Optimization and Simulation

Discrete Events Simulation

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Simulation of a system

Keep track of variables

- Time variable t: amount of time that has elapsed.
- Counter variables: count events having occurred by t
- System state variables.

Events

- List of future events sorted in chronological order
- Process the next event:
  - remove the first event in the list,
  - update the variables,
  - generate new events, if applicable (keep the list sorted),
  - collect statistics.

Discrete Event Simulation: an example

Cloe at Satellite

- Cloe has applied to be a waiter at Satellite
- According to her experience, she pretends to be able to serve in average one customer per minute.
- In order to make the decision to hire Cloe or not, the manager wants to know:
  - In average, how much time will a customer wait after her arrival, until being served?
  - If Cloe will need extra hours to serve everybody?

SATELLITE
bar · concerts · cafés-théâtres

Discrete Event Simulation: an example

Context

- When a customer arrives, she is served if Cloe is free. Otherwise, she joins the queue.
- Customers are served using a "first come, first served" logic.
- When Cloe has finished serving a customer,
  - she starts serving the next customer in line, or
  - waits for the next customer to arrive if the queue is empty.
- The amount of time required by Cloe to serve a customer is a random variable X_s with pdf f_s.
- The amount of time between the arrival of two customers is a random variable X_a with pdf f_a.
- Satellite does not accept the arrival of customers after time T.

Discrete Event Simulation: an example

Variables

Time:	t	
Counters:	N_A	number of arrivals
	N_D	number of departures
System state:	n	number of customers in the system

Event list

- Next arrival. Time: t_A
- Service completion for the customer currently being served. Time: t_D (∞ if no customer is being served).
- The bar closes. Time: T.

List management

- The number of events is always 3 in this example.
- We just need to update the times, and keep them sorted.

Initialization

Variables

- Time: $t = 0$.
- Counters: $N_A = N_D = 0$.
- State: $n = 0$.
- First event: arrival of first customer: draw r from f_a.
- Events list:
  - $t_A = r$,
  - $t_D = \infty$,
  - T (bar closes).

Statistics to collect

- $A(i)$ arrival of customer i.
- $D(i)$ departure of customer i.
- T_p time after T that the last customer departs.

Case 1: arrival of a customer

If $t_A = \min(t_A, t_D, T)$

- Time $t = t_A$: we move along to time t_A.
- Counter $N_A = N_A + 1$: one more customer arrived.
- State $n = n + 1$: one more customer in the system.
- Next arrival:
  - draw r from f_a,
  - $t_A = t + r$.
- Service time: if $n = 1$ (she is served immediately)
  - draw s from f_s,
  - $t_D = t + s$.
- Statistics: $A(N_A) = t$.

Case 2: departure of a customer

If $t_D = \min(t_A, t_D, T)$, $t_D < t_A$

- Time $t = t_D$: we move along to time t_D.
- Counter $N_D = N_D + 1$: one more customer departed.
- State $n = n - 1$: one less customer in the system.
- Service time: if $n = 0$, then $t_D = \infty$. Otherwise,
    - draw s from f_s,
    - $t_D = t + s$.
- Statistics: $D(N_D) = t$.

Case 3: after hours

If $T < \min(t_A, t_D)$

1. Customers are still waiting: $n > 0$
   - Time $t = t_D$: we move along to time t_D.
   - Counter $N_D = N_D + 1$: one more customer departed.
   - State $n = n - 1$: one less customer in the system.
   - Service time: if $n > 0$, then
     - draw s from f_s,
     - $t_D = t + s$.
   - Statistics: $D(N_D) = t$.
2. No more customers: $n = 0$
   - Statistics: $T_p = \max(t - T, 0)$.

An instance

Scenario

- Service time: exponential with mean 1.0
- Inter-arrival time: exponential with mean 1.0
- Closing time: 10.0

An instance (ctd.)

Event	t	NA	ND	n	tA	tD	T
Arrival	0.94	1	0	1	1.48	3.22	10.0
Arrival	1.48	2	0	2	2.01	3.22	10.0
Arrival	2.01	3	0	3	3.16	3.22	10.0
Arrival	3.16	4	0	4	3.44	3.22	10.0
Departure	3.22	4	1	3	3.44	3.49	10.0
Arrival	3.44	5	1	4	3.81	3.49	10.0
Departure	3.49	5	2	3	3.81	3.91	10.0
Arrival	3.81	6	2	4	7.22	3.91	10.0
Departure	3.91	6	3	3	7.22	5.84	10.0
Departure	5.84	6	4	2	7.22	5.88	10.0
Departure	5.88	6	5	1	7.22	6.49	10.0
Departure	6.49	6	6	0	7.22	∞	10.0
Arrival	7.22	7	6	1	7.42	7.38	10.0

...

An instance (ctd.)

Event	t	NA	ND	n	tA	tD	T
...							
Departure	7.38	7	7	0	7.42	∞	10.0
Arrival	7.42	8	7	1	8.58	8.42	10.0
Departure	8.42	8	8	0	8.58	∞	10.0
Arrival	8.58	9	8	1	9.64	9.91	10.0
Arrival	9.64	10	8	2	10.7	9.91	10.0
Departure	9.91	10	9	1	10.7	10.7	10.0
After hours	10.7	10	10	0	10.7	10.7	10.0
Finish	10.7	10	10	0	10.7	10.7	10.0

An instance (ctd.)

Statistics for each customer (rounded)

Cust.	Arrival	Departure	Time
1	0.94	3.22	2.28
2	1.48	3.49	2.02
3	2.01	3.91	1.9
4	3.16	5.84	2.68
5	3.44	5.88	2.45
6	3.81	6.49	2.68
7	7.22	7.38	0.165
8	7.42	8.42	1.0
9	8.58	9.91	1.33
10	9.64	10.7	1.02

An instance (ctd.)

Aggregate indicators

- Average time in the system: 1.75
- Cloe leaves Satellite at 10.7

Realizations

- This represents one draw from the random variables.
- Multiple draws are necessary.
- Remember the pitfalls of simulation.

Another instance

Scenario: Cloe works faster

- Service time: exponential with mean 0.2
- Inter-arrival time: exponential with mean 1.0
- Closing time: 10.0

Another instance (ctd.)

Event	t	NA	ND	n	tA	tD	T
Arrival	1.02	1	0	1	3.14	1.38	10.0
Departure	1.38	1	1	0	3.14	∞	10.0
Arrival	3.14	2	1	1	6.97	3.25	10.0
Departure	3.25	2	2	0	6.97	∞	10.0
Arrival	6.97	3	2	1	7.08	7.26	10.0
Arrival	7.08	4	2	2	7.24	7.26	10.0
Arrival	7.24	5	2	3	10.0	7.26	10.0
Departure	7.26	5	3	2	10.0	8.32	10.0
Departure	8.32	5	4	1	10.0	8.51	10.0
Departure	8.51	5	5	0	10.0	∞	10.0
Finish	10.0	5	5	0	10.0	∞	10.0

Another instance (ctd.)

Statistics for each customer (rounded)

Cust.	Arrival	Departure	Time
1	1.02	1.38	0.355
2	3.14	3.25	0.11
3	6.97	7.26	0.296
4	7.08	8.32	1.24
5	7.24	8.51	1.27

Aggregate indicators

- Average time in the system: 0.654
- Cloe leaves Satellite at 10.0.
- He stops working at 8.51.

General framework

$$Z = h(X, Y, U) + \varepsilon_z$$

State variables X

- Time
- Number of customers in the system

External input Y

Arrival of customers

Control U

Serving customers

General framework

Indicators Z

- Time of each customer in the system.
- Average time in the system.
- Time at which Cloe leaves Satellite.

Statistics

- Numbers reported above are based on one instance.
- Insufficient to draw any conclusion (remember Kid City)
- Their distribution has to be investigated.
- Many realizations are necessary.

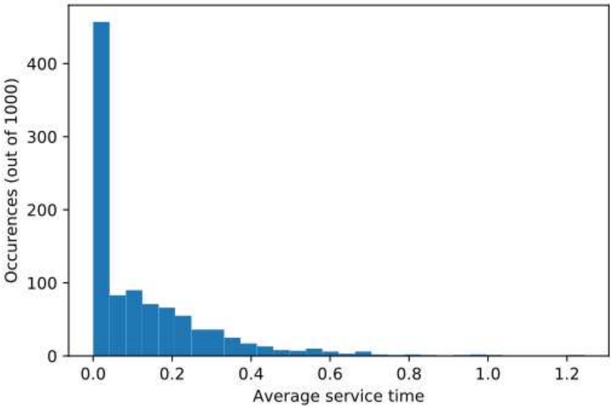
Statistics

Possible confusion in terminology

- The desired indicator Z may be a statistic from the simulator:
  - Mean time spent in the system
  - Maximum time spent in the system
  - Number of customers spending more than α min. in the system
- Still, each of them is a random variable, and statistics must be considered.
  - 5% quantile of the mean time spent in the system
  - Mean of the maximum time spent in the system
  - Mean of the mean time spent in the system
  - Standard deviation of the mean time spent in the system
  - Standard deviation of the number of customers spending more than α in the system
- Drawing histograms is highly recommended

Statistics

Average time spent in the system (service time: 0.2, arrival: 1.0)



Mean: 0.13, %days>0.4: 6.9

Statistics

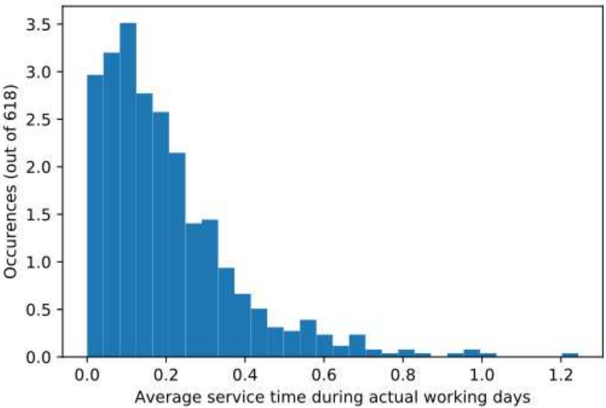
Arrival rate: $\lambda = 0.1$

$$\Pr(\text{first customer arrives before } T) = 1 - e^{-\lambda T} = 63.2\%$$

In our simulation, 618 days out of 1000.

Statistics: remove empty days

Average time spent in the system (service time: 0.2, arrival: 1.0)



Mean: 0.20, %days>0.4: 11.2%

Conclusion

Strengths of discrete event simulation

- Decomposition of a complex system into simple subsystems.
- Easy to mimick a real system

Challenges

- Importance of book-keeping.
- Easy to be overwhelmed by generated data. Careful statistical analysis is needed.
- Importance to distinguish between an indicator and the statistics of its distribution.