Lec 8

Warm up

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\vec{x} \mapsto x_1 \vec{v}_1 + x_2 \vec{v}_2$$

1) Standard matrix of T?

2) Image of $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ under T?

1) $T(\vec{e_1}) = 1 \cdot \vec{v_1} + 0 \cdot \vec{v_2} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$T(\vec{e_2}) = \vec{v_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

2) $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$A \in \mathbb{R}^{m \times n}, \quad A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$$

$\mathbb{R}^{m \times n} = \{$ all matrices of size $m \times n$, with real entries $\}$

Def: $A, B \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}$

(i) $A = B$, $A_{ij} = B_{ij}$ for all $1 \le i \le m$, $1 \le j \le n$

(ii) $A+B=C$, $\quad C_{ij}=A_{ij}+B_{ij}$, $\quad$ "

(iii) $cA=B$, $\quad cA_{ij}=B_{ij}$, $\quad$ "

$$A, B \in \mathbb{R}^{m\times n}$$

$$C = a_1 A + a_2 B \in \mathbb{R}^{m\times n}$$

Matrix multiplication.

proposal: $A, B \in \mathbb{R}^{m \times n}$

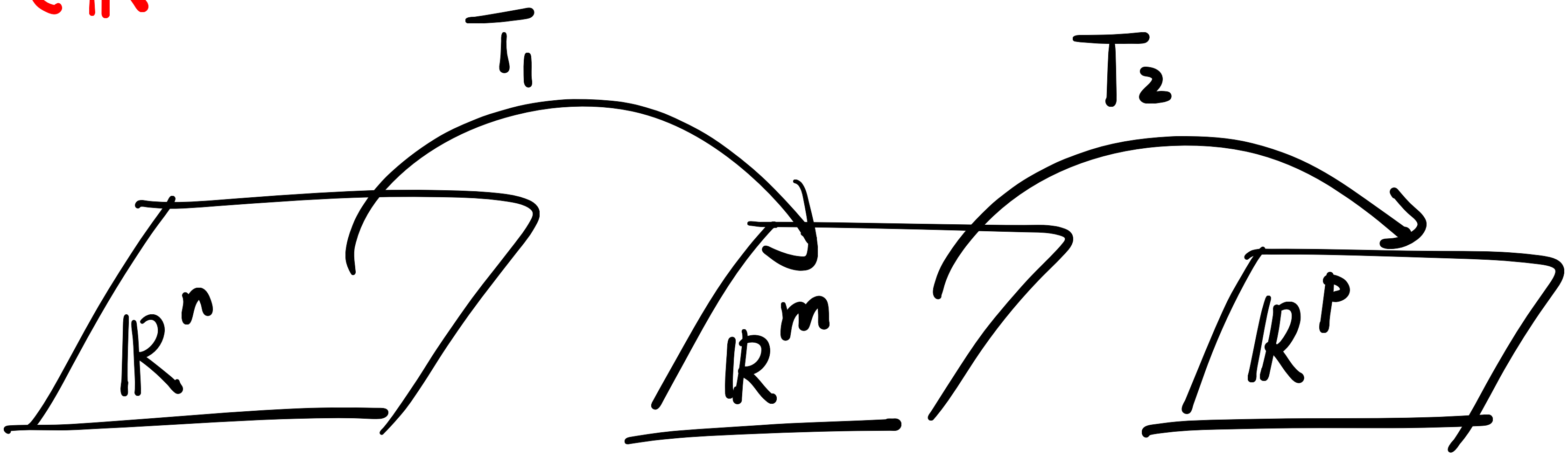
$$(AB)_{ij} = A_{ij} B_{ij}, \quad \begin{array}{l} 1 \leq i \leq m \\ 1 \leq j \leq n. \end{array}$$ ✗

lin trans. $\longleftrightarrow$ matrix

Composition of trans.

$T_1 : \mathbb{R}^n \to \mathbb{R}^m, \quad T_2 : \mathbb{R}^m \to \mathbb{R}^p$

$B \in \mathbb{R}^{m \times n}$ $\qquad$ $A \in \mathbb{R}^{p \times m}$



$\vec{x} \in \mathbb{R}^n \qquad (T_2 \circ T_1)(\vec{x}) := T_2(T_1(\vec{x}))$

$$\underbrace{T_2 \circ T_1}_{\text{standard matrix?}} : \mathbb{R}^n \longrightarrow \mathbb{R}^p$$

Want : $\vec{x} \in \mathbb{R}^n$

$$\underbrace{(AB)}_{\substack{\cap \\ \mathbb{R}^{p\times n}}}\vec{x} := \underbrace{A(\underbrace{B\vec{x}}_{\mathbb{R}^m})}_{\mathbb{R}^p}$$

$$C = AB = [\vec{c}_1 \cdots \vec{c}_n], \quad \vec{c}_i \in \mathbb{R}^p$$

$$\vec{c}_i = (AB)\vec{e}_i, \quad \vec{e}_i = \begin{array}{cc} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{c} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \end{array} \end{array}$$

$$:= A(B\vec{e}_i)$$

$$B = [\vec{b}_1 \cdots \vec{b}_n], \quad \vec{b}_i \in \mathbb{R}^m$$

$$B\vec{e}_j = \vec{b}_j$$

$$\boxed{\vec{c}_j = A\vec{b}_j}$$

$$A = [\vec{a}_1 \cdots \vec{a}_m], \quad \vec{a}_i \in \mathbb{R}^p$$

$$\vec{c}_j = \vec{a}_1 B_{1j} + \cdots + \vec{a}_m B_{mj}$$

Write out the entries

$$(AB)_{ij} = [\vec{C_j}]_i$$

$$= A_{i1} B_{1j} + \cdots + A_{im} B_{mj}$$

$$\boxed{:= \sum_{k=1}^{m} A_{ik} B_{kj}}$$

Def: $0 = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{m \times n}$

zero matrix

identity matrix

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdot 0 \\ 0 & & \ddots & \\ 0 & \cdot & 0 & 1 \end{bmatrix}$$

<u>Def</u> Matrix transpose

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

$$A^T = \begin{bmatrix} A_{11} & \cdots & A_{m1} \\ \vdots & & \vdots \\ A_{1n} & \cdots & A_{mn} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

More concisely

$$A_{ij} = (A^T)_{ji} \qquad \begin{array}{l} 1 \le i \le m \\ 1 \le j \le n \end{array}$$

<u>Def</u> Matrix power

$$A^k := \underbrace{AA \cdots A}_{k}$$

$A \in \mathbb{R}^{n \times n}$

square matrix

Ex. $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $A^3 = ?$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{1} & 0 \\ 2 & 1 \end{bmatrix}$$

$$A^3 = A \cdot A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \overset{?}{=} A^2 \cdot A$$

Is it true $AB = BA$?

<u>Thm</u>. 1) $(A^T)^T = A$

2) $(A+B)^T = A^T + B^T$

3) $c \in \mathbb{R}$, $(cA)^T = c A^T$

4) $(AB)^T = B^T A^T$

Proof as exercise.