(a)

$A: X(X'X)^{-1}$
$B: X'$

$$\text{trace}(P) = \text{trace}(X(X'X)^{-1}X')$$
$$= \text{trace}(X'X(X'X)^{-1}) \quad \text{since } \text{trace}(AB) = \text{trace}(BA)$$
$$= \text{trace}(I_k) = k$$

$$\text{trace}(M) = \text{trace}(I_n - P) = \text{trace}(I_n) - \text{trace}(P) = n-k$$

(b)

if P is positive semidefinite, $\forall\ \underline{x} \in \mathbb{R}^n \setminus \{0\}$, $\underline{x}^T P \underline{x} \geq 0$

since $P^2 = P$ & $P^T = P \Rightarrow \underline{x}^T P \underline{x} = \underline{x}^T P^2 \underline{x} = (\underline{x}^T P)(P\underline{x}) = (\underline{x}^T P^T)(P\underline{x}) = \|P\underline{x}\|^2 \geq 0$

$\Rightarrow P$ is positive semidefinite #

$M = I_n - P \quad M^2 = (I_n - P)(I_n - P) = I_n - 2P + P = I_n - P = M$

$\Rightarrow M^2 = M$, $M^T = I_n^T - P^T = I_n - P \Rightarrow$ by same way above we can show that

$\underline{x}^T M \underline{x} = \|M\underline{x}\|^2 \geq 0 \Rightarrow M$ is positive semi-definite #

2.

$$E[\hat{\sigma}_Y^2] - \sigma_Y^2 = E\left[\frac{1}{n}\sum_{i=1}^n (Y_i - \bar{Y})^2\right] - \sigma_Y^2 = \frac{1}{n}\sum_{i=1}^n E[(Y_i - \bar{Y})^2] - \sigma_Y^2$$

$$= \frac{1}{n}\sum_{i=1}^n E[(Y_i - \mu - (\bar{Y} - \mu))]^2 - \sigma_Y^2$$

$$= \frac{1}{n}E\left[\left(\sum_{i=1}^n (Y_i - \mu)^2 - 2(\bar{Y} - \mu)\sum_{i=1}^n (Y_i - \mu) + \sum_{i=1}^n (\bar{Y} - \mu)^2\right]\right] - \sigma_Y^2$$

$$= \cancel{\frac{n}{n}\sigma_Y^2} - E\left[\frac{2}{n}(\bar{Y} - \mu)\sum_{i=1}^n (Y_i - \mu) + \frac{n}{n}(\bar{Y} - \mu)^2\right] - \cancel{\sigma_Y^2}$$

$$= E\left[-\frac{2}{n}(\bar{Y} - \mu)\cdot n \cdot \frac{1}{n}\sum_{i=1}^n (Y_i - \mu) + (\bar{Y} - \mu)^2\right] = -E[(\bar{Y} - \mu)^2] = -\text{Var}[\bar{X}] = -\frac{\sigma^2}{n} \#$$

$-\frac{\sigma^2}{n} \neq 0 \Rightarrow \hat{\sigma}_Y^2$ is biased

3、

(1)

$$\hat{\beta} = (X_1' X_1)^{-1} X_1' Y$$

```r
# Calculate X'X
X_transpose_X <- t(x) %*% x

beta_hat <- solve(X_transpose_X) %*% t(x) %*% y
print("beta_hat")
print(beta_hat)
```

```
[1] "beta_hat"
              [,1]
        0.01079947
x_dfy   0.99791023
x_infl -0.55424292
x_svar -0.44950368
x_tms   0.23354924
x_tbl   0.14438939
x_dfr   0.06759062
```

(2)

$$X_1 = \begin{bmatrix} 1 \\ x\_dfy \\ x\_infl \\ x\_svar \end{bmatrix} \qquad M_1 = I_{504} - X_1 (X_1' X_1)^{-1} X_1'$$

$$X_2 = \begin{bmatrix} x\_tms \\ x\_tbl \\ x\_dfr \end{bmatrix} \qquad M_2 = I_{504} - X_2 (X_2' X_2)^{-1} X_2'$$

$$\Rightarrow \hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} ((M_2 X_1)'(M_2 X_1))^{-1} (M_2 X_1)'(M_2 Y) \\ ((M_1 X_2)'(M_1 X_2))^{-1} (M_1 X_2)'(M_1 Y) \end{bmatrix}$$

```r
x1 <- cbind(1, x_dfy, x_infl, x_svar)
x2 <- cbind(x_tms, x_tbl, x_dfr)

m1 <- diag(nrow(x1)) - x1 %*% solve(t(x1) %*% x1) %*% t(x1)
m2 <- diag(nrow(x2)) - x2 %*% solve(t(x2) %*% x2) %*% t(x2)

beta_hat1 <- solve(t(m2 %*% x1) %*% m2 %*% x1) %*% t(m2 %*% x1) %*% m2 %*% y
beta_hat2 <- solve(t(m1 %*% x2) %*% m1 %*% x2) %*% t(m1 %*% x2) %*% m1 %*% y

beta_hat_fwl <- matrix(t(c(beta_hat1, beta_hat2)))
print("beta_hat_fwl")
print(beta_hat_fwl)
```

```
[1] "beta_hat_fwl"
            [,1]
[1,]  0.01079947
[2,]  0.99791023
[3,] -0.55424292
[4,] -0.44950368
[5,]  0.23354924
[6,]  0.14438939
[7,]  0.06759062
```

4.

```
#compute R square of a set of variables
compute_R2 <- function(x, y) {
  # Calculate beta_hat
  beta_hat <- solve(t(x) %*% x) %*% t(x) %*% y

  # Calculate y_hat
  y_hat <- x %*% beta_hat

  # Calculate R^2
  R2 <- 1 - sum((y - y_hat)^2) / sum((y - mean(y))^2)

  return(R2)
}

rsquare_list <- c()

for(i in 1:7) {
    x_subset <- x[, 1:i]
    R2 <- compute_R2(x_subset, y)
    rsquare_list <- c(rsquare_list, R2)
}

png("./HW3/plot.png")
plot(1:7, rsquare_list, type = "o", xlab = "j", ylab = "R^2", main = "R^2 vs. set of j")
dev.off()
```

R^2 result

```
r$> rsquare_list
[1] 0.00000000 0.01537088 0.01545069 0.02152345 0.02250799 0.03008154 0.03059011
```