HW 8

5.37 Equation 5.7.2: $\psi_j - \sum_{k=1}^{j} \phi_k \psi_{j-k} = \theta_j$ for $j = 0, 1, 2 \ldots$

In our examples $p = 1$, $q = 2$.

$\Rightarrow \psi_0 = \theta_0 = 1$

$\psi_1 = \phi_1 \psi_0 + \theta_1 = \frac{1}{2} \times 1 + \frac{5}{6} = \frac{4}{3}$

$\psi_2 = \phi_1 \psi_1 + \phi_2 \psi_0 + \theta_2 = \frac{1}{2} \times \frac{4}{3} + 0 + \frac{1}{6} = \frac{5}{6}$.

$\psi_3 = \phi_1 \psi_2 + \phi_2 \psi_1 + \phi_3 \psi_0 + \theta_3 = \frac{1}{2} \times \frac{5}{6} + 0 + 0 + 0 = \frac{5}{12}$.

$\psi_j = \phi_1 \psi_{j-1}$, for $j \geq 4$

5.38 Equation 5.7.1: $\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}$

In our example, $\psi_j = \phi_1^{j-2} \psi_2 = \phi_1^{j-2}(\theta_2 + \theta_1 \phi_1 + \phi_1^2)$ for $j \geq 2$.

$\sum_{j=2}^{\infty} \psi_j \psi_{j+h} = \sum_{j=2}^{\infty} \phi_1^{j-2}(\theta_2 + \theta_1 \phi_1 + \phi_1^2) \cdot \phi_1^{j-2+h}(\theta_2 + \theta_1 \phi_1 + \phi_1^2)$, where $h \geq 0$.

$= \sum_{j=2}^{\infty} \phi_1^{2(j-2)} \phi_1^h (\theta_2 + \theta_1 \phi_1 + \phi_1^2)^2 \quad (*)$

$\because 0 < \phi_1 < 1$

$\therefore (*) = \lim_{j \to \infty} \phi_1^{2(j-2)} \phi_1^h (\theta_2 + \theta_1 \phi_1) + \phi_1^2)^2 = \frac{\phi_1^h}{1-\phi^2}(\theta_2 + \theta_1 \phi_1 + \phi_1^2)^2$

$\because \psi_0 = 1$, $\psi_1 = \theta_1 + \phi_1$

$\therefore \sum_{j=0}^{\infty} \psi_j \psi_{j+h} = \psi_h + \psi_1 \psi_{1+h} + \frac{\phi_1^h}{1-\phi^2}(\theta_2 + \theta_1 \phi_1 + \phi_1^2)^2$

$= \psi_h + (\phi_1 + \theta_1)\psi_{h+1} + \frac{\phi_1^h}{1-\phi^2}(\theta_2 + \theta_1 \phi_1 + \phi_1^2)^2$

$\therefore \gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} = \sigma^2(\psi_h + (\phi_1 + \theta_1)\psi_{h+1} + \frac{\phi_1^h}{1-\phi^2}(\phi_1^2 + \phi_1 \theta_1 + \theta_2)^2)$

Using (5.37) to deduce general formula for $\gamma(h)$:

$\gamma(0) = \sigma^2(1 + \frac{4}{3}(\phi_1 + \theta_1) + \frac{1}{1-\phi^2}(\phi_1^2 + \phi_1 \theta_1 + \theta_2)^2)$

$\gamma(1) = \sigma^2(\theta_1 + \phi_1 + (\phi_1^2 + \phi_1 \theta_1 + \theta_2)(\phi_1 + \theta_1) + \frac{\phi_1}{1-\phi^2}(\phi_1^2 + \phi_1 \theta_1 + \theta_2)^2)$

$h \geq 2, \gamma(h) = \sigma^2(\phi_1^{h-2}(\phi_1^2 + \phi_1 \theta_1 + \theta_2) + (\phi_1 + \theta_1)\phi_1^{h-1}(\phi_1^2 + \phi_1 \theta_1 + \theta_2) + \frac{\phi_1^h}{1-\phi^2}(\phi_1^2 + \phi_1 \theta_1 + \theta_2)^2)$

$= \sigma^2(\phi_1^{h-2}(\theta_2 + \theta_1 \phi_1 + \phi_1^2)[1 + \phi_1^{h+1} + \phi_1^h \theta_1 + \frac{\phi_1^2}{1-\phi^2}(\phi_1^2 + \phi_1 \theta_1 + \theta_2))$

5.40 equation (5.8.1): $\gamma(k) - \phi_1 \gamma(k-1) - \cdots - \phi_p \gamma(k-p) = \sigma^2 \sum_{j=0}^{q-k} \theta_{j+k} \psi_j$

$k=0$: $\gamma(0) = \sigma^2 \sum_{j=0}^{2} \theta_j \psi_j = \sigma^2 (\theta_0 \psi_0 + \theta_1 \psi_1 + \theta_2 \psi_2)$

$= \sigma^2 (1 + \frac{5}{6} \times \frac{4}{3} + \frac{1}{6} \times \frac{5}{6})$

$= \sigma^2 (1 + \frac{10}{9} + \frac{5}{36}) = \frac{81}{36} \sigma^2$

$k=1$: $\gamma(1) - \phi_1 \gamma(0) = \sigma^2 \sum_{j=0}^{1} \theta_{j+1} \psi_j$

$\gamma(1) = \frac{1}{2} \times \frac{81}{36} \sigma^2 + \sigma^2 (\theta_1 \psi_0 + \theta_2 \psi_1)$

$= \sigma^2 (\frac{81}{72} + \frac{5}{6} + \frac{1}{6} \times \frac{4}{3})$

$= \sigma^2 (\frac{81}{72} + \frac{60}{72} + \frac{2}{9})$

$= \frac{157}{72} \sigma^2$

$k=2$: $\gamma(2) - \phi_1 \gamma(1) = \sigma^2 \sum_{j=0}^{0} \theta_{j+2} \psi_j$

$\gamma(2) = \frac{1}{2} \times \frac{157}{72} \sigma^2 + \sigma^2 \theta_2 \psi_2$

$\gamma(2) = \frac{157}{144} \sigma^2 + \sigma^2 \frac{1}{6} \times \frac{5}{6}$

$= \frac{177}{144} \sigma^2$

$k>2$: $\gamma(k) - \phi_1 \gamma(k-1) = \sigma^2 \sum_{j=0}^{2-k} \theta_{j+k} \psi_j$

$\gamma(k) = \frac{1}{2} \gamma(k-1) + 0 = \frac{1}{2} \gamma(k-1)$.

This implies that for $k \leq q$, $\gamma(k)$ relies on initial conditions, and for $k > q$, $\gamma(k)$ recursively relies on previous γ's.

General formula for $\gamma(k)$ in this example:

$\gamma(k) = (\frac{1}{2})^{k-2} \gamma(2) = (\frac{1}{2})^{k-2} \frac{177}{144} \sigma^2$.

6.1 $f(\lambda)=\sum_{k=-\infty}^{\infty} r(k)e^{-i\lambda k}$

$= r(0)e^{-i\cdot 0\cdot\lambda} + r(1)e^{-i\cdot 1\cdot\lambda} + r(-1)e^{-i(-1)\lambda}$

$= r(0) + r(1)(e^{-i\lambda}+e^{i\lambda})$

$= r(0) + r(1)[\cos(-\lambda)+i\sin(-\lambda)+\cos\lambda+i\sin\lambda]$

$= r(0) + r(1)(2\cos\lambda)$

$\Rightarrow \frac{f(\lambda)}{r(0)} = 1+\rho(1)2\cos\lambda$. Hence, $f(\lambda)$ is proportional to $1+2\rho(1)\cos\lambda$

Note that $r(0)>0$ and $|\rho(1)|\le\frac{1}{2}$.

$\Rightarrow -1\le 2\rho(1)\le 1 \iff -1\le 2\rho(1)\cos\lambda\le 1 \Rightarrow \frac{f(\lambda)}{r(0)} = 1+2\rho(1)\cos\lambda \ge 0$.

Therefore, the function is nonnegative.

Suppose $1+2\rho(1)\cos\lambda=0$, then $\rho(1)=\pm\frac{1}{2}$.

$\Rightarrow 1\pm\cos\lambda=0$.

$\cos\lambda=\pm 1$

$\lambda = 0$ or π.

6.6. For an MA(q) process, $r(h)=0$ for $|h|>q$.

$f(\lambda)=r(0)+\sum_{h=-q}^{q} r(h)e^{i\lambda h} = r(0)+\sum_{h=-q}^{q} r(h)[\cos(h\lambda)+i\sin(h\lambda)]$

$= r(0)+\sum_{h=-q}^{q} r(h)\cos(h\lambda) + r(h)\cdot i\sin(h\lambda)$

$= r(0)+\sum_{h=-q}^{q} r(h)\cos(h\lambda) + \sum_{h=0}^{q} i\,r(h)\underbrace{[\sin(h\lambda)+\sin(-h\lambda)]}_{=0}$

$= r(0)+\sum_{h=-q}^{q} r(h)\cos(h\lambda)$

$\frac{f(\lambda)}{r(0)} = 1+\sum_{h=-q}^{q}\rho(h)\cos(h\lambda)$

$= 1+2\sum_{k=1}^{q}\rho(k)\cos(\lambda k)$.

6.31 For AR(2) process $\phi(B)X_t = W_t$, where $\phi(x) = 1 + 2\phi x + \phi^2 x^2$.

$f_X(\lambda) = \sigma^2 \frac{1}{|\phi(e^{i\lambda})|^2}$

Suppose $Y_t = \psi(B) X_t$, then $f_Y(\lambda) = f_X(\lambda) |\psi(e^{i\lambda})|^2$

$= \sigma^2 \frac{|\psi(e^{i\lambda})|^2}{|\phi(e^{i\lambda})|^2}$, $\lambda \in [-\pi, \pi]$

$\Rightarrow \frac{|\psi(e^{i\lambda})|^2}{|\phi(e^{i\lambda})|^2}$ is constant for $\lambda \in [-\pi, \pi]$

Let $\psi = \phi$, then f_Y is constant. and $Y_t = \psi(B) X_t = \phi(B) X_t$

$= (1 + 2\phi B + \phi^2 B^2) X_t$

So the filter should be $Y_t = X_t + \phi^2 X_{t-2} + 2\phi X_{t-1}$ and $Y_t \sim WN(0, \sigma^2)$.