# QUESTION 2: Simulation of power curves

To investigate the power of the Sign test for detecting deviations from the null hypothesis $H_0 : \mu = 0$ against the alternative $H_1 : \mu > 0$, we simulate random samples from the Gumbel distribution using the Inverse Transformation Method.

**Generating Gumbel Random Variables**

The cumulative distribution function (CDF) of the Gumbel distribution with scale parameter set to 1 and location parameter $\mu + c$, where $c = log(log(2))$ is given by

$$F(x) = e^{-e^{-(x-\mu-c)}}, \quad where\ c = log(log(2))$$

To generate random variables from this distribution, we use the Inverse Transformation Method. Specifically, if $U \sim Uniform(0, 1)$, then setting $F(x) = U$ and solving for $X$ gives the inverse CDF:

$$F^{-1}(u) = \mu - c - log(-log(u))$$

This equation allows us to generate Gumbel random variables by:

1. Drawing $U$ from a $Uniform$ distribution on the interval $(0, 1)$.
2. Applying the transformation:
$$X = \mu - c - log(-log(U))$$

This is done by the function `generate_gumbel`, which takes the parameters `mu`, `c` and `n` (number of independent observations to generate). It returns $n$ observations from the Gumbel distribution.

**Hypothesis testing**

We simulate $n = 13$ independent observations from the Gumbel distribution and perform the Sign test to evaluate the following hypotheses:

$$H_0 : \mu = 0\ vs\ H_1 : \mu > 0$$

The Sign test is appropriate here because it tests the median of the distribution. Under the null hypothesis, the median is 0. If the observations are consistently greater than 0, we would expect to reject $H_0$ more frequently, reflecting greater power.

This test is performed by our function `sign_test` , with input parameters `X` and `alpha`, and gives as whether the null hypothesis was rejected.

**Objective of the simulation**

The goal of this simulation study is to estimate the power of the Sign test, defined as the probability of rejecting $H_0$ when the alternative hypothesis is true.

We calculate the power for a range of $\mu$ values from 0 to 2. Since the median of the Gumbel distribution is $\mu$, as $\mu$ increases above 0, we expect the probability of rejecting $H_0$ to increase. This leads to a power curve that illustrates how sensitive the Sign test is to different values of $\mu$.

We have chosen the grid for $\mu$ with a step of 0.1, since a step size of 0.1 provides a fine enough resolution to capture the trend in the power curve without excessive computational cost.

On the other hand, we have performed 1000 repetitions for every $\mu$ because it balances precision and computational cost effectively. Increasing the repetitions further would only marginally improve precision but require significantly more processing time. Moreover, with 1000 repetitions the standard error of the estimated power is approximately:

$$SE = \sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.5(1-0.5)}{1000}} = 0.0158$$

which is a reasonable precision for power estimation.

On the following plot, we observe how the the power of the Sign test increases as we increase $\mu$.

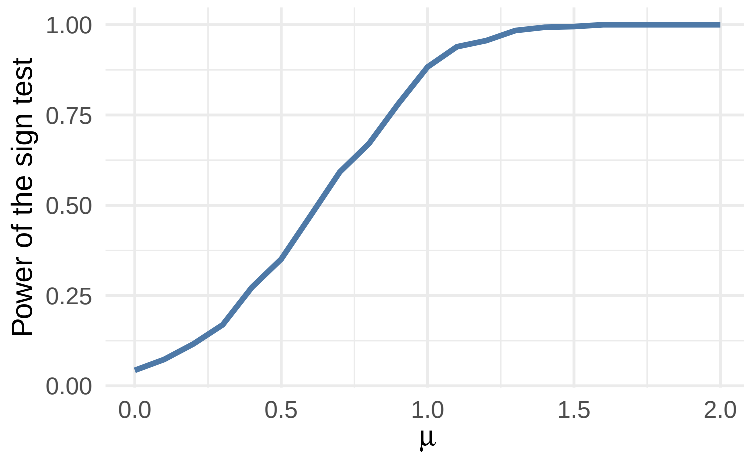


Figure 1: Power Curve of the Sign Test

**Conclusion**

The power curve of the Sign test, as shown in Figure 1, demonstrates how the test's ability to reject the null hypothesis ($H_0 : \mu = 0$) increases as the true value of $\mu$ becomes more positive. For small values of $\mu$ the power is low, indicating a low probability of correctly rejecting the null hypothesis when the true median is only slightly greater than 0. But as $\mu$ increases, the power rapidly rises, reaching approximately 0.85 when $\mu$ is around 1.

For values of $\mu$ greater that 1.5, the power approaches 1 indicating that the Sign test is almost certain to reject the null hypothesis when the true median is much larger than 0.

Overall, this power curve illustrates that the Sign test is most effective for detecting moderate to large positive deviations in the median but has limited power for detecting small effects.

# APPENDIX

**Question 2**

```
library(ggplot2)

# Inverse CDF function
f_inv <- function(u, mu, c){
  mu + c - log(-log(u))
}

# Generate n observations from a Gumbel distribution with mu = mu
generate_gumbel <- function(mu, n, c){
  # generate random uniform
  U <- runif(n)
```

```
  # inverse CDF function for the Uniform random distribution
  X <- f_inv(U, mu, c)
  return(X)
}

# Perform sign test over sample X (Gumbel distribution generated) and returns if H0: mu = 0 is rejected
sign_test <- function(X, alpha){
  # count positives
  positives <- sum(X > 0)
  # perform sign test
  test <- binom.test(positives, length(X), p = 0.5, alternative = "greater")
  # returns TRUE if the test is rejected
  return(test$p.value < alpha)
}

# Parameter initialization
n <- 13
mu <- 0
c <- log(log(2))
nreps <- 1000
alpha <- 0.05
mu_vals <- seq(from = 0, to = 2, by = 0.1)
power <- numeric(length(mu_vals))

# Perform nreps tests for each value of mu
for(i in 1:length(mu_vals)){
  count <- 0
  for(j in 1:nreps){
    # generate sample of size n from a gumbel distribution
    X <- generate_gumbel(mu_vals[i], n, c)
    # perform sign test
    test <- sign_test(X, alpha)
    # update count (adds 1 if the test is rejected)
    count <- count + test
  }
  # calculate the power for each mu
  power[i] <- count / nreps
}

# plot of the power for each mu
plot_data <- data.frame(mu = mu_vals, power = power)
ggplot(plot_data, aes(x = mu, y = power)) +
  geom_line(col = "#4E79A7", linewidth = 1) +
  theme_minimal() +
  labs(x = expression(mu), y = "Power of the sign test")
```