I_S:

$$\frac{\partial \Pi}{\partial I_S} = z_S(r - c - c_T) - h_S + p_{LS}$$

For $\partial\Pi/\partial I_S > 0$, the optimal response of the supplier is $I_S^* = D - I_B$, because I_S is bounded by the constraint **??** and $I_B, I_S \geq 0$. Take the optimal response of I_S^* and integrate it into the problem gives:

$$\Pi(I_B; I_S^*) = (1 - z_S)(r - c)D + z_S((r - c - c_T)(D - I_B) + (r - c)I_B) - h_B I_B - h_S(D - I_B)$$

solve for the optimal I_B with the first-order condition:

$$\frac{\partial \Pi(I_B; I_S^*)}{\partial I_B} = c_T \cdot z_S - h_B + h_S$$

if $\partial\Pi(I_B; I_S^*)/\partial I_B > 0$,it implies that profit incrementally increases with an increase in I_B, and the optimal I_B^* is equal to the demand D, bounded by the constraint **??** and $I_S \geq 0$.In this senoria, $I_S^* = 0$. Conversely, should the derivative $\partial\Pi(I_B; I_S^*)/\partial I_B \leq 0$, the optimal $(I_S, I_B)^* = (0, D)$

For $\partial\Pi/\partial I_S \leq 0$,the the optimal response of the supplier is $I_S^* = 0$, constrained by the non-negative requirement of I_S, plug this response function we obtain:

$$\Pi(I_B; I_S^*) = (1 - z_S)(r - c)D + z_S(r - c)I_B - h_B I_B - h_S(D - I_B)$$
$$\frac{\partial \Pi(I_B; I_S^*)}{\partial I_B} = z_S(r - c) - h_B + h_S$$

Assuming $\partial\Pi(I_B; I_S^*)/\partial I_B < 0$, the resulting optimal I_B^* is zero. This scenario leads to the third optimal solution pair $(I_S, I_B)^* = (0, 0)$. Conversely, if the inequality does not hold, the optimal solution defaults to $(I_S, I_B)^* = (0, D)$.

To synthesize the preceding analysis, we establish the optimal solution as delineated in Proposition1.

□

A.2 Proof of Proposition 2

Proof. To solve problem 4, we can decompose the problem into two sub-problems with constraints $I_S \leq D$ and $I_S \geq D$, denoted by P1 and P2 respectively.

$$P1: \max_{I_S \geq 0} \pi_S(I_S; p, w) = (1 - z_S)(r - c)D + z_S(w - c - c_T)I_S - h_S I_S - p(D - I_S) \quad s.t. \quad I_S \leq D$$

to solve P1, take the derivative of I_S, we obtain

$$\frac{\partial \pi_S}{\partial I_S} = z_S(w - c - c_T) - h_S + p$$

for the condition $\frac{\partial \Pi}{\partial I_S} > 0$, we derive that $h_S < z_S(w - c - c_T) + p$. In this scenario, the optimal response for the supplier is $I_S^* = D$, since it is limited by the demand constraint D. Conversely, if this condition does not hold, the optimal response is zero.

$$P2: \max_{I_S \geq 0} \pi_S(I_S; p, w) = (1 - z_S)(r - c)D + z_S(w - c - c_T)D - h_S I_S \quad s.t. \quad I_S \geq D$$

Furthermore, in subproblem P2, the derivative of I_S is $-h_S$, which is negative. This characteristic dictates that the optimal solution is $I_S^* = 0$.

Integrating the solutions from these two subproblems, we arrive at the optimal solution as presented in Proposition 2. □

A.3 Proof of Proposition 3

Proof. To solve the problem 11, again we decompose the problem into two sub-problems with constraints $h_S \geq z_S(w-c-c_T)+p$ and $h_S \leq z_S(w-c-c_T)+p$ first, denoted by P3 and P4 respectively.

Take the optimal response of the supplier $I_S^* = D$ and the constraint from the condition 2, we obtain such subproblem P3:

$$P3: \max_{I_B,w,p\geq 0} \Pi(I_B,w,p;I_S^*=D) = (1-z_S)(r-c)D + z_S(w-c-c_T)I_S - h_S I_S - p(D-I_S)$$

$$\begin{aligned} s.t. \quad & I_B \leq D - I_S^* \\ & h_S \leq z_S(w-c-c_T)+p \end{aligned}$$

to solve P3 we easily obtain $I_B^* = 0$ because of the constraint $0 \leq I_B \leq D - I_S^*$. Now take the optimal response $I_S^* = 0$ and we construct aoother subproblem P4:

$$P4: \max_{I_B,w,p\geq 0} \Pi(I_B,w,p;I_S^*=0) = (1-z_S)(r-c)D + z_S(r-w)I_B - h_B I_B - p_{LS}(D-I_B)$$

$$\begin{aligned} s.t. \quad & I_B \leq D - I_S^* \\ & h_S \geq z_S(w-c-c_T)+p \end{aligned}$$

To solve P4 take the derivative of I_B:

$$\frac{\partial \pi_B}{\partial I_B} = z_S(r-w) + p_{LS} - h_B$$

let $\partial\pi_B/\partial I_B \leq 0$ we have $h_B \geq z_S(r-w)+P_{LS}$, where the optimal $I_B^* = 0$, otherwise equal to D. □

A.4 Proof of Proposition ??

Proof. Given $f(\xi) = \beta\exp(-\beta\xi)$, which is the density function of an exponential distribution with $\beta > 0$, we need to determine the concavity of the original function with respect to Q by finding the second derivative.

First, we apply Leibniz's rule to find the first derivative:

The first derivative $g'(Q)$ is derived from the upper limit of the integral with respect to Q, yielding:

$$g'(Q) = (1-z_S)\cdot s\cdot(I_S+I_B)\cdot\beta\exp(-\beta Q)$$

Now, we take the derivative of $g'(Q)$ to find the second derivative $g''(Q)$:

$$g''(Q) = (1-z_S)\cdot s\cdot(I_S+I_B)\cdot\beta\cdot(-\beta)\exp(-\beta Q)$$

Since $\exp(-\beta Q)$ is always positive, and β is also positive, $g''(Q)$ is always negative. This indicates that the original function is concave with respect to Q.

Therefore, we can conclude that the given problem is concave in Q. □

References