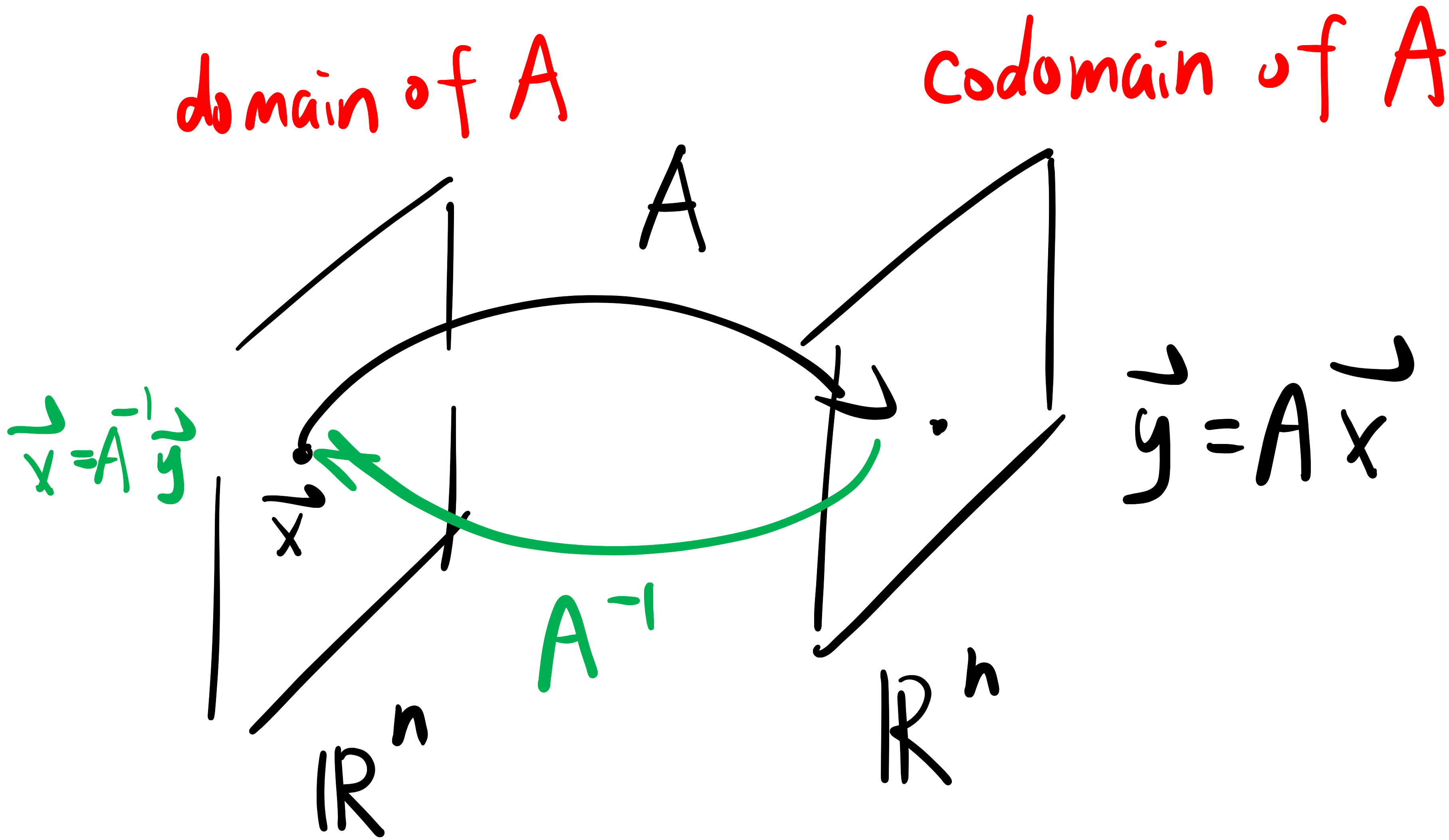
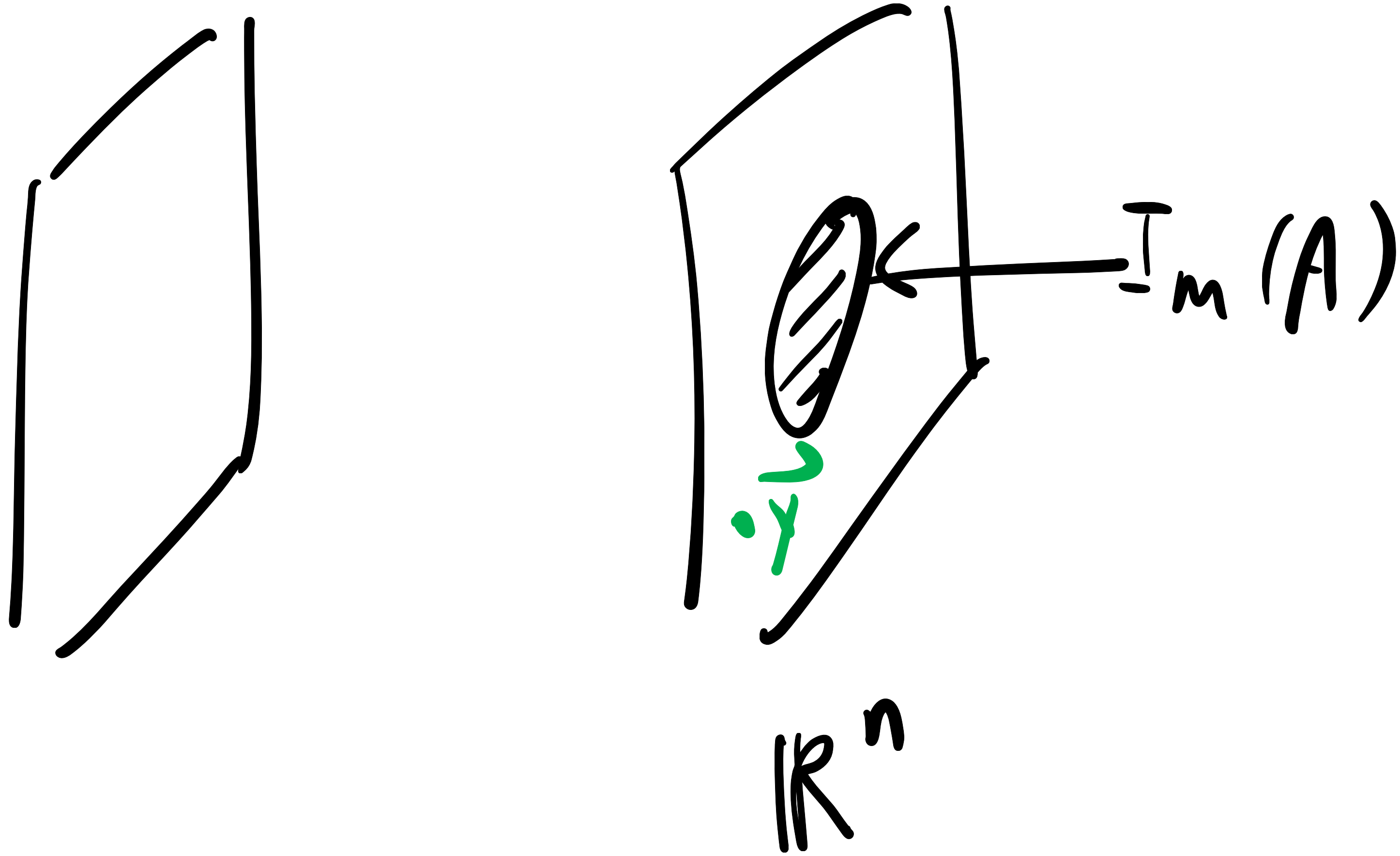
Lec 9.

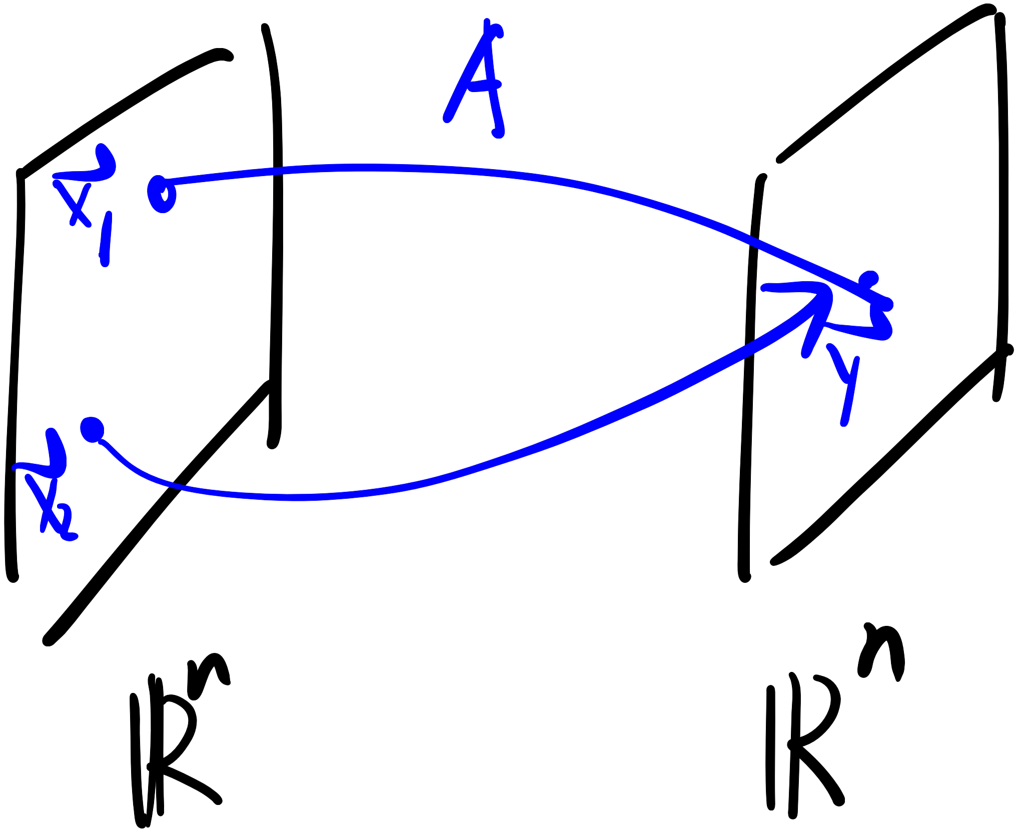
Matrix inverse





To define A^{-1} $\longrightarrow$ A $\begin{cases} \text{onto} \\ \text{one-to-one} \end{cases}$

$\Rightarrow$ A bijective.



Def $A \in \mathbb{R}^{n \times n}$ is invertible

if there is $C \in \mathbb{R}^{n \times n}$ s.t.

$$AC = I_n \quad (\text{and } CA = I_n) \to \text{optional}$$

C is called the inverse of A.

denoted by A^{-1}.

Think $AA^{-1} = I_n$ as composition of mappings.

Ex. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

want: $AC = I_2$, $C = [\vec{c_1} \ \vec{c_2}]$

$$A[\vec{c_1} \ \vec{c_2}] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{cases} Ac_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{e_1} \\ Ac_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{e_2} \end{cases}$$

$$\left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underbrace{}_{A} \qquad \vec{e}_1 \quad \vec{e}_2$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\underbrace{}_{RREF} \qquad \vec{c}_1 \quad \vec{c}_2$$

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\| $$

$$A^{-1}$$

Ex. $A = \begin{bmatrix} 2 & 5 \\ -2 & -7 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ -2 & -7 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 2 & 5 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & \frac{5}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & \frac{7}{4} & \frac{5}{4} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right] \to A^{-1}.$$

Thm. A invertible

$\iff \text{RREF} = I_n$

Two facts: $A \in \mathbb{R}^{n \times n}$ invertible

① For any $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a unique sol.

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$

$$\parallel\!\!|$$

$$(A^{-1}A)\vec{x}$$

$$\parallel$$

$$\vec{x}$$

② $B, C \in \mathbb{R}^{n\times n}$

$$AB = AC \Rightarrow A^{-1}(AB) = A^{-1}(AC)$$

$$\Rightarrow (A^{-1}A)B = (A^{-1}A)C$$

$$\Rightarrow B = C.$$

say scalar $a \neq 0$, $x, y \in \mathbb{R}$

$$ax = ay \Rightarrow x = y.$$

Think: Can you come up w. nonzero A

$$AB = AC \not\Rightarrow B = C.$$

<u>Thm</u>.

(1) $A \in \mathbb{R}^{n \times n}$ invertible

$$(A^{-1})^{-1} = A$$

(2) $A, B \in \mathbb{R}^{n \times n}$ invertible

$$(AB)^{-1} = B^{-1}A^{-1}$$

Pf of (2):

Check

$$(AB)\cdot \underbrace{B^{-1}A^{-1}}_{C} = A(\underbrace{BB^{-1}}_{I_n})A^{-1}$$

$$= A\cdot I_n\cdot A^{-1} = AA^{-1} = I_n$$

$\square$

$A \in \mathbb{R}^{n \times n}$ invertible. A^{-1} is unique.

(1) $A \cdot A^{-1} = I_n$ sol. to lin sys. is unique.

(2) Assume A_1, A_2 are both inverses of A

$$A_1 = A_1 \cdot I_n = A_1 \cdot (AA_2)$$

$$= (A_1 A) A_2 = A_2 \qquad \square$$