STATISTICS FOR BUSINESS AND ECONOMICS

CHAPTER 9 HYPOTHESIS TESTING-PART3

本章重點

- 9.5. Population Proportion
  - Test statistic.
  - One-tailed test.
  - Two-tailed test.
- 9.6. Hypothesis testing and Decision making
- 9.7. Calculating the probability of Type II errors
- 9.8. Determining the Sample size for a Hypothesis test about a Population mean

Population Proportion

- One-tailed tests
  - Lower tail test:
    - $H_0: p \geq p_0$
    - $H_a: p < p_0$
  - Upper tail test:
    - $H_0: p \leq p_0$
    - $H_a: p > p_0$
- Two-tailed tests
  - $H_0: p = p_0$
  - $H_a: p \neq p_0$
  - Hypothesis tests about a population proportion are based on the difference between the sample proportion $\bar{p}$ and the hypothesized population proportion p_0 (similar to the case about population mean)

Population Proportion

- **Ex.**, 20% of the players at Pine Creek were women. In an effort to increase the proportion of women players, Pine Creek implemented a special promotion designed to attract women golfers. One month after the promotion was implemented, the course manager requested a statistical study to determine whether the proportion of women players at Pine Creek had increased.
  - Step1-**Develop null and alternative hypotheses**:
    - The objective of the study is to determine whether the proportion of women golfers increased:
      - $H_0: p \leq 20\%$.
      - $H_a: p > 20\%$.
        - **The hypothesized population proportion is** $p_0 = 20\%$.
- If the sample result reject H_0, we can conclude that the proportion of women golfers increased and the promotion was beneficial.

Population Proportion

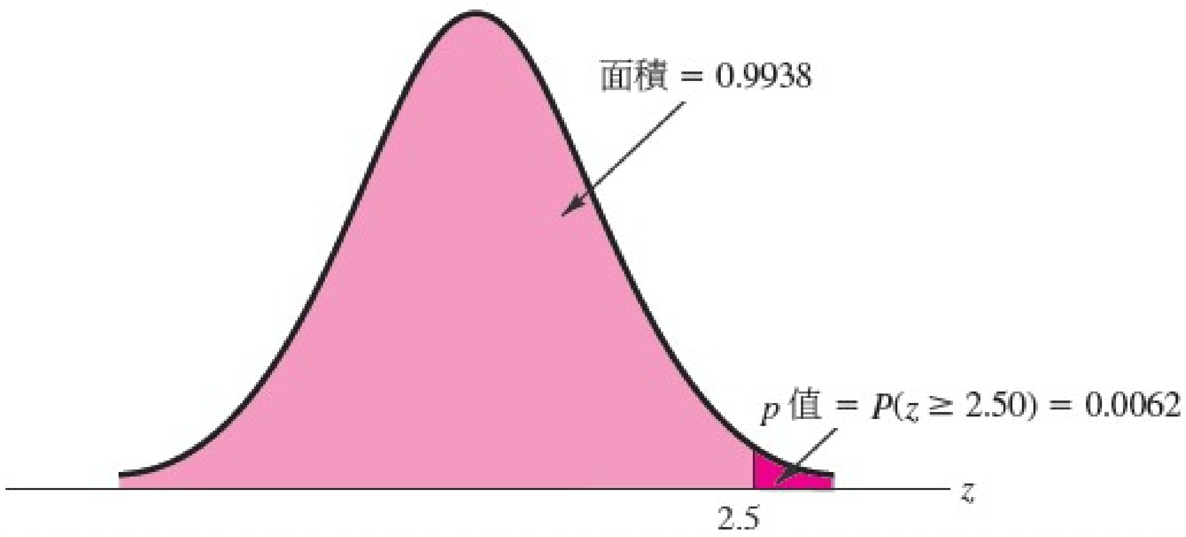
- **Step2-Determine the level of significance α:**
  - The course manager specified that a level of significance $\alpha = 0.05$.
- **Step3-Collect sample data and calculate the test statistic:**
  - Test statistic:
    - The sample proportion $\bar{p}$ is an point estimator of the population parameter p.
    - When the null hypothesis is true as an equality $p = 0.20$, the expected value of $\bar{p}$ equals the hypothesized value , that is $E(\bar{p}) = p_0$.
    - When the sample size is n:
      - The standard error of $\bar{p}$ is:
        - $\sigma_{\bar{p}} = \sqrt{\frac{p_0(1-p_0)}{n}}$
      - The sampling distribution of $\bar{p}$ can be approximately by a normal distribution. Under these conditions, the quantity::
        - $z = \frac{\bar{p}-p_0}{\sigma_{\bar{p}}} = \frac{\bar{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$ has a standard normal distribution.

Population Proportion

- Suppose a random sample of 400 players was selected, and that 100 of the players were women. The proportion of women golfers in the sample is:
- $\bar{p} = \frac{100}{400} = 0.25$
- The test statistic is:
  - $z = \frac{\bar{p}-p_0}{\sigma_{\bar{p}}} = \frac{\bar{p}-p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.25-0.20}{\sqrt{\frac{0.20(1-0.20)}{400}}} = \frac{0.05}{0.02} = 2.50$
  - Sine it is an upper tail test, and the test statistic is positive, therefore, p-value is $P(z \geq 2.50) = 1 - P(z < 2.50) = 1 - 0.9938 = 0.0062$
- Rejection rule:
  - ***p*-value approach:**
    - p-value is $0.0062 < 0.05$, reject H_0.
  - **Critical value approach:**
    - The critical value corresponding to an area of 0.05 in the upper tail of a normal probability distribution is $z_{0.05} = 1.645$.
    - If $z \geq 1.645$, reject H_0. Since $z = 2.50 \geq 1.645$, we reject H_0.

母體比例-拒絕法則-p值法

圖 9.7 Pine Creek 問題之假設檢定的 p 值



母體比例假設檢定彙總

表 9.4 母體比例假設檢定的彙總表

	左尾檢定	右尾檢定	雙尾檢定
假設	$H_0: p \geq p_0$ $H_a: p < p_0$	$H_0: p \leq p_0$ $H_a: p > p_0$	$H_0: p = p_0$ $H_a: p \neq p_0$
檢定統計量	$z = \dfrac{\bar{p} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}}$	$z = \dfrac{\bar{p} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}}$	$z = \dfrac{\bar{p} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}}$
拒絕法則：p 值法	若 p 值 $\leq \alpha$，則拒絕 H_0	若 p 值 $\leq \alpha$，則拒絕 H_0	若 p 值 $\leq \alpha$，則拒絕 H_0
拒絕法則：臨界值法	若 $z \leq -z_\alpha$，則拒絕 H_0	若 $z \geq z_\alpha$，則拒絕 H_0	若 $z \leq -z_{\alpha/2}$ 或 $z \geq z_{\alpha/2}$，則拒絕 H_0

Hypothesis Testing and Decision Making

- Usually, we set our **corresponding decisions or actions** as alternative hypothesis H_a to **control the probability of Type I error by determining the level of significance** α. This can control the costs of making wrong decisions.
- However, if the purpose of a hypothesis test is to make a decision when H_0 is true and a different decision when H_a is true, the decision maker may want to or be forced to **control the probability of making a Type II error**.

Hypothesis Testing and Decision Making

- Ex., a quality control manager must decide to accept a shipment of batteries from a supplier or to return the shipment because of poor quality. Assume that design specifications require batteries from the supplier to have a mean useful life of at least 120 hours. To evaluate the quality of an incoming shipment, a sample of 36 batteries will be selected and tested. On the basis of the sample, a decision must be made to accept the shipment of batteries of to return it to the supplier because of poor quality. The null and alternative hypotheses are:
  - H_0: $\mu \geq 120$
  - H_a: $\mu < 120$
    - If H_0 is rejected , the alternative hypothesis H_a is concluded to be true. This conclusion indicates that the appropriate action is to return the shipment to the supplier.
    - However, if H_0 is not rejected, the decision maker must still determine what action should betaken. Thus, without directly concluding that is true, but merely by not rejecting it, the decision maker will have made the decision to accept the shipment as being of satisfactory quality.

Calculating the probability of Type II errors

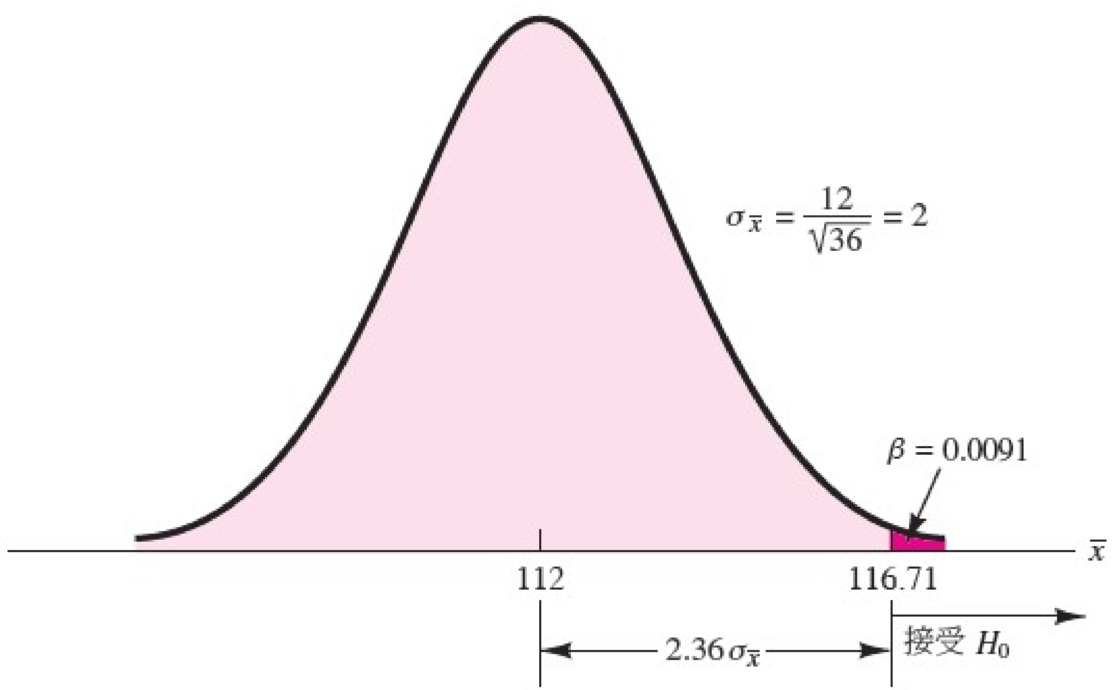
- Assume that $\alpha = 0.05$, the population standard deviation is $\sigma = 12$, and the sample size is 36. Now, we assume that $\bar{x}$ is unknown (we have not sampled yet)
  - The test statistic is: $z = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} = \frac{\bar{x}-12}{12/\sqrt{36}}$
  - According to critical value approach, the rejection rule for this lower tail test is: if $z \leq -1.645$, reject H_0.
  - That is, we will make the decision to reject H_0 if $\bar{x} \leq 116.71$:
    - $z \leq -1.645 \rightarrow \frac{\bar{x}-120}{\frac{12}{\sqrt{36}}} \leq -1.645 \rightarrow \bar{x} \leq 116.71$
    - If $\bar{x} \leq 116.71$, reject H_0, return the batteries to the supplier.
    - If $\bar{x} > 116.71$, do not reject H_0 and accept the shipment of the supplier.

Calculating the probability of Type II errors

- The probability of making a Type II error:
  - If the true shipment mean is less than 120 hours (H_0 is incorrect), but the test statistic result leads us to not reject H_0: $\mu \geq 120$, then we are making a Type II error.
- First, we assume that the true population mean $\mu = 112$.
  - What is the probability of not rejecting H_0: $\mu \geq 120$?(denote the probability of making a Type II error as β)
  - That is, if the true population mean $\mu = 112$, **what is the probability that we obtain a sample mean** $\bar{x} > 116.71$?
    - From the sampling distribution of $\bar{x}$ when $\mu = 112$, **what is the area in the upper tail** $\bar{x} > 116.71$?
    - $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{116.71-11}{12/\sqrt{36}} = 2.36$
    - $P(\bar{x} > 116.71) = P(z > 2.36) = 1 - 0.9909 = 0.0091.$
    - When $\boldsymbol{\mu = 112}$**, the probability of making a Type II error:** $\boldsymbol{\beta = 0.0091}$.

Calculating the probability of Type II errors

圖 9.8 μ = 112 時，發生型 II 錯誤的機率



Calculating the probability of Type II errors

- If we assume that the true population mean $\mu = 115$.
  - What is the probability of not rejecting $H_0\colon\ \mu \geq 120$?
  - That is, if the true population mean $\mu = 115$, **what is the probability that we obtain a sample mean $\bar{x} > 116.71$?**
    - From the sampling distribution of $\bar{x}$ when $\mu = 115$, **what is the area in the upper tail $\bar{x} > 116.71$?**
    - $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{116.71-1}{12/\sqrt{36}} = 0.86$
    - $P(\bar{x} > 116.71) = P(z > 0.86) = 1 - 0.8051 = 0.1949$.
    - When $\boldsymbol{\mu = 115}$**, the probability of making a Type II error:** $\boldsymbol{\beta = 0.1949}$.

Calculating the probability of Type II errors

表 9.5 電池驗收問題的假設檢定中，發生型 II 錯誤的機率

μ 值	$z = \dfrac{116.71 - \mu}{12/\sqrt{36}}$	型 II 錯誤的機率 (β)	檢定力 ($1-\beta$)
112	2.36	0.0091	0.9909
114	1.36	0.0869	0.9131
115	0.86	0.1949	0.8051
116.71	0.00	0.5000	0.5000
117	−0.15	0.5596	0.4404
118	−0.65	0.7422	0.2578
119.999	−1.645	0.9500	0.0500

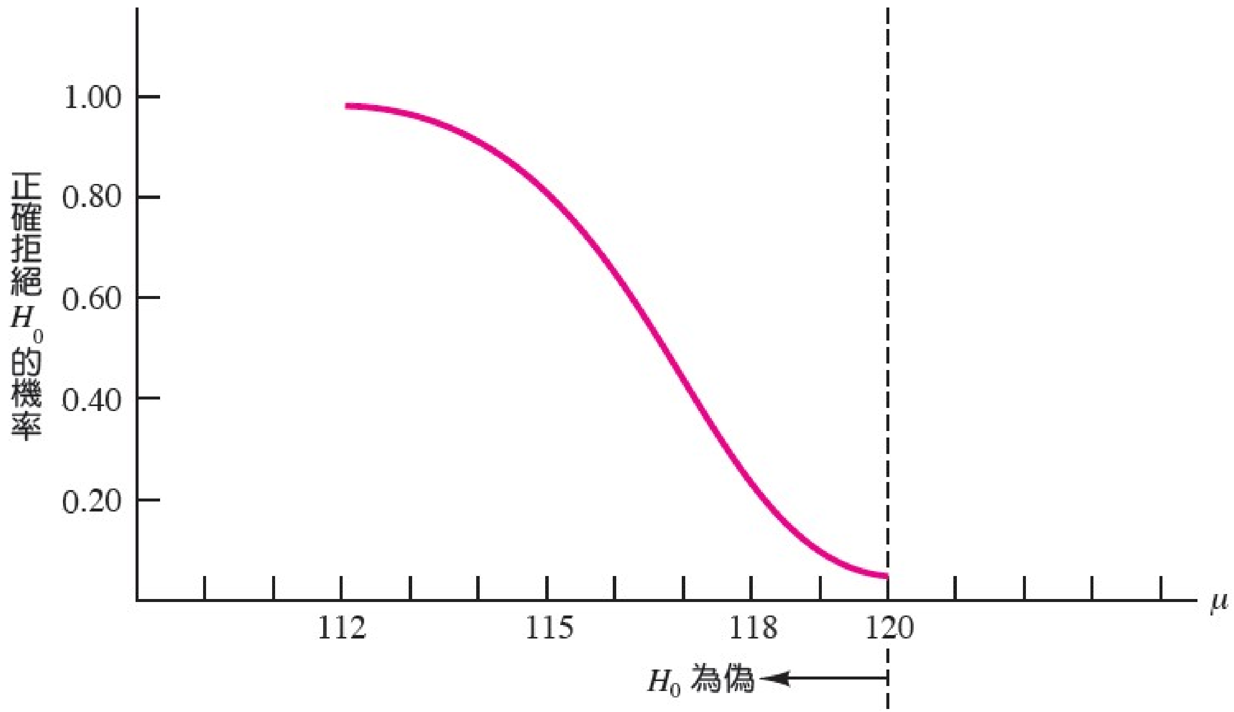
- The probabilities of making a Type II error are different for a variety of values of μ less than 120.
- As μ increases toward 120, the probability increases toward an upper bound of 0.95.
- As μ decreases to values farther below 120, the probability diminishes.
  - When the true population mean is closer to $\mu = 120$ (μ_0), the probability of making a Type II error is higer.

Calculating the probability of Type II errors

- The probability of **correctly rejecting** H_0 when it is **false** is called **the power of the test** (檢定力, or the power).
- For any particular value of μ, the power is $1-\beta$.
  - The probability of correctly rejecting the null hypothesis is 1 minus the probability of making a Type II error.
  - The power associated with each value of μ is shown graphically in Figure 9.9. Such a graph is called a **power curve** (檢定力曲線).

Calculating the probability of Type II errors

圖 9.9 電池驗收問題的假設檢定之檢定力曲線



Calculating the probability of Type II errors

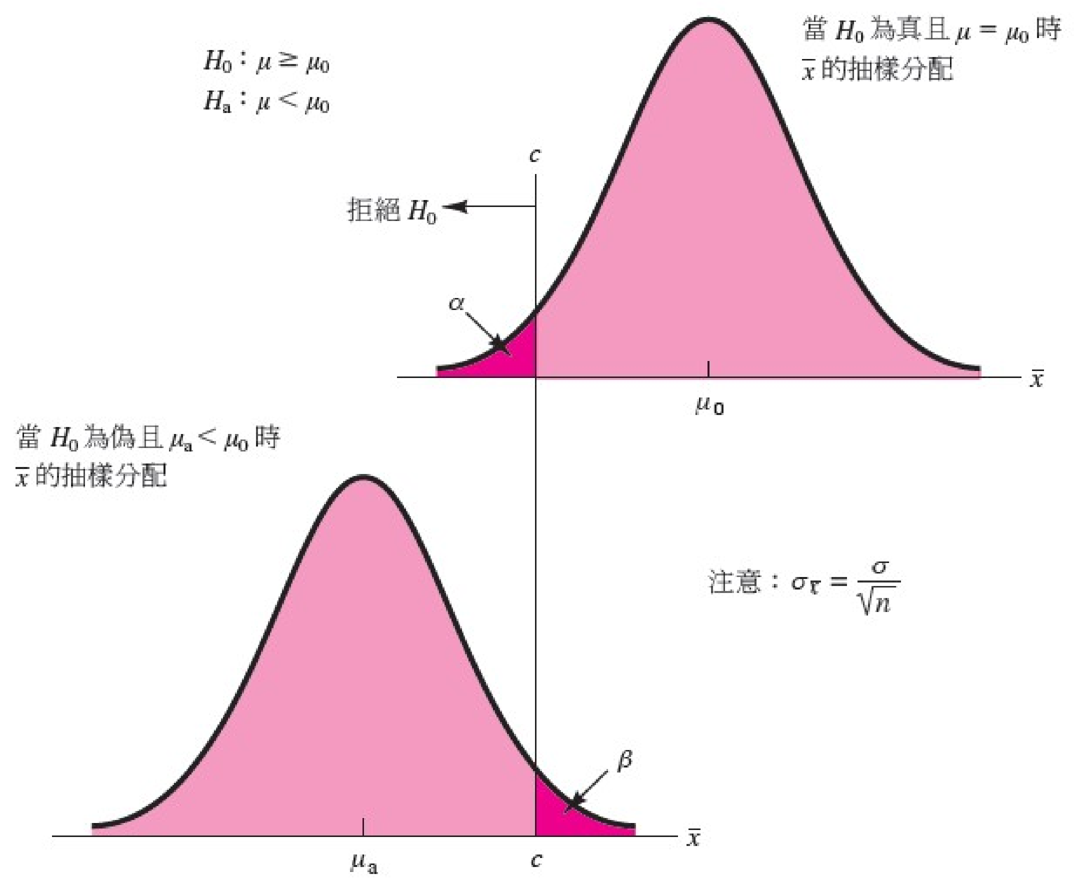
- Calculating β:
  - Develop the null and alternative hypotheses.
  - Use the level of significance α and the critical value approach to determine the critical value and the rejection rule for the test.
  - Use the rejection rule to solve for the value of the sample mean corresponding to the critical value of the test statistic.
  - Use the results above to state the values of the sample mean that lead to the acceptance of H_0. These values define the **acceptance region** (**接受區域**) for the test.
  - Use **the sampling distribution of $\bar{x}$ for a value of μ satisfying the alternative hypothesis** (在某個假定符合H_a狀況下的μ，畫出$\bar{x}$抽樣分配), and the acceptance region, to compute the probability that the sample mean will be in the acceptance region. This probability is the probability of making a Type II error at the chosen value of μ.

Determining the sample size for a hypothesis test about a population mean

- The level of significance specified by the user determines the probability of making a Type I error for the test. By controlling the sample size, the user can also control the probability of making a Type II error. For example, a lower tail test about a population mean is:
  - $H_0: \mu \geq \mu_0$
  - $H_a: \mu < \mu_0$
    - Assume that when $\bar{x} \leq c$ (**rejection rule**), we reject H_0, **the probability of Type I error** is α.
    - With z_α representing the z value corresponding to an area of α in the upper tail of the standard normal distribution, we compute c using the following formula::
      - $c = \boldsymbol{\mu_0} - z_\alpha \frac{\sigma}{\sqrt{n}}$
    - When $\bar{x} > c$ (acceptance region, 接受區域), we do not reject H_0. There is a **probability of making a Type II error**.
    - With z_β representing the z value corresponding to an area of β in the upper tail of the standard normal distribution , we have c:
      - $c = \boldsymbol{\mu_a} + z_\beta \frac{\sigma}{\sqrt{n}}$ ($\boldsymbol{\mu_a}$ is a hypothesized true mean and $\boldsymbol{\mu_a} < \mu_0$)

Determining the sample size for a hypothesis test about a population mean

圖 9.10 特定的型 I (α) 錯誤及型 II (β) 錯誤的水準下決定樣本大小



Determining the sample size for a hypothesis test about a population mean

- What we want to do is to select a value for c so that when we reject $\boldsymbol{H_0}$, the probability of a Type I error is equal to the chosen value of α, and when we accept $\boldsymbol{H_0}$ the probability of a Type II error is equal to the chosen value of β.
- Therefore, the following equation must be true:
  - $\mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} = \mu_a + z_\beta \frac{\sigma}{\sqrt{n}}$
  - To determine the required sample size, we solve for the $\sqrt{n}$ as follows::
    - $\mu_0 - \mu_a = z_\beta \frac{\sigma}{\sqrt{n}} + z_\alpha \frac{\sigma}{\sqrt{n}}$
    - $\mu_0 - \mu_a = \frac{(z_\alpha + z_\beta)\sigma}{\sqrt{n}}$
- Finally, the sample size for a one-tailed hypothesis test about a population mean is:
  - $n = \frac{(z_\alpha + z_\beta)^2 \sigma^2}{(\mu_0 - \mu_a)^2}$
    - In a two-tailed hypothesis test, replace z_α with $z_{\alpha/2}$

Determining the sample size for a hypothesis test about a population mean

- **Ex.**, The design specification for the shipment of batteries indicated a mean useful life of at least 120 hours for the batteries. Shipments were rejected if H_0: $\mu \geq 120$ was rejected:
  - H_0: $\mu \geq 120$
  - H_a: $\mu < 120$
- Assume that the quality control manager makes the following statements:
  - **Type I error:** if the mean life of the batteries in the shipment is $\mu = 120$, I am willing to risk an $\alpha = 0.05$ probability of rejecting the shipment.
  - **Type II error:** If the mean life of the batteries in the shipment is $\mu = 115$, under the specification, I am willing to risk a $\beta = 0.10$ probability of accepting the shipment.

Determining the sample size for a hypothesis test about a population mean

- According to the standard normal probability table, $z_\alpha = z_{0.05} = 1.645$, $z_\beta = z_{0.10} = 1.28$, and $\mu_0 = 120$, $\mu_a = 115$, and $\sigma = 12$.
- The recommended sample size is:
  - $n = \frac{(1.645+\ .28)^2 12^2}{(120-115)^2} = 49.3$
  - Rounding up, we recommend a sample size of 50.
- **Note:**
  - One two of the three values are known (α, β, and n), the other can be computed.
  - For a given level of significance α, increasing the sample size will reduce β.
  - For a given sample size, decreasing α will increase β, whereas increasing α will decrease β.
    - **It suggests that one should not choose unnecessarily small values for the level of significance** α (不要一味降低顯著水準α, 在樣本大小固定下, 顯著水準越小代表型II錯誤機率越大).