1.

since Σ is diagonal

$\Rightarrow$ Covariance of $Y_i, Y_j = E[(Y_i - E[Y_i])(Y_j - E[Y_j])] = 0$ for every $i \neq j$

$E[(Y_i - E[Y_i])(Y_j - E[Y_j])] = E[Y_i Y_j] - E[Y_i]E[Y_j] = 0 \Rightarrow E[Y_i Y_j] = E[Y_i] \cdot E[Y_j]$

choose $m=2$ $\Rightarrow$ when Y_1, Y_2 has zero covariance $\Rightarrow$ Σ is digonal

①

suppose Y_1 is random variable with uniform distribution in interval $[-1,1]$

Then we set $Y_2 = SY_1$, S is another random variable s.t. $S = \begin{cases} 1 & p=\frac{1}{2} \\ -1 & p=\frac{1}{2} \end{cases}$

Clearly, $S \perp Y_1 \Rightarrow E[Y_2] = E[SY_1^2] = E[S]E[Y_1^2] = 0$

Also, $E[Y_1] = 0 \Rightarrow E[Y_1] \cdot E[Y_2] = 0 \Rightarrow E[Y_1 Y_2] = E[Y_1] \cdot E[Y_2] \Rightarrow$ Covariance of $Y_1, Y_2 = 0$ $\Rightarrow$ Σ is diagonal

However, since $P[Y_2 > \frac{1}{2} \mid Y_1 > \frac{1}{2}] = \frac{1}{2} \neq P[Y_2 > \frac{1}{2}] = \frac{1}{4}$ $\Rightarrow$ Y_1, Y_2 aren't independent

$\Rightarrow$ Y isn't a vector of independent even when Σ is diagonal

②

$Y_1 \sim N(0,1)$, $Y_2 = Y_1^2 - 1$

$E[Y_1 Y_2] = E[Y_1^3 - Y_1] = 0 - 0 = 0$ $E[Y_1] \cdot E[Y_2] = 0 \cdot E[Y_1] = 0$ $\Rightarrow Cov(Y_1, Y_2) = 0$ $\Rightarrow$ Σ is diagonal

however, $P(Y_2 > \delta^2 - 1 \mid Y_1 > 0) = \dfrac{P(Y_1 > \delta)}{P(Y_1 > 0)} = 2P(Y_1 > \delta) \neq P(Y_2 > \delta^2 - 1) = \frac{1}{2}$

$\Rightarrow$ Y_1, Y_2 aren't independent even when Σ is diagonal

2、

$Y \sim N(\mu, \Sigma) \Rightarrow$ Y's elements are jointly gaussian

$$\Rightarrow f_Y(y) = \frac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} \exp(-\tfrac{1}{2}(y-\mu)^T \Sigma^{-1} (y-\mu))$$

if Σ is diagonal with diagonal element denotes $\sigma_1^2, \sigma_2^2 \cdots \sigma_m^2$, respectively

$$\Rightarrow f_Y(y) = \frac{1}{(2\pi)^{n/2} \cdot \prod_{i=1}^{m} \sigma_i} \cdot \exp(-\tfrac{1}{2} \cdot \sum_{i=1}^{m} \sigma_i^{-2} \cdot (y_i - \mu)^2)$$

$$= \prod_{i=1}^{m} \frac{1}{(2\pi)^{1/2} \cdot \sigma_i} \cdot \exp(-\tfrac{1}{2} \cdot (\tfrac{y_i-\mu}{\sigma_i})^2)$$

$= \prod_{i=1}^{m} f_{Y_i}(y_i) \Rightarrow$ it is proved that all the elements in Y are independent if $Y \sim N(\mu, \Sigma)$ #

3、

By #2, we can simply recognize that σ_i is the std of each random variable Y_i

$\Rightarrow \Sigma = \sigma^2 I_m$ is equivalent to the fact that Y_i has $Var(Y_i) = \sigma^2$, $i = 1, 2 \cdots m$

square of mean

$E[Y'Y] = E[\sum_{i=1}^{m} Y_i^2] = m(\sigma^2 + 0^2) = m\sigma^2$ #

$3\sigma^4$ for each

$E[(Y'Y)^2] = E[(\sum_{i=1}^{m} Y_i^2)^2] = E[2\sum_{i=1}^{m}\sum_{j=1}^{i-1}(Y_iY_j)^2] + \sum_{i=1}^{m} E[Y_i^4]$

for each i, j $E[(Y_iY_j)^2] = E[Y_i^2] \cdot E[Y_j^2] = \sigma^4$

since Y_i, Y_j are independent

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_i^2 y_j^2 \cdot f(y_i, y_j)\, dy_i dy_j = \int_{-\infty}^{\infty} y_i^2 f(y_i) dy_i \cdot \int_{-\infty}^{\infty} y_j^2 f(y_j) dy_j = E[Y_i^2] \cdot E[Y_j^2]$$

$\Rightarrow E[(Y'Y)^2] = E[(\sum_{i=1}^{m} Y_i^2)^2] = E[2\sum_{i=1}^{m}\sum_{j=1}^{i-1}(Y_iY_j)^2] + \sum_{i=1}^{m} E[Y_i^4] = (m(m-1) + 3m)\sigma^4 = m(m+2)\sigma^4$ #

4h

by definition $E[Y] = \sum_{i=1}^{n} P(A_i) \cdot E[Y|A_i]$, $\sum_{i=1}^{n} P(A_i) = 1$

$\Rightarrow \alpha_1$ should be $P(X_1 > 0.015) = 0.091$ α_2 should be $P(X_1 \leq 0.015) = 0.909$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\frac{46}{504} \qquad\qquad \frac{458}{504}$$

$\Rightarrow$ $\beta_{11} = P[X_1 > 0.015, X_2 > 0.02] = 0.056$ $\beta_{11} = P[X_1 > 0.015, X_2 \leq 0.02] = 0.036$

$\beta_{21} = P[X_1 \leq 0.015, X_2 > 0.02] = 0.478$ $\beta_{21} = P[X_1 \leq 0.015, X_2 \leq 0.02] = 0.430$

Also, $E[Y|A] = E[Y|A \cap B_1] \cdot P(B_1|A) + E[Y|A \cap B_2] \cdot P(B_2|A)$

$\Rightarrow$ $\gamma_1 = P(X_1 > 0.015 | X_3 > -4) = 0.135$

$\gamma_2 = P(X_1 \leq 0.015 | X_3 > -4) = 0.865$

5.

$$E[(m(x) - X'b)^2]$$

$$= E[(m(x) - X'b)'(m(x) - X'b)]$$

$$= E[(m(x))^2 - b'X \cdot m(x) - m(x)'X'b + b'XX'b]$$

$$= E[(m(x))^2] - 2b'E[X \cdot m(x)] + b'E[XX']b$$

derive by b, argmin should satisfy that

$$\nabla_b E[(m(x) - X'b)^2] = -2E[X\, m(x)] + 2E[XX']b = 0$$

$\Rightarrow$ we can obtain b by solving $E[XX']b = E[X\, m(x)]$

since $E[XX']$ is positive-definite $\Rightarrow E[XX']^{-1}$ exists

$$\Rightarrow b = E[XX']^{-1} \cdot E[X\, m(x)]$$

$$= E[XX']^{-1} \cdot E[X \cdot E[Y|X]]$$

$$E[E[X \cdot Y|X]] = \int_{-\infty}^{\infty} E[X \cdot Y | X = x] \cdot f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot y \cdot f_{Y|X}(y|x)\, dy\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot y \cdot f_{Y|X}(y|x) \cdot f_X(x)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x \cdot y \cdot f(x, y)\, dy\, dx$$

$$= E[X \cdot Y]$$

$$\Rightarrow b = E[XX']^{-1} \cdot E[X\, E[Y|X]] = E[XX']^{-1} \cdot E[X \cdot Y] \ \#$$

6.

pdf of x : $f(t;3) = \frac{\Gamma(2)}{\sqrt{3\pi}\,\Gamma(1.5)} \left(1+\frac{t^2}{3}\right)^{-2} = \frac{2}{\sqrt{3}} \cdot \frac{1}{\pi} \cdot \left(1+\frac{t^2}{3}\right)^{-2}$

(1)

we need degree of freedom to make X possess larger variance, which enable X to reach a wider range than normal distribution could. It make X be able to endure the outliers.

(2)

linear projection of Y on X follows the least square error

$\Rightarrow$ for random variable X, Y, coefficient β is $\frac{Cov[X,Y]}{Var[X]}$

since $Y = \frac{1}{1+X^4}$ is non-linear, Y has a complicated distribution

$\Rightarrow$ it is hard to compute $cov[X,Y]$

Thus I use python to sample 100000 pts of X and estimate the $\hat{\beta}$ based on the sample points.

I got $\beta \stackrel{\wedge}{=} -0.00041$, which means that linear relationship between X, Y is very weak.

```
import numpy as np
from scipy.stats import t

# Define the function for Y = 1 / (1 + X^4)
def Y_func(X):
    return 1 / (1 + X**4)

# Generate random samples for X from t-distribution with 3
np.random.seed(0)
num_samples = 100000
X_samples = t.rvs(df=3, size=num_samples)

# Calculate the corresponding Y values
Y_samples = Y_func(X_samples)

# Estimate the necessary expectations
E_X = np.mean(X_samples)
E_Y = np.mean(Y_samples)
E_X2 = np.mean(X_samples**2)
E_XY = np.mean(X_samples * Y_samples)

# Calculate the linear regression coefficient b
b = (E_XY - E_X * E_Y) / (E_X2 - E_X**2)
```