Lec 5.

Warm up

Which of the following vectors is first to be in the span of previous vectors?

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix}$$

$$\vec{v}_5 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \end{bmatrix}$$

$$x_1 \vec{v}_1 + \cdots + x_n \vec{v}_n = 0$$

what is the smallest n s.t. eq. has a nontrivial sol?

$$\left[\begin{array}{ccccc|c} 2 & -1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 2 & 2 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[\text{ex.}]{\text{REF}} \left[\begin{array}{ccccc|c} \boxed{1} & 0 & 1 & 1 & 2 & 0 \\ 0 & \boxed{1} & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & -4 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right] \longrightarrow x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3 = 0$$

$$(x_3 \neq 0)$$

free variable

$$\rightarrow \vec{v}_3 = \left(-\frac{x_1}{x_3}\right)\vec{v}_1 + \left(-\frac{x_2}{x_3}\right)\vec{v}_2 \in \operatorname{span}\{\vec{v}_1, \vec{v}_2\}.$$

Thm A set of vectors $\{\vec{v}_1, \cdots, \vec{v}_k\}$ is lin. dep.

$\Longleftrightarrow$ at least one of the vectors is a lin. comb. of the rest of the vectors.

Caution not any vector

Ex. $\{\vec{v}_1, \vec{v}_2\}$ lin. indep.

Consider $\{\vec{v}_1, \vec{v}_2, \underbrace{2\vec{v}_2}_{\vec{v}_3}\}$.

lin. dep. $0\cdot\vec{v}_1 + (-2)\vec{v}_2 + \vec{v}_3 = 0$.

If try $\vec{v}_1 = x_1\vec{v}_2 + x_2\vec{v}_3 = (x_1 + 2x_2)\vec{v}_2$

$\Rightarrow \vec{v}_1 + (-x_1 - 2x_2)\vec{v}_2 = 0 \to$ contradiction.

$\underline{\text{Pf}}$: $\Leftarrow$ say $1 \le p \le k$

$$\vec{v}_p = \sum_{\substack{i=1 \\ i \neq p}}^{k} c_i \vec{v}_i$$

$$\Rightarrow c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + (-1)\vec{v}_p + \cdots + c_k\vec{v}_k = \vec{0}$$

$\rightarrow$ nontrivial sol to hom. eq. $\Rightarrow$ lin. dep.

$\Rightarrow$ there exists $\begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \neq \vec{0}$ s.t.

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = 0.$$

Say $c_p \neq 0 \quad (1 \le p \le k)$

$$\Rightarrow \vec{v}_p = \sum_{\substack{i=1 \\ i \neq p}}^{k} \left(-\frac{c_i}{c_p}\right) \vec{v}_i \qquad \square$$

Ex. $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

① $\vec{v}_3$ in $\operatorname{span}\{\vec{v}_1, \vec{v}_2\}$?

② $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$?

① whether exists x_1, x_2 s.t.

$$\vec{v_3} = x_1\vec{v_1} + x_2\vec{v_2} \quad ?$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{REF}} \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{-1} & 1 \\ 0 & 0 & \boxed{3} \end{bmatrix}$$

$x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3} = 0$ only trivial sol.

$\rightarrow$ $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ lin. indep.

answer is NO.

② Whether for any $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3$.

$\left[\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3 \mid \vec{b} \right]$ has a sol?

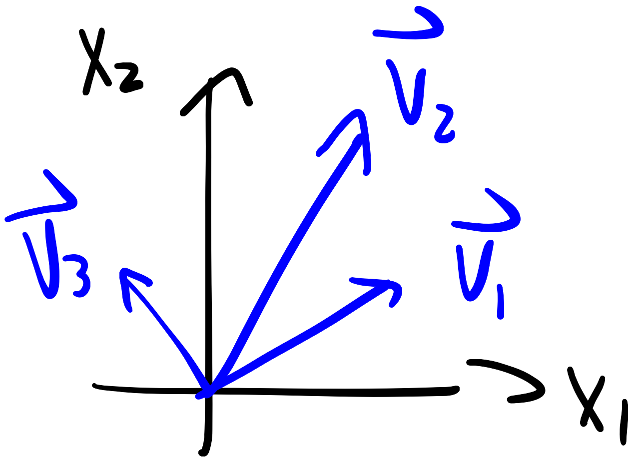
Yes $\longrightarrow$ $\mathbb{R}^3 \subseteq \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$\Rightarrow \mathbb{R}^3 = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

Thm. $\{\vec{v}_1, \cdots, \vec{v}_k\}$, $\vec{v}_i \in \mathbb{R}^n$, $k > n$

is lin. dep.

Geometrically. $(\mathbb{R}^2)$



Pf: Aug. matrix

$$\begin{array}{cc} & k \\ n & \left[\begin{array}{cccc|ccc} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n & \vec{v}_{n+1} & \cdots & v_k \end{array}\right] \end{array}$$

at most n pivoting columns

$\Rightarrow$ at least $k-n$ free var.

$\Rightarrow$ lin. dep.