1.

a.

$$E[Y_i^2] = E[(X_i'\beta + e_i)'(X_i'\beta + e_i)]$$

$$= E[\beta' X_i X_i' \beta + \beta' X_i e_i + e_i' X_i' \beta + e_i' e_i]$$

$$= \beta' E[X_i X_i'] \beta + \beta' E[X_i e_i] + E[e_i' X_i'] \beta + E[e_i^2]$$

since $E[X_i e_i] = E[X_i E[e_i | X_i]] = E[X_i \cdot 0] = 0 = 0' = E[e_i' X_i']$

$$\Rightarrow E[Y_i^2] = \beta' E[X_i X_i'] \beta + E[e_i^2].$$

b.

$$\mathrm{var}[Y_i] = E[Y_i^2] - E[Y_i]^2$$

$$= \beta' E[X_i X_i'] \beta + E[e_i^2] - (E[X_i'\beta])^2 - 2E[e_i]E[X_i'\beta] - (E[e_i])^2$$

$\rightarrow$ covariance matrix of $X_i$

$$= \beta'(E[X_i X_i'] - E[X_i] \cdot E[X_i]')\beta + E[e_i^2] - (E[e_i])^2 - 2 \cdot 0$$

$$= \beta' \mathrm{Var}[X_i] \beta + \mathrm{Var}[e_i] \ \#$$

2.

$\mathrm{Var}[\tilde{\beta}] = E[\mathrm{Var}[\tilde{\beta}|X]] + \mathrm{Var}[E[\tilde{\beta}|X]] = E[\mathrm{Var}[\tilde{\beta}|X]] = \sigma^2 E[AA']$ for some $A$ dependent to $X$

$$\mathrm{Var}[\hat{\beta}] = E[\mathrm{Var}[\hat{\beta}|X]] + \mathrm{Var}[E[\hat{\beta}|X]] = E[\mathrm{Var}[\hat{\beta}|X]] = \sigma^2 E[(X'X)^{-1}]$$

since $\mathrm{Var}[e] = \sigma^2 I$ & $E[e|X] = 0$

By Gaussian-Markov theroem, $\mathrm{Var}[\tilde{\beta}|X] - \mathrm{Var}[\hat{\beta}|X]$ is positive semidefinite

$\Rightarrow \sigma^2 AA' - \sigma^2 (X'X)^{-1}$ is PSD

$$\Rightarrow \mathrm{var}[\tilde{\beta}] - \mathrm{var}[\hat{\beta}] = E[\mathrm{Var}[\tilde{\beta}|X] - \mathrm{Var}[\hat{\beta}|X]] = \sigma^2(E[AA'] - E[(X'X)^{-1}])$$

$= \sigma^2$ (integration of All kinds of $AA' - (XX')^{-1}$ multiplies its pdf weight)

is definitely positive semidefinite since pdf weight $> 0$ & $AA' - (X'X)^{-1}$ is positive definite #

3.

(a) for $\hat{\beta}_{GLS}$ we consider $\Omega^{-1/2} Y = \Omega^{-1/2} X \beta + \Omega^{-1/2} e$

then we get OLS of this linear regression is $\hat{\beta}_{GLS} = \left((\Omega^{-1/2}X)'(\Omega^{-1/2}X)\right)^{-1} \left((\Omega^{-1/2}X)'(\Omega^{-1/2}Y)\right)$

$$= (X'\Omega^{-1}X)^{-1} \cdot X'\Omega^{-1}Y$$

$\Omega = cov(e) = E[ee^T|X]$ is positive definite since $E[e_i|X_i] = \sigma X_i > 0$
$= \sigma^2 \cdot Var[x]$

(b)

$$var[\hat{\beta}_{GLS}|x] - var[\hat{\beta}|x] = (x'\Omega^{-1}x)^{-1} - \sigma^2(x'x)^{-1}$$

$$Var[\beta_{GLS}] - var[\hat{\beta}] = E[(x'\Omega^{-1}x)^{-1}] - \sigma^2 E[(x'x)^{-1}]$$

proof it is positive semidefinite by showing $\frac{1}{\sigma^2}E[(x'x)] - E[x'\Omega^{-1}x]$ is PSD

since $\Omega^{-1}$ indicates the variability of error, if $\Omega \neq \sigma^2 I$ then $z'E[x'\Omega^{-1}x]z$ is less then $z'E[x'x]z$ for $z \in \mathbb{R}^n - \{\underline{0}\}$ $\Rightarrow$ $var[\hat{\beta}_{GLS}] - var[\hat{\beta}]$ is PSD #

4.

(a)

$\beta_2 = 0 \quad Y_i = X_{1i}'\beta_1 + e_i \quad$ with $\quad E[ee'|X] = \sigma^2 I_n \;,\; E[e|X] = 0$

$\Rightarrow$ it is a constrained regression

$\hat{\gamma} = \beta_1 + (X_1'X_1)^{-1} X_1' e$

$E[\hat{\gamma}|X] = \beta_1$ since $E[e|X] = 0$ #

$Var[\hat{\gamma}|X] = Var[(X_1'X_1)^{-1} X_1' e] = \sigma^2 (X_1'X_1)^{-1}$ since $Var[e|X] = \sigma^2 I_n$ #

(b)

$\beta_2 \neq 0 \Rightarrow Y_i = X_{1i}\beta_1 + X_{2i}\beta_2 + e_i$

$E[\hat{\gamma}|X] = E[(X_1'X_1)^{-1} X_1' (X_1\beta_1 + X_2\beta_2 + e) | X]$

$= \beta_1 + E[(X_1'X_1)^{-1} X_1'X_2 \beta_2 | X] + 0$ #

$Var[\hat{\gamma}|X] = \sigma^2 (X_1'X_1)^{-1}$ #