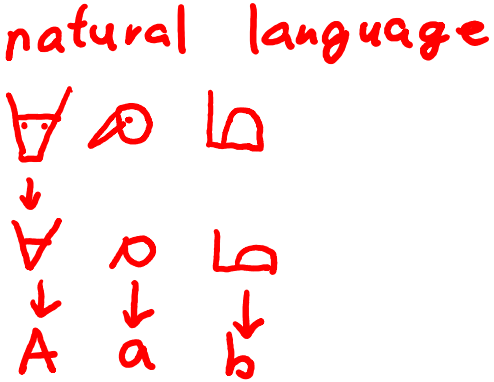
VII. Data Compression (A)

♦壓縮的通則：

entropy (亂度; 熵)

利用資料的一致性

資料越一致的資料，越能夠進行壓縮



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- 酒井善則，吉田俊之原著，原島博監修，白執善編譯，“影像壓縮術”，全華印行, 2004.
- 戴顯權，“資料壓縮 Data Compression,” 旗標出版社, 2007.
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◉ 7-A 壓縮的哲學：

(1) 利用資料的一致性，規則性，與可預測性

(exploit redundancies and predictability, find the compact or sparse representation)

(2) 通常而言，若可以用比較精簡的自然語言來描述一個東西，那麼也就越能夠對這個東西作壓縮

Q: 最古老的壓縮技術是什麼？

(3) 資料越一致，代表統計特性越集中

包括 Fourier transform domain, histogram, eigenvalue …….. 等方面的集中度

Data type	Compression technique	Compression rate
Audio 1D	MPEG3 *.wav → *.mp3	1/3
Image 2D	JPEG *.bmp → *.jpg	for gray images : 1/10 color images: 1/20 4:2:0
Video 3D	MPEG4 H264, H265 AV1	*.mpg *.mpeg *.mp4 *.avi *.wmv *.mov	for gray videos: 1/25 color videos : 1/50

image before compression $3000 \times 4000 \times 3 = 36M$ JPEG → 1.8M

$2.4M \times 126K$

video $(768 \times 1024 \times 3) \times (60 \times 70\ min) \times 30 = 300G$ MPEG → 6G

For a video, there are 30 / 60 frames per second

思考：如何對以下的資料作壓縮

Article: 常用字 / 常用字母 e, t ; q, x, z ; moss

Song:
① Concentration $f_0, 2f_0, 3f_0 \cdots$ more at the frequencies
② For each note, the frequency is fixed.
③ Fundamental frequencies are $f_0 \cdot 2^{\frac{k}{12}}$
④ beat, intervals are $T \cdot 2^k$, $k = -1, 0, 1, 2, \cdots$
⑤ repeated melody

Cartoon or Mark:
① The color / intensity is fixed within a region
② edges can be approximated by lines or arcs

Compression: Original signal → Compact representation + residual information

◉ 7-B Compression for Images

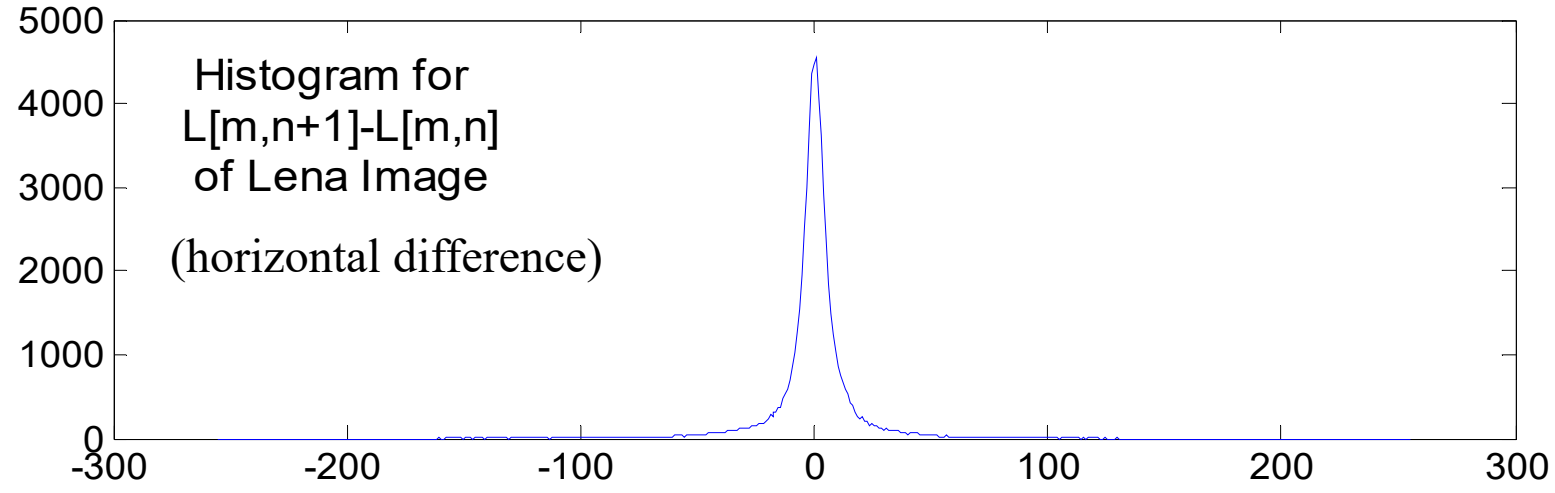
- 影像的「一致性」：

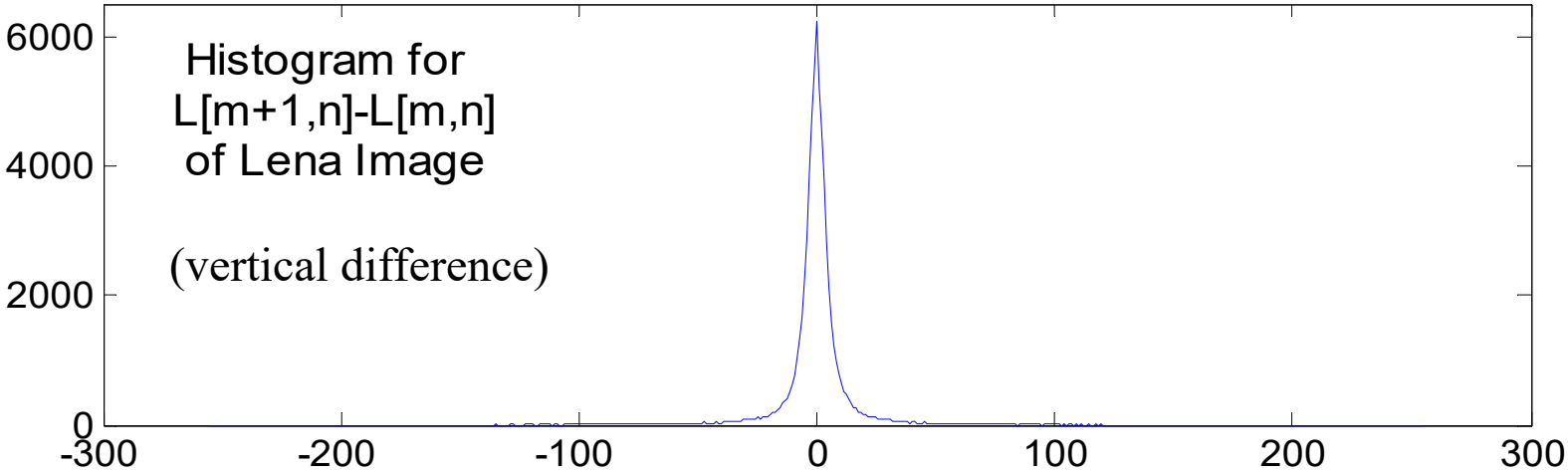
Space domain: 每一點的值，會和相鄰的點的值非常接近

$$F[m, n+1] \approx F[m, n], \qquad F[m+1, n] \approx F[m, n]$$

Frequency domain: 大多集中在 **低頻** 的地方。

Lena Image 在 space domain 上的一致性





Histogram: 直方圖

一個 vector 或一個 matrix 當中，有多少點會等於某一個值

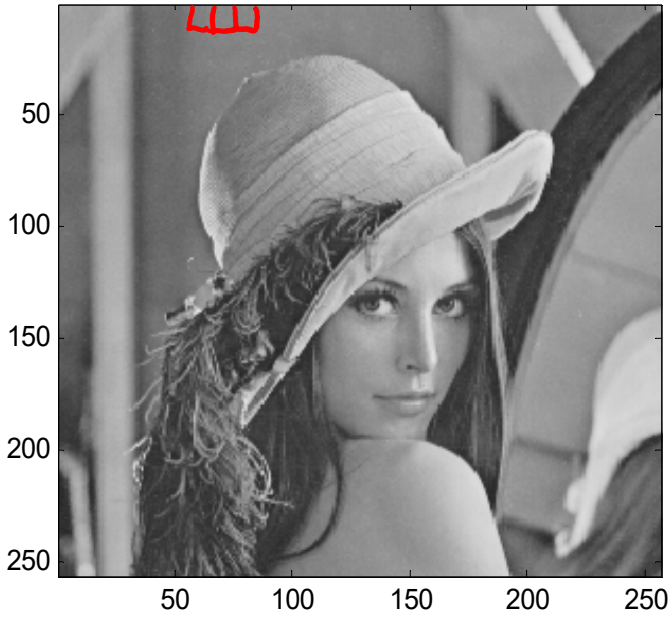
例如：$x[n] = [1\ 2\ 3\ 4\ 4\ 5\ 5\ 3\ 5\ 5\ 4]$

則 $x[n]$ 的 histogram 為

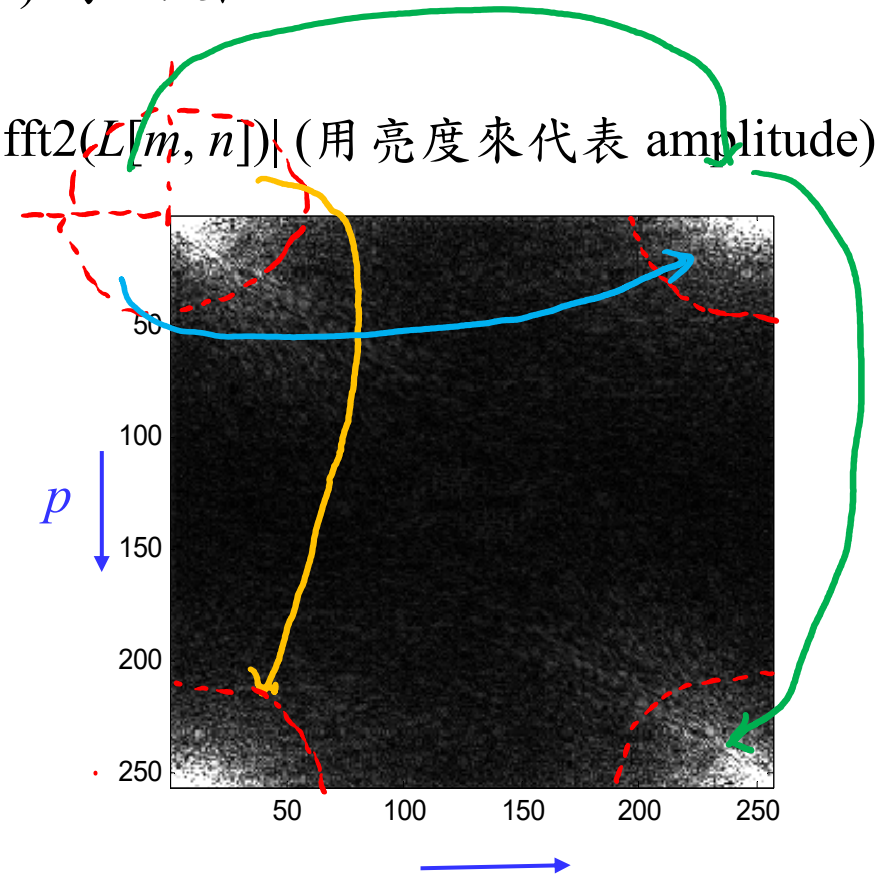
$h[1] = 1, h[2] = 1, h[3] = 2, \quad h[4] = 3, h[5] = 4$

Lena Image 頻譜 (frequency domain) 的一致性

$L[m, n]$



|fft2($L[m, n]$)| (用亮度來代表 amplitude)



$L_F[p,q] = L_F[p+M,q] = L_F[p,q+N] = L_F[p+M,q+N]$

$$L_F[p,q] = fft2\{L[m,n]\} = \sum_{m=1}^{M}\sum_{n=1}^{N} L[m,n] e^{-j2\pi\frac{pm}{M}} e^{-j2\pi\frac{qn}{N}}$$

$e^{-j2\pi\frac{p+M}{M}m}$

$= e^{-j2\pi\frac{pm}{M}} e^{-j2\pi m}$

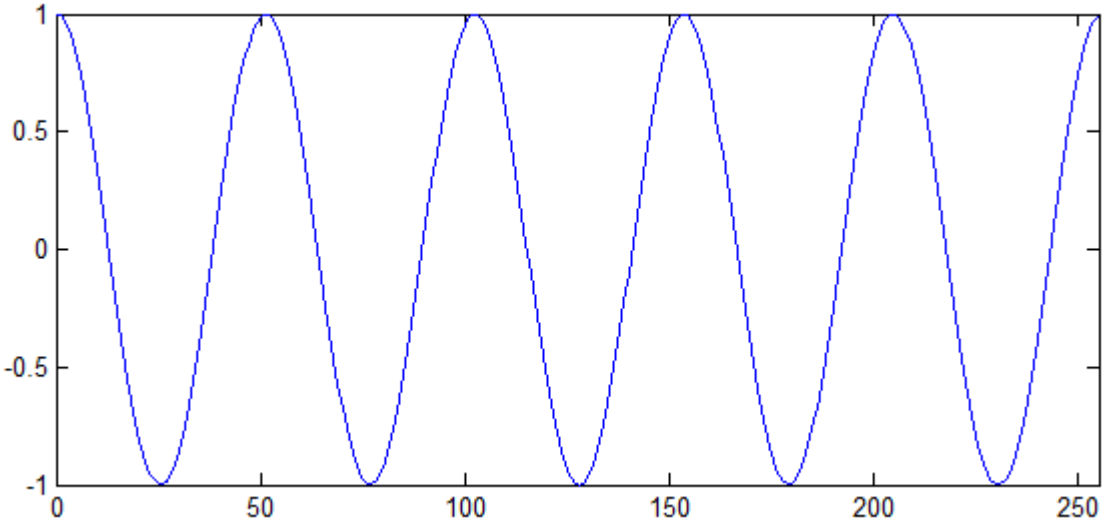
$= e^{-j2\pi\frac{pm}{M}}$

$$L[m,n] = \frac{1}{MN}\sum_{m=1}^{M}\sum_{n=1}^{N} L_F[p,q] e^{j2\pi\frac{pm}{M}} e^{j2\pi\frac{qn}{N}}$$

影像的「頻率」：frequency in the space domain

$e^{j2\pi\frac{pm}{M}}$　　從 $m = 0$ 至 $m = M$-1 之間有 p 個週期

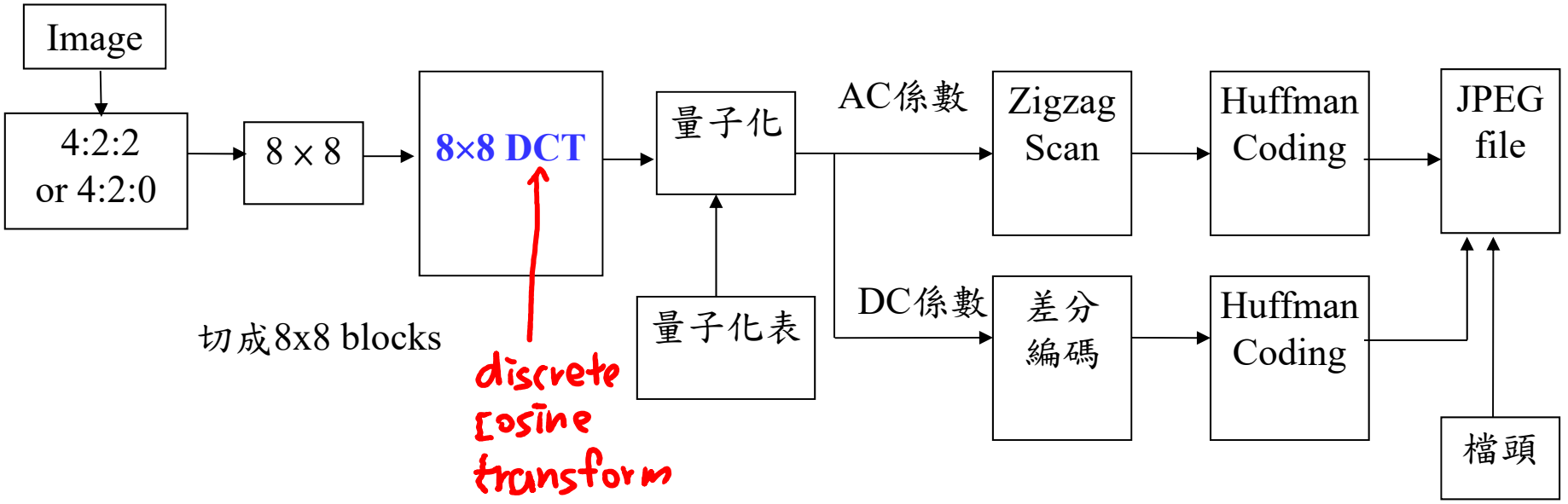
$p = 5$　　$\mathrm{Re}\{e^{j2\pi\frac{pm}{M}}\}$



larger p : more variation in the space domain

◉ 7.C JPEG Standard

Process of JPEG Image Compression



- 主要用到四個技術：(1) 4:2:2 or 4:2:0 (和 space domain 的一致性相關)

  (2) 8 × 8 DCT and zig zag (和 frequency domain 的一致性相關)

  (3) 差分編碼 (和 space domain 的一致性相關)

  (4) Huffman coding (和 lossless 編碼技術相關)

JPEG： 影像編碼的國際標準　　全名： Joint Photographic Experts Group

JPEG 官方網站： http://www.jpeg.org/

參考論文：G. K. Wallace, “The JPEG still picture compression standard,” *IEEE Transactions on Consumer Electronics,* vol. 38, issue 1, pp. 18-34, 1992.

JPEG 的 FAQ 網站： http://www.faqs.org/faqs/jpeg-faq/

JPEG 的 免費 C 語言程式碼：
http://opensource.apple.com/source/WebCore/WebCore-1C25/platform/image-decoders/jpeg/

一般的彩色影像，可以壓縮 20 倍。

簡單的影像甚至可以壓縮超過 30 倍。

- 壓縮的技術分成兩種

lossy compression techniques

無法完全重建原來的資料

Examples: DFT, DCT, KLT (with quantization and truncation),
4:2:2 or 4:2:0, polynomial approximation

壓縮率較高

lossless compression techniques

可以完全重建原來的資料

Examples: binary coding, Huffman coding, arithmetic coding,
Golomb coding

壓縮率較低

◉ 7-D 4:2:2 and 4:2:0

double(imread('file name'))

$$\begin{bmatrix} Y \\ C_b \\ C_r \end{bmatrix} = \begin{bmatrix} 0.299 & 0.587 & 0.114 \\ -0.169 & -0.331 & 0.500 \\ 0.500 & -0.419 & -0.081 \end{bmatrix} \begin{bmatrix} R \\ G \\ B \end{bmatrix}$$

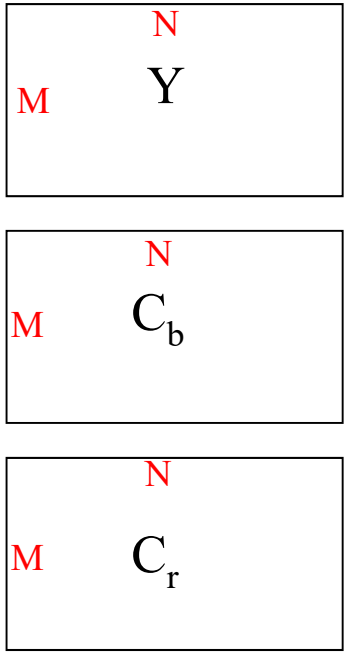
R: red, G: green, B: blue

Y: 亮度, C_b: 0.565$(B-Y)$, ← complementary of blue

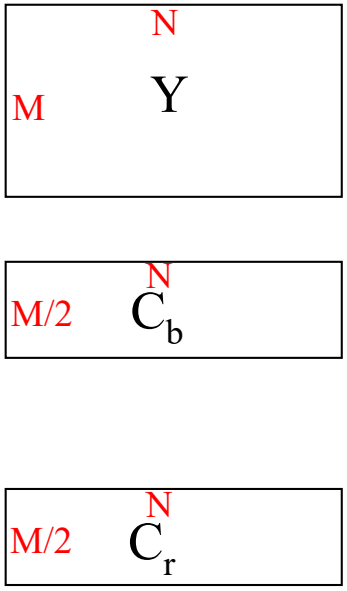
C_r: 0.713$(R-Y)$, ← complementary of red

3MN

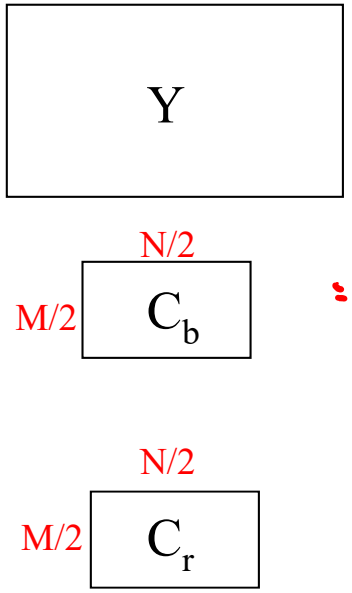
4 : 4 : 4



4 : 2 : 2



4 : 2 : 0



If $\widetilde{C_b}(m,n) = C_b(2m,2n)$ to recover

$C_b(2m,2n+1) = \frac{1}{2}(C_b(2m,2n) + C_b(2m,2n+2)) = \frac{1}{2}(\widetilde{C_b}(m,n) + \widetilde{C_b}(m,n+1))$

$C_b(2m+1,n) = \frac{1}{2}(C_b(2m,n) + C_b(2m+2,n))$

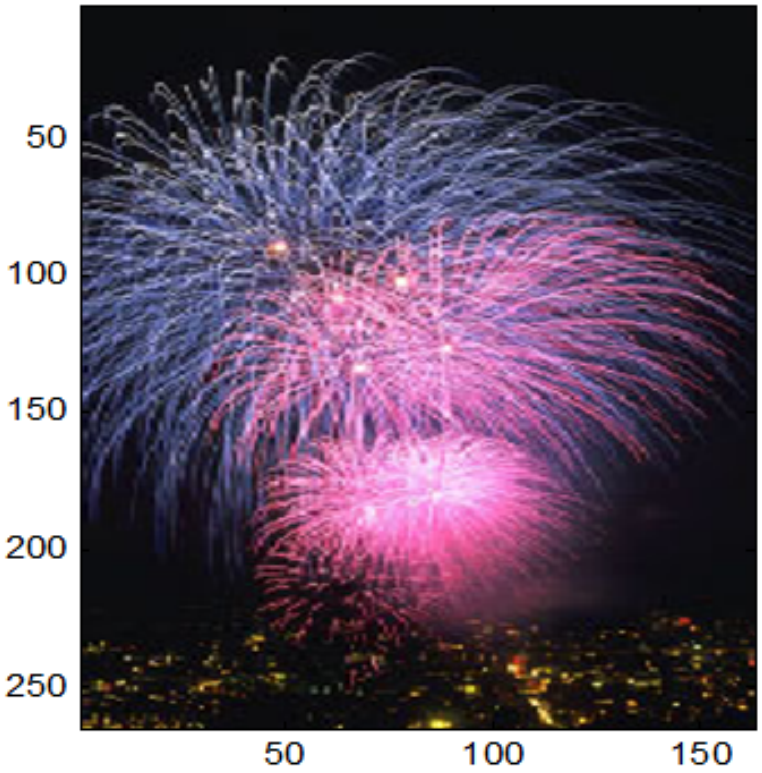
$MN + \frac{MN}{4} + \frac{MN}{4} = \frac{3}{2}MN$

24 bits/pixel → 16 bits/pixel → 12 bits/pixel

同樣使資料量省一半的(b)(d)圖，(d)圖和原來差不多，
然而(b)圖邊緣會有失真現象。

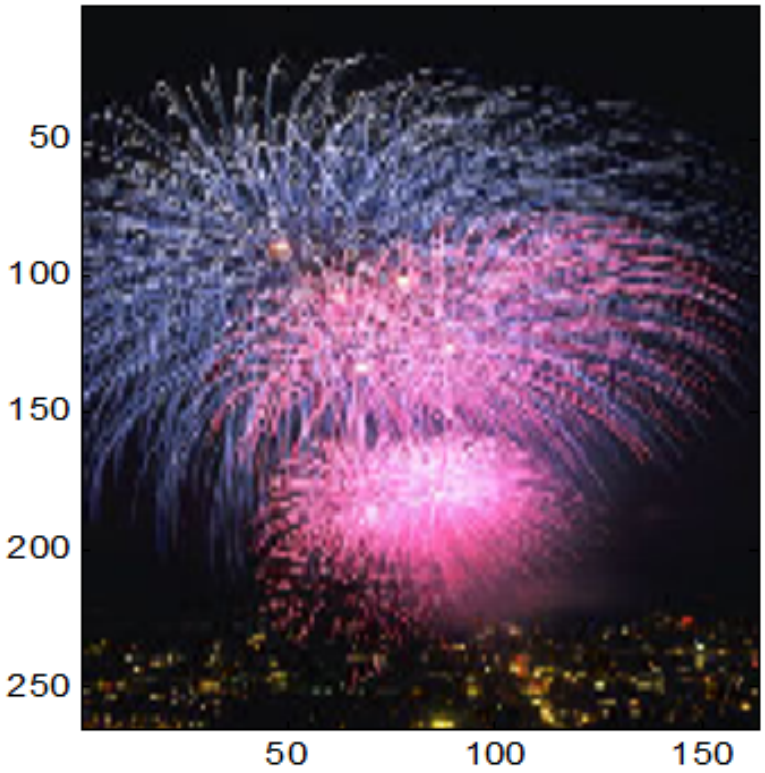
還原時，用 interpolation 的方式

原圖



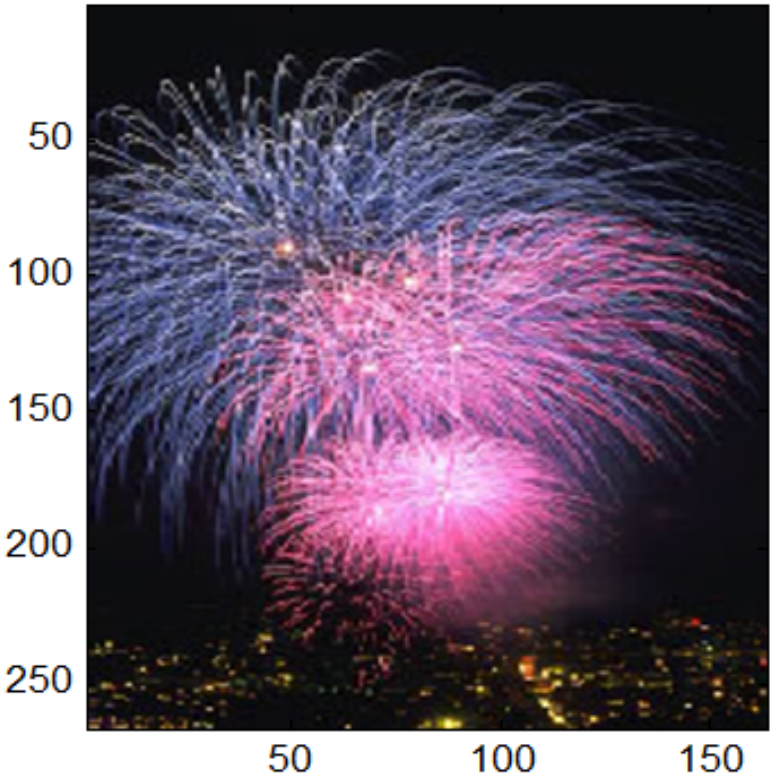
(a)

直接在縱軸取一半的pixels 再還原



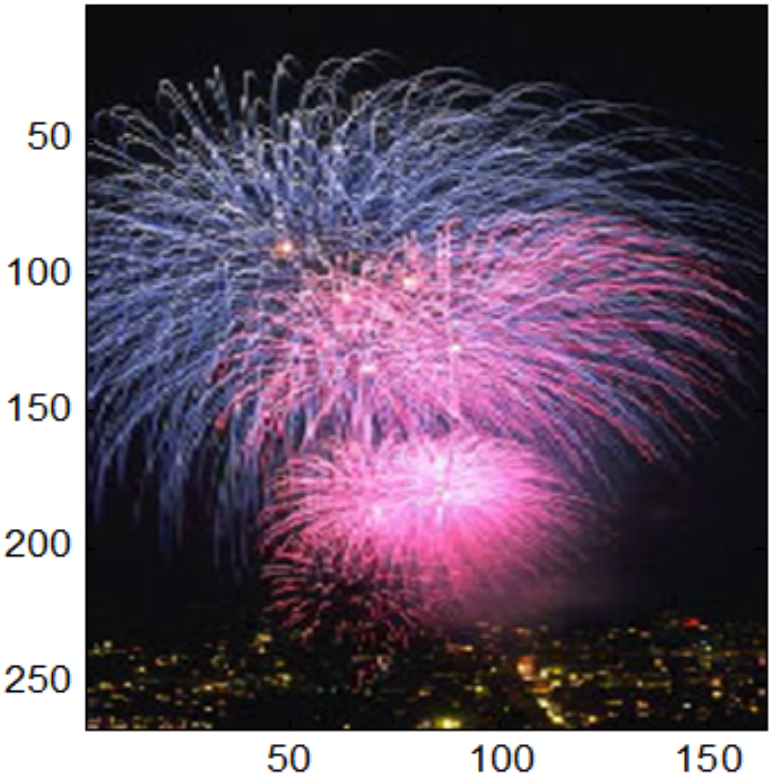
(b)

4 : 2: 2



(c)

4 : 2: 0



(d)

◉ 7-E Optimal Transform--KLT

複習：DFT 的優缺點

- **Karhunen-Loeve Transform (KLT)**
  (similar to Principal component analysis (PCA))

It is optimal, but dependent on the input

經過轉換後，能夠將影像的能量分佈變得最為集中

分析影像的主要成分，第二主要成份，第三主要成份，.....

- 1-D Case $X[u]=\sum_{n=0}^{N-1} x[n]K[u,n]$

$X = Kx$ $\quad x = [x[0], x[1], \cdots, x[N-1]]^T$

$K[u, n] = e_n[u]$ $(K = [e_0, e_1, e_2, \ldots, e_{N-1}]^T$

$\mathbf{e_n}$ 為 covariance matrix **C** 的 eigenvector

C 為 N×N matrix

mean

$$C[m, n] = \text{cov}(x[m], x[n]) = E\left[(x[m]-\overline{x[m]})(x[n]-\overline{x[n]})\right]$$

Note: cov 代表covariance

$\rho^{|m-n|}$

when $\rho \to 1$

KLT → DCT

KLT 的理論基礎：

經過 KLT 之後，當 $u_1 \neq u_2$ 時，$X[u_1]$ 和 $X[u_2]$ 之間的 covariance 必需近於零 (即 decorrelation)

即 $\text{cov}(X[u_1],\ X[u_2]) = E\left[(X[u_1]-\overline{X[u_1]})(X[u_2]-\overline{X[u_2]})\right] = 0$

If we set

$$\mathbf{x} = \left[x[0], x[1], x[2], \cdots, x[N-1]\right]^T \qquad \mathbf{X} = \left[X[0], X[1], X[2], \cdots, X[N-1]\right]^T$$

and

$$\mathbf{C} = E\left((\mathbf{x}-\bar{\mathbf{x}})(\mathbf{x}-\bar{\mathbf{x}})^T\right) \qquad \text{where} \quad \bar{\mathbf{x}} = E(\mathbf{x})$$

$$\mathbf{C}_\mathbf{X} = E\left((\mathbf{X}-\bar{\mathbf{X}})(\mathbf{X}-\bar{\mathbf{X}})^T\right) \quad \text{where} \quad \bar{\mathbf{X}} = E(\mathbf{X})$$

then

$$\mathbf{C}[m,n] = corr\left(x(m), x(n)\right) \qquad \mathbf{C}_\mathbf{X}[u_1,u_2] = corr\left(X(u_1), X(u_2)\right)$$

$\mathbf{C}_\mathbf{X}$ should be a diagonal matrix.

Also note that

$$\mathbf{X} = \mathbf{K}\mathbf{x} \qquad \bar{\mathbf{X}} = \mathbf{K}\bar{\mathbf{x}}$$

$$\mathbf{C}_{\mathbf{X}} = E\left((\mathbf{X}-\bar{\mathbf{X}})(\mathbf{X}-\bar{\mathbf{X}})^T\right)$$

Since $\mathbf{X} = \mathbf{K}\mathbf{x}$ $\bar{\mathbf{X}} = \mathbf{K}\bar{\mathbf{x}}$

$$\mathbf{C}_{\mathbf{X}} = E\left((\mathbf{K}\mathbf{x}-\mathbf{K}\bar{\mathbf{x}})(\mathbf{K}\mathbf{x}-\mathbf{K}\bar{\mathbf{x}})^T\right) = E\left(\mathbf{K}(\mathbf{x}-\bar{\mathbf{x}})(\mathbf{x}-\bar{\mathbf{x}})^T\mathbf{K}^T\right)$$

$$= \mathbf{K}E\left((\mathbf{x}-\bar{\mathbf{x}})(\mathbf{x}-\bar{\mathbf{x}})^T\right)\mathbf{K}^T = \mathbf{K}\mathbf{C}\mathbf{K}^T$$

To make $\mathbf{C}_{\mathbf{X}}$ a diagonal matrix, the KLT transform matrix $\mathbf{K}$ should diagonalize $\mathbf{C}$. Suppose that the rows of $\mathbf{K}$ are the eigenvectors of $\mathbf{C}$ is:

$$\mathbf{C} = \mathbf{E}\mathbf{D}\mathbf{E}^{\mathbf{T}}$$

where each column of $\mathbf{E}$ is an orthonormalized eigenvector of $\mathbf{C}$ and each diagonal entry of the diagonal matrix $\mathbf{D}$ is the eigenvalue, then we can set

$$\mathbf{K} = \mathbf{E}^{\mathbf{T}}$$

Then

$$\mathbf{C}_{\mathbf{X}} = \mathbf{E}^{\mathbf{T}}\mathbf{E}\mathbf{D}\mathbf{E}^{\mathbf{T}}\mathbf{E} = \mathbf{D}$$

is a diagonal matrix.

- 2-D Case $$X[u,v] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x[m,n]K[u,m]K[v,n]$$

KLT 缺點: dependent on image
(不實際，需要一併記錄 transform matrix)

Reference

W. D. Ray and R. M. Driver, “Further decomposition of the Karhunen-Loeve series representation of a stationary random process,” *IEEE Trans. Inf. Theory*, vol. 16, no. 6, pp. 663-668, Nov. 1970.

◉ 7-F Suboptimal Transform-- DCT

Suboptimal, but independent of the input

- **DCT: Discrete Cosine Transform**

$$F[u,v]=\frac{2C[u]C[v]}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}f[m,n]\cos\frac{(2m+1)u\pi}{2N}\cos\frac{(2n+1)v\pi}{2N}$$

$$C[0]=1/\sqrt{2}\quad,\ C[u]=1 \text{ for } u\neq 0$$

IDCT: inverse discrete cosine transform

$$f[m,n]=\frac{2}{N}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F[u,v]C[u]C[v]\cos\frac{(2m+1)u\pi}{2N}\cos\frac{(2n+1)v\pi}{2N}$$

對於大部分的影像而言， DCT 能夠近似 KLT (near optimal)

尤其是當 $\text{corr}\{f[m,n],f[m+\tau,n+\eta]\}=\rho^{|\tau|}\rho^{|\eta|}$， $\rho\to 1$ 時

有 fast algorithm

Advantage: (1) independent of the input (2) near optimal (3) real output

DCT

$F[u,v] = \sqrt{\frac{2}{N}} C[u] \sum_{m=0}^{M-1} F_1[m,v] \cos\left(\frac{2m+1}{2M} u\pi\right)$ $v = 0, 1, \cdots N-1$

where $F_1[m,v] = \sqrt{\frac{2}{N}} C[v] \sum_{n=0}^{N-1} f[m,n] \cos\left(\frac{2n+1}{2N} v\pi\right)$ $m = 0, 1, \cdots M-1$

$$F[u,v] = \frac{2C[u]C[v]}{N} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] \cos\frac{(2m+1)u\pi}{2M} \cos\frac{(2n+1)v\pi}{2N}$$

$$C[0] = 1/\sqrt{2} \quad , C[u] = 1 \text{ for } u \neq 0$$

$[u, v] = [0, 0]$: DC term

$$F[0,0] = \frac{1}{N} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n]$$

$u \neq 0$ or $v \neq 0$: AC terms

借用電路學的名詞

$m = 0, 1, \cdots M-1$

$M \times N$ DCT: M times of 1D N-point DCT + N times of 1D M-point DCT

complexity

$M \times N \log N + N \times M \log M = MN(\log N + \log M)$

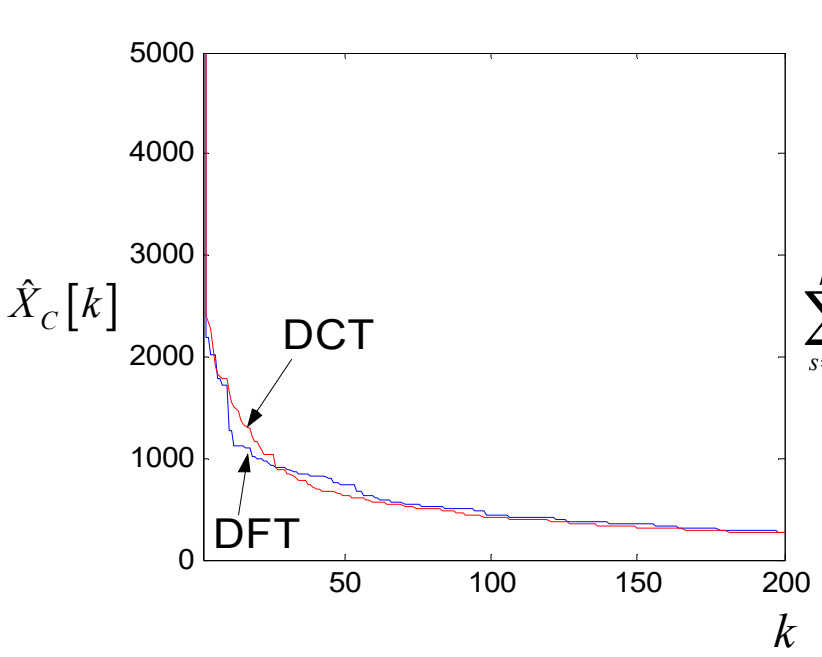
$= MN \log MN$

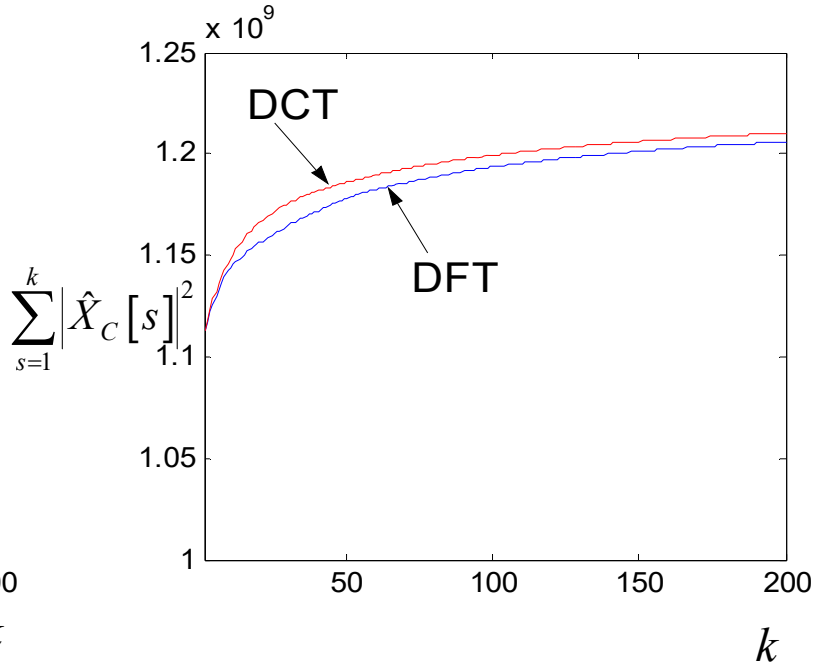
左圖：將 DFT，DCT 各點能量(開根號)由大到小排序

右圖：累積能量

DCT output

$$|X_C[p,q]| \xrightarrow{sort} \hat{X}_C[k] \qquad \hat{X}_C[1] \geq \hat{X}_C[2] \geq \hat{X}_C[3] \geq \cdots\cdots$$





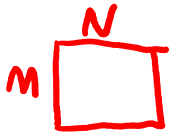
Energy concentration at low frequencies: KLT > DCT > DFT

通常，我們將影像切成 8 × 8 的方格作DCT

Why: (1) The characteristics of an image vary with the location

(2) The memory requirement is reduced

(3) Low complexity $\Theta(MN \log MN) \xrightarrow{\text{reduce}} \Theta(MN)$



N-point 1D DCT

complexity = $\Theta(N \log N)$

MxN 2D DCT

complexity = $\Theta(MN \log MN)$

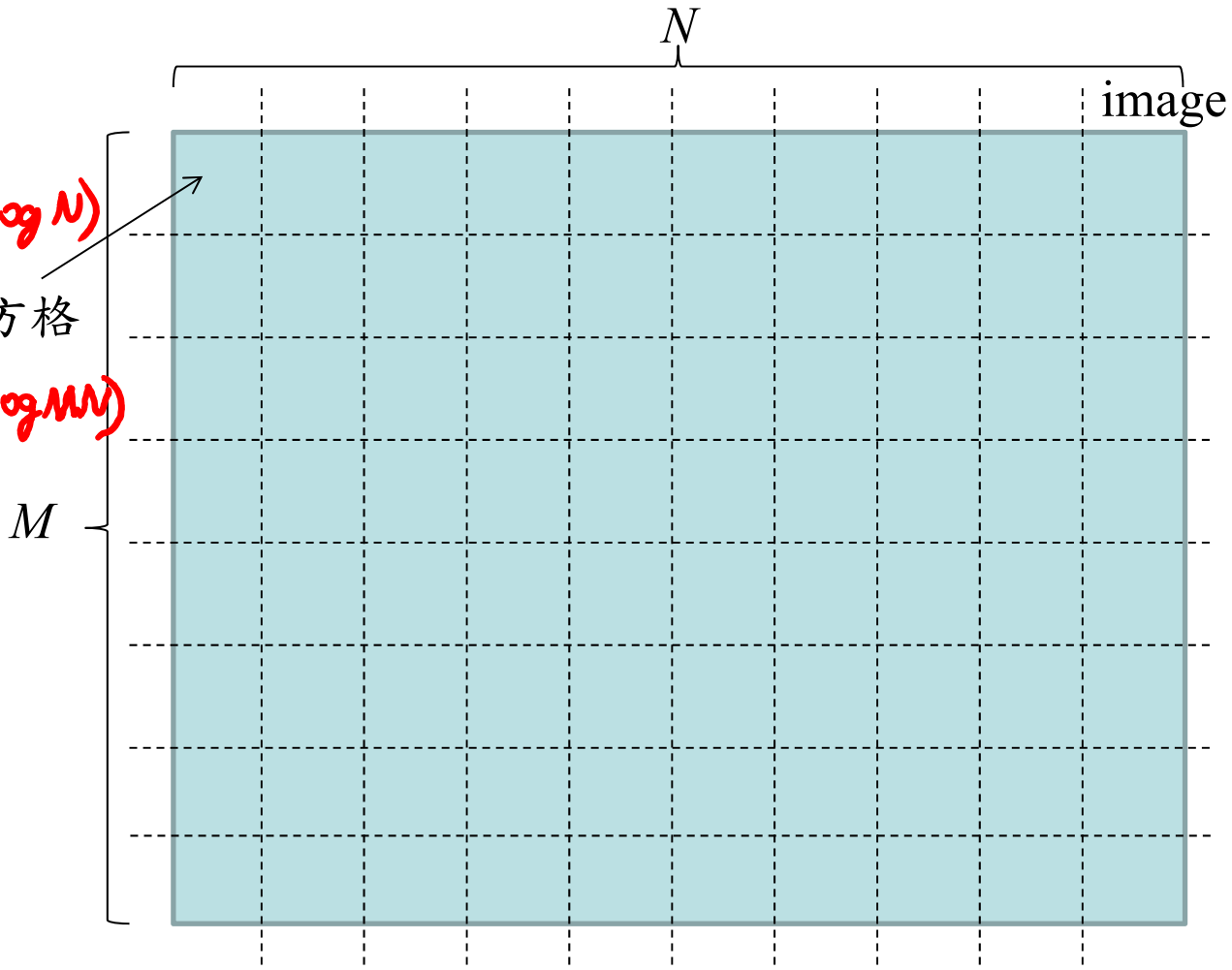
with 8x8 blocks

complexity =

$\frac{MN}{64} \times 64 \log 64$

$= MN \log 64$

$\Theta(MN)$



References

[1] N. Ahmed, T. Natarajan, and K. R. Rao, "Discrete cosine transform," *IEEE Trans. Comput.*, vol. C-23, pp. 90-93, Jan 1974.

[2] K. R. Rao and P. Yip, *Discrete Cosine Transform, Algorithms, Advantage, Applications*, New York: Academic, 1990.

VIII. Data Compression (B)

◉ 8-A Differential Coding for DC Terms, Zigzag for AC Terms

這兩者可視為 JPEG Huffman coding 的前置工作

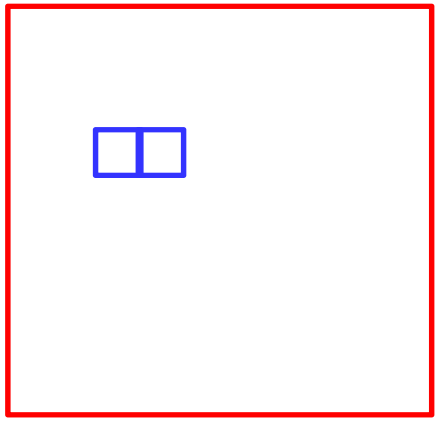
Differential Coding (差分編碼)

mean of the $(i,j)^{th}$ block × constant

If the DC term of the $(i,\ j)^{th}$ block is denoted by $DC[i,j]$, then

encode $DC[i,j] - DC[i,j\text{-}1]$

instead of $DC[i,j]$



(也是運用 space domain 上的一致性)

Zigzag scanning

將 2D 的 8x8 DCT outputs 變成 1D 的型態

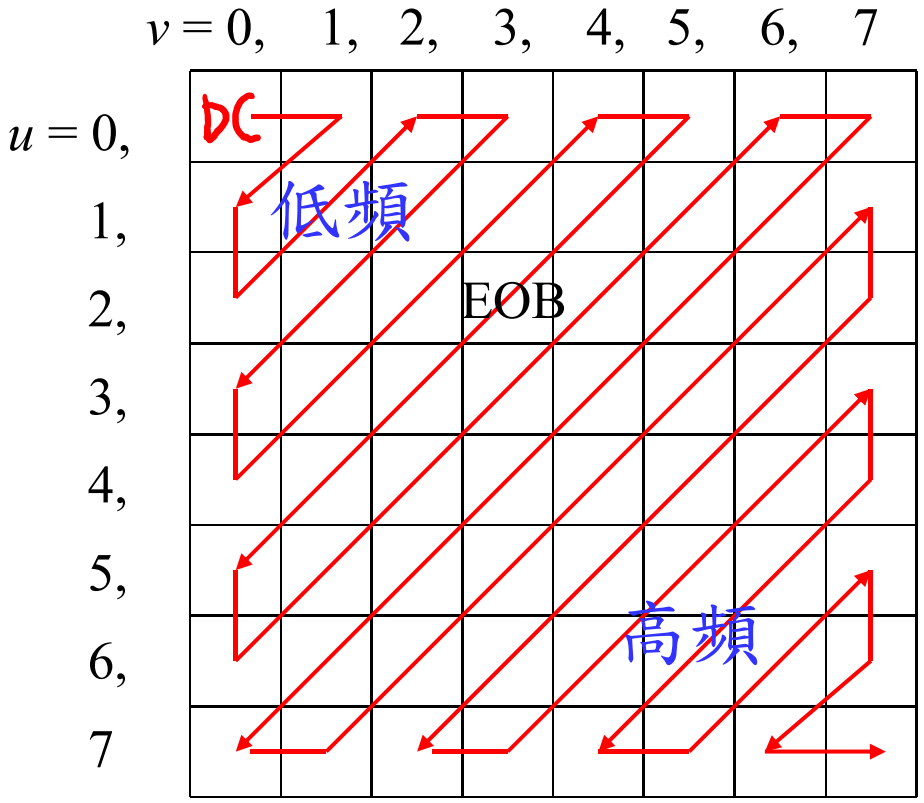
但按照 “zigzag” 的順序 (能量可能較大的在前面)

8x8 DCT output:

$C[u, v]$

$u = 0, 1, \ldots, 7$

$v = 0, 1, \ldots, 7$



EOB (end of block):

指後面的高頻的部分經過 quantization 之後皆為0

63 AC terms

(也是運用 frequency domain 上的一致性)

◉ 8-B Lossless Coding

Lossless Coding: The original data can be perfectly recovered

Example:

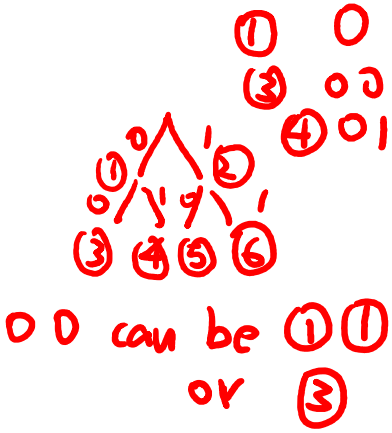
direct coding method

Huffman coding

Arithmetic coding

Shannon–Fano Coding, Golomb coding, Lempel–Ziv, ...

◉ 8-C Lossless Coding: Huffman Coding



- **Huffman Coding 的編碼原則: (Greedy Algorithm)**

(1) 所有的碼皆在 Coding Tree 的端點，再下去沒有分枝
(滿足一致解碼和瞬間解碼)

(2) 機率越大的， code length 越短；機率越小的， code length 越長

(3)假設 S_1 是第 L 層的 node，S_2是第 L+1 層的 node

則 $P(S_1) \geq P(S_2)$ 必需滿足

不滿足以上的條件則將 S_2 往上推一層

原始的編碼方式：

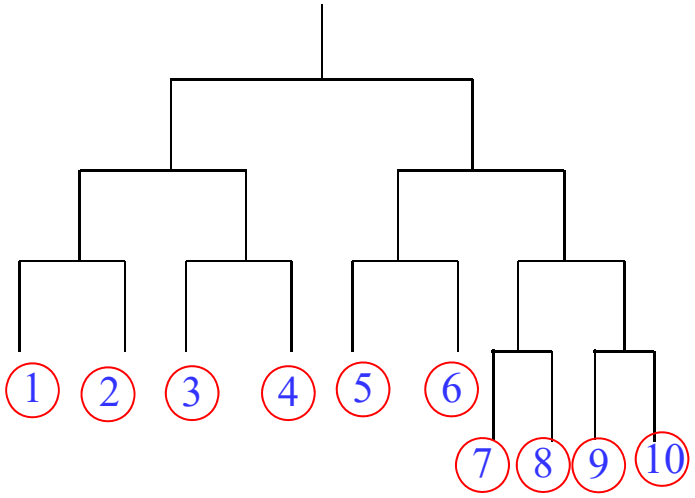
若 data 有 M 個可能的值，使用 k 進位的編碼，
則每一個可能的值使用 floor($\log_k M$) 或 ceil($\log_k M$) 個 bits 來編碼

floor: 無條件捨去

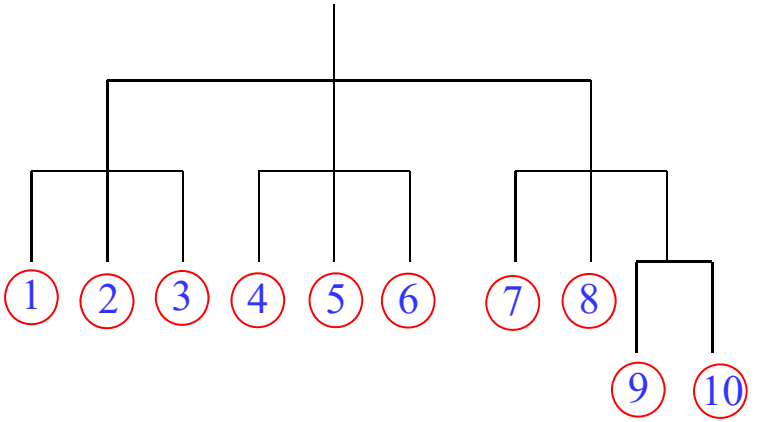
ceil: 無條件進位

Example:

若有 8 個可能的值，在2進位的情形下，需要 3 個 bits



若有 10 個可能的值，在3進位的情形下，需要 2 個或 3 個 bits



Example:

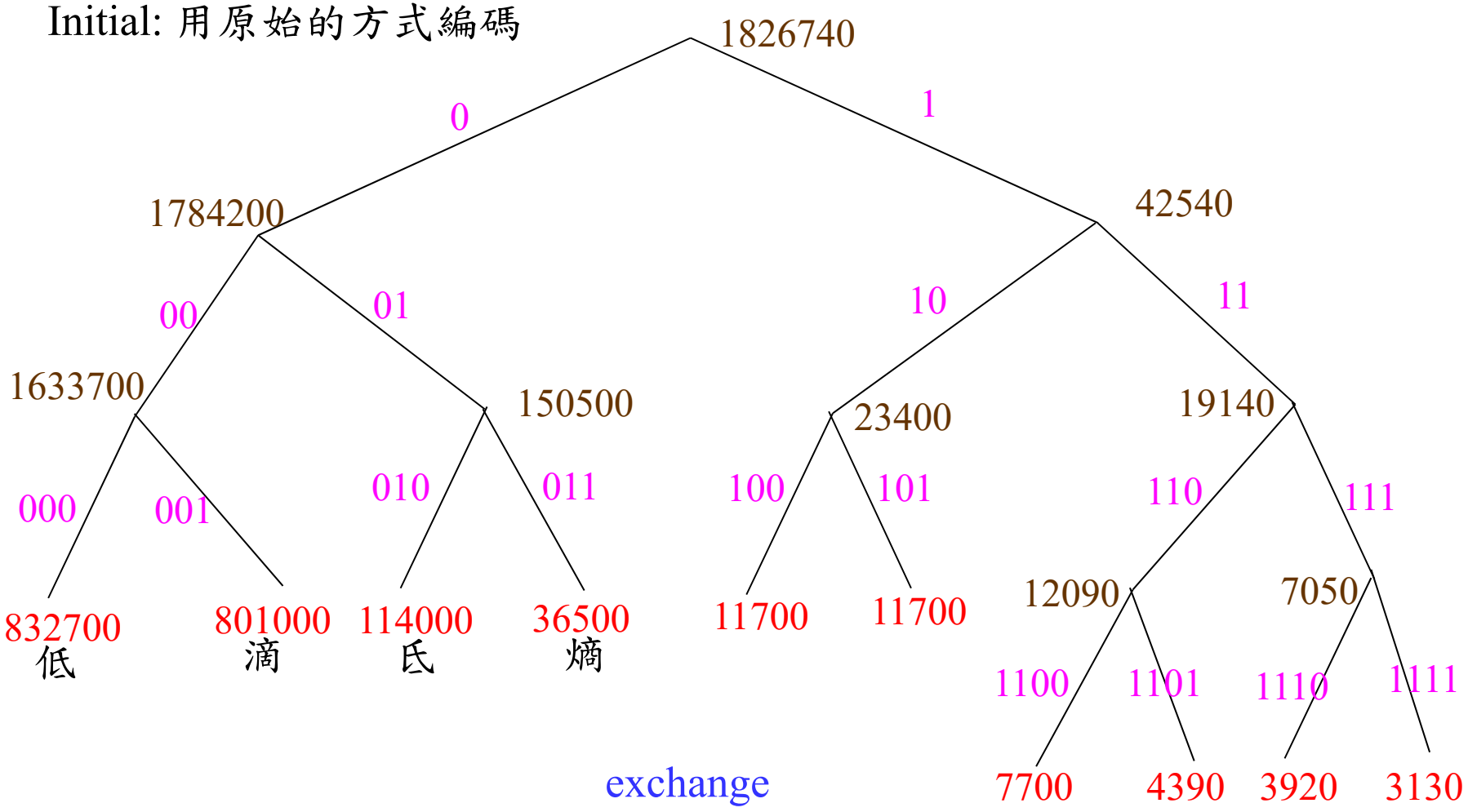
低	滴	氐	羝	鞮
832700	801000	114000	7700	4390
磾	祗	葥	墑	熵
3920	11700	11700	3130	36500

ㄉ

他們 2進位的Huffman Code 該如何編

Huffman coding

Initial: 用原始的方式編碼



exchange

average code length = 3.0105

The rules of the Huffman coding process.

(1) Process the case with lower probability first

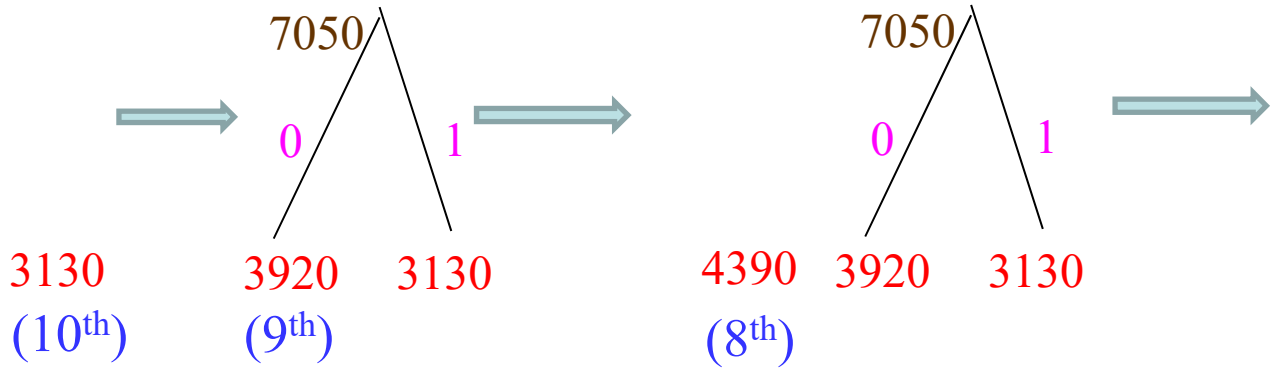
(2) If the node in the lower layer has higher probability, then move it upward.

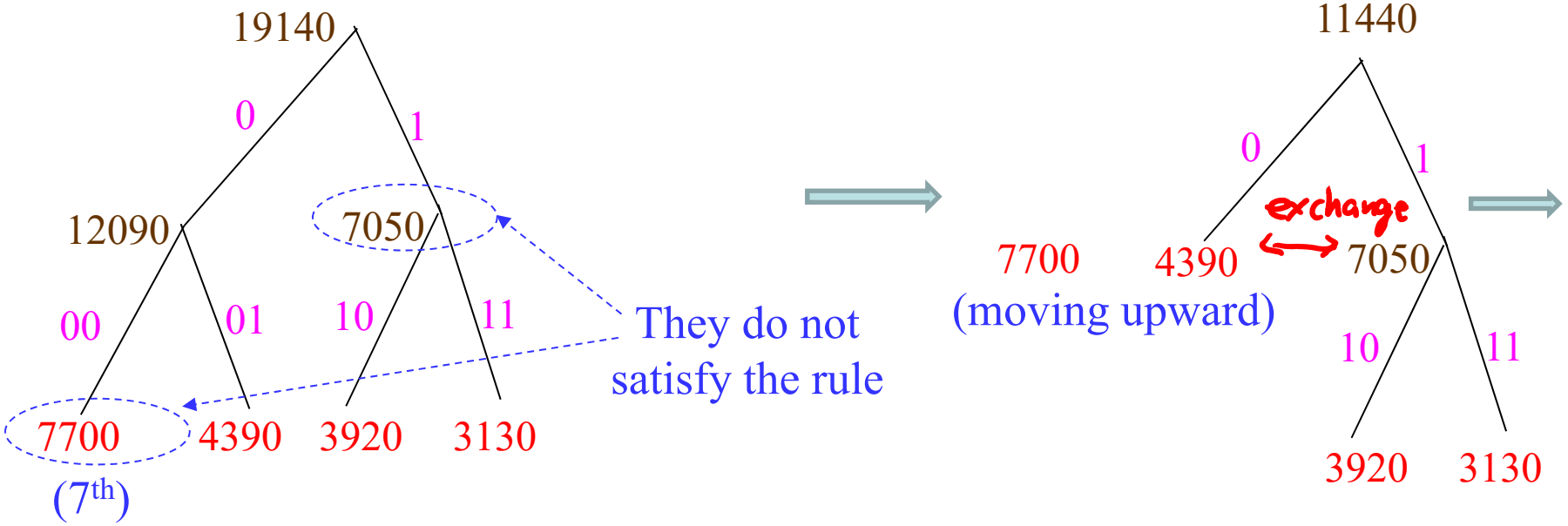
(3) For the node to be moved upward, if it has a partner, then move the partner, too.

(4) Re-sort after moving upward.

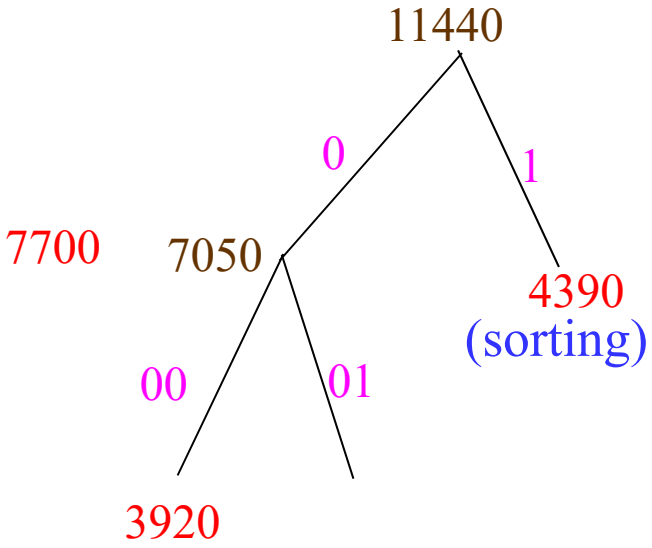
Huffman coding

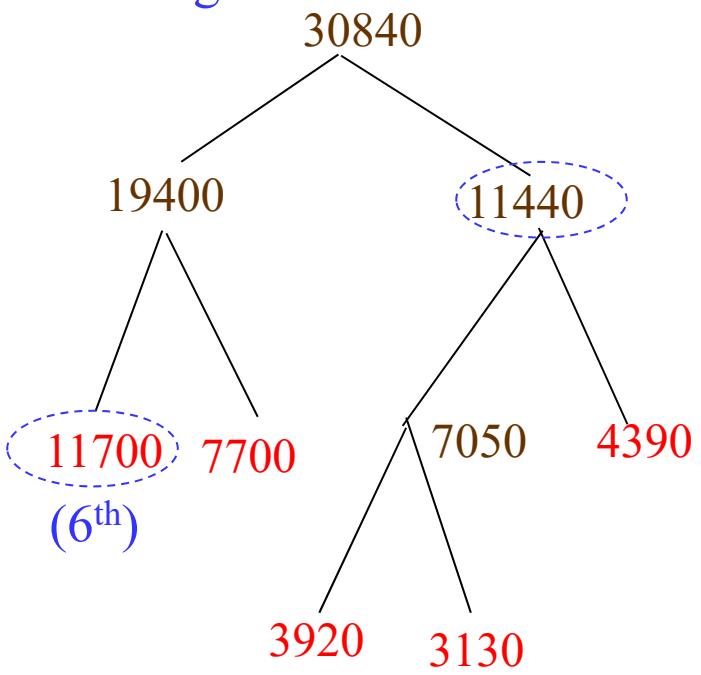
由機率低的開始編碼，一步一步加進機率高的

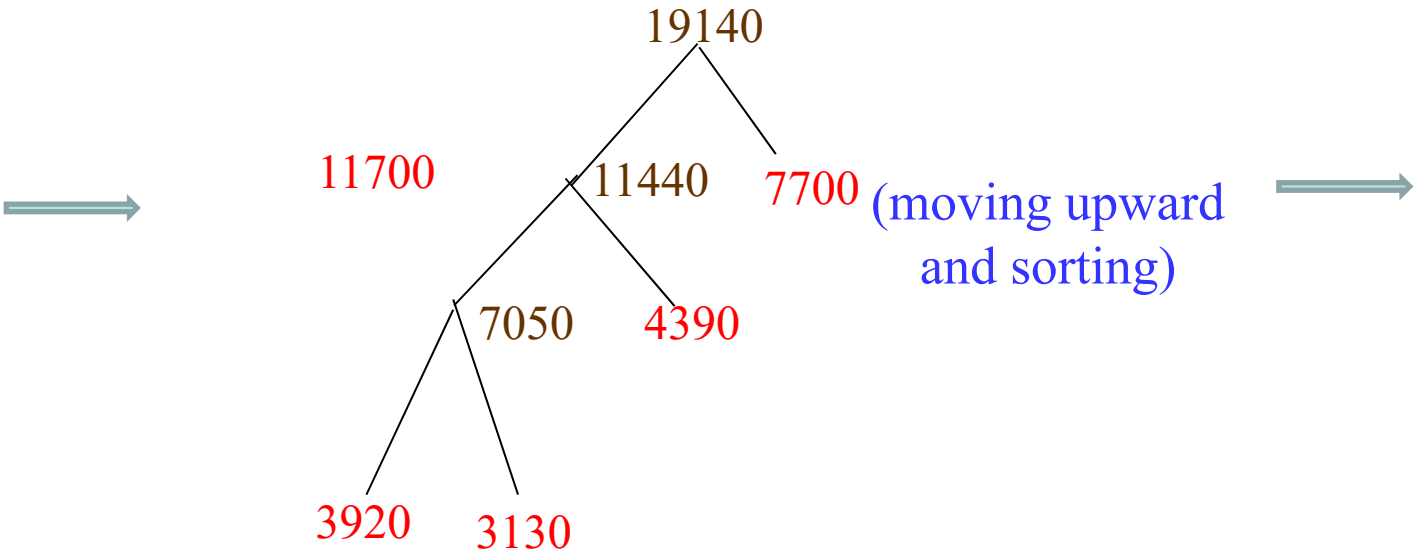




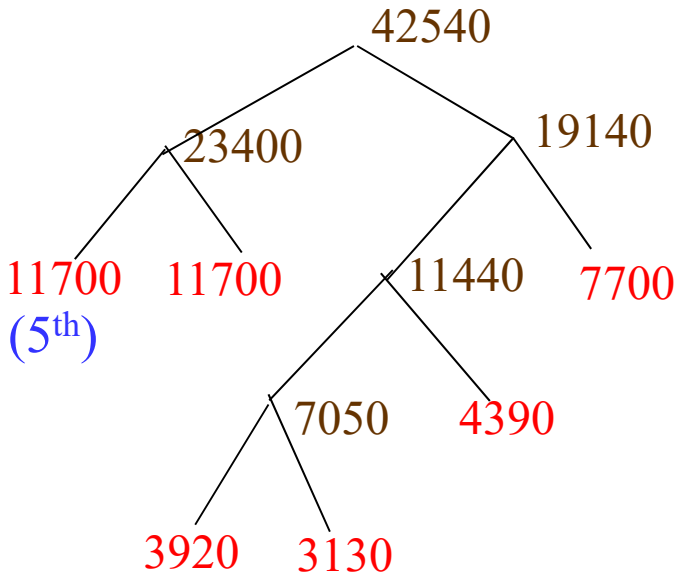
Huffman coding





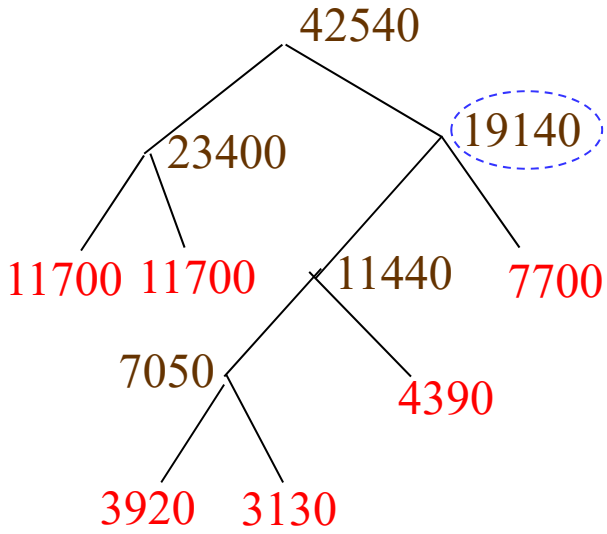


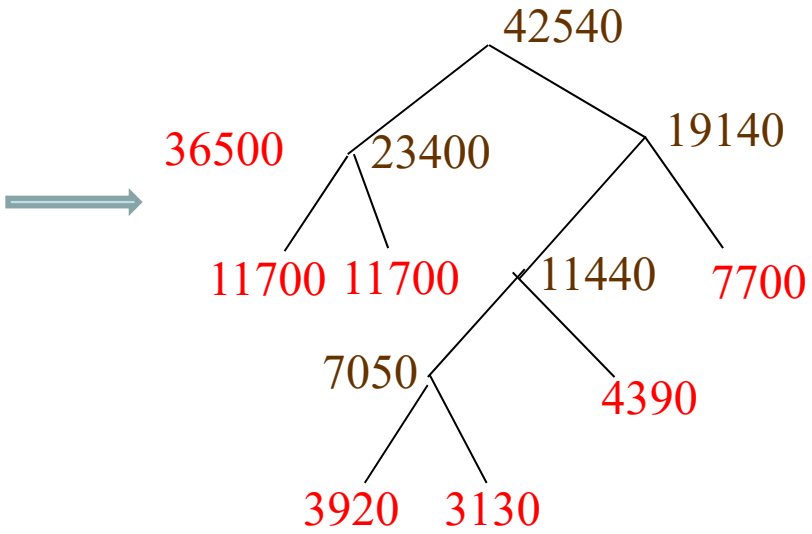
Huffman coding

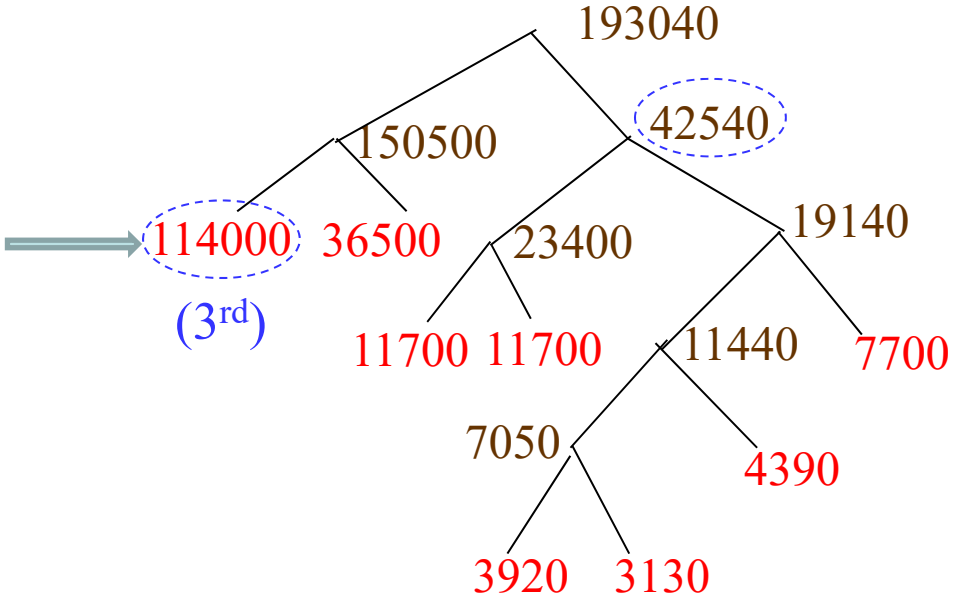


(5th)

(4th)

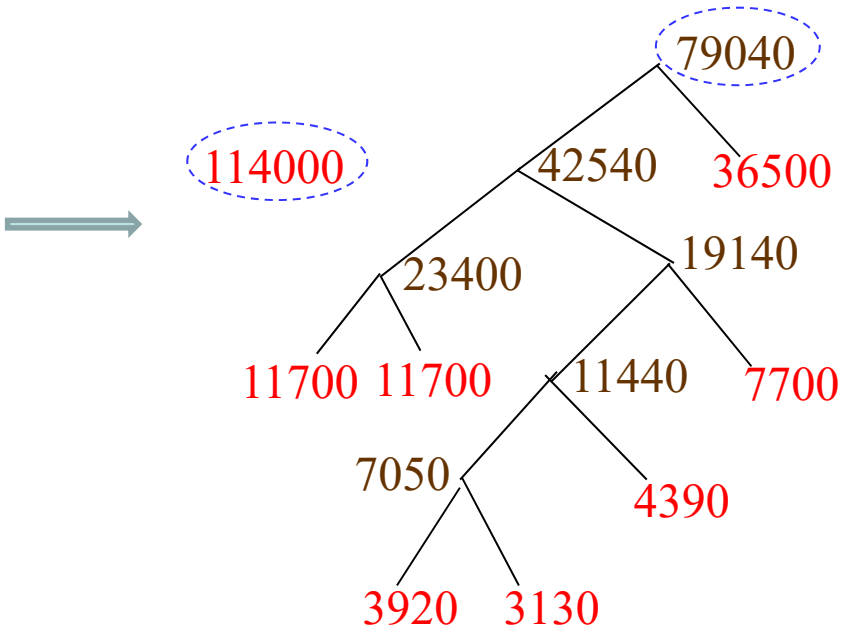




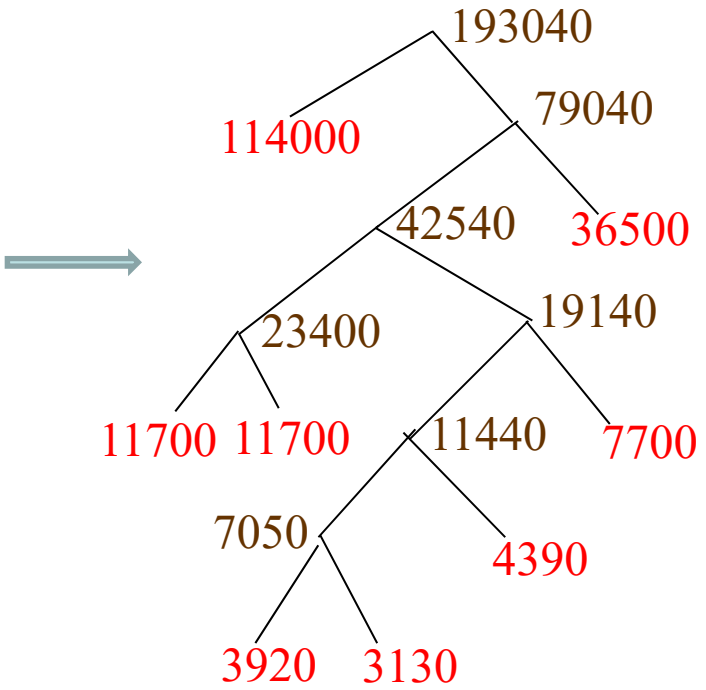


(3rd)

Huffman coding

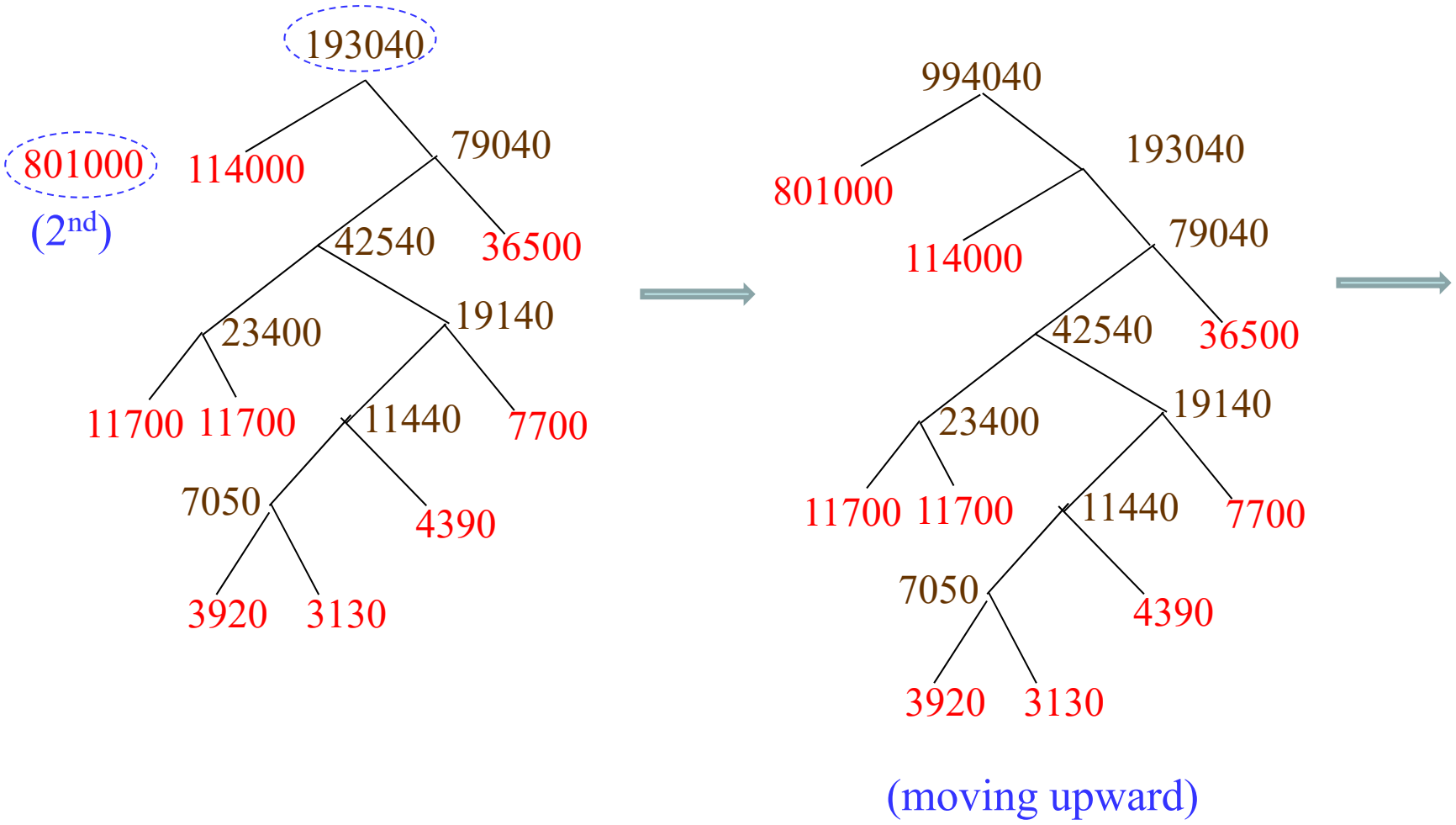


(moving upward and sorting)

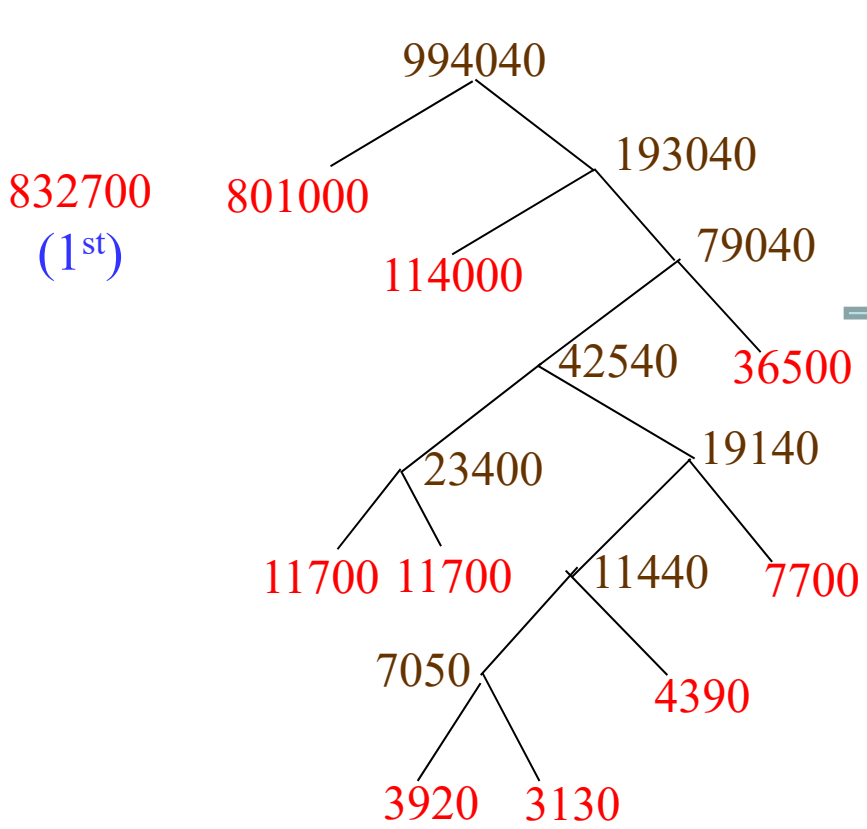


(moving upward again)

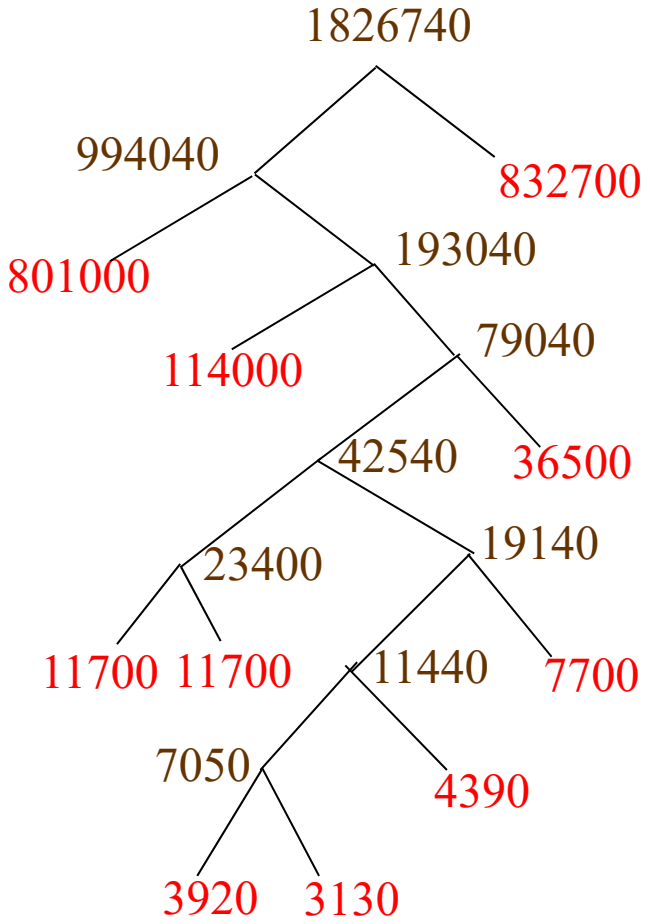
Huffman coding



Huffman coding

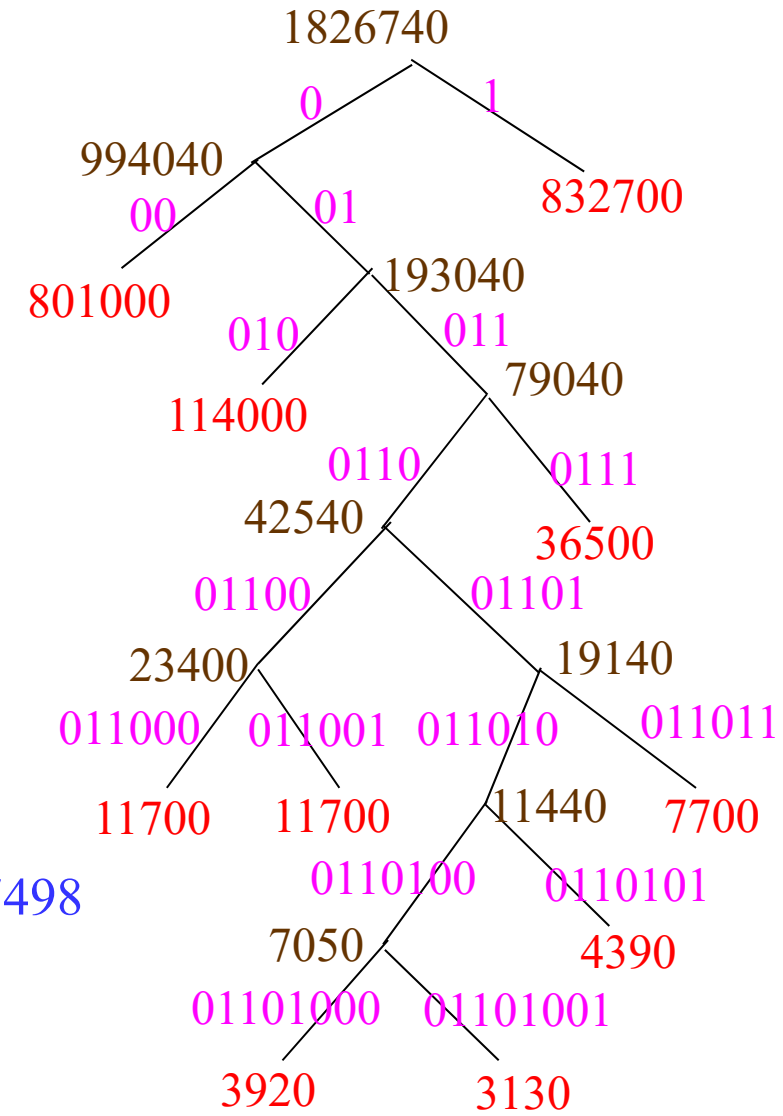


(no paring)



(moving upward
and sorting)

Huffman coding by the greedy algorithm



average code length = 1.7498

entropy/log(2) = 1.5830

思考： 郵遞區號是多少進位的編碼？

電話號碼的區域碼是多少進位的編碼？

中文輸入法是多少進位的編碼？

如何用 Huffman coding 來處理類似問題？

◉ 8-D Entropy and Coding Length

- Entropy 熵；亂度 (Information Theory)

註：此處 log 即 ln
和 $\log_{10}$ 不同

$$entropy = \sum_{j=1}^{J} P(S_j) \ln \frac{1}{P(S_j)}$$

P: probability

$\frac{entropy}{\ln 2} = 1$

① $P(S_0) = 1$, entropy = 0

② $P(S_0) = P(S_1) = 0.5$, entropy = 0.6931

$0.5 \ln \frac{1}{0.5} + 0.5 \ln \frac{1}{0.5} = 1 \times \ln 2 = \ln 2$

③ $P(S_0) = P(S_1) = P(S_2) = P(S_3) = 1/4$, entropy = 1.3863

$(0.25 \ln \frac{1}{0.25}) \times 4 = \ln 4 = 2 \ln 2$

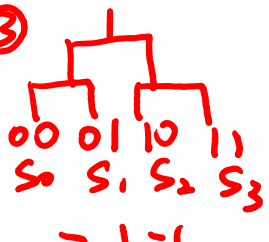
④ $P(S_0) = 0.7$, $P(S_1) = P(S_2) = P(S_3) = 0.1$, entropy = 0.9404

$\frac{entropy}{\ln 2} = 2$

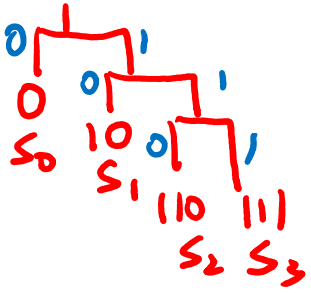
同樣是有 4 種組合，機率分佈越集中，亂度越少

② 

1 bit

③ 

2 bits

④ 

$1 \times 0.7 + 2 \times 0.1 + 3 \times 0.2 = 1.5$ bits

$P(S_0) = 0.7$

$P(S_1) = P(S_2) = P(S_3) = 0.1$

$\frac{entropy}{\ln 2} = \frac{0.9404}{0.6931} = 1.36$

- Huffman Coding 的平均長度

$$mean(L)=\sum_{j=1}^{J}P(S_j)L(S_j)$$ $P(S_j)$: S_j 發生的機率， $L(S_j)$: S_j 的編碼長度

- **Shannon 編碼定理：**

$$\frac{entropy}{\ln k}\le mean(L)\le\frac{entropy}{\ln k}+1$$ 若使用 k 進位的編碼

- Huffman Coding 的 total coding length $b=mean(L)N$ N: data length

$$ceil\left(N\frac{entropy}{\ln k}\right)\le b\le floor\left(N\frac{entropy}{\ln k}+N\right)$$

都和 entropy 有密切關係

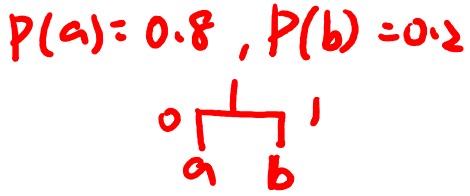
ceil: 無條件進位， floor: 無條件捨去

Entropy: 估計 coding length 的重要工具

$$\frac{entropy}{\ln k} \cong \text{average bit length}$$

$$N\frac{entropy}{\ln k} \cong \text{total bit length}$$

◉ 8-E Arithmetic Coding



- **Arithmetic Coding (算術編碼)**

Huffman coding 是將每一筆資料分開編碼

Arithmetic coding 則是將多筆資料一起編碼，因此壓縮效率比 Huffman coding 更高，近年來的資料壓縮技術大多使用 arithmetic coding

K. Sayood, *Introduction to Data Compression*, Chapter 4: Arithmetic coding, 3rd ed., Amsterdam, Elsevier, 2006

編碼

若 data X 有 M 個可能的值 ($X[i] = 1, 2, \ldots,$ or M)，使用 k 進位的編碼，且

P_n : the probability of $x = n$ (from prediction)

$$S_0 = 0, \quad S_n = \sum_{j=1}^{n} P_j$$

現在要對 data X 做編碼，假設 length(X) = N

Algorithm for arithmetic encoding

initiation: $lower = S_{X[1]-1}$ $upper = S_{X[1]}$

for $i = 2 : N$

$$lower = lower + S_{X[i]-1} \times (upper - lower)$$

$$upper = lower + S_{X[i]} \times (upper - lower)$$

end

(continue)…

interval: (lower, upper)

Suppose that

$$lower \le C \cdot k^{-b} < (C+1) \cdot k^{-b} \le upper$$

where C and b are integers (b is as small as possible), then the data X can be encoded by

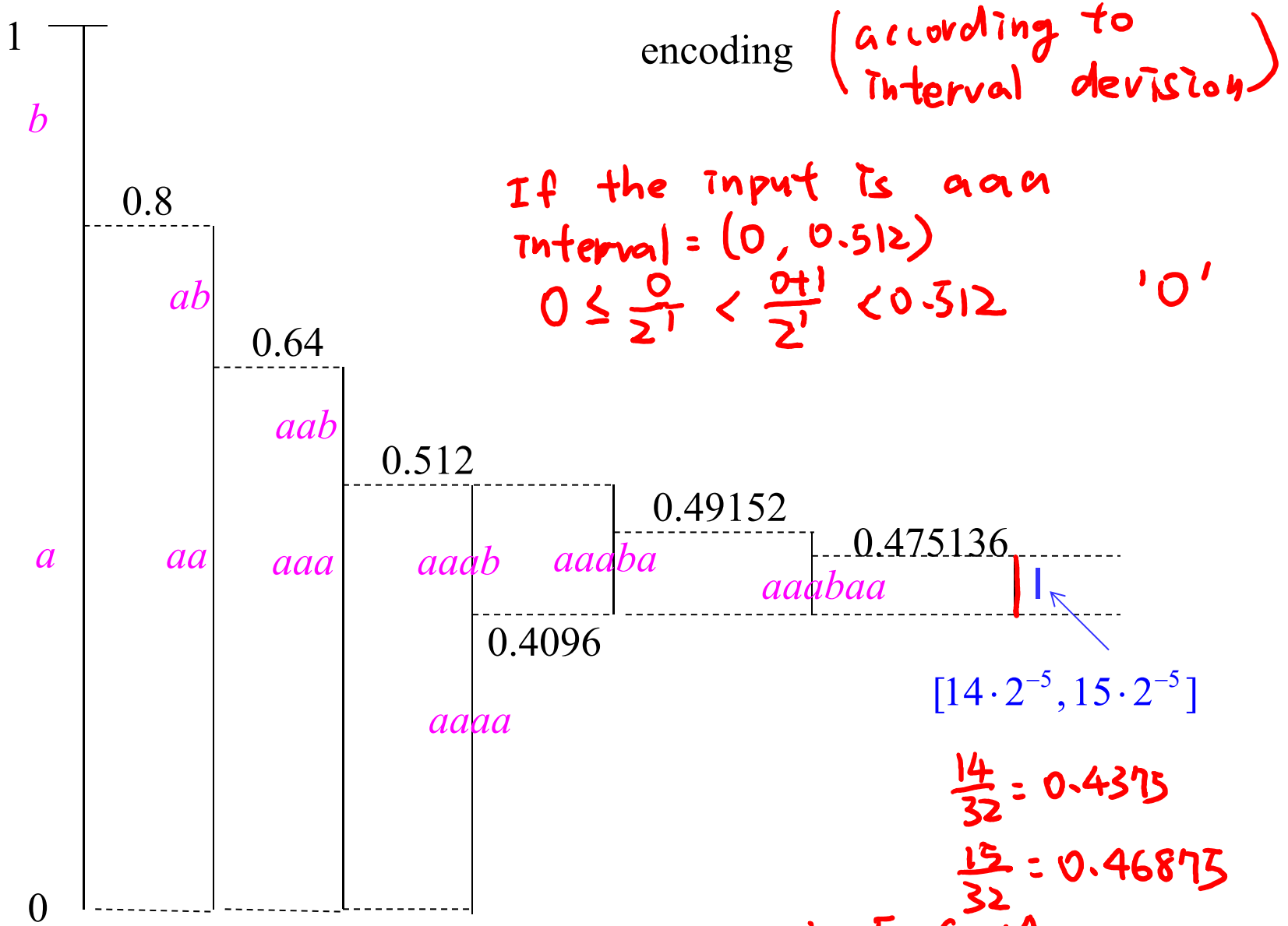
$$C_{(k,b)}$$

where $C_{(k,b)}$ means that using k-ary (k 進位) and b bits to express C.

(註： Arithmetic coding 還有其他不同的方式，以上是使用其中一個較簡單的 range encoding 的方式)

P(a)=0.8, P(b)=0.2 aaabaa

encoding (according to interval devision)



If the input is aaa

interval = (0, 0.512)

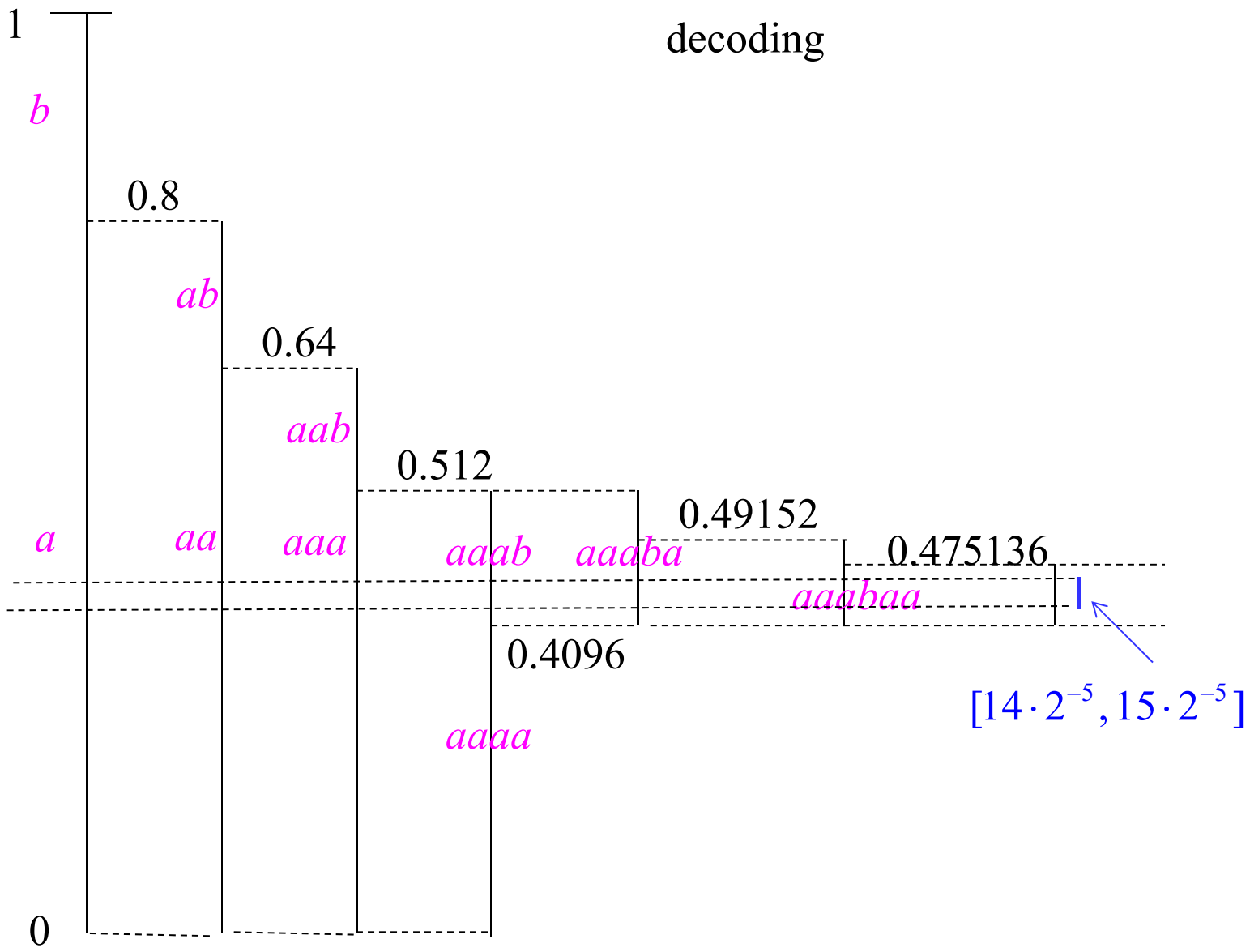
$0 \le \frac{0}{2^1} < \frac{0+1}{2^1} < 0.512$ '0'

$\frac{14}{32} = 0.4375$

$\frac{15}{32} = 0.46875$

b = 5, C = 14

14 → 5 bits → 01110


decoding
1
0
b
0.8
ab
0.64
aab
0.512
aaab
0.4096
aaaa
a
aa
aaa
0.49152
aaaba
0.475136
aaabaa
$[14 \cdot 2^{-5}, 15 \cdot 2^{-5}]$

Example:

假設要對 X 來做二進位 ($k = 2$) 的編碼

且經由事先的估計，$X[i] = a$ 的機率為 0.8, $X[i] = b$ 的機率為 0.2

$\Rightarrow$ $P_1 = 0.8, \quad P_2 = 0.2, \qquad S_0 = 0, \quad S_1 = 0.8, \quad S_2 = 1$

若實際上輸入的資料為 $X = a\ a\ a\ b\ a\ a$

Initiation ($X[1] = a$): $lower = 0, \quad upper = 0.8$

When $i = 2$ ($X[2] = a$): $lower = 0, \quad upper = 0.64$

When $i = 3$ ($X[3] = a$): $lower = 0, \quad upper = 0.512$

When $i = 4$ ($X[4] = b$): $lower = 0.4096, \quad upper = 0.512$

When $i = 5$ ($X[5] = a$): $lower = 0.4096, \quad upper = 0.49152$

When $i = 6$ ($X[6] = a$): $lower = 0.4096, \quad upper = 0.475136$

由於 $lower = 0.4096, \quad upper = 0.475136$

$$lower \le 14 \cdot 2^{-5} < 15 \cdot 2^{-5} \le upper$$

$$0.4375 \qquad 0.46875$$

所以編碼的結果為

$$14_{(2,5)} = 01110$$

2 進位　5 個 bits

解碼

假設編碼的結果為 Y, length(Y) = b

其他的假設，和編碼 (see page 317) 相同　　使用 k 進位的編碼

Algorithm for arithmetic decoding

initiation:　$lower = 0$　　$upper = 1$　　$j = 1$

$lower\ 1 = 0$　　$upper\ 1 = 1$

for i = 1 : N　　% loop 1

check = 1;

while check = 1　　% loop 2

if there exists an n such that

$lower + (upper\text{-}lower)S_{n-1} \leqq$ lower 1　and

$lower + (upper\text{-}lower)S_n \geqq$ upper 1　are both satisfied,

then

check = 0;

(continue)….

else

$upper\,1 = lower\,1 + (Y[j]+1)k^{-j}$

$lower\,1 = lower\,1 + Y[j]k^{-j}$

$j = j+1$

end

end % end of loop 2

$X(i) = n;$

$lower = lower + (upper\text{-}lower)S_{n\text{-}1}$

$upper = lower + (upper\text{-}lower)S_n$

end % end of loop 1

Coding Length for Arithmetic Coding

假設 P_n 是預測的 $X[i]=n$ 的機率

Q_n 是實際上的 $X[i]=n$ 的機率

(也就是說，若 length(X) = N, X 當中會有 $Q_n N$ 個 elements 等於 n)

則

$$upper-lower=\prod_{m=1}^{M} P_m^{Q_m N}$$

$\prod$ ：連乘符號

另一方面，由於 (from page 318)

$$k^{-b} \le upper-lower < (2k)k^{-b}$$

$$-\log_k (upper-lower) \le b < -\log_k (upper-lower)+1+\log_k 2$$

$$ceil\left(-N\sum_{m=1}^{M} Q_m \log_k P_m\right) \le b \le floor\left(-N\sum_{m=1}^{M} Q_m \log_k P_m+\log_k 2\right)+1$$

$$ceil\left(-N\sum_{m=1}^{M}Q_m \log_k P_m\right) \le b \le floor\left(-N\sum_{m=1}^{M}Q_m \log_k P_m + \log_k 2\right)+1$$

在機率的預測完全準確的情形下， $Q_m = P_m$

Total coding length b 的範圍是

$$ceil\left(-N\sum_{m=1}^{M}P_m \log_k P_m\right) \le b \le floor\left(-N\sum_{m=1}^{M}P_m \log_k P_m + \log_k 2\right)+1$$

$$ceil\left(N\cdot\frac{entropy}{\ln k}\right) \le b \le floor\left(N\cdot\frac{entropy}{\ln k} + \log_k 2 + 1\right)$$

for binary
$k=2$
upper bound =
$floor\left(N\frac{entropy}{\ln 2}+2\right)$

(Compared to page 314)

Arithmetic coding 的 total coding length 的上限比 Huffman coding 更低

◉ 8-F MPEG

MPEG： 動態影像編碼的國際標準　　全名： Moving Picture Experts Group

MPEG standard： http://www.iso.org/iso/prods-services/popstds/mpeg.html

MPEG 官方網站： http://mpeg.chiariglione.org/

人類的視覺暫留： 1/24 second

一個動態影像，每秒有 30個或 60個畫格 (frames)

例子：

Pepsi 的廣告

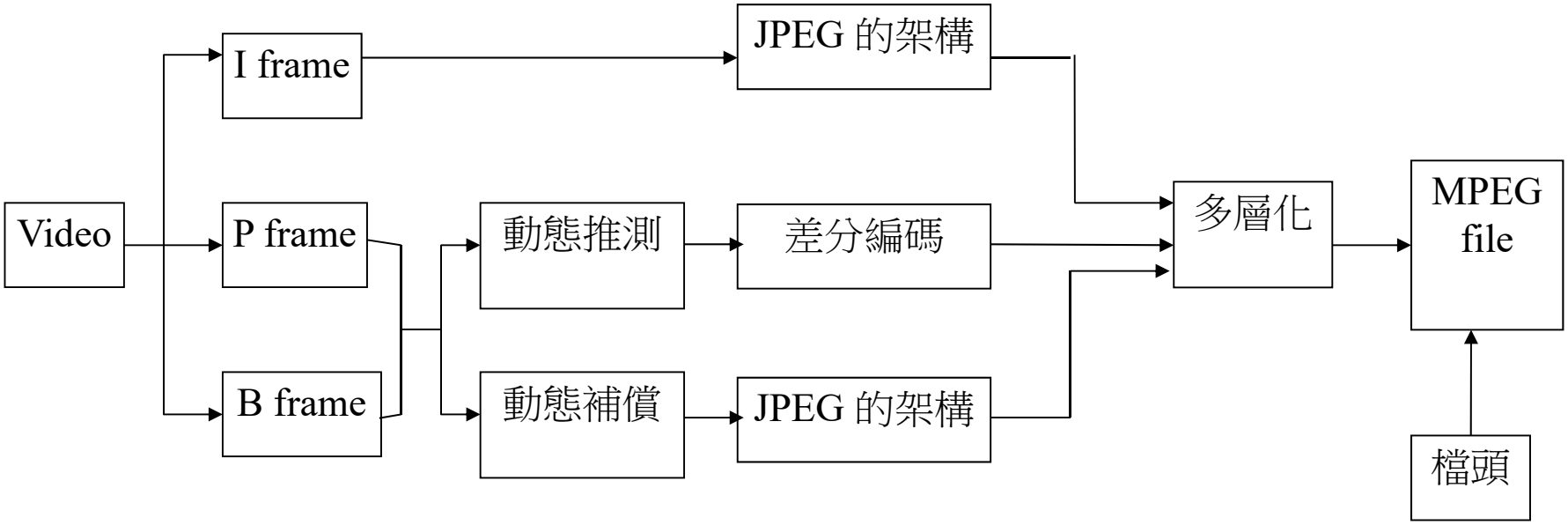
Size: 160 × 120　Time: 29 sec　一秒 30 個 frames

若不作壓縮：　160 × 120 × 29 × 30 × 3 = 50112000 = 47.79 M bytes。

經過 MPEG壓縮：　1140740 = 1.09 M bytes。

只有原來的 2.276%。

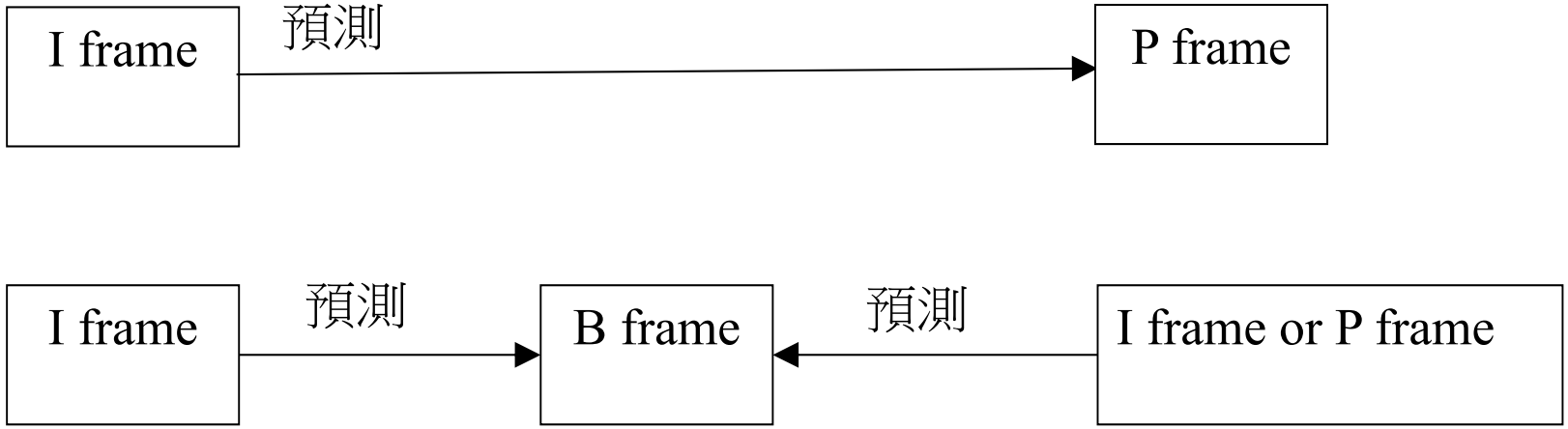
• Flowchart of MPEG Compression



I frame (Intra-coded picture): 作為參考的畫格

P frame (Predictive-coded picture): 由之前的畫格來做預測

B frame (Bi-directionally predictive-coded picture): 由之前及之後的畫格來做預測



I frame: The coding method is the same as that of the still image.

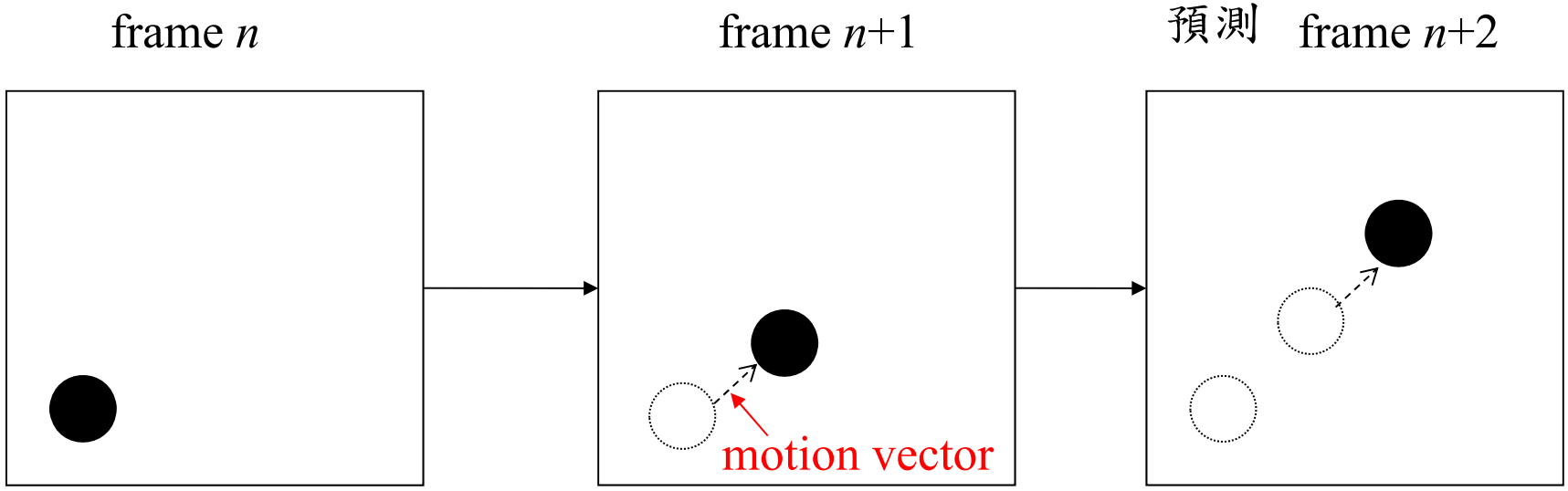
P and B frames: The prediction residues is encoded.

- **動態影像之編碼**

  原理：不同時間，同一個 pixel 之間的相關度通常極高
  只需對有移動的 objects 記錄 “motion vector”

- 動態補償 (Motion Compensation)

  時間相近的影像，彼此間的相關度極高



$F[m, n, t]$: 時間為 t的影像

如何由 $F[m, n, t]$， $F[m, n, t+\Delta]$ 來預測 $F[m, n, t+2\Delta]$?

(1) 移動向量 $V_x(m, n), V_y(m, n)$

(2) 預測 $F[m, n, t+2\Delta]$ ：

$$F_p[m, n, t+2\Delta] = F[m - V_x(m, n), n - V_x(m, n), t+\Delta]$$

(3) 計算 「預測誤差」

$$E[m, n, t+2\Delta] = F[m, n, t+2\Delta] - F_p[m, n, t+2\Delta]$$

對預測誤差 $E[m, n, t+2\Delta]$ 做 編碼

◉ 8-G Data Compression 未來發展的方向

Two important issues:

Q1: How to further improve the compression rate

Q2: How to develop a compression algorithm whose compression rate is acceptable and the buffer size / hardware cost is limited