Lec 15. Warm up

Ex. Is T isomorphism?

$$T: \mathbb{P}_2 \to \mathbb{R}^3$$

$$p(x) \mapsto \begin{bmatrix} p'(0) \\ p'(1) \\ p'(2) \end{bmatrix}$$

$p(x) = a_0 + a_1 x + a_2 x^2$, $p'(x) = a_1 + 2a_2 x$

$$T(p) = \begin{bmatrix} a_1 \\ a_1 + 2a_2 \\ a_1 + 4a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

no pivot $\Rightarrow$ a_0 free. No.

$T: S \to S$

$$(a_1, a_2, a_3, \cdots) \mapsto (0, a_1, a_2, a_3, \cdots)$$

one-to-one, but not onto.

because $(1, 0, 0, 0 \cdots) \notin \text{Image}(T)$

not isomorphism

$T: S \to S$

$$(a_1, a_2, a_3, \cdots) \mapsto (a_2, a_3, a_4, \cdots)$$

onto. but not one-to-one. NOT isomorphism.

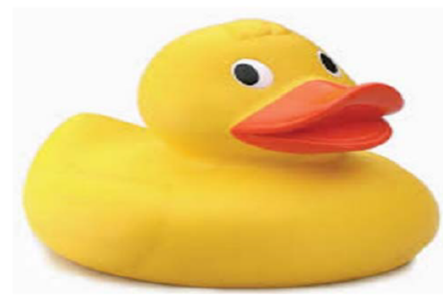
Inf. dim vector space.

Change of coordinate

$$\begin{tikzcd} & V \arrow[dl, shift left, "P_B^{-1}"] \arrow[dr, shift left, "P_C^{-1}"] & \\ \mathbb{R} \arrow[ur, shift left, "P_B"] & & \mathbb{R} \arrow[ul, shift left, "P_C"] \end{tikzcd}$$

 $= \vec{v}$

$\mathcal{B} = \{$  $\}$

$C = \{$  $\}$

$[\vec{v}]_{\mathcal{B}} = 2$

$[\vec{v}]_C = -\frac{1}{2}$

$$[\vec{v}]_C = -\frac{1}{4}[\vec{v}]_B$$

Ex. $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ w.r.t. $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Find $[\vec{v}]_{\mathcal{C}}$ $\quad$ $\mathcal{C} = \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$.

$$\vec{v} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\|$$

$$\vec{v} = \left([\vec{v}]_{\mathcal{C}}\right)_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \left([\vec{v}]_{\mathcal{C}}\right)_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Solve $\left[\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & 2 & 0 \end{array}\right] \to [\vec{v}]_c = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$

Combine <u>2 steps</u> into

<u>1 step</u>

$$
\begin{tikzcd}
 & V \arrow[dr, "P_C^{-1}"] & \\
\mathbb{R}^n \arrow[ur, "P_B"] \arrow[rr, "P_C^{-1} P_B =: P_{C \leftarrow B}"] & & \mathbb{R}^n
\end{tikzcd}
$$

$$[\vec{v}]_B \qquad\qquad\qquad\qquad [\vec{v}]_C$$

$$[\vec{v}]_C = \underset{C \leftarrow B}{P} \; [\vec{v}]_B .$$

$$\rotatebox[origin=c]{90}{$\in$}$$

$$\mathbb{R}^{n \times n}$$

$$\underset{C \leftarrow B}{P} \; [\vec{v}]_B = P_C^{-1}\left(P_B [\vec{v}]_B \right)$$

Compute $\underset{C \leftarrow B}{P}$ as a standard matrix $\mathbb{R}^n \longrightarrow \mathbb{R}^n$

Recall $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

standard matrix

$$A = \left[T(\vec{e}_1) \cdots T(\vec{e}_n) \right]$$

$\vec{e_i} \in \mathbb{R}^n$. $T(\vec{e_i}) \in \mathbb{R}^m$

<u>Therefore</u>

$B = \{\vec{b}_1, \dots, \vec{b}_n\}$

$C = \{\vec{c}_1, \dots, \vec{c}_n\}$

$$\underset{C \leftarrow B}{P} = \left[P_C^{-1} P_B(\vec{e}_1) \cdots P_C^{-1} P_B(\vec{e}_n) \right]$$

$$P_B(\vec{e}_1) = \vec{b}_1 \Rightarrow P_C^{-1} P_B(\vec{e}_1) = [\vec{b}_1]_C$$

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} [\vec{b}_1]_C & \cdots & [\vec{b}_n]_C \end{bmatrix}$$

similarly

$$\underset{B \leftarrow C}{P} = \begin{bmatrix} [\vec{c}_1]_B & \cdots & [\vec{c}_n]_B \end{bmatrix}$$

Thm. $\underset{C\leftarrow B}{P} \; \underset{B\leftarrow C}{P} = I_n$

$$\iff \underset{B\leftarrow C}{P} = \left(\underset{C\leftarrow B}{P}\right)^{-1}$$

Thm. V has basis A, B, C

$$\underset{C\leftarrow A}{P} \; \underset{A\leftarrow B}{P} = \underset{C\leftarrow B}{P}$$

$$
\begin{tikzcd}
& V \arrow[d, shift right] \arrow[dr] & \vec{v} \\
\mathbb{R}^n \arrow[ur] \arrow[r, "P_{A\leftarrow B}"'] & \mathbb{R}^n \arrow[u, shift right] \arrow[r, "P_{C\leftarrow A}"'] & \mathbb{R}^n \\
\mathcal{B} \arrow[rr, bend right, "P_{C\leftarrow B}"'] & A & C
\end{tikzcd}
$$