Simulation laboratory 4: Variance reduction

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Goals

Variance reduction:

- Understand two different variance reduction techniques.
- Increase the precision of vehicle queue simulation result.

Implementation:

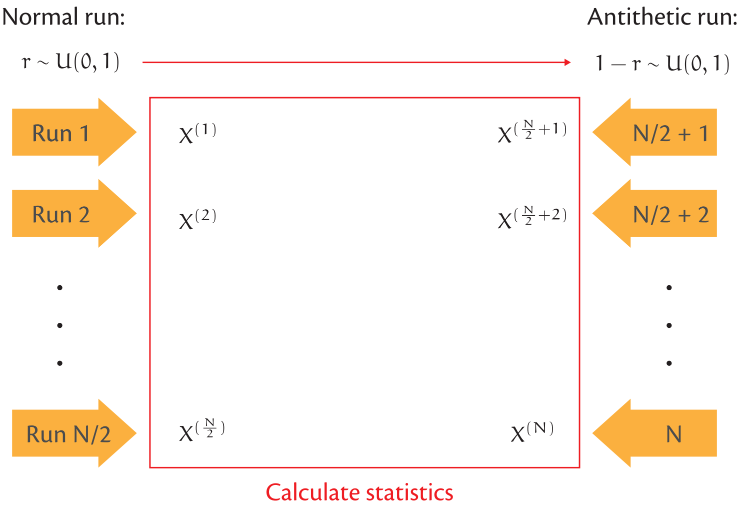
1. Antithetic draws
2. Control variates

1. Antithetic draws
2. Control variates
3. My results

Variance reduction

- What we do: approximate mean by MC simulations using iid draws
- Why it works: LLN
- But ... in general independence is not needed for LLN
- We can reduce spread of average estimates by negative correlations between draws!

Antithetic draws



Antithetic draws

Implementation:

- Modify the function `simulate`:
  - Keywords: `scenario`, `u`
  - Return: `times, queues`
- Conduct statistical analysis of maximum queue length using code developed in the previous lab.

Antithetic draws workflow

1. For $r = 1, \ldots, N/2$:
   1. Independent simulation:
      1. Generate array of uniform random numbers u.
      2. Run `simulate(scenario, u)` and obtain max-queue-length $q_{\max}^{\text{ind}}$.
   2. Antithetic simulation:
      1. Set $u = 1 - u$.
      2. Run `simulate(scenario, u)` and obtain max-queue-length $q_{\max}^{\text{ant}}$.
      3. Compute $q_{\max}^{(r)} = \frac{q_{\max}^{\text{ind}} + q_{\max}^{\text{ant}}}{2}$.
2. Analyse the statistics of $[q_{\max}^{(1)}, \ldots, q_{\max}^{(\frac{N}{2})}]$

Keep draws

- Draws for events $\{$Generation, Arrival, Departure$\}$ are exploited for antithetic run
- The order of events could change between the two runs.
- Normal and Antithetic runs should be executed separately.

Control variates

Run 1 $\quad X^{(1)} \longrightarrow$ $Z^{(1)} = X^{(1)} + c^{(1)}(Y^{(1)} - \mu)$

Control variate draw

Control variate expectation

Run 2 $\quad X^{(2)} \longrightarrow$ $Z^{(2)} = X^{(2)} + c^{(2)}(Y^{(2)} - \mu)$

$\vdots$

Run N $\quad X^{(N)} \longrightarrow$ $Z^{(N)} = X^{(N)} + c^{(N)}(Y^{(N)} - \mu)$

Calculate statistics

Recalculating parameter $c^{(r)}$ for updated sample with new draw

Control variates

Implementation:

- Modify the function `simulate`:
  - Keywords: `scenario`,
  - Return: `times, queues, service_time_mean`
- Conduct statistical analysis of maximum queue lengths.

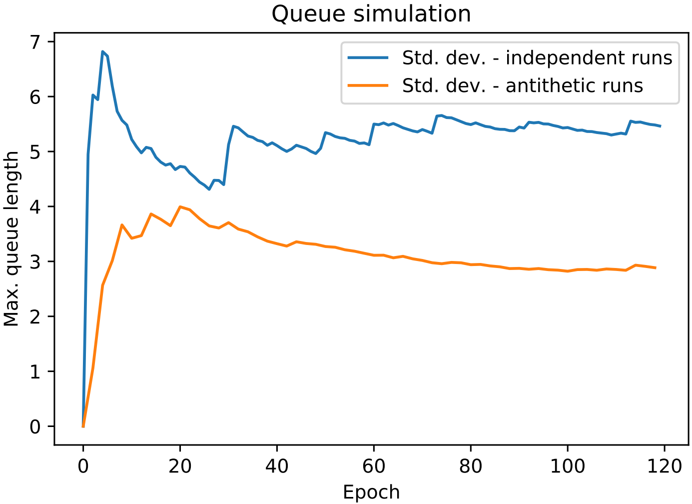
Control variates workflow

1. For $r = 1, \ldots, N$:
    1. Run `simulate` and output the control variate `control` $(= Y^{(r)})$.
    2. Obtain max-queue-length $q_{\max}$ $(= X^{(r)})$.
    3. Execute the function `controlled_mean`:
        1. Calculate variance $\mathrm{Var}(Y)$ and covariance $\mathrm{Cov}(X, Y)$
        2. Define the constant $c^* = -\mathrm{Cov}(X, Y)/\mathrm{Var}(Y)$
        3. Calculate $Z = X + c^*(Y - \mu)$
        4. Obtain the average and variance of $Z = [Z^{(1)}, \ldots, Z^{(r)}]$

Control variates

- Choose the control variate (Y) as you prefer. The higher the correlation is, the bigger the variance reduction is.
- Example: $Y =$ mean of service time at bottleneck. Mean $\mathbb{E}[Y] = \mu$ is known!

Antithetic draws



Control variates