Lec 3.

Warm-up

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 0 = 1 \end{cases}$$

Write in matrix form.

Is it REF? RREF?

Is it consistent?

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

REF ✓

RREF ✓

inconsistent.

Write lin. sys. in terms of column vectors.

<u>Def</u> An n-vector (a.k.a. vector of size n) is an ordered list of n numbers.

written as

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Notation.

$\mathbb{R}^n$: set of all n-vectors w.

real components, i.e. $x_i \in \mathbb{R}$, $i=1,\cdots,n$

$\mathbb{C}^n$: " " " " "

complex " " $x_i \in \mathbb{C}$, $i=1,\cdots,n$

$\mathbb{R} \subset \mathbb{C}$, $\mathbb{R}^n \subset \mathbb{C}^n$

Convention

$$\vec{x} = \mathbf{x} \text{ ``}\equiv\text{''} x$$

What can we do w. vectors?

1) Add 2 vectors. component-wise

Ex. $\begin{bmatrix} 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

2) Scale a vector by a number. component-wise

Ex. $\frac{1}{2}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$

In general. $\vec{x}, \vec{y} \in \mathbb{R}^n$, $a, b \in \mathbb{R}$

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$a\vec{x} + b\vec{y} = \begin{bmatrix} ax_1 + by_1 \\ \vdots \\ ax_n + by_n \end{bmatrix}$$

Special vectors

1) zero vector $\qquad \vec{0} \equiv \mathbf{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ "=" 0

2) $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$-\vec{x} \equiv (-1)\vec{x} = \begin{bmatrix} -x_1 \\ \vdots \\ -x_n \end{bmatrix}$

Caution:

1) Cannot add vectors of different sizes

2) Cannot multiply 2 vectors.

<u>Vector eq</u>:

Ex. $\begin{cases} 2x_1 + 3x_2 + 5x_3 = 1 \\ x_1 \qquad\qquad - x_3 = 0 \end{cases}$

Augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 1 \\ 1 & 0 & -1 & 0 \end{array}\right].$$

$\vec{a}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\vec{a}_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ $\vec{a}_3 = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 = \vec{b}$$

vector eq.

linear combination of $\vec{a}_1, \vec{a}_2, \vec{a}_3$.

Q: can $\vec{b}$ be written as the linear combination of $\vec{a_1}, \vec{a_2}, \vec{a_3}$?

Homogeneous lin. sys.

$$A = [\vec{a_1} \cdots \vec{a_n}]$$

$$x_1 \vec{a_1} + \cdots + x_n \vec{a_n} = \vec{0}$$

1. $\vec{x} = \vec{0}$ is always a sol. $\rightarrow$ trivial sol.

any non-zero sol: non-trivial sol.

2. If $\vec{x}$ is a non-trivial sol,

then $c\vec{x}$ is also a sol. $c \in \mathbb{R}$.

Ex. $A = \begin{bmatrix} 2 & -5 & 8 \\ -2 & -4 & 1 \\ 4 & -1 & 7 \end{bmatrix}$

Aug. matrix

$$\left[\begin{array}{ccc|c} 2 & -5 & 8 & 0 \\ -2 & -4 & 1 & 0 \\ 4 & -1 & 7 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & -5 & 8 & 0 \\ 0 & -9 & 9 & 0 \\ 0 & 9 & -9 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & -5 & 8 & 0 \\ 0 & -9 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$-x_2 + x_3 = 0 \quad \Rightarrow \quad x_2 = x_3$$

$$x_1 = -\frac{3}{2} x_3$$

Sol: $\begin{cases} x_1 = -\frac{3}{2} x_3 \\ x_2 = x_3 \\ x_3 \text{ is a free variable.} \end{cases}$

Sol. set $\left\{ \left(-\frac{3}{2}x_3, x_3, x_3\right) \middle| \, x_3 \in \mathbb{R} \right\}$

$$= \left\{ x_3 \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 1 \end{bmatrix} \middle| \, x_3 \in \mathbb{R} \right\}$$

parametric form of sol. to lin. sys.

Hom. lin. sys. has a nontrivial sol.

$\Leftrightarrow$ " " " " inf " "

$\Leftrightarrow$ lin. sys. has at least one free variable.