lab1_q3

Helena Llorens Lluís (hllor282), Yi Yang (yiyan338)

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3. Bayesian inference for the rate parameter in the Poisson distribution.

We are going to derive the expression that the posterior pdf $p(\lambda \mid y, \sigma)$ is proportional to. The observations are from the Poisson distribution with rate parameter $\lambda > 0$. The prior distribution of $\lambda > 0$ is the **half-normal distribution** with prior pdf:

$$p(\lambda|\sigma) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \exp\left(-\frac{\lambda^2}{2\sigma^2}\right), \ \lambda \geq 0$$

From bayesian theorem, we know that the posterior is propotional the product of likelihood and the prior:

$$p(\theta \mid \text{Data}) \propto p(\text{Data} \mid \theta)\, p(\theta)$$
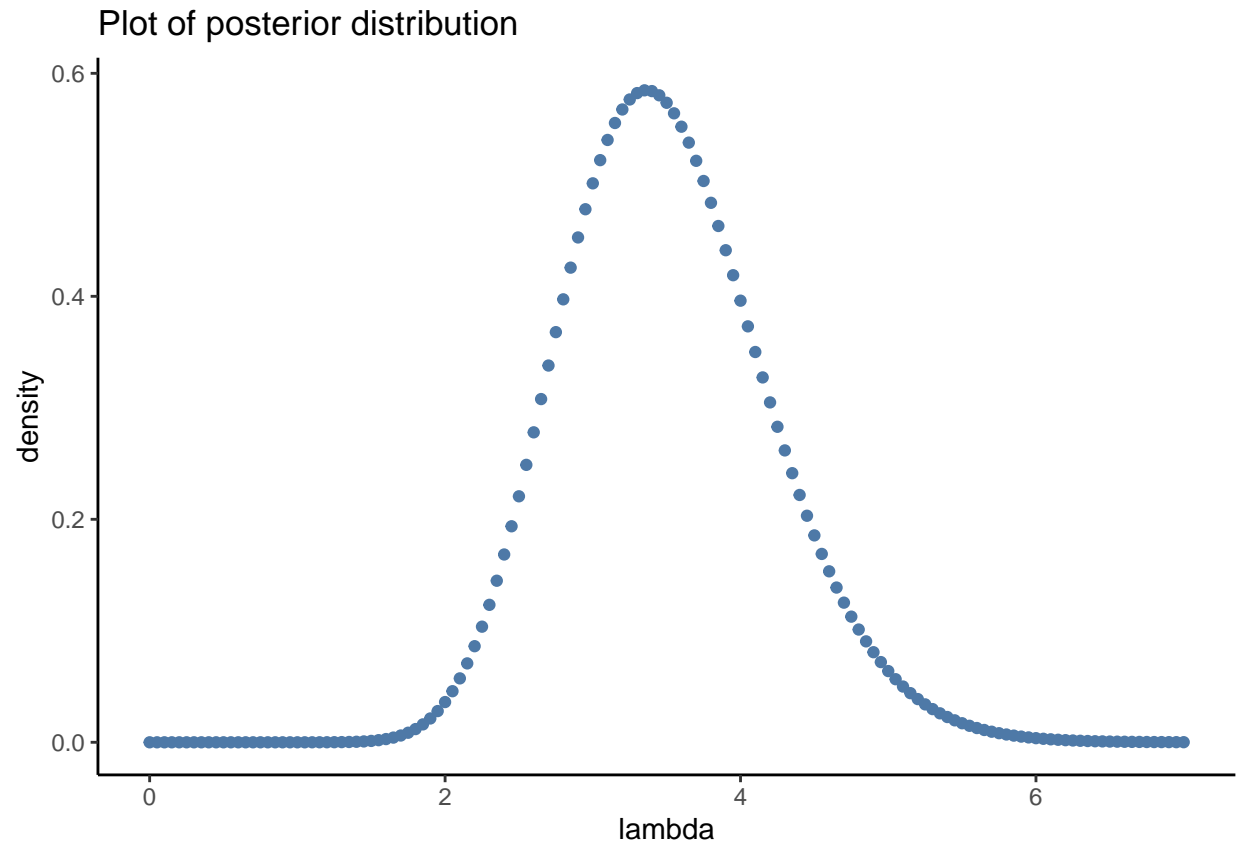
Then we could derive the expression of the posterior pdf :

$$p(\lambda \mid y, \sigma) \propto \lambda^{\sum y_i} \exp\left(-n\lambda - \frac{\lambda^2}{2\sigma^2}\right), \quad \lambda \geq 0$$

We select a grid of $\lambda > 0$ values from 0 to 7 and set the increment of the value equals to 0.5. We compute the posterior values and then use the following formulate to normalize the values:

$$\text{Normalized posterior} = \frac{p(\lambda)}{\sum p(\lambda) \cdot \Delta\lambda}$$

The plot of the posterior distribution is shown as follows:

Plot of posterior distribution


```r
mode_index <- which.max(normalized)
mode <- lambda_grid[mode_index]
cat("The approximate posterior mode of lambda is",mode,"\n")
```

```
## The approximate posterior mode of lambda is 3.35
```

Appendix

assignment 3

```r
#(a)
y <- c(0,2,5,5,7,1,4)
sigma <- 5

#the expression of posterior distribution unnormalized
posterior <- function(lambda,sigma,y){
  s <- sum(y)
  n <- length(y)
  res <- lambda^(s)*exp(-n*lambda-(lambda^2)/(2*sigma^2))
  return(res)
}
```

```r
lambda_grid <- seq(0,7,by=0.05)

unnormalized <- posterior(lambda_grid,sigma,y)
normalized <- unnormalized/sum(unnormalized*0.05)

#plot(x=lambda_grid,y=normalized,type = "p")
df <- data.frame(lambda=lambda_grid,density=normalized)
ggplot(data=df,mapping = aes(x=lambda,y=density))+
  geom_point(color="#4E79A7")+
  labs(title = "Plot of posterior distribution ")+
  theme_classic()

#(b)
mode_index <- which.max(normalized)
mode <- lambda_grid[mode_index]
cat("The approximate posterior mode of lambda is",mode,"\n")
```