Lec 13.

Warm up

$V = W = \mathbb{P}_3$, $\quad T : V \to W$

$$f \mapsto \frac{df}{dx}$$

① T is lin. trans.

② Image(T), Null(T)

③ Basis

① Take $p(x) \in \mathbb{P}_3$,

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$T(p)(x) = \frac{dp}{dx}(x) = a_1 + 2a_2 x + 3a_3 x^2 \in \mathbb{P}_2 \subset \mathbb{P}_3$$

T is well defined.

linearity follows from that of $\frac{d}{dx}$

say $\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$

② $\text{Image}(T) = \mathbb{P}_2$

$T(p) = 0 \Rightarrow a_1 = a_2 = a_3 = 0$

$\text{Null}(T) = \{ a_0 \mid a_0 \in \mathbb{R} \} = \mathbb{P}_0$

③ basis for $\text{Image}(T)$: $\{1, x, x^2\}$

" " $\text{Null}(T)$: $\{1\}$.

Ex. $V = W = \mathbb{R}^{2\times 2}$.

$T: V \to W$

$A \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} A$

Repeat prev questions.

① T defined via matrix multiplication resulting in 2×2 matrix in W.

T is well defined.

Linearity follows from that of mat-mult.

② $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$T(A) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$$

$$\text{Image}(T) = \left\{ \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \middle| \ c, d \in \mathbb{R} \right\}.$$

$$\mathrm{Null}(T) = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}.$$

$$\mathrm{Image}(T) = \mathrm{Null}(T) \subset V$$

$$\cap$$

$$W$$

coincidence.

③ Basis $= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}.$

$T: V \to W$ lin. trans.

1) Image(T) is a subspace of W

T is surjective/onto if

Image$(T) = W$.

2) Null(T) is a subspace of V

T is injective/one to one if

$\mathrm{Null}(T) = \{\vec{0}\}$ in V

injective & surjective $\longrightarrow$ bijective.

(when T is bijective, T is called an isomorphism between V and W.)

Ex. Is $1+x-x^3$ in

$\operatorname{span}\{1+x^3, 1-2x, x^2-x^3\}$?

If in span,

$$1+x-x^3 = a_1(1+x^3) + a_2(1-2x)$$
$$+ a_3(x^2-x^3)$$
$$= (a_1+a_2) - 2a_2 x + a_3 x^2 + (a_1-a_3)x^3$$

A linear sys.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

has sol ?

Aug $\rightarrow$ REF.

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 1 \\ 0 & \boxed{-2} & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & -\frac{5}{2} \end{array}\right].$$

No sol $\Rightarrow$ not in span.

Ex. $V = \mathbb{P}_2$ Find all vectors in V of the form

$$f_3(x) = a_0 + a_1 x + a_2 x^2$$

s.t. $\{1, x, f_3(x)\}$ forms a basis of $\mathbb{P}_2$.

Want:

$$\operatorname{span}\{1,\ x,\ a_0+a_1x+a_2x^2\} = \mathbb{P}_2 .$$

$$\{1,\ x,\ a_0+a_1x+a_2x^2\} \text{ lin. indep.}$$

$$[1,\ x,\ f_3] = [1,\ x,\ x^2]\underbrace{\begin{bmatrix} \boxed{1} & 0 & a_0 \\ 0 & \boxed{1} & a_1 \\ 0 & 0 & \boxed{a_2} \end{bmatrix}}_{\mathbb{R}^{3\times 3}}$$

$$[\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3]$$

$a_2 \neq 0 \Rightarrow$ each row/col has a pivot

$\Rightarrow$ basis (exer to fill steps).