# Lec 6.

## Warm up

$A = [\vec{a}_1 \ \ \vec{a}_2]$, $\vec{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

① Columns of A lin. indep.? YES.

② " " " span $\mathbb{R}^2$? NO.

③ " " " span $\mathbb{R}^3$? NO.

| Linear indep. | span |
| --- | --- |
| $\{\vec{v}_1, \dots, \vec{v}_k\}$ lin. indep. in $\mathbb{R}^n$<br>$\iff [\vec{v}_1, \dots, \vec{v}_k \mid \vec{0}]$ has unique, trivial sol $\vec{0}$<br>$\Rightarrow k \le n$ | $\{\vec{v}_1, \dots, \vec{v}_k\}$ spans $\mathbb{R}^n$<br>$\iff$ for any $\vec{b} \in \mathbb{R}^n$ $\vec{b} = x_1 \vec{v}_1 + \dots + x_k \vec{v}_k$<br>$\iff [\vec{v}_1 \cdots \vec{v}_k \mid \vec{b}]$ always has sol<br>$\Rightarrow k \ge n$ |
| deleting vectors helps | adding vectors helps. |

# Linear transformation

$$Ax = b.$$

Focus on properties of $A$.

---

Map / Mapping / Transformation.

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$\vec{x} \in \mathbb{R}^n \mapsto T(\vec{x}) \in \mathbb{R}^m$$

Matrix transformation.

$A$ of size $m \times n$.

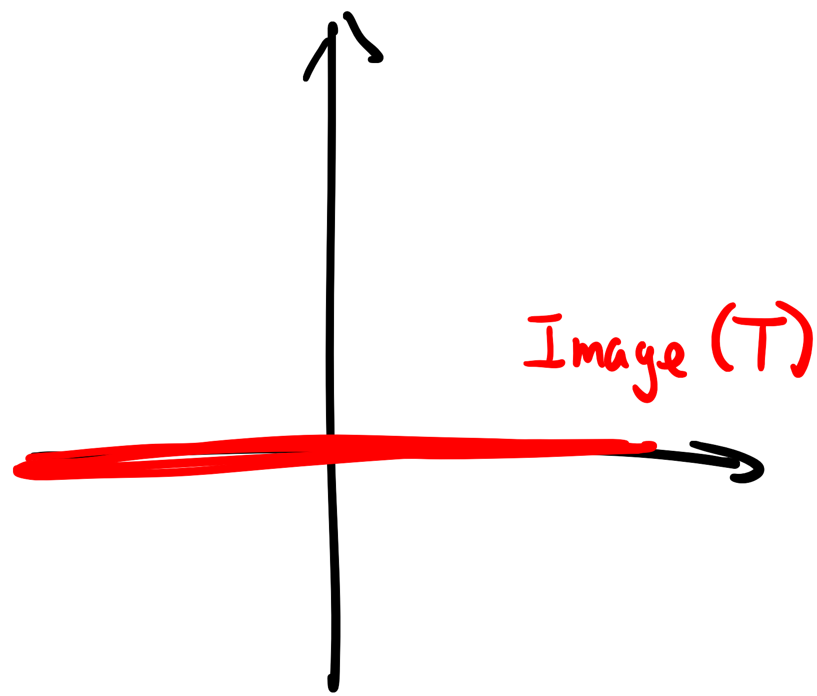
$$T(\vec{x}) := A\vec{x} := x_1 \vec{a_1} + \cdots + x_n \vec{a_n}$$

$$\text{Image}(T) = \{ \vec{b} \in \mathbb{R}^m \mid \text{there exists } \vec{x} \in \mathbb{R}^n \text{ s.t. } \vec{b} = T(\vec{x}) \}.$$

Ex. $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $T : \mathbb{R} \to \mathbb{R}^2$

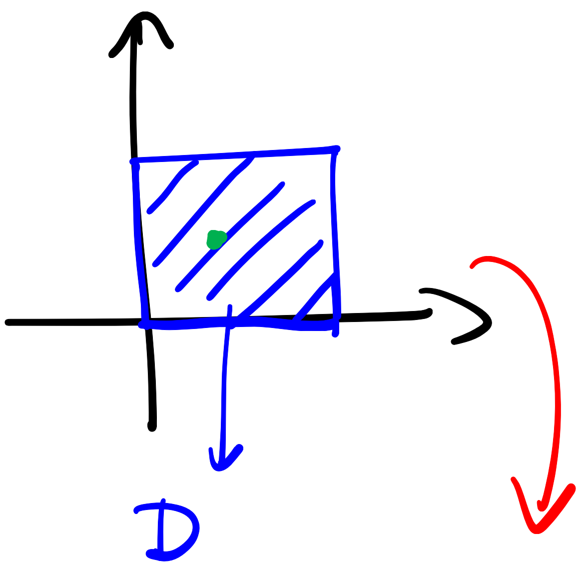
$x \mapsto Ax = \begin{bmatrix} x \\ 0 \end{bmatrix}$

Image $(T)$ ?

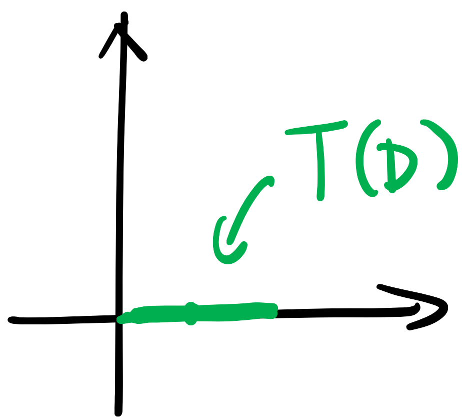


Ex. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$,

$$T(\vec{x}) = A\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$



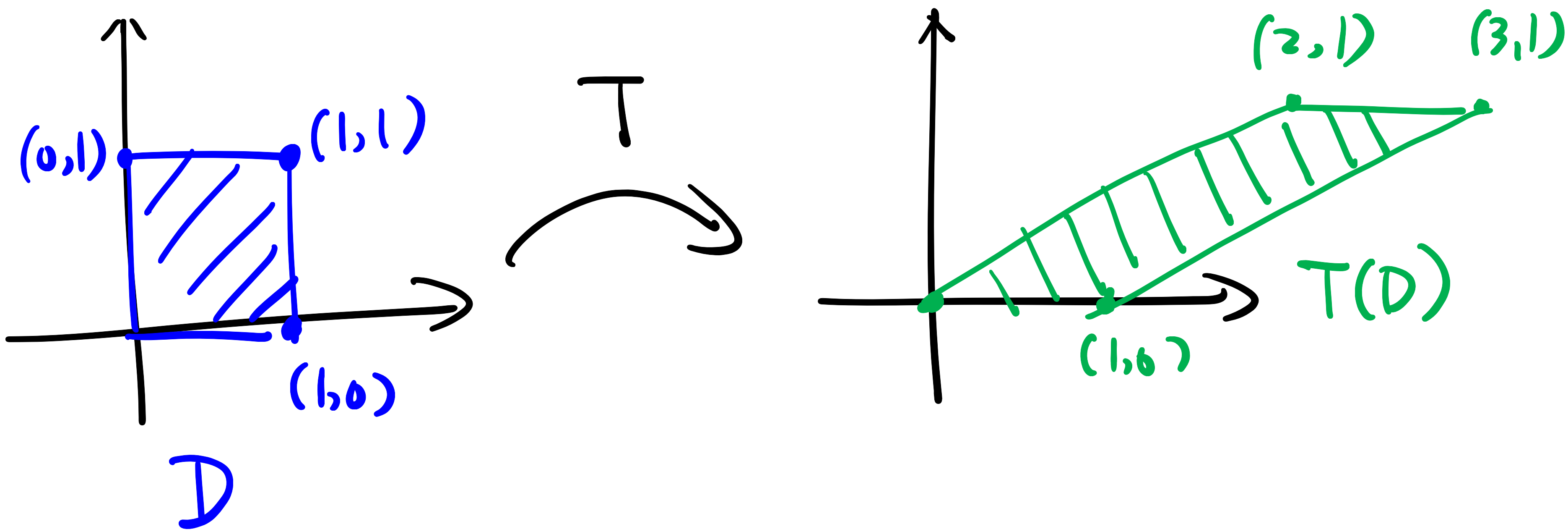

what $T(D) := \{ \vec{b} \mid \text{there exists } \vec{x} \in D \text{ s.t. } T(\vec{x}) = \vec{b} \}$




projection.

Ex. shearing.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \end{bmatrix}$$



## Properties of matrix trans.

$\vec{u}, \vec{v} \in \mathbb{R}^n$ , $A$ $m \times n$ , $c \in \mathbb{R}$

(1) $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$

(2) $A(c\vec{u}) = c(A\vec{u})$.

Linearity. $\forall c_1, c_2 \in \mathbb{R}, \vec{u}, \vec{v} \in \mathbb{R}^n$

$$A(c_1\vec{u} + c_2\vec{v}) = c_1(A\vec{u}) + c_2(A\vec{v})$$